Rigidity of Almost-Isometric Universal Covers

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ABSTRACT. Almost-isometries are quasi-isometries with multiplicative constant one. Lifting a pair of metrics on a compact space gives quasi-isometric metrics on the universal cover. Under some additional hypotheses on the metrics, we show that there is no almost-isometry between the universal covers. We also show that Riemannian manifolds that are almost-isometric have the same volume growth entropy. We then establish various rigidity results as applications.

1. INTRODUCTION

Quasi-isometries are the natural morphisms in asymptotic geometry. Their definition involves both an additive constant ≥ 0 and a multiplicative constant ≥ 1. *Bi-Lipschitz maps* are quasi-isometries with additive constant equal to zero; we define *almost-isometries* as quasi-isometries with multiplicative constant equal to one. When looking at a general inequality, it is often important that we understand the equality case. Thus, a natural problem is to identify conditions that force quasi-isometric spaces to be either bi-Lipschitz equivalent or almost isometric.

For discrete spaces, bi-Lipschitz maps coincide with *bijective* quasi-isometries. In [Wh], quasi-isometry classes of maps that contain a bijective quasi-isometry are characterized for uniformly discrete metric spaces of bounded geometry. In particular, quasi-isometric finitely generated groups that are nonamenable are bi-Lipschitz equivalent. In contrast, there exist separated nets in $\mathbb{R}^2$ that are quasi-isometric but not bi-Lipschitz equivalent to $\mathbb{Z}^2$ [BK, Mc]. Quasi-isometric finitely generated groups that are not bi-Lipschitz equivalent first appeared in [Dy]. While those first examples were not finitely presented, examples of type F_n for each n appear in [DPT].

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Existing results about almost-isometries primarily concern equivalence classes of metrics on a fixed space, where two metrics are equivalent when the *identity map* is an almost-isometry. For instance, pairs of $\mathbb{Z}^n$-equivariant metrics on $\mathbb{R}^n$ whose ratio tends to one as distances tend to infinity are equivalent by [Bu]. Analogous results hold for metrics periodic under Gromov hyperbolic and Heisenberg groups [Kr] or under toral relatively hyperbolic groups [Fuj]. The equivalence classes of left-invariant metrics on non-elementary Gromov hyperbolic groups are studied in [Fur] where the *Marked Length Spectrum (MLS) Rigidity Conjecture* is reformulated as follows.

Given negatively curved Riemannian metrics g_0 and g_1 on a compact manifold M, the identity map $(\tilde{M}, \tilde{g}_0) \to (\tilde{M}, \tilde{g}_1)$ between the universal Riemannian coverings is an almost-isometry if and only if $(\tilde{M}, \tilde{g}_0)$ and $(\tilde{M}, \tilde{g}_1)$ are isometric.

In view of the resolution of the MLS Conjecture in dimension two [Cr, Ot], and its expected validity in higher dimensions, the following question is quite natural.

Question 1.1. *If g_0 and g_1 are two negatively curved Riemannian metrics on a compact manifold M, can the Riemannian universal coverings $(\tilde{M}, \tilde{g}_0)$ and $(\tilde{M}, \tilde{g}_1)$ be almost isometric without being isometric?*

In the above question, non-identity and non-equivariant almost-isometries are allowed (thus generalizing Furman's reformulation of the MLS conjecture). Our focus in this paper is to show that under suitable rigidity hypotheses on the metrics g_i, the answer is "no"—though in general, the answer is "yes" (see [LSvL] for some two-dimensional examples).

Theorem 1.2. *Let G be a group acting geometrically on a proper* $\mathrm{CAT}(-1)$ *space X (distinct from $\mathbb{R}$), and let Y be another proper* $\mathrm{CAT}(-1)$ *space having the geodesic extension property and connected spaces of directions. Assume the following:*

- *Y is almost-isometrically rigid.*
- *The G-action on X is marked length spectrum rigid.*

Then, X and Y are almost isometric if and only if there is a coarsely onto isometric embedding of X into Y.

The rigidity properties required of the space Y and the G-space X are defined in Section 2. Note that we are not assuming a G-action on Y (in particular, there is no equivariance assumption on the almost-isometry). As a concrete application of the methods behind Theorem 1.2, we mention the following.

Corollary 1.3. *Let (M, g_0) be a closed locally symmetric space modeled on a quaternionic hyperbolic space, or on the Cayley hyperbolic plane, and let g_1 be a negatively curved Riemannian metric on M. Then, $(\tilde{M}, \tilde{g}_0)$ and $(\tilde{M}, \tilde{g}_1)$ are almost isometric if and only if (M, g_0) and (M, g_1) are isometric.*

The ideas behind Theorem 1.2 also yield some rigidity results for Fuchsian buildings (see Corollary 3.9). After discussing some preliminaries in Section 2, we prove Theorem 1.2 and its corollaries in Section 3.

In Section 4, we relate the presence of almost-isometries with dynamical invariants. Recall that the *upper volume entropy* of a complete Riemannian manifold X is defined to be

$$h_{\mathrm{vol}}^{+}(X) := \limsup_{r\to\infty} \frac{\ln \mathrm{Vol}(B_p(r))}{r},$$

where $B_p(r)$ denotes the ball of radius r centered at a chosen fixed basepoint p. Similarly, the *lower volume entropy* is defined to be

$$h_{\mathrm{vol}}^{-}(X) := \liminf_{r\to\infty} \frac{\ln \mathrm{Vol}(B_p(r))}{r}.$$

These quantities are independent of the chosen point $p \in X$, and in the case where X is a Riemannian cover of a compact manifold, one has that $h_{\mathrm{vol}}^{+}(X) = h_{\mathrm{vol}}^{-}(X)$ (see [Ma]); this common value is then called the *volume entropy* of X, and is denoted $h_{\mathrm{vol}}(X)$. In general, the upper and lower volume entropies can differ, even for Riemannian covers of finite volume manifolds [Na].

Theorem 1.4. *Let M_1, M_2 be complete Riemannian manifolds having bounded sectional curvatures. If M_1 is almost isometric to M_2, then $h_{\mathrm{vol}}^{+}(M_1) = h_{\mathrm{vol}}^{+}(M_2)$, and $h_{\mathrm{vol}}^{-}(M_1) = h_{\mathrm{vol}}^{-}(M_2)$. In particular, if the M_i are Riemannian covers of compact manifolds, then $h_{\mathrm{vol}}(M_1) = h_{\mathrm{vol}}(M_2)$.*

In fact, the proof of Theorem 1.4 only uses the property that r-balls in M_i have volume uniformly bounded above and below by positive constants. This property is a consequence of having bounded sectional curvatures by [Bi, Gün].

Corollary 1.5. *Let (M, g) be a Riemannian cover of a compact manifold. If $h_{\mathrm{vol}}(M) > 0$, then for any positive $\lambda \neq 1$, the manifolds (M, g) and $(M, \lambda g)$ are* not *almost-isometric.*

For a closed Riemannian manifold (M, g), the volume growth entropy h_{vol} of its universal covering and the topological entropy h_{top} of its geodesic flow satisfy $h_{\mathrm{vol}} \leq h_{\mathrm{top}}$ [Ma]. Equality holds for metrics without conjugate points [FM], a class of metrics including the nonpositively curved metrics, but is in general strictly larger [Gu].

Corollary 1.6. *Let g_0 and g_1 be Riemannian metrics on a closed manifold M. If the universal coverings $(\tilde{M}, \tilde{g}_0)$ and $(\tilde{M}, \tilde{g}_1)$ are almost isometric, then $h_{\mathrm{top}}(g_0) = h_{\mathrm{top}}(g_1)$.*

As a final application of Theorem 1.4, we mention the following result.

Corollary 1.7. *Let M be a closed n-manifold equipped with Riemannian metrics g_0 and g_1 for which the universal coverings $(\tilde{M}, \tilde{g}_0)$ and $(\tilde{M}, \tilde{g}_1)$ are almost-isometric. Further, assume that the metrics satisfy any of the following conditions:*

(1) *$n = 2$, g_0 is a flat metric, and g_1 is arbitrary.*
(2) *$n = 2$, g_0 is a real hyperbolic metric, and g_1 satisfies* $\mathrm{Vol}(g_0) \geq \mathrm{Vol}(g_1)$.

(3) *$n \geq 3$, g_0 is a negatively curved locally symmetric metric, and g_1 satisfies* $\mathrm{Vol}(g_0) \geq \mathrm{Vol}(g_1)$.
(4) *$n \geq 5$, g_0 is an irreducible, higher rank, nonpositively curved locally symmetric metric, and g_1 is conformal to g_0 and satisfies* $\mathrm{Vol}(g_0) \geq \mathrm{Vol}(g_1)$.
(5) *$n \geq 6$, g_0 is a locally symmetric metric modeled on a product of negatively curved symmetric spaces of dimension ≥ 3 (suitably normalized), and g_1 is any metric satisfying* $\mathrm{Vol}(g_0) \geq \mathrm{Vol}(g_1)$.

Then, the universal covers $(\tilde{M}, \tilde{g}_0)$ and $(\tilde{M}, \tilde{g}_1)$ are isometric. In particular, (M, g_0) is isometric to (M, g_1) by Mostow rigidity in cases (3)–(5).

While the rigidity results Corollary 1.3 and Corollary 1.7(3) both apply to locally symmetric metrics g_0 modeled on quaternionic hyperbolic space or on the Cayley hyperbolic plane, the former requires the metric g_1 to be negatively curved, while the latter requires g_1 to have volume majorized by that of g_0.

A discussion of rigidity results for almost-isometries between metric trees (Theorem 4.3 and Corollary 4.4) appears at the end of Section 4. Section 5 concludes the paper with some remarks and open questions.

2. PRELIMINARIES

Throughout, parentheses are suppressed according to the following notational convention. Given a function $\varphi : X \to Y$ between sets and an element $x \in X$, the image $\varphi(x) \in Y$ is frequently denoted by φx. Similarly, if $\psi : Y \to Z$ is a function, the composite function $\psi \circ \varphi : X \to Z$ is frequently denoted by $\psi\varphi$.

Quasi-isometries and almost-isometries. This subsection reviews basics concerning quasi-isometries.

