Characteristic Classes: Homework Set # 2.

The next few problems are designed to give you some practice working with Stiefel-Whitney numbers.

(1) For $f : X \to Y$ a continuous map, $c \in H^i(Y)$, and $\alpha \in H_i(X)$, show that one has an equality: $\langle f^* c, \alpha\rangle = \langle c, f_* \alpha\rangle$. Here we have that $f^* : H^*(Y) \to H^*(X)$ and $f_* : H_*(X) \to H_*(Y)$ are the maps induced by f on cohomology and homology respectively.

(2) Let M, N be a pair of oriented n-manifolds. Show that the degree of a map $f : M \to N$ can be computed as follows: if $\mu_M \in H_n(M)$ and $\mu_N \in H_n(N)$ are the generators of M, N given by the orientations, then we have $f_*\mu_M = deg(f) \cdot \mu_N$.

(3) For M, N a pair of manifolds, we say that $f : M \to N$ is *tangential* if $f^*\tau_N = \tau_M$. Show that if M, N are orientable n-manifolds representing different classes in the bordism groups $\mathcal{N}_n$, then any tangential map $f : M \to N$ must have **even** degree.

(4) Use Pontrjagin-Thom's result to show that a non-orientable n-manifold can never bound an $(n + 1)$-manifold. Show that every *oriented* 3-manifold M^3 bounds a 4-manifold. The later result was first demonstrated by Rohlin in 1951 (easy geometric proofs now also exist).

The next few problems focus on the Grassmann manifolds $G_n(\mathbb{R}^m)$ and the canonical associated $\mathbb{R}^n$-bundle γ^n_m.

(5) Show that for the Grassmann manifold $M = G_n(\mathbb{R}^m)$, we have a bundle isomorphism $\tau_M \cong Hom(\gamma^n_m, \gamma^\perp)$, where $\gamma^\perp$ is the orthogonal complement of γ^n_m in the trivial bundle ϵ^m.

(6) For a smooth imbedding $f : M \to \mathbb{R}^m$, show how the embedding can be used to construct a section of the bundle $Hom(\tau_M, Hom(\tau_M, \nu)) \cong Hom(\tau_M \otimes \tau_M, \nu)$, where ν is the normal bundle to the immersion f. This section is called the *second fundamental form* of M. [Hint: use the previous exercise.]

(7) For $X, Y \in G_n(\mathbb{R}^m)$, define $\angle(X, Y)$ to be the maximum, over all unit vectors $u \in X, v \in Y$, of the angle between u and v. Show that $\angle$ defines a metric on the topological space $G_n(\mathbb{R}^m)$, and that the "orthogonal complement" map $f : G_n(\mathbb{R}^{n+k}) \to G_k(\mathbb{R}^{n+k})$ given by $f(X) = X^\perp$ yields an isometry between the associated metric spaces.

(8) Let $i : G_n(\mathbb{R}^{n+k}) \to G_n(\mathbb{R}^\infty)$ denote the inclusion map. Show that for $k > p$, the induced map

$$i^* : H^p(G_n(\mathbb{R}^\infty); \Lambda) \to H^p(G_n(\mathbb{R}^{n+k}); \Lambda)$$

is an isomorphism on cohomology (with any coefficients). [Hint: recall that $G_n(\mathbb{R}^{n+k})$ is a subcomplex of the CW-complex $G_n(\mathbb{R}^\infty)$, and use cellular cohomology.]

(9) Show that the correspondence $f : X \to \mathbb{R}^1 \oplus X$ defines an embedding of the Grassmann manifold $G_n(\mathbb{R}^m)$ into $G_{n+1}(\mathbb{R} \oplus \mathbb{R}^m) = G_{n+1}(\mathbb{R}^{m+1})$, and that f is covered by a bundle map $\epsilon^1 \oplus \gamma^n_m \to \gamma^{n+1}_{m+1}$. Show that f carries the r-dimensional cell in $G_n(\mathbb{R}^m)$ associated to a given partition $i_1 \dots i_s$ of r onto the r-dimensional cell in $G_{n+1}(\mathbb{R}^{m+1})$ which corresponds to the same partition.

(10) Show that the *number* of distinct Stiefel-Whitney numbers $w_1^{r_1} \cdots w_n^{r_n}$ for an n-dimensional manifold is equal to $p(n)$. What does this tell you about the unoriented bordism group $\mathcal{N}_n$ in dimension n?