Differential Topology: Homework Set # 4.

1) Manifolds with boundary. This exercise is to help you get some practice working with manifolds with boundar.

(i) If $U \subset \mathbb{R}^k$ and $V \subset H^k$ are neighborhoods of 0, show that U and V are *not* diffeomorphic to each other.

(ii) Show that the square $S = [0,1] \times [0,1]$ is *not* a smooth manifold with boundary.

(iii) Suppose that M is a manifold with boundary, and $p \in \partial M$. Given a smooth parametrization $\phi : U \to M$ ($U \subset \mathbb{R}^k$) with $\phi(0) = p$, define the *upper half space* $H_pM \subset T_pM$ to be the image of $H^k \subset \mathbb{R}^k$ under $d\phi_0 : \mathbb{R}^k \cong T_0\mathbb{R}^k \to T_pM$. Show that this subset of T_pM is well defined. Tangent vectors in H_pM will be called *inward pointing*, while vectors in $T_pM - H_pM$ will be called *outward pointing*.

(iv) For M a manifold with boundary, and $p \in \partial M$, show that there are precisely two unit vectors in T_pM which are perpendicular to $T_p(\partial M)$ (with respect to the inner product on the ambient $\mathbb{R}^n$), with one contained in H_pM, and the other contained in $T_pM - H_pM$. The one in H_pM is called the *inward unit normal vector* to the boundary, and the other one is called the *outward unit normal vector*.

(v) For $M \subset \mathbb{R}^n$ a manifold with boundary, and $p \in \partial M$, let $\vec{n}_p \in T_pM \subset T_p\mathbb{R}^n \cong \mathbb{R}^n$ be the outward unit normal vector. This defines a map from ∂M to $\mathbb{R}^n$, given by $p \mapsto \vec{n}_p$. Show that this map is smooth.

(vi) For M a manifold with boundary, and $p \in \partial M$, show that there exists a neighborhood U of p, and a smooth function $f : U \to \mathbb{R}$, with the property that (a) $f(x) = 0$ if and only if $x \in \partial U$, and (b) for points $x \in \partial U$, $df_x(\vec{n}_x) > 0$.

2) Variations on Retraction theorem and Brouwer fixed point theorem. Recall that we established in class the smooth version of the Brouwer fixed point theorem: every smooth map $f : \mathbb{D}^n \to \mathbb{D}^n$ has a fixed point. The argument relied on the retraction theorem: there are no retractions from a compact manifold with boundary to its boundary. This exercise has you consider related notions.

(i) Give an example of a non-compact manifold M with non-empty boundary, and a retraction $r : M \to \partial M$.

(ii) For M a manifold with non-empty boundary, assume that V is a smooth vector field on M, with the property that V is non-zero and *outward pointing* at every boundary point $p \in \partial M$. Show that V must vanish at some point $q \in Int(M)$. [Hint: a smooth vector field locally gives rise to a particularly nice system of differential equations.]

(iii) Show that in the Brouwer fixed point theorem, the fixed point might not be an interior point. Conclude that the analogous fixed point theorem *fails* for smooth maps $f : Int(\mathbb{D}^n) \to Int(\mathbb{D}^n)$.

(iv) The Weierstrass approximation theorem states that given a *continuous* map $f : \mathbb{D}^n \to \mathbb{D}^n$, one can find for each $\epsilon > 0$ a *polynomial map* $p : \mathbb{R}^n \to \mathbb{R}^n$ such that $|f(x) - p(x)| < \epsilon$ for every $x \in \mathbb{D}^n$. Show that the polynomial p can be chosen to additionally satisfy the property that $p(\mathbb{D}^n) \subset \mathbb{D}^n$.

