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Relative cubulation of relative strict hyperbolization

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Abstract

We prove that many relatively hyperbolic groups obtained by relative strict hyperbolization admit a cocompact action on a CAT(0) cubical complex. Under suitable assumptions on the peripheral subgroups, these groups are residually finite and even virtually special. We include some applications to the theory of manifolds, such as the construction of new non-positively curved Riemannian manifolds with residually finite fundamental group, and the existence of non-triangulable aspherical manifolds with virtually special fundamental group.

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Contents

1 INTRODUCTION

Hyperbolization procedures were introduced by Gromov in [21] as a way to construct aspherical manifolds which allows for more flexibility than the classical methods coming from Lie theory. Roughly speaking, a hyperbolization procedure is defined by the choice of a *hyperbolizing cell* and of a class of combinatorial complexes to be hyperbolized, and it consists in replacing the cells of a complex with copies of the chosen hyperbolizing cell. The resulting space is often called a *hyperbolized complex*, and it is a metric space of non-positive curvature in the sense of [7]; in particular, it is aspherical. In this paper, we consider the *strict* hyperbolization procedure introduced by Charney and Davis in [9], which can produce a space of strictly negative curvature.

A main feature of any hyperbolization procedure is that links of cells remain essentially unchanged under hyperbolization. In particular, hyperbolizing a triangulation of a closed (pseudo-)manifold results in a closed aspherical (pseudo-)manifold. This has been used to construct examples of closed aspherical (pseudo-)manifolds that display various pathological features; see [9, 11, 41] for a few examples. For some applications, one may want to preserve a certain subcomplex of the original complex, that is, ensure that the hyperbolized complex contains a certain prescribed subspace. This can be arranged via *relative* hyperbolization procedures, which consist in coning off the desired subcomplex, performing a non-relative hyperbolization, and then removing a neighborhood of the cone point. The link of the cone point is homeomorphic to the desired subcomplex. See [1, 5, 10, 12, 31, 34, 39, 46, 50] for a few examples of this strategy.

Since the spaces obtained from hyperbolization are aspherical, a natural problem is to understand what their fundamental groups look like. As one might expect, the fundamental group of a space obtained via Charney–Davis strict hyperbolization is a hyperbolic group. In [33], the authors showed that these hyperbolic groups are virtually special. In particular, all the closed aspherical manifolds obtained via Charney–Davis strict hyperbolization have linear, hence residually finite, fundamental group.

Similarly, the fundamental group of a space obtained via the relative version of Charney–Davis strict hyperbolization is a relatively hyperbolic group, with peripheral structure given by the fundamental group of the desired subcomplex; see [6]. In this paper, we aim at extending the results of [33] to the relative setting. To this end, we consider the relatively hyperbolic groups that arise from relative strict hyperbolization and construct certain actions on CAT(0) cubical complexes. Let us fix some objects and notations that appear in the statement of our main results.

Let K be a finite simplicial complex and L a subcomplex. Assume that the complex $\mathrm{C}(K, L)$ obtained by coning off each component of L is homogeneous and without boundary (i.e., each simplex is contained in an n-simplex, and each $(n-1)$-simplex is contained in at least two n-simplices). Note that this implies in particular that L is homogeneous of dimension $n-1$ and without boundary. As a motivating example, consider the case in which K is a triangulation of a compact manifold with boundary and L is the induced triangulation of the boundary. This is a common setup in many contexts in which relative hyperbolization procedures are used, and will be a standing assumption throughout this paper. Let $\mathcal{R}(K, L)$ denote the relative strict hyperbolization of K with respect to L; see [6, 12] or Section 3 for details. There is a natural π_1-injective embedding of L into $\mathcal{R}(K, L)$. Let $\mathcal{P}$ be a set of representatives of the conjugacy classes of the fundamental groups of the components of L. Then, the group $G = \pi_1(\mathcal{R}(K, L))$ is known to be relatively hyperbolic with respect to $\mathcal{P}$. Our main result is the following.

