Differential Topology: Homework Set # 5.

1) Intersection and winding mod 2.

(i) Assume $f : N \to M$ between compact manifolds is homotopic to a constant map, and $W \subset M$ is a closed submanifold satisfying $\dim(N) + \dim(W) = \dim(M)$. Show that if $\dim(N) > 0$, then one automatically has $I_2(f, W) = 0$. If N is a point, for which W will $I_2(f, W) \neq 0$?

(ii) Show that mod 2 intersection theory is vacuous in contractible manifolds: if M is contractible, and $\dim(M) > 0$, then $I_2(f, W) = 0$ for every map $f : N \to M$, where N is compact, W is closed, and $\dim(N) + \dim(W) = \dim(M)$.

(iii) Prove that if M is a closed, contractible manifold, then M is just a point. [Hint: use part (ii).]

(iv) Consider the function $F : \mathbb{C} \to \mathbb{C}$ defined by $F(z) = z^9 + \sin(||z||^2)[1 - 4z^3]$. Show that for each complex number $w \in \mathbb{C}$, the equation $F(z) = w$ has a solution.

2) Borsuk-Ulam theorem.

(i) Given a smooth map $f : S^1 \to S^1$, show that there exists a smooth map $g : \mathbb{R} \to \mathbb{R}$ with the property that $f(cos(t), sin(t)) = \big(\cos(g(t)), \sin(g(t))\big)$, and $g(2\pi) = g(0) + 2\pi \cdot q$ for some integer q. Show that the map g allows us to compute the degree of f as follows: $\deg_2(f) = q \mod 2$.

(ii) Show that if $f : S^1 \to S^1$ maps antipodal points to antipodal points, then $\deg_2(f) = 1$. [Hint: use the expression from (i) to calculate the degree.]

(iii) Show that the Borsuk-Ulam theorem is equivalent to the following statement: if $f : S^k \to S^k$ maps antipodal points to antipodal points, then $\deg_2(f) = 1$.

(iv) Let $p_1, \dots, p_n$ be real homogenous polynomials in $n + 1$ variables, all of odd order. Prove that the associated functions on $\mathbb{R}^{n+1}$ simultaneously vanish along some line through the origin.

3) Linking numbers of submanifolds. This problem is designed to help you test your knowledge of intersection theory, by analyzing the useful notion of linking. Given a pair of pairwise disjoint compact oriented manifolds $M^m, N^n \subset \mathbb{R}^k$ satisfying $m + n = k - 1$, $\partial M = \partial N = \emptyset$, form the *linking map* $M \times N \to \mathbb{R}^k$ defined by $(p, q) \mapsto \frac{p-q}{||p-q||} \in S^{k-1}$. Define the linking number $Lk(M, N)$ to be the degree of the linking map.

(i) Show that $Lk(M, N) = (-1)^{(m-1)(n-1)} Lk(N, M)$.

(ii) If M is homotopic to a point in $\mathbb{R}^k - N$, or if M bounds an oriented compact manifold in $\mathbb{R}^k - N$, then show that $Lk(M, N) = 0$.

(iii) Consider the submanifold with boundary $A_k \subset \mathbb{R}^3$ obtained by embedding the cylinder $I \times S^1$ into $\mathbb{R}^3$ with k-twists. Show that if $\gamma, \gamma' \subset \mathbb{R}^3$ are the two corresponding boundary curves with induced orientations, then $Lk(\gamma, \gamma') = k$.

(iv) Show that if A_k, A_l are as in (iii), and $|k| \neq |l|$, then *there is no diffeomorphism* $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ *taking* A_k *to* A_l.

(v) Show that if $M^n \subset \mathbb{R}^{n+1}$ is a compact submanifold with $\partial M = \emptyset$, and $p, q \in \mathbb{R}^{n+1} - M^n$ are an arbitrary pair of points, then p and q lie in the same component of $\mathbb{R}^{n+1} - M^n$ if and only if $Lk(\{p, q\}, M) = 0$.

4) Hopf invariant. The notion of linking number was used by Hopf to study maps from S^3 to S^2. Given such a smooth map, let $p \neq q \in S^2$ be a pair of distinct regular values for the map f. Then we know that $f^{-1}(p), f^{-1}(q)$ are a pair of disjoint, compact, oriented, 1-dimensional submanifolds of S^3. Taking $r \notin S^3 - (f^{-1}(p) \cup f^{-1}(q))$, we can use stereographic projection from r to view the curves $f^{-1}(p), f^{-1}(q)$ as lying in $\mathbb{R}^3$, and hence we can compute their linking number. We define the Hopf invariant of f to be the integer $H(f) := Lk(f^{-1}(p), f^{-1}(q))$.

(i) There are a number of choices appearing in our definition: the choice of regular points a, b, and the choice of point r. Show that the integer $H(f)$ is independent of all these choices.

(ii) Show that if $f_0, f_1 : S^3 \to S^2$ are smoothly homotopic, then $H(f_0) = H(f_1)$. Show that constant maps have Hopf invariant zero.

(iii) Show that if $g : S^3 \to S^3$ has degree p, then for any map $f : S^3 \to S^2$, we have that $H(f \circ g) = p \cdot H(f)$.

(iv) Show that if $h : S^2 \to S^2$ has degree q, then for any map $f : S^3 \to S^2$, we have that $H(h \circ f) = q^2 \cdot H(f)$

(v) Show that if $S^3 \subset \mathbb{C}^2$ is the unit sphere, and one identifies complex projective space $\mathbb{C}P^1$ with S^2 in the usual manner, then the *Hopf map* $f : S^3 \to S^2$ given by $(w, z) \mapsto [w, z] \in \mathbb{C}P^1 \equiv S^2$ has Hopf invariant one.

(vi) Using the previous parts of this exercise, show that one can associate to each integer $n \in \mathbb{Z}$, a map $f_n : S^3 \to S^2$, with the property that for $n \neq m$, the maps f_n and f_m are not homotopic to each other.

These ideas of Hopf were a major breakthrough when they appeared in 1930. They provided the very first instance where some non-trivial information was obtained about homotopy classes of maps from a higher dimensional manifold to a lower dimensional one. It was subsequently shown (as suggested by (vi) above) that $\pi_3(S^2) \cong \mathbb{Z}$, i.e. that the Hopf invariant classifies maps $S^3 \to S^2$ up to homotopy. A similar construction can be applied to maps from $S^{2n-1} \to S^n$ (so that one has the correct dimensions to define linking). It is relatively easy to see that the Hopf invariant of maps to odd higher-dimensional spheres is zero (why?). In the case where $n = 2k$ is even, things get interesting: Hopf managed to show in every even dimension the existence of maps $S^{4k-1} \to S^{2k}$ with non-zero Hopf invariant. But he noticed that, except in dimensions $2, 4$, and 8, the maps he constructed always had an *even* Hopf invariant. This led to the famous *Hopf invariant one* problem: conjecturally, those are the only dimensions in which there exist a map with Hopf invariant one (along with the exceptional odd-dimensional case $n = 1$). This was eventually resolved in a major paper by Adams in 1960, exploiting the structure of higher cohomology operations. A considerably simpler proof was given by Adams and Atiyah in 1966, using methods from the relatively new field of topological K-theory.