Differential Forms Homework set.

(1) Show that the wedge product $\wedge : \Omega^p(M) \otimes \Omega^q(M) \to \Omega^{p+q}(M)$ descends to a well-defined operation on cohomology. Show that, if M is oriented with $\partial M = \emptyset$, then integration of top-dimensional forms $\int_M : \Omega^n_c(M) \to \mathbb{R}$ descends to a well-defined operation on cohomology. What goes wrong if M is allowed to have boundary?

(2) Let $S^n(r) \subset \mathbb{R}^{n+1}$ be the n-dimensional sphere of radius r, and define the n-form:

$$\omega := \frac{1}{r}\sum_{i=1}^{n+1}(-1)^{i-1}x_i dx_1 \cdots \widehat{dx_i} \cdots dx_{n+1}$$

Compute $\int_{S^n} \omega$ where S^n is the unit sphere, and conclude that ω is not exact. Viewing the radius $r : \mathbb{R}^{n+1} \to \mathbb{R}$ as a function, verify that $dr \wedge \omega = dx_1 \cdots dx_{n+1}$. Use this to give an explicit representative for the generator $[1] \in H^n(S^n)$ for the top dimensional de Rham cohomology of the unit sphere S^n.

(3) In the proof of the Poincaré Lemma for compactly supported cohomology, we introduced a mapping $\pi_* : \Omega^*_c(M \times \mathbb{R}) \to \Omega^{*-1}_c(M)$, which was defined as follows:

$$\pi_* : (\pi^*\phi)f(x,t) \mapsto 0$$

$$\pi_* : (\pi^*\phi)f(x,t)dt \mapsto \phi\int_{-\infty}^{\infty} f(x,t)dt$$

where $\phi \in \Omega^*(M)$, and $f \in C^\infty_c(M \times \mathbb{R})$. Verify that π_* is a chain map, i.e. has the property that $d\pi_* = \pi_* d$, and hence that it does indeed descend to a well-defined map $\pi_* : H^*_c(M \times \mathbb{R}) \to H^{*-1}_c(M)$.

(4) The Künneth formula for compactly supported cohomology states that for manifolds M, N with finite good covers, one has:

$$H^*_c(M \times N) \cong H^*_c(M) \otimes H^*_c(N).$$

In the case where M, N are orientable, show that this can be deduced from the corresponding Künneth formula for de Rham cohomology. In the general case, use a Mayer-Vietoris argument to establish the Künneth formula.

(5) Recall that in dimension two, orientable manifolds are precisely the surfaces S_g of genus $g \geq 0$. The Euler characteristic of S_g is given by $\chi(S_g) = 2 - 2g$, and in particular is always *even.* As an application of Poincaré Duality, show that if M is a closed orientable manifold, of dimension $n = 4k + 2$ congruent to two mod four, then the Euler characteristic $\chi(M)$ has to be even.

(6) Let $U \subset \mathbb{R}^n$ be an open set, and define the *Hodge star operator* $* : \Omega^p(U) \to \Omega^{n-p}(U)$ by setting:

$$* : dx_{\pi(1)} \wedge \ldots dx_{\pi(p)} \mapsto Sign(\pi) \cdot dx_{\pi(p+1)} \wedge \ldots \wedge dx_{\pi(n)}$$

when $\pi \in S_n$ is an arbitrary permutation, and extending linearly. Verify that $* \circ * = (-1)^{n(n-p)}$. Now define the codifferential $\delta : \Omega^p(U) \to \Omega^{p-1}(U)$ by setting $\delta(\omega) = (-1)^{np+n-1}(* \circ d \circ *)(\omega)$. Verify that $\delta \circ \delta = 0$. For a differential form $\omega = f \cdot dx_{i_1} \wedge \ldots \wedge dx_{i_p}$, with $1 \leq i_1 < \ldots < i_p \leq n$, verify the following formula for $\delta(\omega)$:

$$\delta(\omega) = \sum_{\nu=1}^{p}(-1)^\nu \frac{\partial f}{\partial x_{i_\nu}} dx_{i_1} \wedge \ldots \wedge \widehat{dx_{i_\nu}} \wedge \ldots \wedge dx_{i_p}$$