Let (X, d_X) and (Y, d_Y) be metric spaces. Given constants $K \geq 1$ and $C \geq 0$, a map $\varphi : X \to Y$ is (K,C)-*quasi-isometric* if, for every $x_1, x_2 \in X$,

$$\frac{1}{K} d_X(x_1, x_2) - C \leq d_Y(\varphi x_1, \varphi x_2) \leq K d_X(x_1, x_2) + C. \tag{2.1}$$

The map $\varphi : X \to Y$ is C-*coarsely onto* if, for each $y \in Y$, there exists $x \in X$ with

$$d_Y(\varphi x, y) \leq C. \tag{2.2}$$

Note that the map $\varphi : X \to Y$ is a (K,C)-*quasi-isometry* when it is both (K,C)-quasi-isometric and C-coarsely onto. The spaces X and Y are *quasi-isometric* when such a quasi-isometry exists. Note that when $C = 0$, quasi-isometries reduce to bi-Lipschitz maps.

A *coarse inverse* to a (K,C)-quasi-isometric map $\varphi : X \to Y$ is a (K,C)-quasi-isometric map $\psi : Y \to X$ satisfying

$$d_X((\psi\varphi)x, x) \leq C d_Y((\varphi\psi)y, y) \leq C \tag{2.3}$$

for every $x \in X$ and for every $y \in Y$. If $\varphi : X \to Y$ is (K,C)-quasi-isometric with a coarse inverse $\psi : Y \to X$, then (2.3) implies (2.2) for both φ and ψ so that they are both (K,C)-quasi-isometries.

Conversely, given a (K,C)-quasi-isometry $\varphi : X \to Y$, one uses (2.2) to define a map $\psi : Y \to X$ satisfying

$$d_Y((\varphi\psi)y, y) \leq C \tag{2.4}$$

for each $y \in Y$. The triangle inequality, (2.1), and (2.4) imply that, for each $y_1, y_2 \in Y$ and $x \in X$,

$$\frac{1}{K}d_Y(y_1,y_2) - \frac{3C}{K} \leq d_X(\psi y_1, \psi y_2) \leq K d_Y(y_1,y_2) + 3KC,$$
$$d_X((\psi\varphi)x, x) \leq 2KC.$$

Therefore, $\psi : Y \to X$ is $(K, 3KC)$-quasi-isometric, and if we let $\bar{C} = 3KC$, both φ and ψ are $(K,\bar{C})$-quasi-isometric and coarse inverses of each other.

Define two maps $f, g : X \to Y$ to be equivalent if $\sup_{x\in X} d_Y(fx, gx) < \infty$, and denote this equivalence relation by $f \sim g$. The discussion in the previous paragraph is summarized by the following: given any quasi-isometry $\varphi : X \to Y$, there exists a quasi-isometry $\psi : Y \to X$ with the property that $\varphi\psi \sim \mathrm{Id}_X$ and $\psi\varphi \sim \mathrm{Id}_Y$. Equivalence classes of self-quasi-isometries of X form a group, denoted by $\mathrm{QI}(X)$. Quasi-isometries $\varphi : X \to Y$ between spaces induce isomorphisms $\mathrm{QI}(X) \cong \mathrm{QI}(Y)$.

In the special case when $K = 1$, we replace everywhere the adjective *quasi-* with *almost-*. In particular, the discussion above yields the following result.

Lemma 2.1. *Let $\varphi : X \to Y$ be a $(C/3)$-almost-isometry. Then, there exists a C-almost-isometry $\psi : Y \to X$ satisfying $d_X((\psi\varphi)x, x) \leq C$ and $d_Y((\varphi\psi)y, y) \leq C$ for every $x \in X$ and $y \in Y$.*

Since almost-isometries are special cases of quasi-isometries, the $\sim$ equivalence relation restricts to an equivalence relation on almost-isometries. Compositions of almost-isometries are almost-isometries, and by Lemma 2.1, coarse inverses of almost-isometries are almost-isometries. Therefore, equivalence classes of almost-isometries form a subgroup $\mathrm{AI}(X)$ of $\mathrm{QI}(X)$ with a canonical homomorphism $\mathrm{Isom}(X) \to \mathrm{AI}(X)$. Almost-isometries $\varphi : X \to Y$ induce isomorphisms $\mathrm{AI}(X) \cong \mathrm{AI}(Y)$.

Example 2.2. If X is a compact metric space, then any two maps have finite distance, so $\mathrm{QI}(X)$ and $\mathrm{AI}(X)$ are trivial. In contrast, $\mathrm{Isom}(X)$ can be quite nontrivial. In particular, $\mathrm{Isom}(X) \to \mathrm{AI}(X)$ need not be injective.

If $X = \mathbb{R}^n$ with the Euclidean metric, then $\mathrm{Isom}(X) = \mathbb{R}^n \rtimes O(n)$ where $O(n) = \{A \in GL(\mathbb{R}^n) \mid A^T A = \mathrm{Id}\}$ denotes the orthogonal group and where $(v, A) \in \mathrm{Isom}(X)$ acts via $w \mapsto Aw + v$. The natural homomorphism $\mathrm{Isom}(X) \to \mathrm{AI}(X)$ has kernel given by the translations $\mathbb{R}^n$ and image isomorphic to $O(n)$.

(Quasi)-isometries of* CAT(-1) *spaces. This subsection summarizes the basic theory of isometries, quasi-isometries, and boundary maps of CAT(-1) spaces; the reader is referred to [BH, Chapter II.6] for more details.

Throughout, X denotes a CAT(-1) metric space that is *proper*: *all metric balls are compact*. A group G acting on X acts *geometrically* provided the G-action on X is isometric, proper, free, and co-compact.

(Bounded) isometries. Note that, for $I \in \operatorname{Isom}(X)$, the displacement function $d_I : X \to \mathbb{R}$ is defined by $d_I(x) = d(Ix, x)$. The isometry I is defined to be a *bounded isometry* if d_I is a bounded function. The translation length of I, denoted by $\tau(I)$, is defined by $\tau(I) = \inf_{x\in X} d_I(x)$. The set of points where d_I achieves its infimum is denoted $\operatorname{Min}(I)$. An isometry I is *semi-simple* if $\operatorname{Min}(I) \neq \emptyset$. If G acts geometrically on X, then every $g \in G$ acts via a semi-simple isometry. For a semi-simple isometry I,

$$\tau(I) = \lim_{n\to\infty} \frac{d(x, I^n x)}{n}, \tag{2.5}$$

where $x \in X$ is an arbitrary point [BH, II.6, Exercise 6.6(1)].

Lemma 2.3. *Let X be a* CAT(-1) *space, not isometric to $\mathbb{R}$. Then, X has no nontrivial bounded isometries.*

Proof. Assume that $I \in \operatorname{Isom}(X)$ is bounded. Then, the displacement function d_I is bounded and convex [BH, II.6, Proposition 6.2(3)], and hence constant. In particular, $\operatorname{Min}(I) = X$.

If this constant is positive, then $\operatorname{Min}(I)$ splits isometrically as a metric product $Y \times \mathbb{R}$, for some convex subset $Y \subset X$ [BH, II.6, Theorem 6.8(4)]. Note that Y cannot consist of a single point (since X is not isometric to $\mathbb{R}$), nor can it have more than one point (for otherwise, X contains an isometric copy of $[0, \varepsilon] \times \mathbb{R}$, and so is not CAT$(-1)$). Conclude that $d_I \equiv 0$ and that $I = \mathrm{Id}_X$. □

Corollary 2.4. *Let X be a* CAT(-1) *space, not isometric to $\mathbb{R}$, and let also $g, h \in \operatorname{Isom}(X)$. If $\sup_{x\in X} d(gx, hx) < \infty$, then $g = h$. In particular, for such spaces, the natural map $\operatorname{Isom}(X) \to \mathrm{AI}(X)$ is injective.*

Proof. Apply Lemma 2.3 to gh^{-1}. □

***Boundary structure of* X.** The boundary ∂X of X consists of the set of equivalence classes of geodesic rays in X, where two rays are equivalent if they are at bounded Hausdorff distance. There is a natural topology on ∂X, where two geodesic rays based at $x_0 \in X$ are close provided they stay close for a long period of time. A quasi-isometry $\varphi : X \to Y$ induces a homeomorphism $\varphi^\partial : \partial X \to \partial Y$.

A pair of maps $f, g : X \to X$ are at distance at most L when $d(fx, gx) \le L$ for every $x \in X$. The boundary at infinity detects whether maps are at bounded distance apart. More precisely, we have the following well-known result.

Proposition 2.5. *Let* (X, d) *be a complete simply connected* $\mathrm{CAT}(-1)$ *metric space, having the geodesic extension property, and with the property that the space of directions at each point is connected. For each* $K \geq 1$ *and* $C > 0$*, there exists a constant* $L := L(K, C) > 0$ *with the following property: if* F *is a* (K, C)*-quasi-isometry of* X *and* I *is an isometry of* X *with boundary maps* $F^{\partial} \equiv I^{\partial}$*, then* F *and* I *are at distance at most* L*.*

We were unable to locate a proof in the literature, other than in the special case where (X, d) is a negatively curved Riemannian manifold (which was shown by Pansu [Pa, Lemma 9.11, p. 39]). For the convenience of the reader, we provide a proof that closely follows Pansu's Riemannian argument.

Proof. Without loss of generality, we assume that $I = \mathrm{Id}_X$. Let $p \in X$ be an arbitrary point, and consider the geodesic segment η from p to $F(p)$, whose length we would like to uniformly control. The segment defines a point x_- in the space of directions S_p at the point p. From the geodesic extension property, we can extend this geodesic beyond p, which defines a second point x_+ on S_p. In terms of the Alexandrov angular metric $\angle_p$ on the space of directions S_p (see [BH, Definition II.3.18, p. 190]), we have that $\angle_p(x_+, x_-) = \pi$ (as they correspond to a geodesic through p). Since the space of directions S_p is connected, continuity now implies the existence of a point $y_- \in S_x$ with the property that $\angle_p(x_+, y_-) = \pi/2 = \angle_p(x_-, y_-)$. Let γ be a geodesic segment terminating on p, and representing y_-. By the geodesic extension property, we can extend γ to a bi-infinite geodesic $\hat{\gamma}$. The continuation of γ defines a second point $y_+ \in S_p$; again, we have $\angle_p(y_+, y_-) = \pi$. We now claim that the point p coincides with the projection point of $F(p)$ on the geodesic $\hat{\gamma}$.