Theorem A. *The group $G = \pi_1(\mathcal{R}(K, L))$ acts on a* CAT(0) *cubical complex $C(\widehat{X_\Gamma})$ by isometries and satisfying the following properties:*

(1) $G \backslash C(\widehat{X_\Gamma})$ *is compact.*
(2) *Each $P \in \mathcal{P}$ acts elliptically on $C(\widehat{X_\Gamma})$.*
(3) *For each cube C of $C(\widehat{X_\Gamma})$,* $\mathrm{Stab}_G(C)$ *is either maximal parabolic, or else is full relatively quasi-convex, hyperbolic, and virtually compact special.*

This set up does not automatically fit in the available literature about cubulation of relatively hyperbolic groups, such as [16, 17, 24, 45]. Indeed, the action of G on $C(\widehat{X_\Gamma})$ is very far from proper, because most vertices have infinite stabilizer. This action is not relatively geometric in the sense of [16, 17], and not even weakly relatively geometric in the sense of [24]. The problem here does not arise from the parabolics, but rather from points that are away from the fixed points of the parabolics. It is the same lack of properness already encountered in [33], which required a criterion obtained by Groves and Manning in [25] to promote an improper action of a hyperbolic group to a proper one (on a different cubical complex). A relative analog of [25] is not available yet in full generality. However, one can obtain the following.

Theorem B. *Let $G = \pi_1(\mathcal{R}(K, L))$ and $\mathcal{P}$ be as above. Then, the following hold.*

(1) *If each $P \in \mathcal{P}$ is residually finite, then G is residually finite and each $P \in \mathcal{P}$ is separable.*
(2) *If each $P \in \mathcal{P}$ is hyperbolic and virtually compact special, then G is hyperbolic and virtually compact special.*

Here, (1) follows from a more general result contained in Appendix A by Groves and Manning (see Theorem A.4). In particular, Theorem B shows that relative strict hyperbolization is unlikely to provide a negative answer to [42, Problem 6.6], which is the relative analog of the well-known question of Gromov about residual finiteness of hyperbolic groups. A result analogous to Theorem B has recently been obtained by Avramidi, Okun, and Schreve for a different relative hyperbolization procedure, obtained by combining the Davis reflection group trick with the Charney–Davis strict hyperbolization; see [1, Theorem D].

We conclude the introduction with a brief description of some applications to the theory of manifolds.

1.1 | Aspherical manifolds with residually finite fundamental groups

In Section 5.1, we obtain examples of closed aspherical manifolds which have residually finite fundamental group. This is based on a construction that we call *hyperbolized mapping torus*, and which has already been considered for instance in [39, 46]. It consists in taking a manifold M, hyperbolizing $M \times [-1, 1]$ relatively to the boundary, and then gluing the two boundary components to get a closed manifold $\mathcal{T}(M)$. We show that certain properties of the fundamental group of M (such as residual finiteness) are inherited by the fundamental group of $\mathcal{T}(M)$.

By choosing the manifold M appropriately, one can thus obtain examples which are "new," in the sense that they are not homotopy equivalent to manifolds for which residual finiteness of the fundamental group was previously known: see Theorem 5.3 (non-positively curved, dimension $n \geqslant 6$) and Remark 5.5 (negatively curved, dimension $n \geqslant 9$). These examples are obtained starting from lattices in $\mathrm{SL}(3, \mathbb{R})$ and in $\mathrm{Sp}(1, 2) = \mathrm{Isom}(\mathbb{H}^2_{\mathbb{H}})$, respectively.

For $n \geqslant 5$, we can also use this construction to obtain some closed Riemannian manifolds having negative curvature and virtually compact special fundamental group; see Theorem 5.8. These can be obtained starting from Gromov–Thurston manifolds or strictly hyperbolized manifolds. On the other hand, it is worth mentioning that there are closed aspherical manifolds that do not virtually fiber over the circle and whose fundamental group is hyperbolic and virtually compact special; see [1, Theorem A] for odd-dimensional examples.