(7) Continuing with the notation from the previous exercise, we now define the *Laplace operator* $\Delta : \Omega^p(U) \to \Omega^p(U)$ via the equation $\Delta = \delta \circ d + d \circ \delta$. Check that the Laplacian of a differential form $\omega = f dx_{i_1} \wedge \ldots \wedge dx_{i_p}$ satisfies:

$$\Delta(\omega) = -\big(\sum_{\nu=1}^{p} \frac{df}{dx_{i_\nu}^2}\big) dx_{i_1} \wedge \ldots \wedge dx_{i_p}$$

A p-form $\omega \in \Omega^p(U)$ is said to be *harmonic* provided that $\Delta(\omega) = 0$. Conclude that ω is harmonic if and only if $*(\omega)$ is harmonic.

Both the Hodge dual and the Laplacian can be defined on closed orientable (Riemannian) manifolds. In this setting, the harmonic forms provide some particularly useful representatives for cohomology classes. More precisely, one can consider the subcomplex $\mathcal{H}^*(M) \subset \Omega^*(M)$ consisting of harmonic forms. It is easy to check that d leaves the subcomplex $\mathcal{H}^*(M)$ invariant, and restricts to the zero map on $\mathcal{H}^*(M)$. This implies that there is a natural induced map $\mathcal{H}^p(M) \to H^p(M)$. The *Hodge theorem* asserts that this natural map is an isomorphism of vector spaces. In other words, every cohomology class can be represented by a harmonic form, and this representation is unique. The Hodge star then provides an isomorphism between the vector spaces $\mathcal{H}^p(M)$ and $\mathcal{H}^{n-p}(M)$ (a "dual" formulation of Poincaré duality).

The next few problems have to do with relative de Rham cohomology. For $N \subset M$ a submanifold, we form the relative de Rham complex by setting:

$$\Omega^q(M, N) := \Omega^q(M) \oplus \Omega^{q-1}(M)$$

equipped with the differential $d(\omega, \tau) = (d\omega, \omega|_N - d\tau)$ (check that $d^2 = 0$). The homology of this chain complex is called the *relative de Rham cohomology* of the pair (M, N). Note that if $N = \emptyset$, then one has $\Omega(M, \emptyset) = \Omega(M)$, and hence $H^*(M, \emptyset) = H^*(M)$. The obvious short exact sequence $0 \to \Omega^{*-1}(N) \to \Omega^*(M, N) \to \Omega^*(M) \to 0$ induces a long exact sequence in cohomology (see Bott-Tu, pgs 78-79 for details):

$$\ldots \to H^{i-1}(N) \to H^i(M, N) \to H^i(M) \to H^i(N) \to \ldots$$

Similarly, one can define the *compactly supported relative cohomology* $H_c^*(M, N)$ by using the complex $\Omega_c^*(M, N) = \Omega_c^*(M) \oplus \Omega_c^{*-1}(M)$.

(8) If $H^n \subset \mathbb{R}^n$ denotes the upper half space $x_n \geq 0$, and $\mathbb{R}^{n-1} \subset H^n$ denotes its boundary $x_n = 0$, compute (a) the cohomology $H^*(H^n)$, (b) the compactly supported cohomology $H_c^*(H^n)$, and (c) the relative cohomology $H^*(H^n, R^{n-1})$, and (d) the compactly supported relative cohomology $H_c^*(H^n, \mathbb{R}^{n-1})$.

(9) From your computations in (9), what seems to be the relationship between the relative cohomology groups of $(M, \partial M)$ and the cohomology groups of M? Prove your conjecture (this is the de Rham version of *Lefschetz duality*). [Warning: this is the most involved of the problems here. Depending on the approach you take, you might need to show various preliminary results, which could include establishing Mayer-Vietoris for relative cohomology, variations on the 5-lemma, integration pairings on relative cohomology, etc.]