To see this, recall that in a $\mathrm{CAT}(-1)$ space, there is uniqueness of the projection point $\rho(q)$ of a point q onto a closed convex subset C. Moreover, the point $\rho(q)$ is characterized by the following property: the angle at $\rho(q)$ between the geodesic segment from $\rho(q)$ to q and any other geodesic segment originating at $\rho(q)$ *in the set* C is at least $\pi/2$ (see, e.g., [BH, Proposition II.2.4, p. 176]). We apply this criterion to the convex set $\hat{\gamma}$, and the point $F(p)$. Locally near p, there are precisely two geodesics segments in $\hat{\gamma}$, corresponding to the pair of directions $y_+, y_- \in S_p$. We already know that $\angle_p(y_-, x_-) = \pi/2$, so it suffices to verify that $\angle_p(y_+, x_-) \geq \pi/2$. But this is clear, for otherwise the triangle inequality would force a contradiction:

$$\pi = \angle_p(y_+, y_-) \leq \angle_p(y_+, x_-) + \angle_p(x_-, y_-) < \frac{\pi}{2} + \frac{\pi}{2}.$$

Thus, p is indeed the closest point to $F(p)$ on the geodesic $\hat{\gamma}$.

Now, apply the map F to obtain the (K, C)-quasi-geodesic $F \circ \hat{\gamma}$. From the stability theorem for quasi-geodesics (see [BH, Theorem III.H.1.7, p. 401]), there is a uniform constant $L := L(K, C)$, depending only on the constants K, C for the quasi-geodesic, with the property that $F \circ \hat{\gamma}$ is at Hausdorff distance $\leq L$ from the geodesic with same endpoints on ∂X, which is $\hat{\gamma}$. It follows that the point

$F(p) \in F \circ \hat{y}$ is at distance $\leq L$ from $\hat{y}$. But from the discussion in the previous paragraph, this implies $d(p, F(p)) \leq L$, as desired. □

In fact, there is some additional *metric* structure on ∂X: fixing a basepoint $x \in X$; define the visual metric

$$d_{\partial X}(p, q) = e^{-(p|q)_x},$$

where $p, q \in \partial X$, and $(p|q)_X$ denotes the Gromov product of the pair of points with respect to the basepoint x (see [Bo1, Section 2.5] for details). While the metric $d_{\partial X}$ depends on the choice of basepoint x, changing basepoints gives bi-Lipschitz equivalent metrics, and hence the *bi-Lipschitz class* of the metric $d_{\partial X}$ is well defined.

Fixing such metrics on $\partial X, \partial Y$, the behavior of a quasi-isometry $\varphi : X \to Y$ is closely related to the metric properties of the induced map $\varphi^\partial : \partial X \to \partial Y$. Most relevant for our purposes is work of Bonk and Schramm, who showed that if φ is an almost-isometry, then φ^∂ is a bi-Lipschitz map [BS, proof of Theorem 6.5]; that is, there is a constant $\lambda > 1$ with the property that, for all $x, y \in \partial X$, we have

$$\lambda^{-1} \cdot d_{\partial X}(x, y) \leq d_{\partial Y}(\varphi^\partial(x), \varphi^\partial(y)) \leq \lambda \cdot d_{\partial X}(x, y).$$

Conversely, if φ^∂ is a bi-Lipschitz map, then φ is at bounded distance from an almost-isometry [BS, Theorems 7.4 and 8.2]. In particular, boundary maps induce an isomorphism $\mathrm{AI}(X) \cong \mathrm{BiLip}(\partial X)$.

Rigidity statements. In the statement of our Main Theorem, our hypotheses involve some rigidity statements concerning the spaces X, Y. We define these rigidity statements in this subsection for the convenience of the reader.

Definition 2.6. A metric space Y is *quasi-isometrically rigid* (QI-rigid) if each quasi-isometry of Y is at bounded distance from an isometry of Y. In other words, the canonical homomorphism $\mathrm{Isom}(Y) \to \mathrm{QI}(Y)$ is surjective. A metric space is *almost-isometrically rigid* (AI-rigid) if every almost-isometry of Y is at bounded distance from an isometry. In other words, the canonical homomorphism $\mathrm{Isom}(Y) \to \mathrm{AI}(Y)$ is surjective.

A celebrated result of Pansu [Pa] shows that the quaternionic hyperbolic space $H^n_{\mathbb{H}}$ (of real dimension $4n$) and the Cayley hyperbolic plane $H^2_{\mathbb{O}}$ (of real dimension 16) are both QI-rigid (and hence AI-rigid). In contrast, we have the following result.

Lemma 2.7. *For any $n \geq 2$, real hyperbolic space $H^n_{\mathbb{R}}$ is* not *AI-rigid. In other words, there exist almost-isometries $\varphi : H^n_{\mathbb{R}} \to H^n_{\mathbb{R}}$ that are* not *at bounded distance from any isometry.*

Proof. From the discussion in the previous section, one can think of this entirely at the level of the metric structure on the boundary at infinity. Choosing the disk model for $H^n_{\mathbb{R}}$ and the basepoint x to be the origin, the metric $d_{\partial H^n_{\mathbb{R}}}$

on $\partial H^n_{\mathbb{R}} = S^{n-1}$ is conformal to the standard (round) metric on the sphere—in fact, $d_{\partial H^n_{\mathbb{R}}}(p, q)$ is half the (Euclidean) length of the (Euclidean) segment joining p to q (see [Bo1, Example 2.5.9]). Recalling that almost-isometries induce bi-Lipschitz maps [BS, Theorem 6.5], while isometries induce conformal maps [Bo1, Corollaire 2.6.3], the lemma follows immediately from the fact that there exist bi-Lipschitz maps $\varphi^\partial : S^{n-1} \to S^{n-1}$ that are not conformal. $\square$

Similarly, one can show that the complex hyperbolic space $H^n_{\mathbb{C}}$ (of real dimension $2n$) is *not* AI-rigid. Let us mention a few further examples.

Example 2.8. Consider $\mathbb{R}$ with the standard metric. From the discussion in Example 2.2, we have that the image of $\mathrm{Isom}(\mathbb{R})$ inside $\mathrm{QI}(\mathbb{R})$ is a copy of $\mathbb{Z}_2$ (with non-trivial element represented by the map σ defined via $\sigma(x) = -x$). For any $\lambda > 0$, the map $\mu_\lambda : x \mapsto \lambda x$ is a quasi-isometry, and if $\lambda \neq \lambda'$, then $\mu_\lambda \not\sim \mu_{\lambda'}$. Thus, $\mathrm{QI}(\mathbb{R})$ has cardinality at least as large as the continuum, and the map $\mathrm{Isom}(\mathbb{R}) \to \mathrm{QI}(\mathbb{R})$ is far from being surjective.

On the other hand, assume $\varphi : \mathbb{R} \to \mathbb{R}$ is a C-almost-isometry. Up to composing with σ, we may assume that φ preserves the two ends of $\mathbb{R}$, and up to composing with a translation, we may assume $\varphi(0) = 0$. Let us estimate the distance from φx to x for a generic $x \in \mathbb{R}$. First, if $x > 0$ is sufficiently large, we have that $\varphi x > 0$ (since φ preserves the ends of $\mathbb{R}$), and since φ is a C-almost-isometry, $|\varphi x - x| = \big|\,|\varphi x - \varphi 0| - |x - 0|\,\big| \leq C$. An identical argument shows that if $x < 0$ is sufficiently negative, then $|\varphi x - x| \leq C$. This leaves an R-neighborhood B of the fixed point 0 (for some R). But for $x \in B$, we know that φx has distance at most $R + C$ from the origin, so the triangle inequality gives $|x - \varphi x| \leq 2R + C$. It follows that $\sup_{x \in \mathbb{R}}(\varphi x, x) \leq 2R + C$, and hence $\varphi \sim \mathrm{Id}_{\mathbb{R}}$. This shows that every almost-isometry of $\mathbb{R}$ lies at finite distance from an isometry. Hence, $\mathbb{R}$ is an example of an AI-rigid space that is *not* QI-rigid.

Example 2.9. As a somewhat more sophisticated example, consider now the case of $\mathbb{R}^2$ with a flat metric. We claim that $\mathbb{R}^2$ is an AI-rigid space, that is, that every self almost-isometry is at bounded distance from an isometry. To see this, we start with $F \in \mathrm{AI}(\mathbb{R}^2)$ arbitrary, and try to find a *standard form* almost-isometry at bounded distance from F. Note that, by composing with a translation, we may assume $F(0) = 0$, and at the cost of a bounded perturbation, we can also assume that F is continuous. We will find it convenient to work in polar coordinates (r, θ).

Since $F(0) = 0$, we see that F maps the circle $r = R$ into the annular region $R - C \leq r \leq R + C$. Performing a radial projection of the image onto the circle of radius R results in a new map at bounded distance from F (hence, a new almost-isometry), which has the additional property that F maps each circle about the origin to itself. So without loss of generality, we may assume that F has the form $F(r, \theta) = (r, f(r, \theta))$ for some continuous function f; we let $\alpha : \mathbb{R}^+ \to \mathbb{R}$ denote the function $\alpha(r) := f(r, 0)$. Now consider the points $(r, 0)$ on the ray $\theta = 0$, and observe that each of these gets sent to a point $(r, \alpha(r))$. On the circle $S(R)$ of radius $r = R$ centered at the origin, the map φ is at bounded distance from the rotation by an angle $\alpha(R)$—moreover, the distance between the two maps is bounded independently of the radius R. It follows that the map F is at bounded distance from the map $(r,\theta) \mapsto (r,\theta+\alpha(r))$.