(v) Use (iii) to establish the continuous version of the Brouwer fixed point theorem: every continuous map $f : \mathbb{D}^n \to \mathbb{D}^n$ has a fixed point. [Hint: replace f by a suitable smooth map F]

Parts (iv) and (v) above give an alternate route to the continuous Brouwer fixed point theorem. The standard proof of this result goes via algebraic topology. Yet another proof makes use instead of Sperner's Lemma from combinatorics (a particularly slick argument, if you're interested in combinatorics).

3) Applications of Brouwer's fixed point theorem. This exercise will have you look at various applications of Brouwer's fixed point theorem.

(i) Show that if a manifold M *without boundary* supports a non-vanishing vector field, then it supports a self-map with no fixed points.

(ii) Show that if a manifold M (possibly with boundary) is diffeomorphic to $S^1 \times N$, then M supports self-maps with no fixed points. Conclude that $\mathbb{D}^n$ does *not* split as a product $S^1 \times N^{n-1}$, where N^{n-1} is an $(n-1)$-dimensional manifold with boundary.

(iii) Use the Brouwer fixed point theorem to establish the following result of Frobenius: if A is an $(n \times n)$-matrix with real, non-negative entries, then A has a real non-negative eigenvalue. [Hint: use A to define a suitable self-map $f : S^{n-1} \to S^{n-1}$ having the property that f maps the "first quadrant" of the sphere into itself, and apply problem #1(v) .]

4) Fixed point theorems in normed vector spaces. Brouwer's theorem ensures fixed points for self-maps $f : \mathbb{D}^n \to \mathbb{D}^n$. For applications to functional analysis, it is important to consider self-maps on more general spaces. We say a space X has the *fixed point property* if every self map of X has a fixed point.

For V be a (real) vector space, and $F = \{v_1, \ldots, v_r\} \subset V$ a finite subset, define the *convex hull* of F to be:

$$Conv(F) := \{w \in V \mid w = \sum t_i v_i, \text{where } t_i \geq 0, t_1 + \cdots t_r = 1\}$$

We say that a subset $X \subset V$ is convex if for every $p, q \in X$, we have $Conv(\{p, q\} \subset X$. In this exercise, V will be a vector space equipped with a *norm*, and we will be interested in the fixed point property for convex subsets of V.

(i) Show that if X is a compact, convex subset of a *finite* dimensional Hilbert space, then X has the fixed point property.

(ii) Show that if X is the unit ball in infinite dimensional Hilbert space l_2, then X is convex, but there exist self maps $\sigma : X \to X$ with no fixed points. Show that X is *not* compact, by exhibiting a sequence in X with no convergent subsequence.

(iii) Given a *compact* subset $K \subset V$ of the normed vector space V, show that K is "almost finite dimensional", in the following sense: given $\epsilon > 0$, there exists a finite subset $F \subset K$, and a map $P : K \to Conv(F)$ satisfying $d(P(x), x) < \epsilon$ for all $x \in K$.

[Hint: take $F = \{v_1, \ldots v_r\}$ to be a finite ϵ-net in K, define ϕ_i to be the linear ϵ-bump function centered at v_i, set $P(x) = \left[\sum \phi_i(x) v_i\right] / \sum \phi_i(x)$, and check that this has the desired properties.]

(iv) Show that if $X \subset V$ is a closed, convex subset, and $f : X \to X$ has the property that $f(X)$ has compact closure, then f has a fixed point. Conclude that compact convex subsets of normed real vector spaces have the fixed point property.

[Hint: Use (iii) above to reduce to the finite dimensional case.]

This gives one of the extensions of Brouwer's fixed point theorem, originally due to Schauder. This result is important, as it can be used to give a (topological!) proof of the classic Cauchy-Peano theorem (the most elementary existence result on solutions to differential equations).

Other variations on Brouwer's theorem include Kakutani's work on set valued mappings (which was the cornerstone for Nash proof of a "Nash equilibrium" in economics), Michael's work on continuous selections (an important result at the intersection of set theoretic topology and geometric topology), and Lefschetz's fixed point theorem (a smooth version of which we will discuss in a few weeks).