# Differential Topology: Homework Set # 2.

**1) Submersions and submanifolds.** Use submersions to establish that the following spaces are smooth manifolds. We use the notation $M_n(\mathbb{C})$ to denote $(n \times n)$-matrices with complex entries (which we can naturally identify with $\mathbb{C}^{n^2} \cong \mathbb{R}^{2n^2}$), $I_n \in M_n(\mathbb{C})$ to denote the identity matrix, and $A^T$ to denote the transpose of a matrix $A$.

(i) The subspace $SL_n(\mathbb{C}) \subset M_n(\mathbb{C})$, consisting of all matrices $A$ satisfying $\det(A) = 1$.

(ii) The subspace $SO_n(\mathbb{C}) \subset M_n(\mathbb{C})$, consisting of all matrices $A$ satisfying both $A^T A = I_n$ and $\det(A) = 1$.

(iii) The subspace $Sp_{2n}(\mathbb{C}) \subset M_n(\mathbb{C})$, consisting of all matrices $A$ satisfying the equation $A^T J A = J$, where $J$ is the $(2n \times 2n)$-matrix given by $J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$.

(iv) Show that the set of $(z_1, z_2, z_3, z_4, z_5) \subset \mathbb{C}^5 \equiv \mathbb{R}^{10}$ satisfying the pair of equations:

$$z_1^5 + z_2^3 + z_3^2 + z_4^2 + z_5^2 = 0$$

$$||z_1||^2 + ||z_2||^2 + ||z_3||^2 + ||z_4||^2 + ||z_5||^2 = \epsilon$$

forms a 7-dimensional submanifold in $\mathbb{R}^{10}$ (for $\epsilon > 0$).

The manifolds considered in (i)-(iii) are some of the most important families of complex Lie groups. In fact, the tangent spaces of these complex Lie groups exhaust *all but five* of the complex (non-trivial) simple Lie algebras. Example (iv) yields interesting manifolds: for $\epsilon$ close enough to zero, the resulting manifold is homeomorphic to $S^7$, but is *not* diffeomorphic to $S^7$. This is one of the simplest description of an *exotic sphere*.

**2) Transversality.** Recall that transversality is a condition on a pair of submanifolds which says that the tangent spaces of the submanifolds at each point of intersection span out the entire ambient tangent space. Infinitesimally, this says that the two submanifolds do not "line up along a lower dimensional subspace".

(i) For $N_1, N_2$ a pair of transversal submanifolds of $M$, show that for $p \in N_1 \cap N_2$, we have that $T_p(N_1 \cap N_2) = T_pN_1 \cap T_pN_2$.

(ii) Extend (i) by showing that if $f : N \to M$ is transverse to a submanifold $M' \subset M$, then the tangent space to the submanifold $N' := f^{-1}(M')$ at a point $p \in N'$ is given by $T_pN' = df_p^{-1}(T_{f(p)}M')$.

(iii) Given a pair of smooth maps $f : M_1 \to M_2$ and $g : M_2 \to M_3$, assume that $g$ is transversal to $N \subset M_3$. Show that $f$ is transversal to $g^{-1}(N)$ if and only if the composite $g \circ f$ is transversal to $N$.

**3) Homotopy.** Recall that a pair of smooth maps $f_0, f_1 : M^k \to N^l$ are said to be *smoothly homotopic* provided that there exists a smooth map (called a smooth homotopy) $F : [0,1] \times M^k \to N^l$ having the property that $F|_{\{0\} \times M^k} \equiv f_0$ and $F|_{\{1\} \times M^k} \equiv f_1$. We denote by $f_t : M^k \to N^l$ the map $F|_{\{t\} \times M^k}$. We write $f \sim g$ if $f, g$ are smoothly homotopic maps. Smooth homotopy is a precise formulation of the intuitive notion of "smoothly deforming" one map into another.

(i) Show that if $f_0, f_1$ are smoothly homotopic maps, then there is a smooth homotopy $F$ satisfying the additional property that the associated $f_t$ satisfy $f_t \equiv f_0$ for $0 \leq t \leq 1/3$ and $f_t \equiv f_1$ for $2/3 \leq t \leq 1$.

(ii) Show that $\sim$ is a transitive relation: i.e. if $f \sim g$ and $g \sim h$, then $f \sim h$.

(iii) Show that if $M$ is a connected manifold, then $M$ is *arcwise connected*: given any two points $p, q \in M$, there exists a smooth curve $\gamma : [0,1] \to M$ satisfying $\gamma(0) = p$, $\gamma(1) = q$.

(iv) A manifold $M$ is *contractible* if the identity map on $M$ is smoothly homotopic to a point map $M \to p$ for some $p \in M$. Show that if $M$ is contractible, then any two maps $f_0, f_1$ from an arbitrary manifold $N$ into $M$ are automatically smoothly homotopic.

(v) A manifold $M$ is *simply connected* if any pair of smooth maps $f_0, f_1 : S^1 \to M$ are smoothly homotopic. Show that any contractible manifold is simply connected.

(vi) Show that for $k > 1$, the sphere $S^k$ is simply connected. [Hint: use Sard's theorem and stereographic projection.]

**4) Stability of Morse functions.** Show that Morse functions on compact manifolds are stable, by establishing the following:

(i) Let $f$ be a smooth function on an open set $U \subset \mathbb{R}^n$. For each $x \in U$, denote by $H(x)$ the Hessian of $f$ at the point $x$. Prove that $f$ is Morse if and only if the following inequality holds for all $x \in U$:

$$0 < det(H)^2 + \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i}\right)^2$$

(ii) Use (i) to show that if $f_t$ is a smoothly homotopic family of functions on $\mathbb{R}^n$, and $f_0$ is Morse on a neighborhood of a compact set $K \subset \mathbb{R}^n$, then $f_t$ is Morse on $K$ for $t$ sufficiently close to 0.

(iii) Use (ii) to show that if $f$ is a Morse function on a compact manifold $M$, and $f_t$ is a smooth homotopy of $f$, then $f_t$ is Morse for $t$ sufficiently close to 0.

(iv) Where is compactness of $M$ used? What would you need to assume about the Morse function in order to obtain an analogous statement for non-compact $M$?