Next, let us focus on properties of the map α. The ray $\theta = 0$ maps under the almost-isometry F to the path $(r,\alpha(r))$. We now estimate the angle $\rho(s,t)$ ($s < t$) from the origin between the points $(s,\alpha(s))$ and $(t,\alpha(t))$—which is obviously $\alpha(t)-\alpha(s)$—via the law of cosines:

$$\cos(\rho(s,t)) := \frac{s^2+t^2-\|(t,\alpha(t))-(s,\alpha(s))\|^2}{2st}.$$

But since the map F is a K-almost isometry, we have the estimate

$$t-s-K \le \|(t,\alpha(t))-(s,\alpha(s))\| \le t-s+K,$$

which upon substitution gives the estimate

$$1-\frac{K^2+2K(t-s)}{2st} \le \cos(\rho(s,t)) \le 1-\frac{K^2-2K(t-s)}{2st}.$$

These bounds tend to 1 as $s < t$ both tend to infinity. Moreover, for any $\varepsilon > 0$, we can find an s_0 with the property that, for any $t > s_0$, the lower bound is at least $1-\varepsilon$. This implies that $\alpha(r)$ has a limit. Let α_∞ denote the limit $\lim_{r\to\infty}\alpha(r)$, and observe that, at the cost of composing with a rotation by $-\alpha_\infty$, we may as well assume that $\lim_{r\to\infty}\alpha(r) = 0$. We have thus reduced the problem to the following special case: let $F : (r,\theta) \mapsto (r,\theta+\alpha(r))$ be a K-almost-isometry, where $\alpha : \mathbb{R}^+ \to \mathbb{R}$ is a continuous map with $\lim_{r\to\infty}\alpha(r) = 0$. We need to show that this map F is at bounded distance from the identity map—it is sufficient to prove that, for r sufficiently large, $\alpha(r) \le K'/r$ (for some constant K').

Consider the pair of points $(r_1, 0)$ and (r_2,θ) on the plane, and their image under the K-almost-isometry. The distance between the two pairs of points is easily calculated from the law of cosines, and the K-almost-isometry condition gives the following estimate:

$$\left|\sqrt{r_1^2+r_2^2-2r_1r_2\cos(\theta+\Delta\alpha(r_1,r_2))}-\sqrt{r_1^2+r_2^2-2r_1r_2\cos(\theta)}\,\right| \le K,$$

which can be rewritten as

$$\left|\sqrt{1-\frac{2r_1r_2}{r_1^2+r_2^2}\cos(\theta+\Delta\alpha(r_1,r_2))}-\sqrt{1-\frac{2r_1r_2}{r_1^2+r_2^2}\cos(\theta)}\,\right| \le \frac{K}{\sqrt{r_1^2+r_2^2}},$$

where $\Delta\alpha$ is the difference function that is associated with α, that is, $\Delta\alpha(s,t) = \alpha(t)-\alpha(s)$. Now, fix a $0 < \lambda < 1$, and specialize the above equation to the case where $r_2 = r$ and $r_1 = \lambda r$ (r will be taken to tend to infinity), and θ is a fixed constant chosen so that $\sin(\theta) \neq 0$. We obtain

$$\left| \sqrt{1 - \frac{2\lambda}{1+\lambda^2}\cos(\theta + \varphi(r))} - \sqrt{1 - \frac{2\lambda}{1+\lambda^2}\cos(\theta)} \right| \leq \frac{K}{r\sqrt{1+\lambda^2}}$$

where $\varphi(r) := \Delta\alpha(\lambda r, r) = \alpha(r) - \alpha(\lambda r)$ tends to 0 as $r \to \infty$. Using the sum-angle formula for cosine, and a Taylor approximation for the terms involving $\varphi(r)$, we can rewrite the left-hand side as

$$\left| \sqrt{1 - \frac{2\lambda}{1+\lambda^2}\cos(\theta) + \varphi(r)\frac{2\lambda\sin(\theta)}{1+\lambda^2} + o(\varphi(r))} - \sqrt{1 - \frac{2\lambda}{1+\lambda^2}\cos(\theta)} \right|.$$

Recalling that λ, θ are fixed, while $\varphi(r) \to 0$ as $r \to \infty$, we can use a Taylor expansion for the function $g(x) = \sqrt{a+x} \approx \sqrt{a} + x/2\sqrt{a} + o(x)$. After substituting, the left-hand side further reduces, and we obtain

$$\left| \varphi(r)\left(\frac{\lambda\sin(\theta)}{(1+\lambda^2)\sqrt{1 - 2\lambda\cos(\theta)(1+\lambda^2)^{-1}}} \right) + o(\varphi(r)) \right| \leq \frac{K}{r\sqrt{1+\lambda^2}},$$

which gives us the asymptotic estimate $|\varphi(r)| \leq K''/r$ (for r sufficiently large), where K'' is a constant satisfying

$$K'' > K\frac{\sqrt{1+\lambda^2 - 2\lambda\cos(\theta)}}{\lambda\sin(\theta)}.$$

Finally, recalling that $\varphi(r) := \alpha(r) - \alpha(\lambda r)$, that $\lim_{s\to\infty}\alpha(s) = 0$, and that $0 < \lambda < 1$, we can use a telescoping sum to obtain the estimate

$$\begin{aligned} |\alpha(r)| = \lim_{s\to\infty} |\alpha(s) - \alpha(r)| &\leq \sum_{i=0}^{\infty} |\alpha(\lambda^{-i-1}r) - \alpha(\lambda^{-i}r)| \\ &\leq \sum_{i=0}^{\infty} \frac{K''}{\lambda^{-i}r} = \frac{K''}{r(1-\lambda)}. \end{aligned}$$

Since K'', λ are fixed constants, this gives the desired asymptotic estimate on the rotation function $\alpha(r)$, completing the argument.

Example 2.10. Consider $H^2_{\mathbb{R}}$ with the standard hyperbolic metric of constant curvature -1. Taking a compact set $K \subset H^2_{\mathbb{R}}$, perturb the metric slightly in the compact set K, and call the resulting Riemannian manifold X. If the perturbation is small enough, X will be negatively curved, and one can arrange for $\mathrm{Isom}(X)$ to be trivial.

Let $\varphi : H^2_{\mathbb{R}} \to X$ be the identity map, and note that φ is an almost-isometry from $H^2_{\mathbb{R}} \to X$ (though there are no isometries from $H^2_{\mathbb{R}}$ to X). It follows that $\mathrm{AI}(X) \cong \mathrm{AI}(H^2_{\mathbb{R}})$, and we know from Lemma 2.3 that the map $\mathrm{Isom}(H^2_{\mathbb{R}}) \hookrightarrow \mathrm{AI}(H^2_{\mathbb{R}})$ is injective. Hence, the group $\mathrm{AI}(X)$ contains a copy of $PSL(2,\mathbb{R})$, and the map $\mathrm{Isom}(X) \to \mathrm{AI}(X)$ fails to be surjective.

Definition 2.11. A complete CAT(-1) space X equipped with a geometric G-action $\rho : G \to \mathrm{Isom}(X)$ is *marked length spectrum rigid* (MLS-rigid) provided that, anytime we are given a complete CAT(-1) space Y equipped with a geometric G-action $i : G \to \mathrm{Isom}(Y)$, and the translation lengths satisfy $\tau(\rho(g)) = \tau(i(g))$ for each $g \in G$, there then exists a (ρ, i)-equivariant isometric embedding $X \hookrightarrow Y$.

Remark 2.12. When considering the MLS-rigidity question, one can also formulate versions where, rather than allowing an arbitrary CAT(-1) space Y, one restricts to a certain subclass $\mathcal{F}$ of CAT(-1) spaces. In this case, we say that X is MLS-rigid within the class $\mathcal{F}$. For instance, if X is a negatively curved Riemannian manifold, it is reasonable to focus on the case where Y is also a negatively curved Riemannian manifold. In this case, the conclusion forces the embedding to be surjective, and hence the equivariant embedding is automatically an isometry from X to Y. This is the context of the classical MLS conjecture.

3. PROOF OF THEOREM 1.2 AND APPLICATIONS

Throughout this section, we assume that X and Y satisfy the hypotheses of Theorem 1.2. Let us briefly sketch out the main steps of the proof. First, we use the almost-isometry between X and Y to transfer the isometric G-action on X to an almost-isometric G-action on Y. Using the property that Y is almost-isometrically rigid, one can straighten the almost-isometric G-action on Y to a *genuine* isometric G-action on Y. We then verify that this new isometric action on Y is also geometric. Such a construction of a geometric G-action on Y is likely well known—we include the details for the convenience of the reader. Now with respect to this new action on Y, one can construct an *equivariant* almost-isometry between X and Y. It is easy to check that these two actions have the same translation lengths, so from the marked length rigidity of X, we obtain the isometric embedding $X \hookrightarrow Y$. We now give the details of the proof.

Pushing forward the action. As X and Y are almost-isometric, there exists a $(C/3)$-almost-isometry $\varphi : X \to Y$. In particular, φ is a C-almost-isometry. By Lemma 2.1, there exists a C-almost-isometry coarse inverse $\psi : Y \to X$ satisfying

$$d_X((\psi\varphi)x, x) \le C,\ d_Y((\varphi\psi)y, y) \le C,$$

for every $x \in X$ and $y \in Y$.

Recall here that $G < \mathrm{Isom}(X)$ acts properly discontinuously, freely, and cocompactly on X. For $g \in G$, define the map $\bar{g} : Y \to Y$ by $\bar{g} = \varphi g \psi$. In other words, $\bar{g}$ is chosen to make the following diagram commute:

$$\begin{tikzcd} Y \arrow[r, "\psi"] \arrow[d, "\bar{g}"'] & X \arrow[d, "g"] \\ Y & X. \arrow[l, "\varphi"] \end{tikzcd}$$

Lemma 3.1. *For each $g \in G$, $\bar{g}$ is a $3C$-almost-isometry of Y.*

Proof. Let $y_1, y_2 \in Y$. We verify

$$\begin{aligned} d_Y(\bar{g}y_1, \bar{g}y_2) &= d_Y((\varphi g \psi)y_1, (\varphi g \psi)y_2) \le d_X(g\psi y_1, g\psi y_2) + C \\ &= d_X(\psi y_1, \psi y_2) + C \le d_Y(y_1, y_2) + 2C. \end{aligned}$$

A symmetric argument shows that $d_Y(\bar{g}y_1, \bar{g}y_2) \ge d_Y(y_1, y_2) - 2C$, giving us that

$$d_Y(y_1, y_2) - 3C \le d_Y(\bar{g}(y_1), \bar{g}(y_2)) \le d_Y(y_1, y_2) + 3C.$$

It remains to show that $\bar{g}$ is $3C$-coarsely onto. For $y \in Y$, we let $y' = \overline{g^{-1}}y = \varphi g^{-1}\psi y$. Then,

$$\begin{aligned} &d_Y(\bar{g}y', y) \\ &\quad = d_Y((\varphi g \psi)(\varphi g^{-1}\psi y), y) \\ &\quad \le d_X(\psi\varphi g \psi\varphi g^{-1}\psi y, \psi y) + C \\ &\quad \le d_X((\psi\varphi)(g\psi\varphi g^{-1}\psi y), g\psi\varphi g^{-1}\psi y) + d_X(g\psi\varphi g^{-1}\psi y, \psi y) + C \\ &\quad \le d_X(g\psi\varphi g^{-1}\psi y, \psi y) + 2C \\ &\quad = d_X((\psi\varphi)(g^{-1}\psi y), g^{-1}\psi y) + 2C \\ &\quad \le 3C. \end{aligned}$$

The first inequality comes from ψ being a C-almost-isometry, the second is the triangle inequality, and the third and fourth both come from $\psi\varphi \sim \mathrm{Id}_X$. This completes the proof of the lemma. ◻

As Y is AI-rigid, Lemma 3.1 and Proposition 2.5 yield a constant $L > 0$ such that for each $g \in G$, there is a unique isometry $i(g) \in \mathrm{Isom}(Y)$ satisfying

$$d_Y(\bar{g}y, i(g)y) \le L$$

for every $y \in Y$. It is important to note that the constant L is *independent* of the choice of element g (this is used in the proof of Lemma 3.4). This defines a map $i : G \to \mathrm{Isom}(Y)$.

