Differential Topology: Homework Set # 1.

1) Intrinsic definition of manifolds. In class, we gave two definitions of k-dimensional manifolds M^k: an extrinsic definition (viewing M^k as a subset of $\mathbb{R}^n$), and an intrinsic definition (a second countable, Hausdorff, topological space locally homeomorphic to $\mathbb{R}^k$, and satisfying a suitable transition law).

(i) Show that a manifold in the extrinsic sense is automatically a manifold in the intrinsic sense.

(ii) Give an example of a topological space which is not Hausdorff, but satisfies all the remaining conditions of the intrinsic definition of a manifold.

(iii) Give an example of a topological space which is not second countable, but satisfies all the remaining conditions of the intrinsic definition of a manifold.

2) Velocity vector definition of tangent space. Given a smooth manifold $M^k \subset \mathbb{R}^n$, a *parametrized smooth curve* in M is a smooth map $\gamma : \mathbb{R} \to \mathbb{R}^n$, whose image is entirely contained in M. Note that $T_0\mathbb{R} = \mathbb{R}$, and $T_{\gamma(0)}\mathbb{R}^n = \mathbb{R}^n$. Hence the differential (at the origin) of the smooth map γ can be viewed as a map $d\gamma_0 : \mathbb{R} \to \mathbb{R}^n$. The *velocity vector* of γ at the origin, denoted by $\dot{\gamma}(0)$ is defined to be $d\gamma_0(1) \in \mathbb{R}^n$.

(i) Given a point $x \in M$, show that every vector in T_xM is the velocity vector of some parametrized smooth curve in M.

(ii) Conversely, show that for any parametrized smooth curve γ, we have that $\dot{\gamma}(0)$ actually lies in $T_{\gamma(0)}M$.

Conclude that the tangent space T_xM can be identified with the space of velocity vectors of parametrized smooth curves γ in M satisfying $\gamma(0) = x$.

3) Product manifolds. Let $M^k \subset \mathbb{R}^n$ and $N^l \subset \mathbb{R}^m$ be a pair of smooth manifolds.

(i) Show that $M \times N \subset \mathbb{R}^n \times \mathbb{R}^m = \mathbb{R}^{n+m}$ is a $(k+l)$-dimensional smooth manifold.

(ii) Show that for $x \in M$, $y \in N$, we have a natural identification $T_{(x,y)}(M \times N) = T_xM \times T_yN$.

(iii) Let $f : M \times N \to M$ be the projection onto the first factor, i.e. $f : (x, y) \mapsto x$. Show that the corresponding differential
$$df_{(x,y)} : T_{(x,y)}(M \times N) = T_xM \times T_yN \to T_xM$$
is given by projection onto the first factor, i.e. $df_{(x,y)} : (u, v) \mapsto u$.

(iv) Fixing $y \in N$, we have an injection $g : M \hookrightarrow M \times N$ given by $g : x \mapsto (x, y)$. Show that the corresponding differential
$$dg_x : T_xM \to T_{(x,y)}(M \times N) = T_xM \times T_yN$$
is given by $dg_x : v \mapsto (v, 0)$.

(v) Given manifolds M, M', N, N', and smooth maps $f : M \to M'$ $g : N \to N'$, show that the obvious product map $f \times g : M \times N \to M' \times N'$ is smooth.

(vi) In the situation discussed in (v) above, show that the differential $d(f \times g)_{(x,y)}$ of the product map is given (with respect to the natural splittings of the tangent spaces) by the product map $df_x \times dg_y$.

4) Generalized Inverse Function Theorem. Let $f : M \to M'$ be a smooth map, which is one to one on a compact submanifold $N \subset M$. Suppose that for every point $x \in N$, we have that the differential

$$df_x : T_x M \to T_{f(x)} M'$$

is an isomorphism.

(i) Show that f maps N diffeomorphically to $f(N)$.

(ii) Show that if we have an *injective* local diffeomorphism $g : X \to Y$ between smooth manifolds, then g is a diffeomorphism of X onto $g(X)$.

(iii) Using (ii), show that the map f discussed above in fact maps an open neighborhood U of $N \subset M$ diffeomorphically onto an open neighborhood of $f(N) \subset M'$. [Hint: you just need to show that f is injective on some neighborhood of N].

(iv) Show that the requirement that N is compact *cannot* be removed, i.e. that there exist counterexamples to (iii) if N is allowed to be non-compact.

Observe that in the special case where N is a point, this reduces to the classical inverse function theorem discussed in class.