Characteristic Classes: Homework Set # 1.

Last quarter, we spent a fair amount of time discussing intersection numbers. In problems (1)-(4), you are asked to interpret various (co)-homological notions in terms of intersection theory. You may assume that M, N are oriented.

(1) Let $N^k \subset M^n$ be a closed submanifold, and $\sum a_i\sigma_i \in C_{n-k}(M)$ a singular chain (with coefficients in $\Lambda = \mathbb{Z}$ or $\Lambda = \mathbb{Z}_2$). Define a suitable notion of transversality for a chain with respect to a submanifold. Explain how, for a transverse chain of complementary dimension, one can extend the notion of intersection number to make sense of $I(N^k, \sum a_i\sigma_i) \in \Lambda$.

(2) Explain why the notion of intersection number defined above respects the boundary operator, i.e. if $\sum a_i\sigma_i \in C_{n-k}(M)$ and $\sum b_i\tau_i \in C_{n-k}(M)$ are homologous to each other, then $I(N^k, \sum a_i\sigma_i) = I(N^k, \sum b_i\tau_i)$.

(3) Explain how this allows us to think of such a k-dimensional closed, oriented submanifold $N^k \subset M^n$ as representing:

- a well-defined element in $H^{n-k}(M;\mathbb{Z}_2)$ in the case where $\Lambda = \mathbb{Z}_2$, or
- a well-defined element in $H^{n-k}(M;\mathbb{Z})/T^{n-k}$ in the case where $\Lambda = \mathbb{Z}$, and T^{n-k} denotes the torsion subgroup of the cohomology group.

(4) Alternatively, since N^k is closed orientable, we know that $H_k(N^k, \Lambda) \cong \Lambda$, and we let $\mu_N \in H_k(N^k, \Lambda)$ denote the fundamental class. Then the image of μ_N under the inclusion $N_k \hookrightarrow M^n$ gives a well defined homology class inside $H_k(M^n, \Lambda)$. In particular, we can think of such a submanifold $N^k \subset M^n$ as representing either (1) an element in $H_k(M)$ by the discussion above, or (2) an element in $H^{n-k}(M)$ by the previous exercise. Explain how the various products on (co)-homology (cup, cap, Kronecker) can be interpreted geometrically in terms of submanifold representatives for the (co)-homology classes.

Here are a few problems concerning the construction of vector bundles. Recall that a Riemannian metric on a vector bundle ξ is a smoothly varying family of positive definite inner products on the fibers of ξ.

(5) Given a submersion $f : M \to N$, show that one can construct a vector bundle κ_f out of the kernels of the linear maps $D_pf : T_pM \to T_{f(p)}N$. If M has a Riemannian metric show that $\tau_M = \kappa_f \oplus f^*\tau_N$.

(6) Given a subbundle $\xi \subset \eta$ of a vector bundle η, define the quotient bundle η/ξ. If η is equipped with a Riemannian metric, show that there is an isomorphism $\eta/\xi \cong \xi^\perp$.

(7) Given a bundle ξ, equipped with a Riemannian metric. Show that ξ is isomorphic to its dual bundle $\xi^* = Hom(\xi, \epsilon^1)$.

The next few problems are for you to get some practice working with Stiefel-Whitney classes.

(8) A manifold M^n is said to admit a field of tangent k-planes if the tangent bundle τ_M admits an $\mathbb{R}^k$-subbundle. Show that $\mathbb{R}P^n$ admits a field of tangent 1-planes if and only if n is odd. Show that $\mathbb{R}P^4$ and $\mathbb{R}P^6$ do *not* admit a field of tangent 2-planes.

(9) If a manifold M^n can be immersed in $\mathbb{R}^{n+1}$, show that the Stiefel-Whitney classes must be of the form

$w_k(\tau_M) = w_1(\tau_M)^k$ (for all k). Show that if $\mathbb{R}P^n$ can be immersed into $\mathbb{R}^{n+1}$, then n must be of the form $2^r - 1$ or $2^r - 2$.

(10) For a pair of vector bundles ξ, η, over possibly different base spaces, define the *product bundle* $\xi \times \eta$ to be the bundle with total space $E(\xi \times \eta) = E(\xi) \times E(\eta)$ and base space $B(\xi \times \eta) = B(\xi) \times B(\eta)$, with the obvious projection map. Recall that, with coefficients in $\mathbb{Z}_2$ and mild hypotheses on the factors (for example, if they are finite dimensional CW-complexes), the Künneth formula allows you to compute the cohomology of a product in terms of the cohomology of the factors, and gives the expected relationship:

$$H^*(X \times Y; \mathbb{Z}_2) \cong H^*(X; \mathbb{Z}_2) \otimes H^*(Y; \mathbb{Z}_2)$$

Note that this equation does not always hold with other coefficients: the failure of these two rings to coincide is measured by the Tor-functors.

Show that the total Stiefel-Whitney classes of the three bundles are related by the formula:

$$w(\xi \times \eta) = w(\xi) \otimes w(\eta)$$

(11) Show that the set $\mathfrak{N}_n$ consisting of all unoriented cobordism classes of smooth closed n-dimensional manifolds can be made into an abelian group. From the discussion in class, this *unoriented cobordism groups* $\mathfrak{N}_n$ is always finite. Show that the group $\mathfrak{N}_n$ is always of the form $(\mathbb{Z}_2)^k$ for some suitable k (which depends on n). Use the manifolds $\mathbb{R}P^2 \times \mathbb{R}P^2$ and $\mathbb{R}P^4$ to show that, in dimension $n = 4$ we have the lower bound $k \geq 2$ (for $\mathbb{R}P^2 \times \mathbb{R}P^2$ you will need to use the previous exercise). Show why, in dimension $n = 4$, we have that $k \leq 5$.

Thom computed the abelian groups $\mathfrak{N}_n$, and in the particular case of dimension $n = 4$, one does indeed have $\mathfrak{N}_4 = (\mathbb{Z}_2)^2$. In fact, the collection $\mathfrak{N}_*$ has the structure of a graded ring, with product given by Cartesian product of manifolds, and Thom actually determined the structure of the graded ring $\mathfrak{N}_*$: it is a free polynomial algebra over $\mathbb{Z}_2$, with very explicit generators (two of which are $\mathbb{R}P^2$ and $\mathbb{R}P^4$).