Lemma 3.2. *The map* $i : G \to \mathrm{Isom}(Y)$ *is a homomorphism.*

Proof. Let $g_1, g_2 \in G$; then, we want to compare the elements $i(g_1g_2)$ and $i(g_1)i(g_2)$ inside $\mathrm{Isom}(Y)$. By Corollary 2.4 (since $Y \neq \mathbb{R}$, as the space of directions of Y is connected), it suffices to show that $i(g_1g_2) \sim i(g_1)i(g_2)$. Thus, we need to estimate the effect of these two isometries on a generic element $y \in Y$. As a preliminary estimate, we have

$$\begin{aligned} d_Y(\overline{g_1g_2}y, \overline{g_1}\,\overline{g_2}y) &= d_Y((\varphi g_1g_2\psi)y, (\varphi g_1\psi)(\varphi g_2\psi)y) \\ &\leq d_X(g_1(g_2\psi y), g_1(\psi\varphi g_2\psi y)) + C \\ &= d_X(g_2\psi y, (\psi\varphi)(g_2\psi y)) + C \leq 2C. \end{aligned}$$

The first inequality uses that φ is a C-almost-isometry, while the second uses that $\psi\varphi \sim \mathrm{Id}_X$. Using this, we can now estimate:

$$\begin{aligned} &d_Y(i(g_1g_2)y, i(g_1)i(g_2)y) \\ &\quad\leq d_Y(i(g_1g_2)y, (\overline{g_1g_2})y) + d_Y(\overline{g_1g_2}y, i(g_1)i(g_2)y) \\ &\quad\leq d_Y(\overline{g_1g_2}y, i(g_1)i(g_2)y) + L \\ &\quad\leq d_Y(\overline{g_1g_2}y, \overline{g_1}\,\overline{g_2}y) + d_Y(\overline{g_1}\,\overline{g_2}y, i(g_1)i(g_2)y) + L \\ &\quad\leq d_Y(\overline{g_1}\,\overline{g_2}y, i(g_1)i(g_2)y) + 2C + L \\ &\quad\leq d_Y(\overline{g_1}(\overline{g_2}y), i(g_1)(\overline{g_2}y)) + d_Y(i(g_1)\overline{g_2}y, i(g_1)i(g_2)y) + 2C + L \\ &\quad\leq d_Y(i(g_1)\overline{g_2}y, i(g_1)i(g_2)y) + 2C + 2L \\ &\quad= d_Y(\overline{g_2}y, i(g_2)y) + 2C + 2L \leq 2C + 3L. \end{aligned}$$

Since this estimate holds for arbitrary $y \in Y$, we conclude $i(g_1g_2) \sim i(g_1)i(g_2)$. Applying Corollary 2.4, this gives us

$$i(g_1g_2) = i(g_1)i(g_2),$$

establishing the lemma. ∎

Verifying the new action is geometric. Now that we have constructed a homomorphism $i : G \to \mathrm{Isom}(Y)$, our next step is to show that this G-action on Y is geometric.

Lemma 3.3. *The homomorphism* $i : G \to \mathrm{Isom}(Y)$ *is injective.*

Proof. Let $g \in G$, and assume that $i(g) = \mathrm{Id}_Y$. By Lemma 2.3, it suffices to show that $g \sim \mathrm{Id}_X$, so we need to estimate how far g moves a generic element $x \in X$. First, observe that for each $y \in Y$,

$$d_Y(\bar{g}y, y) = d_Y(\bar{g}y, i(g)y) \leq L.$$

We can now estimate how far g moves elements of the form ψy:

$$\begin{aligned} d_X(g(\psi y), \psi y) &\leq d_Y(\varphi(g\psi y), \varphi(\psi y)) + C = d_Y(\bar{g}y, \varphi\psi y) + C \\ &\leq d_Y(\bar{g}y, y) + d_Y(y, (\varphi\psi)y) + C \leq L + 2C. \end{aligned}$$

Now, for a generic $x \in X$, we have that $\varphi x \in Y$, so we can estimate

$$\begin{aligned} d_X(gx, x) &\leq d_X(gx, g(\psi\varphi)x) + d_X(g(\psi\varphi)x, x) \\ &= d_X(x, (\psi\varphi)x) + d_X(g(\psi\varphi)x, x) \\ &\leq d_X(g(\psi\varphi)x, x) + C \\ &\leq d_X(g(\psi\varphi)x, (\psi\varphi)x) + d_X((\psi\varphi)x, x) + C \\ &\leq L + 2C + C + C. \end{aligned}$$

This shows $g \sim \mathrm{Id}_X$, so by Lemma 2.3 (and using the hypothesis that $X \neq \mathbb{R}$), we conclude $g = \mathrm{Id}_X$, as claimed. ◻

Lemma 3.4. *The G-action on Y is proper.*

Proof. We argue here by contradiction. If the above is false, then there exist a d_Y-metric ball $B_Y \subset Y$ and an infinite sequence of distinct elements $i(g_j) \in i(G)$ with

$$i(g_j)B_Y \cap B_Y \neq \emptyset$$

for each index j. For each index j, choose $y_j, k_j \in B_Y$ such that $i(g_j)y_j = k_j$. Let $D = \mathrm{diam}(B_Y)$.

We consider the closed d_X-metric ball

$$B_X = \{x \in X \mid d_X(x, \psi y_0) \leq L + 2D + 5C\}.$$

Since the G action on X is proper, we obtain a contradiction by showing that $g_j(\psi y_j) \in g_j B_X \cap B_X$ for each index j. First, note that for each j,

$$d_Y(i(g_j)y_j, y_j) = d_Y(k_j, y_j) \leq D.$$

Next, we estimate how far each g_j displaces the corresponding ψy_j:

$$\begin{aligned} d_X(g_j(\psi y_j), \psi y_j) &\leq d_Y(\varphi(g_j\psi y_j), \varphi(\psi y_j)) + C \\ &= d_Y(\overline{g_j}y_j, \varphi\psi y_j) + C \\ &\leq d_Y(\overline{g_j}y_j, y_j) + d_Y(y_j, (\varphi\psi)y_j) + C \\ &\leq d_Y(\overline{g_j}y_j, y_j) + 2C \\ &\leq d_Y(\overline{g_j}y_j, i(g_j)y_j) + d_Y(i(g_j)y_j, y_j) + 2C \\ &\leq L + D + 2C. \end{aligned}$$

We can now show $\psi y_j \in B_X$ for each j, since

$$\begin{aligned}
d_X(\psi y_0, \psi y_j) &\le d_Y(\varphi(\psi y_0), \varphi(\psi y_j)) + C \\
&\le d_Y((\varphi\psi)y_0, y_0) + d_Y(y_0, \varphi\psi y_j) + C \\
&\le d_Y(y_0, \varphi\psi y_j) + 2C \\
&\le d_Y(y_0, y_j) + d_Y(y_j, (\varphi\psi)y_j) + 2C \\
&\le D + 3C.
\end{aligned}$$

Combining the above two inequalities, we obtain that $g_j(\psi y_j) \in B_X$, since

$$\begin{aligned}
d_X(g_j(\psi y_j), \psi y_0) &\le d_X(g_j(\psi y_j), \psi y_j)) + d_X(\psi y_j, \psi y_0) \\
&\le (L + D + 2C) + (D + 3C).
\end{aligned}$$

We conclude that

$$g_j(\psi y_j) \in g_j B_X \cap B_X$$

for each index j as claimed above. This yields the desired contradiction, and completes the proof of the lemma. □

Corollary 3.5. *The G-action on Y is free.*

Proof. If this is not the case, then there exists a nonidentity element $g \in G$ and a point $y \in Y$ with $i(g)y = y$. Note that G is torsion-free (as the G-action on X is free), and i is injective by Lemma 3.3, so $i(g) \in \mathrm{Isom}(Y)$ also has infinite order. This gives infinitely many elements fixing the point y, contradicting Lemma 3.4. □

Lemma 3.6. *The G-action on Y is co-compact.*

Proof. Since Y is a proper metric space, it suffices to prove there is a closed d_Y-metric ball B_Y such that the $i(G)$-translates of B_Y cover Y. As the G-action on X is co-compact, and X is a proper metric space, there exists $x_0 \in X$ and $R > 0$ such that the G-translates of the closed d_X-metric ball

$$B_X = \{x \in X \mid d_X(x, x_0) \le R\}$$

cover M. Let

$$B_Y = \{y \in Y \mid d_Y(y, \varphi x_0) \le R + 3C + L\}.$$

Fix $y \in Y$. As the G-translates of B_X cover X, there exists $g \in G$ such that $\psi y \in gB_X$, or equivalently, $d_X(gx_0, \psi y) \le R$. We conclude the proof by showing that

$$d_Y(i(g)(\varphi x_0), y) \le R + 3C + L,$$

or equivalently that $y \in i(g)B_Y$. Indeed, we can estimate

$$\begin{aligned}
d_Y(i(g)(\varphi x_0), y) &\leq d_Y(i(g)(\varphi x_0), \bar{g}(\varphi x_0)) + d_Y(\bar{g}\varphi x_0, y) \\
&\leq d_Y(\bar{g}\varphi x_0, y) + L = d_Y((\varphi g \psi)\varphi x_0, y) + L \\
&\leq d_X(\psi(\varphi g \psi \varphi x_0), \psi(y)) + C + L \\
&\leq d_X((\psi\varphi)(g\psi\varphi x_0), g\psi\varphi x_0) + d_X(g\psi\varphi x_0, \psi y) + C + L \\
&\leq d_X(g(\psi\varphi x_0), \psi y) + 2C + L \\
&= d_X(\psi\varphi x_0, g^{-1}(\psi y)) + 2C + L \\
&\leq d_X((\psi\varphi)x_0, x_0) + d_X(x_0, g^{-1}\psi y) + 2C + L \\
&\leq d_X(x_0, g^{-1}\psi y) + 3C + L \\
&= d_X(g x_0, \psi y) + 3C + L \leq R + 3C + L.
\end{aligned}$$

$\square$

Combining Lemma 3.4, Corollary 3.5, and Lemma 3.6, we see that the G-action on Y given by $i : G \to \mathrm{Isom}(Y)$ is geometric.

