Homework on systems of linear differential equations.

1) For each of the following systems of differential equations, find two distinct pairs of solutions u_1, v_1 and u_2, v_2. You should be able to do these using both of the methods discussed in class.

- $u' = u + v$ and $v' = u - 3v$.
- $u' = 2u - v$ and $v' = -2u + 3v$.
- $u' = -u + v$ and $v' = u - v$.
- $u' = -4u + 2v$ and $v' = u - 3v$.
- $u' = 5u + 3v$ and $v' = 2u + 4v$.

2) Assume that u_1, v_1 and u_2, v_2 are two distinct pairs of solutions to a system of differential equations of the form: $u' = au + bv$ and $v' = cu + dv$, with a, b, c, d constants. Show that:

- for any constant C, the pair of functions Cu_1, Cv_1 is also a solution to the same system of differential equations.
- the sums $u_1 + u_2$ and $v_1 + v_2$ also form a solution to the same system of differential equations.

These are analogues of the corresponding results for a *single* linear differential equation, where we know that multiples of solutions are solutions, and sums of solutions are solutions.

3) The first method discussed in class for solving a system of linear differential equations involved changing (1) the system of two first order linear differential equations (involving the functions u,v) into (2) a single second order linear differential equation (involving only the function u, for instance). It is also possible to *reverse* this process: consider a second order differential equation

$$ay'' + by' + cy = 0$$

with constant coefficients a, b, c.

- use a substitution $u = y$, $v = y'$ to rewrite this differential equation as a system of differential equations relating u, v and their first derivatives.
- write out the polynomial whose roots are the eigenvalues of the associated matrix. How does this polynomial relate to the characteristic polynomial of the original second order differential equation?

4) Recall that the motivation for studying systems of differential equations came from studying configurations of two weights attached to springs. The equations modeling such a configuration are actually *2nd order* systems of linear differential equations, i.e. they involve 2nd derivatives of the functions. Consider the 2nd order system: $u'' = 5u + 3v$, $v'' = 2u + 4v$.

- guessing for a solution of the form $u = Ae^{rt}$, $v = Be^{rt}$, write a matrix equation that must be satisfied by the numbers r, A, B.
- adapting the second method discussed in class, find *four* distinct solutions to the system of differential equations given above. The four distinct solutions should all have different exponents r. [Note: the matrix that comes up already appeared in the first problem, so you should already know its eigenvalues and eigenvectors from your earlier work.]