1.2 | Cobordism of manifolds

In Section 5.2, we consider some classical applications of hyperbolization procedures to cobordism of manifolds. In Corollary 5.9, we obtain that a cobordism between triangulable manifolds with residually finite fundamental group can be chosen to have residually finite fundamental group. Similarly, in Corollary 5.11 we obtain that every closed flat manifold bounds geometrically a manifold of pinched negative curvature with residually finite fundamental group. In both cases, the ambient fundamental group is relatively hyperbolic and the fundamental groups of the boundary components are separable.

1.2.1 | Aspherical manifolds that cannot be triangulated

Finally, in Section 5.3 we show that for $n \geqslant 6$ there is a closed aspherical n-manifold that cannot be triangulated and whose fundamental group is hyperbolic and virtually compact special; see Theorem 5.12. These are the manifolds constructed in [10] by gluing together two pieces, one obtained via strict hyperbolization and one obtained via relative strict hyperbolization. Combining the results of [33] and of this paper, we show that these manifolds have virtually special fundamental group. In particular, they have a rich collection of finite covers, and it makes sense to ask if these manifolds are *virtually triangulable*, that is, if they admit a finite cover that can be triangulated. While we do not answer this question, we show that no cover of odd degree can be triangulated. The situation for covers of even degree is more delicate. For instance, the Galewski–Stern 5-manifold from [18] is non-triangulable and non-orientable, but its orientable double cover is triangulable, since all orientable closed 5-manifolds are triangulable by [47].

1.3 | Outline

In Section 2, we collect background notation and terminology. In Section 3, we present the relative strict hyperbolization procedure, introduce the spaces of interest in this paper, and present some lemmas about the action of the relatively hyperbolized groups on them. In Section 4, we construct the dual cubical complex, prove it is CAT(0), and complete the analysis of stabilizers. The proofs of Theorems A and B appear in Section 4.3. In Section 5, we discuss some applications to the theory of manifolds. Appendix A by Groves and Manning contains the proof that a group satisfying the conclusion of Theorem A with respect to residually finite peripherals has separable full relatively quasiconvex subgroups.

2 | PRELIMINARIES

In this section, we fix review some standard background and terminology needed in this paper.

2.1 | Cell complexes

For background on cell complexes, we refer the reader to [7]. A *cell complex* is a topological space X obtained by gluing together cells along their faces, in such a way that each cell embeds in X and the intersection of any two cells is either empty or a cell. A *simplicial complex* is a cell complex obtained by gluing copies of the standard simplex $\triangle^n$. A *cubical complex* is a cell complex obtained by gluing copies of the standard cube $\square^n = [0,1]^n$. The *dimension* of a cell complex is the maximum dimension of its cells. We say that an n-dimensional cell complex is *homogeneous* if every cell is contained in a cell of dimension n, and that it is *without boundary* if every $(n-1)$-cell is contained in at least two different n-cells.

A cell complex X is *piecewise spherical*, *Euclidean*, or *hyperbolic* if its cells can be realized as totally geodesic cells in a round sphere $\mathbb{S}^n$, a Euclidean space $\mathbb{E}^n$, or the (real) hyperbolic space $\mathbb{H}^n$, and the gluing maps can be realized by isometries. If X has finitely many isometry classes of cells then the metrics defined on the cells can be glued together and X is a complete geodesic metric space (see [7, Theorem I.7.50]). In particular, if X is simplicial or cubical then it carries a standard piecewise Euclidean metric.

If X and Y are cell complexes, a continuous function $f : X \to Y$ is a *combinatorial map* if for every cell C of X we have that f is a homeomorphism from C to a cell $f(C)$ of Y. A simplicial n-dimensional complex X is *foldable* if it admits a combinatorial map $f : X \to \triangle^n$ which is injective on each simplex. Such a map will be called a *folding* for X. An analogous definition can be given in the cubical case in terms of a map to $\square^n$. The barycentric subdivision of any cell complex is a foldable simplicial complex. Also, note that the cells of a foldable complex are necessarily embedded.