An equivariant almost-isometry. Let $\Omega \subset X$ be a *strict* fundamental domain for the G-action on X. In other words, Ω consists of a single point from each G-orbit in X. Then, for each $x \in X$, there exist unique $g \in G$ and $\omega \in \Omega$ with $g\omega = x$.

Define $\Phi : X \to Y$ by $\Phi(x) = \Phi(g\omega) := i(g)\varphi\omega$. By construction, Φ is equivariant with respect to the G and $i(G)$ actions on X and Y, respectively.

Lemma 3.7. *The $(G, i(G))$-equivariant map $\Phi : X \to Y$ is a $(5C+2L)$-almost-isometry.*

Proof. Let $x \in X$. There are unique $g \in G$ and $\omega \in \Omega$ such that $x = g\omega$. Then,

$$\begin{aligned}
d_Y(\varphi x, \Phi x) &= d_Y(\varphi g\omega, \Phi g\omega) = d_Y(\varphi g\omega, i(g)\varphi\omega) \\
&\leq d_Y(\varphi g\omega, \bar{g}\varphi\omega) + d_Y(\bar{g}(\varphi\omega), i(g)(\varphi\omega)) \\
&\leq d_Y(\varphi g\omega, \bar{g}\varphi\omega) + L = d_Y(\varphi g\omega, (\varphi g\psi)\varphi\omega) + L \\
&\leq d_X(g\omega, g\psi\varphi\omega) + C + L \\
&= d_X(\omega, (\psi\varphi)\omega) + C + L \leq 2C + L.
\end{aligned}$$

It follows that, for $x_1, x_2 \in X$,

$$\begin{aligned}
d_Y(\Phi x_1, \Phi x_2) &\leq d_Y(\Phi x_1, \varphi x_1) + d_Y(\varphi x_1, \varphi x_2) + d_Y(\varphi x_2, \Phi x_2) \\
&\leq d_Y(\varphi x_1, \varphi x_2) + 4C + 2L \leq d_X(x_1, x_2) + 5C + 2L.
\end{aligned}$$

A similar argument gives the estimate

$$d_X(x_1, x_2) \leq d_Y(\Phi x_1, \Phi x_2) + 5C + 2L.$$

The previous two inequalities show that Φ is a $(5C + 2L)$-almost-isometric map. It remains to show that Φ is $(5C + 2L)$-coarsely onto. Let $y \in Y$. Then, $\psi y \in X$ and

$$\begin{aligned} d_Y(\Phi(\psi y), y) &\leq d_Y(\Phi(\psi y), \varphi(\psi y)) + d_Y(\varphi\psi y, y) \\ &\leq 2C + L + C, \end{aligned}$$

concluding the proof. □

Comparing the marked length spectrum. To summarize, we have constructed a new G-action on Y, given by $i : G \to \mathrm{Isom}(Y)$, which we have shown to be geometric. We have also constructed an *equivariant* almost-isometry Φ from X to Y. We now compare the translation lengths for the G-actions on X and Y. Let $\bar{C} = 5C + 2L$, the almost-isometry constant for the equivariant almost-isometry $\Phi : X \to Y$.

Lemma 3.8. *For every $g \in G$, we have $\tau(g) = \tau(i(g))$.*

Proof. By formula (2.5), for any $x \in X$ we have

$$\begin{aligned} \tau(g) &= \lim_{n\to\infty} \frac{d_X(x, g^n x)}{n} \geq \lim_{n\to\infty} \frac{d_Y(\Phi x, \Phi g^n x) - \bar{C}}{n} \\ &= \lim_{n\to\infty} \frac{d_Y(\Phi x, i(g)^n \Phi x)}{n} = \tau(i(g)). \end{aligned}$$

An identical argument, using a coarse inverse to Φ, gives the reverse inequality. □

Concluding the proof. We now have isometric G-actions on X and Y. We have shown that the action on Y is geometric, and that the two actions have the same translation lengths. Since X, by hypothesis, is marked length spectrum rigid, we conclude that there is an equivariant isometric embedding $\psi : X \to Y$. Finally, to see that ψ is coarsely onto, note that $\partial\psi \equiv \partial\Phi$ (as both these maps are at finite distance from the same orbit map), so ψ and Φ are at bounded distance apart. The first part of the proof of Lemma 3.7 shows Φ and φ are also at bounded distance apart, and hence ψ and φ are at bounded distance apart. Since φ is coarsely onto, it immediately follows that ψ is coarsely onto. This completes the proof of Theorem 1.2.

Application—locally symmetric manifolds. In this subsection, we prove Corollary 1.3, dealing with quaternionic hyperbolic space $H^n_{\mathbb{H}}$ and the Cayley hyperbolic plane $H^2_{\mathbb{O}}$.

Proof of Corollary 1.3. Pansu [Pa] has shown that $H^n_{\mathbb{H}}$ and $H^2_{\mathbb{O}}$ are QI-rigid, and hence AI-rigid. Combining work of Hamenstadt [Ha] and Besson-Courtois-Gallot [BCG], we also know that uniform lattices in the semi-simple Lie groups $Sp(n, 1)$ and $F_{4,-20}$ are marked length spectrum rigid within the class of actions on negatively curved manifolds of the same dimension as the corresponding symmetric space.

Following the notation in our Theorem 1.2, we let $Y = (\tilde{M}, \tilde{g}_0)$ denote the symmetric space, and $X = (\tilde{M}, \tilde{g}_1)$ the universal cover with the exotic metric. Proceeding as in the Main Theorem, we assume there is an almost-isometry $\varphi : X \to Y$. One then uses AI-rigidity of the symmetric space Y to construct a new geometric G-action on Y, so that that the two G-actions have the same marked length spectrum (Lemma 3.8). Finally, we apply marked length rigidity for the G-action *on the symmetric space* Y (rather than on the symmetric space X) to obtain a coarsely onto isometric embedding of Y into X. Since X, Y are complete Riemannian manifolds of the same dimension, such a map provides an isometry between X and Y. Thus, X is also a symmetric space, and so (M, g_1) had to also be locally symmetric, as claimed. ◻

Application—Fuchsian buildings. We start by quickly recalling some of the terminology concerning Fuchsian buildings, which were first introduced by Bourdon [Bo2]. These are two-dimensional polyhedral complexes that satisfy a number of axioms. First, we start with a compact convex hyperbolic polygon $R \subset H^2_{\mathbb{R}}$, with each angle of the form π/m_i for some m_i associated with the vertex ($m_i \in \mathbb{N}$, $m_i \geq 2$). Reflection in the geodesics extending the sides of R generates a Coxeter group W, and the orbit of R under W gives a tessellation of $H^2_{\mathbb{R}}$. Cyclically labeling the vertices of R by the integers $\{1\}, \dots, \{k\}$ (so that the i^{th} vertex has angle π/m_i), and the corresponding edges by $\{1,2\}, \{2,3\}, \dots, \{k,1\}$, one can apply the W action to obtain a W-invariant labeling of the tessellation of $H^2_{\mathbb{R}}$; this labeled polyhedral 2-complex will be denoted A_R, and called the *model apartment*.

A polygonal 2-complex X is called a two-dimensional hyperbolic building if it contains a vertex labeling by the integers $\{1, \dots, k\}$, along with a distinguished collection of subcomplexes $\mathcal{A}$ called the *apartments*. The individual polygons in X will be called *chambers*. The complex is required to have the following properties:

- Each apartment $A \in \mathcal{A}$ is isomorphic, as a labeled polygonal complex, to the model apartment A_R.
- Given any two chambers in X, one can find an apartment $A \in \mathcal{A}$ that contains the two chambers,
- Given any two apartments $A_1, A_2 \in \mathcal{A}$ that share a chamber, there is an isomorphism of labeled 2-complexes $\varphi : A_1 \to A_2$ that fixes $A_1 \cap A_2$.

If in addition each edge labeled $\{i, i+1\}$ has a fixed number q_i of incident polygons, then X is called a *Fuchsian building*. For a Fuchsian building, the combinatorial axioms force some additional structure on the links of vertices: these graphs must be *generalized m-gons* in the sense of Tits. Work of Feit and Higman [FH] then implies that each m_i must lie in the set $\{2, 3, 4, 6, 8\}$. Note that making each polygon in X isometric to R via the label-preserving map produces a CAT(-1)

metric on X. However, a given polygonal 2-complex might have several metrizations as a Fuchsian building: these correspond to varying the hyperbolic metric on R while preserving the angles at the vertices. Any such variation induces a new CAT(-1) metric on X. The hyperbolic polygon R is called *normal* if it has an inscribed circle that touches all its sides—if we fix the angles of a polygon to be $\{\pi/m_1, \dots, \pi/m_k\}$, there is a unique normal hyperbolic polygon with those given vertex angles. We can now state a rigidity result for Fuchsian buildings.

