Characteristic Classes: Homework Set # 3.

(1) Let M^{4n+2} be a complex manifold, of real dimension $4n+2$, and let τ denote its tangent bundle, viewed as a complex vector bundle. Show that if $\tau \cong \bar{\tau}$, where $\bar{\tau}$ is the conjugate complex vector bundle, then M must have zero Euler characteristic.

(2) Assume that you are given a procedure $q(\cdot)$ for associating to any real vector bundle η a cohomology class $q(\eta) \in H^n(B(\eta); \mathbb{Z}_2)$, and assume that you know this procedure is natural (i.e. satisfies $q(f^*\xi) = f^*q(\xi)$ for every continuous map $f : X \to B(\xi)$ and every vector bundle ξ). Show that there exists a unique polynomial Q_k with the property that for every real vector bundle η^k with k-dimensional fibers:

$$q(\eta^k) = Q_k(w_1(\eta), \dots, w_n(\eta))$$

If in addition, the cohomology classes q are *stable* (i.e. satisfies $q(\eta \oplus \epsilon) = q(\eta)$ for every η, where ϵ is the trivial $\mathbb{R}$-bundle), then show that there exists a universal polynomial Q such that $q(\eta) = Q(w_1(\eta), \dots, w_n(\eta))$ holds independent of the dimension of η.

Formulate and prove analogous results for complex vector bundles and for oriented real vector bundles.

(3) In analogy to the construction of the Euler class, consider the following construction associated to a real vector bundle η. Let $u \in H^*(E, E_0; \mathbb{Z}_2)$ be the Thom class, and consider the image cohomology class $\phi(u)$, where ϕ is the composite map:

$$H^*(E, E_0; \mathbb{Z}_2) \to H^*(E; \mathbb{Z}_2) \to H^*(B; \mathbb{Z}_2)$$

where the first map is induced by the inclusion $(E, \emptyset) \subset (E, E_0)$, and the second map is $(\pi^*)^{-1}$ (and $\pi : E \to B$ is the canonical projection). Show that this cohomology class is natural. Express the cohomology class $\phi(u)$ in terms of the Stiefel-Whitney classes of the bundle η (see previous exercise).

(4) In analogy with the construction of the Chern classes, consider the following cohomology classes associated to a real vector bundle η, with fibers of dimension n. Set $v_i(\eta) = 0$ for $i > n$, and $v_n(\eta) := \phi(u) \in H^n(B; \mathbb{Z}_2)$. For $i < n$, inductively define $v_i(\eta) := (\pi_0^*)^{-1} v_i(\eta_0)$, where η_0 is a suitably defined real vector bundle with base space $E_0(\eta)$, and with fibers of dimension $n-1$. Show directly that the collection of cohomology classes v_i satisfy all the axioms of Stiefel-Whitney classes, and hence conclude that $v_i(\eta) = w_i(\eta)$ for all i, η (compare Exercise 2).

(5) Show that the change of coefficient homomorphism $H^*(B; \mathbb{Z}) \to H^*(B; \mathbb{Z}_2)$ maps the total Chern class $c(\omega)$ of a complex vector bundle to the total Stiefel-Whitney class $w(\omega_{\mathbb{R}})$ of the underlying real vector bundle.

(6) For η a real vector bundle, define $\bar{p}_i(\eta) \in H^{4i}(B; \mathbb{Z}_2)$ to be the image of the Pontrjagin classes of η under the change of coefficient morphism $H*(B; \mathbb{Z}) \to H^*(B; \mathbb{Z}_2)$. Show that the cohomology classes $\bar{p}_i$ are natural and stable (see exercise 2), and hence can be expressed as universal polynomials in the Stiefel-Whitney classes. Find the polynomials expressions for $\bar{p}_i$.

(7) Let $\Lambda = \mathbb{Z}[1/2]$. Show that, if $\bar{X} \to X$ is a double cover, with covering transformation $t : \bar{X} \to \bar{X}$, then $H^*(X; \Lambda)$ can be identified with the fixed set of the involution $t^* : H^*(\bar{X}; \Lambda) \to H^*(\bar{X}; \Lambda)$. As an application, compute the cohomology ring $H^*(G_n; \Lambda)$.

The next few problems give you a walkthrough of how to compute the Stiefel-Whitney classes for a tensor product of two real vector bundles. You might find it helpful to look through the proof from class of the Whitney formula for Chern classes.

(8) Assume ξ and η are real vector bundles over the same base space B, with one dimensional fibers, and consider the vector bundle $\xi \otimes \eta$. Note that the fibers of $\xi^1 \otimes \eta^1$ are also one dimensional, hence the only interesting characteristic classes for these bundles are $w_1(\eta), w_1(\xi), w_1(\eta \otimes \xi) \in H^1(B; \mathbb{Z}_2)$. Show that these cohomology classes satisfy the relationship: $w_1(\eta \otimes \xi) = w_1(\eta) + w_1(\xi)$.

(9) Now for ξ^n, η^m real vector bundles with fibers of dimension n, m respectively, defined over the same base space B, one can consider the tensor product bundle $\xi^n \otimes \eta^m$. Show that there exist universal polynomials $P_{n,m}$ (i.e. depending solely on the dimensions n, m) which compute the total Stiefel-Whitney class of the tensor product:

$$w(\xi^n \otimes \eta^m) = P_{n,m}\big(w_1(\xi^n), \ldots, w_n(\xi^n), w_1(\eta^1), \ldots, w_m(\eta^m)\big)$$

(10) Show that the polynomials $P_{n,m}$ can be characterized in the following manner. If $\sigma_1, \ldots, \sigma_n$ are the elementary symmetric functions of the indeterminates $t_1, \ldots, t_n$, and if $\sigma'_1, \ldots, \sigma'_m$ are the elementary symmetric functions of the indeterminates $t'_1, \ldots, t'_m$, then:

$$P_{n,m}(\sigma_1, \ldots, \sigma_n, \sigma'_1, \ldots, \sigma'_m) = \prod_{i=1}^{n} \prod_{j=1}^{m} (1 + t_i + t'_j)$$

[Hint: consider bundles that decompose as Whitney sums of one-dimensional bundles.]

(11) Use the previous problem to show that if ξ, η are a pair of real vector bundles, both having *even* dimensional fibers, then the tensor product bundle $\xi \otimes \eta$ is automatically orientable.