Corollary 3.9. *Let G be a group acting freely and cocompactly on a combinatorial Fuchsian building X. Let d_0 be the metric on X/Γ where each chamber is the normal hyperbolic polygon, and let d_1 be a locally* CAT(-1) *metric, where each polygon has a Riemannian metric of curvature ≤ 1 with geodesic sides. Then, the universal covers $(X, \tilde{d}_0)$ and $(X, \tilde{d}_1)$ are almost-isometric if and only if they are isometric, in which case the isometry can be chosen to be equivariant with respect to the G-actions, and hence $(X/\Gamma, d_0)$ is isometric to $(X/\Gamma, d_1)$.*

Proof. The argument for this is similar to the proof of Corollary 1.3. Let $\varphi : (X, \tilde{d}_0) \to (X, \tilde{d}_1)$ be the almost-isometry between the universal covers. For the Fuchsian building $(X, \tilde{d}_0)$, Xie [Xi] has established QI-rigidity (and hence AI-rigidity). It is important here that, for the $\tilde{d}_0$-metric, all polygons are normal—otherwise QI-rigidity does *not* hold. Using the AI-rigidity, we can construct a new geometric Γ-action on $(X, \tilde{d}_0)$. The Γ-actions on $(X, \tilde{d}_0)$ and $(X, \tilde{d}_1)$ now have the same marked length spectrum (see Lemma 3.8). But Constantine and Lafont [CL] have established that the metric $\tilde{d}_0$ is marked length spectrum rigid within the class of metrics described in the statement of our corollary (thus including $\tilde{d}_1$). This establishes the corollary. ◻

4. AIs AND VOLUME GROWTH

In this section, we establish Theorem 1.4. We start by reminding the reader of a standard packing/covering argument, which allows us to reinterpret volume growth entropy in terms of quantities we can estimate.

Lemma 4.1. *Let M be a Riemannian cover of a compact manifold. Fix a basepoint $p \in M$, a parameter $s > 0$, and define the counting function $N(s,r)$ to be the minimal cardinality of a covering of $B_p(r)$ by balls of radius s. Then, for any choice of s, we have that*

$$h_{\mathrm{vol}}(M) = \lim_{r\to\infty} \frac{\ln(N(s,r))}{r}.$$

Proof. Let $V_s < \infty$ be the maximal volume of a ball of radius s, and $v_s > 0$ be the minimal volume of a ball of radius $s/2$ (so, clearly, $v_s < V_s$). A maximal packing of $B_p(r)$ by disjoint balls of radius $s/2$ induces a covering of $B_p(r)$ by balls of radius s with the same centers. We thus obtain the following bounds:

$$\frac{\mathrm{Vol}(B_p(r))}{V_s} \leq N(s,r) \leq \frac{\mathrm{Vol}(B_p(r))}{v_s}.$$

Since both V_s, v_s are fixed real numbers, taking the log and the limit as $r \to \infty$ yields the lemma. $\square$

Now with Lemma 4.1 in hand, the proof is straightforward. We will use the almost-isometry to relate the counting function $N_1(s, r)$ for the manifold M_1 to the counting function $N_2(s', r')$ for the manifold M_2.

Let $\varphi : M_1 \to M_2$ be the C-almost-isometry. Choose a basepoint $p \in M_1$, and let $q = \varphi(p)$ be the basepoint in M_2. Consider the counting function $N_1(1, r)$ for the manifold M_1. For a given r, let $\{p_1, \ldots, p_N\}$ (where $N := N_1(1, r)$) be the centers of the balls of radius 1 for the minimal covering of $B_p(r)$, and let $q_i := \varphi(p_i)$ be the corresponding image points in M_2.

The covering of $B_p(r)$ by the set of balls $\{B_{p_i}(1)\}_{i=1}^N$ maps over to a covering $\{\varphi(B_{p_i}(1))\}_{i=1}^N$ of the set $\varphi(B_p(r))$. Since φ is an almost-isometry with additive constant C, we have for each i that

$$\varphi(B_{p_i}(1)) \subseteq B_{q_i}(1 + C),$$

and hence we also have a covering $\{B_{q_i}(1 + C)\}_{i=1}^N$ of the set $\varphi(B_p(r))$ by metric balls centered at $\{q_1, \ldots, q_N\}$.

Next, we note that the C-neighborhood of the set $\varphi(B_p(r))$ contains the set $B_q(r - 2C)$. Indeed, we know that $\varphi(M_1)$ is C-dense in M_2, so given an arbitary point $x \in B_q(r - 2C)$, we can find a point $y \in M_1$ with the property that $d_2(\varphi y, x) < C$. Now assume y lies outside of $B_p(r)$. Then, $d_1(y, p) > r$, which would imply

$$d_2(\varphi y, q) = d_2(\varphi y, \varphi p) \geq d_1(y, p) - C > r - C.$$

Since $d_2(\varphi y, x) < C$, the triangle inequality forces $d_2(x, q) > r - 2C$, a contradiction. Thus, we must have $y \in B_p(r)$.

Since the C-neighborhood of $\varphi(B_p(r))$ contains the set $B_q(r - 2C)$, and we have a covering $\{B_{q_i}(1 + C)\}_{i=1}^N$ of the set $\varphi(B_p(r))$ by metric balls, we obtain a corresponding covering $\{B_{q_i}(1 + 2C)\}_{i=1}^N$ of the set $B_q(r - 2C)$ by balls of radius $1 + 2C$. This implies that

$$N_1(1, r) \geq N_2(1 + 2C, r - 2C).$$

Taking the log and the limit as $r \to \infty$, and taking into account Lemma 4.1, we obtain the pair of inequalities

$$h^+_{\mathrm{vol}}(M_1) \geq h^+_{\mathrm{vol}}(M_2), \quad h^-_{\mathrm{vol}}(M_1) \geq h^-_{\mathrm{vol}}(M_2).$$

Applying the same argument to a coarse inverse almost-isometry yields the pair of reverse inequalities, completing the proof of Theorem 1.4.

Remark 4.2. In the special case where the M_i both have metrics of bounded negative sectional curvature, and support compact quotients, one can give an alternate proof of Theorem 1.4 by exploiting the metric structures on the boundaries at infinity. Indeed, fixing a basepoint $p \in M_1$ and corresponding basepoint $q := \varphi(p)$, one can construct metrics on the boundaries at infinity $\partial_\infty M_1$ and $\partial_\infty M_2$. It follows then from work of Bonk and Schramm that the almost-isometry $\varphi : M_1 \to M_2$ induces a bi-Lipschitz homeomorphism $\varphi_\infty : \partial_\infty M_1 \to \partial_\infty M_2$ (see [BS, proof of Theorem 6.5]). In particular, the two boundaries have identical Hausdorff dimension. But Otal and Peigné [OP] have shown that for such manifolds, the Hausdorff dimension of the boundary at infinity coincides with the topological entropy of the geodesic flow on the compact quotient of the M_i (which by Manning [Ma] coincides with the volume growth entropy of the M_i).

Application—rigidity results. We now give a proof of Corollary 1.7.

Proof. We deal with each of the various cases separately.

Case (1). The manifold M is finitely covered by the 2-torus T^2. Lifting the metrics g_0, g_1 to this finite cover, we see that it is enough to deal with the case where $M = T^2$. Then, the metrics $\tilde{g}_0, \tilde{g}_1$ can be viewed as a pair of $\mathbb{Z}^2$-invariant metrics on $\mathbb{R}^2$. Associated with these two periodic metrics, we have a pair of Banach norms on $\mathbb{R}^2$ defined via

$$\|v\|_i := \lim_{r\to\infty} \frac{d_i(0, rv)}{r},$$

where d_i is the distance function associated with the metric g_i. Burago [Bu] showed that the identity map on $\mathbb{R}^2$ provides an almost-isometry from the Banach norm to the original periodic metric; that is, there is a constant C with the property that, for all vectors $v, w \in \mathbb{R}^n$, we have

$$\big|\, \|v - w\|_i - d_i(v, w) \,\big| < C.$$

We note there is an alternate way to view the Banach norm: consider the pointed space $(\mathbb{R}^2, 0)$ with the sequence of metrics given by d_i/n ($n \in \mathbb{N}$), and take the ultralimit. The resulting pointed space, the *asymptotic cone*, is topologically $(\mathbb{R}^2, 0)$, equipped with the corresponding Banach norm (regardless of the choice of ultrafilter). We denote by F_i the unit ball, centered at 0, in the Banach norm $\|\cdot\|_i$.

Now assume we have an almost-isometry $\varphi : (\mathbb{R}^2, d_0) \to (\mathbb{R}^2, d_1)$. Then, passing to the asymptotic cones, we obtain an *isometry*

$$\hat{\varphi} : (\mathbb{R}^2, \|\cdot\|_0) \to (\mathbb{R}^2, \|\cdot\|_1)$$

fixing 0, and sending the unit ball F_0 to the unit ball F_1. Since the geodesics in any Banach norm are straight lines, the map $\hat{\varphi}$ is a linear map. Now, for the flat metric $\tilde{g}_0$, we know that the associated Banach norm is a Euclidean norm (i.e., the unit ball F_0 is an ellipsoid). Since $\hat{\varphi}$ is linear, we have that $\hat{\varphi}(F_0) = F_1$ is also an ellipsoid, and hence that $\| \cdot \|_1$ is a (smooth) Euclidean norm.

By Bangert's [Ba, Theorem 5.3], the periodic minimal geodesics of (T^2, g_1) in any nontrivial free homotopy class of T^2 foliate T^2. By Innami [In] (or [Ba, proof of Theorem 6.1]), the metric g_1 must also be flat.

Cases (2)–(4). By our Theorem 1.4, we have $h_{\mathrm{vol}}(\tilde{g}_0) = h_{\mathrm{vol}}(\tilde{g}_1)$, which immediately implies that $h_{\mathrm{vol}}(\tilde{g}_0) \cdot \mathrm{Vol}(g_0) \geq h_{\mathrm{vol}}(\tilde{g}_1) \cdot \mathrm{Vol}(g_1)$. Locally symmetric metrics uniquely minimize the functional $h_{\mathrm{vol}}(-)^n \cdot \mathrm{Vol}(-)$ in case (2) by Katok [Ka], in case (3) by Besson, Courtois, and Gallot [BCG], and in the conformal class in case (4) by Knieper [Kn]. In each of these cases, we conclude that $\tilde{g}_1 = \lambda \tilde{g}_0$ for some $0 < \lambda < \infty$. Corollary 1.5 implies $\lambda = 1$, completing the proof of Cases (2)–(4).

Case (5). Let us briefly specify the metric g_0—for this metric, the individual negatively curved symmetric spaces factors are scaled as in [CF, Section 2]. Connell and Farb have now shown that the metric g_0 is the unique minimizer for the volume growth entropy on the space of locally symmetric metrics on M. In [CF, Theorem A], they then proceed to show that g_0 is the unique minimizer of the functional $h_{\mathrm{vol}}(-)^n \cdot \mathrm{Vol}(-)$ on the space of all metrics on M. The same argument as in cases (2)–(4) gives the desired conclusion. $\square$

Application—the case of metric trees. While we have primarily focused on Riemannian manifolds, some of our results hold in greater generality. For instance, the proof of Theorem 1.4 did not make any particular use of the fact that our metric was Riemannian. In fact, the very same proof yields the following more general result. For (X, d) a metric space of Hausdorff dimension s, denote by $\mathcal{H}^s$ the s-dimensional Hausdorff measure, and define the upper/lower exponential volume growth rate to be

$$h^+(X, d) := \limsup_{r \to \infty} \frac{\ln(\mathcal{H}^s(B_p(r)))}{r}$$

$$h^-(X, d) := \liminf_{r \to \infty} \frac{\ln(\mathcal{H}^s(B_p(r)))}{r},$$

where $B_p(r)$ is the metric ball of radius r centered at a fixed basepoint $p \in X$ (these are independent of the choice of basepoint). In the case where we have $h^+(X, d) = h^-(X, d)$, we denote the common value by $h(X, d)$, which we call the exponential volume growth rate of X. The proof of Theorem 1.4 in fact establishes the following result.

Theorem 4.3. *Let (X, d_1), (X, d_2) be a pair of metric spaces of Hausdorff dimension s, and assume that there are two-sided bounds on the s-dimensional Hausdorff measure of balls of any given radius. Then, if (X, d_1) is almost isometric to (X, d_2), we must have $h^+(X, d_1) = h^+(X, d_2)$, and $h^-(X, d_1) = h^-(X, d_2)$.*

For an easy example illustrating this more general setting, consider the setting of connected metric graphs. The one-dimensional Hausdorff measure of a $\tilde{d}$-ball of radius r in the graph will then be the sum of the edge lengths of the (portions of) edges inside the ball. If one imposes a lower bound on the length of edges, and an upper bound on the degree of vertices, this easily leads to two-sided bounds on the one-dimensional Hausdorff measure of balls of any given radius. Thus, Theorem 4.3 applies to this class of metric spaces.

Let us give an application of this: consider a finite combinatorial graph X, with the property that each vertex has degree ≥ 3. The universal cover of X is then a combinatorial tree T. One can metrize X in many different ways, by assigning lengths to each edge, and making each edge isometric to an interval of the corresponding length. We let $\mathcal{M}(X)$ be the space of such metrics. Any such metric d lifts to give a $\pi_1(X)$-invariant metric $\tilde{d}$ on the tree T, with lower bounds on the edge lengths and upper bounds on the degree of vertices. In this special case, one has that $h^+(T, \tilde{d}) = h^-(T, \tilde{d})$, and we will denote the common value by $h_{\mathrm{vol}}(d)$. Then, Theorem 4.3 tells us that, for $d_0, d_1 \in \mathcal{M}(X)$ arbitrary, if $(T, \tilde{d}_0)$ is almost-isometric to $(T, \tilde{d}_1)$, then $h_{\mathrm{vol}}(d_0) = h_{\mathrm{vol}}(d_1)$.

We now view h_{vol} as a function on the space $\mathcal{M}(X)$, an open cone inside some large $\mathbb{R}^n$ (where n is the number of edges in X). It is easy to see, from the scaling property of Hausdorff dimension, that

$$h_{\mathrm{vol}}(\alpha \cdot d) = \frac{1}{\alpha} h_{\mathrm{vol}}(d).$$

As such, it is reasonable to impose a normalizing condition (for example, letting $\mathcal{M}_1(X) \subset \mathcal{M}(X)$ be the subspace of metrics whose sum of lengths is $= 1$). The behavior of h_{vol} on the subspace $\mathcal{M}_1(X)$ was studied by Lim in her thesis, and she showed [Li] there is a *unique* metric d_0 that *minimizes* h_{vol}—moreover, she gave an explicit computation of this metric in terms of the degrees at the various vertices of X. Some related work was done by Kapovich and Nagnibeda [KN] and by Rivin [Ri]. In conjunction with Lim's result, our Theorem 4.3 implies the following result.

Corollary 4.4. *Let X be a combinatorial graph, d_0 the metric produced by Lim, and $d_1 \in \mathcal{M}_1(X)$ any metric on X distinct from d_0. Then, $(T, \tilde{d}_0)$ and $(T, \tilde{d}_1)$ are* not *almost isometric.*

Remark 4.5. The reader will undoubtedly wonder as to whether some similar result holds for the Fuchsian buildings discussed in Section 3. While Theorem 4.3 applies to Fuchsian buildings (of course, using two-dimensional Hausdorff measure, and appropriate constraints on the metrics), the behavior of the functional h_{vol} on the corresponding moduli space of metrics is much more mysterious. In particular, (local) minimizers of the functional are not known, and indeed uniqueness of such a minimizer is not known (see Ledrappier and Lim [LL] for some work on this question).

5. Concluding Remarks

Much of the work in this paper was motivated by the following question.

Question 5.1. *Let M be an aspherical manifold with universal cover $\tilde{M}$. Can one find a pair of Riemannian metrics g, h on M, whose lifts to the universal cover $(\tilde{M}, \tilde{g})$, $(\tilde{M}, \tilde{h})$ are almost isometric but* not *isometric?*

Our results in this paper give a number of examples (see Corollaries 1.3, 1.6, 1.7) of pairs of metrics on compact manifolds whose lifts to the universal cover are quasi-isometric, but *not* almost isometric. Thus, any QI between the universal covers must have multiplicative constant greater than 1. One can ask whether there is a "gap" in the multiplicative constant. We suspect such gaps do not exist in general.

Question 5.2. *Can one find an aspherical manifold M and a pair of Riemannian metrics g, h, with the property that the universal covers are (C_i, K_i)-quasi-isometric via a sequence of maps f_i, where $C_i \to 1$, but are* not *almost-isometric?*

In the special case where M is a higher genus surface, and the metrics under consideration are negatively curved, one has complete answers to both of the above questions (see [LSvL]).

In a different direction, we saw in our Theorem 1.4 that the rate of exponential growth is an almost-isometry invariant (though it is *not* a quasi-isometry invariant). At the other extreme, universal covers of infra-nil manifolds, equipped with the lift of a metric, are known to have *polynomial* growth. More precisely, $\mathrm{Vol}(B(r)) \sim C(g) \cdot r^k$ where the integer $k \in \mathbb{N}$ depends only on M, but the constant $C(g)$ depends on the chosen metric g on M. One can ask the following question.

Question 5.3. *Let M be an infra-nil manifold, and g, h a pair of Riemannian metrics on M. Denote by $C(g), C(h) \in (0, \infty)$ the coefficient for the polynomial growth rate of balls in $\tilde{M}$. If $(\tilde{M}, \tilde{g})$ is almost isometric to $(\tilde{M}, \tilde{h})$, does it follow that $C(g) = C(h)$?*

It is easy to see that the estimates appearing in our proof of Theorem 1.4 are too crude to deal with the coefficient of polynomial growth. In the special case where (M, g) is a flat surface, we have an affirmative answer to Question 5.3: our Corollary 1.7 implies that h must also be flat, from which it is immediate that $C(g) = C(h)$. Observe that the exponential volume growth rate can alternatively be interpreted as either an isoperimetric profile, or a filling invariant (in the sense of Brady and Farb [BF]). One could also ask whether one can use these alternate viewpoints to define some new almost-isometry invariants.

We have focused on almost-isometric metrics on the universal cover of a fixed topological manifold M. We could also ask similar questions for a pair of closed smooth manifolds (N^n, g), (M^m, h) where $n \leq m$. For instance, can one find an almost-isometric embedding $(\tilde{N}, \tilde{g}) \to (\tilde{M}, \tilde{h})$ that is not at finite distance from an isometric embedding? In the case where the universal covers are isometric to irreducible (Euclidean) buildings, or to irreducible non-positively curved symmetric spaces of equal rank $r > 1$, recent work of Fisher and Whyte establishes that every almost-isometric embedding is at finite Hausdorff distance from an isometric embedding (see [FW, Corollary 1.8]).

Finally, while our purpose in this paper was mostly the study of *spaces* up to almost-isometry, one can ask similar questions at the level of finitely generated *groups*. One says that a pair of finitely generated groups G, H are almost isometric provided he can find finite symmetric generating sets $S \subset G$, $T \subset H$ so that the corresponding metrics spaces (G, d_S) and (H, d_T) are almost isometric. A basic problem here is to resolve the following problem.

Question 5.4. *Let G, H be a pair of quasi-isometric groups. Must they be almost isometric?*

For instance, it is easy to see that commensurable groups are almost isometric. In general, one suspects that the answer should be "no," though again examples seem elusive. The corresponding question for bi-Lipschitz equivalence was answered in the negative by Dymarz [Dy]. One aspect which seems to make the almost-isometric question harder than the corresponding bi-Lipschitz question lies in the fact that distinct word metrics on a fixed finitely generated group G are not *a priori* almost isometric to each other, whereas they are always bi-Lipschitz equivalent. This means that understanding groups up to AI involves understanding *all* word metrics. For instance, let us specialize to the case where the groups G, H have exponential growth. Then, by varying the possible generating sets for G, H, and looking at the corresponding exponential volume growth rate, we obtain the *growth spectra* $\mathrm{Spec}(G), \mathrm{Spec}(H) \subset (0, \infty)$. An affirmative answer to 5.4 would imply, by Theorem 4.3, that when G, H are quasi-isometric, $\mathrm{Spec}(G) \cap \mathrm{Spec}(H) \neq \emptyset$, a result that seems unlikely to be true in full generality.

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