The *link* of a point p in a cell complex X is defined to be the space of unit vectors at p that point into the cells of X that contain p, and is denoted $\mathrm{lk}(p, X)$. Similarly, the *link* of a cell C in a cell complex X is defined as the space of unit vectors based at an interior point of C which are orthogonal to C and point into the cells that contain C. It is denoted $\mathrm{lk}(C, X)$. Observe that links of points or cells are naturally piecewise spherical complexes, and that if X is simplicial or cubical, then all links are simplicial.

Finally, note that the *cone* $\mathrm{C}(X)$ over a cell complex X admits two natural topologies, namely the cone topology (i.e., the quotient topology coming from the quotient map $X \times [0,1] \to \mathrm{C}(X)$ that defines the cone $\mathrm{C}(X)$), and the metric topology (i.e., the one coming from the fact that $\mathrm{C}(X)$ has a natural structure of a cell complex.) If X is finite, then the two topologies agree, but when X is infinite the cone topology is much finer; in particular, it is not first countable, hence not metrizable. Similarly, if X is a cell complex and Y is a subcomplex, then the *relative cone* $\mathrm{C}(X, Y)$ obtained by attaching the cone over Y to X is endowed with the metric topology. We also say that $\mathrm{C}(X, Y)$ is obtained from X by *coning off* Y.

2.2 | Bounded curvature

We will consider the usual notions of curvature for metric spaces, see [7, §II.1, §III.H.1] for details. A space is *non-positively curved* if it is locally CAT(0), and *negatively curved* if it is locally CAT(k) for some $k < 0$. In this paper "hyperbolic" always means "Gromov hyperbolic" unless otherwise specified. A CAT(0) space is contractible; a CAT(k) space is hyperbolic as soon as $k < 0$. The action of a group on a metric space is *geometric* if it is cocompact, proper, and isometric. A group is CAT(0) (resp. hyperbolic) if it admits a geometric action on a CAT(0) (resp. hyperbolic) space.

We recall the following well-known characterization of non-positive curvature for cubical complexes, see [7, Theorems II.5.20]). Recall that links in a cubical complex are simplicial. A simplicial complex is *flag* if any $k+1$ pairwise adjacent vertices span a k-simplex.

Lemma 2.1 (Gromov's link condition). *Let X be a cubical complex. Then, X is non-positively curved if and only if the link of each vertex is flag.*

Finally, let us recall some standard terminology about relative hyperbolicity; see [30] for details. Let G be a finitely generated group and let $\mathcal{P}$ be a finite collection of subgroups of G. Given a Cayley graph Γ for G, we can construct the *coned-off Cayley graph* $\widehat{\Gamma}$ by adding a vertex v_{gP} for each coset of a subgroup $P \in \mathcal{P}$ and attaching v_{gP} to vertices in gP with edges of length $\frac{1}{2}$. We say that $(G, \mathcal{P})$ is *relatively hyperbolic* if some (every) coned-off Cayley graph $\widehat{\Gamma}$ is hyperbolic and has the bounded coset penetration property. We will also say that G is *hyperbolic relative to* $\mathcal{P}$; see [30] for details and other equivalent definitions. A subgroup $P \in \mathcal{P}$ is a *peripheral subgroup* and $\mathcal{P}$ is a *peripheral structure* on G. A conjugate of a peripheral subgroup is a *maximal parabolic subgroup*, and a *parabolic subgroup* is a subgroup of a maximal parabolic subgroup. Finally, we say that a subgroup $H \subseteq G$ is *relatively quasiconvex* if it is quasiconvex in $\widehat{\Gamma}$, that is, if there is constant $K \geqslant 0$ such that every geodesic in $\widehat{\Gamma}$ with endpoints in H lies at distance at most K from H. (In our setting, this is equivalent to being quasiconvex in the *relative Cayley graph*, obtained by adding $\mathcal{P}$ to the generating set. Indeed, these two graphs are quasi-isometric and induce the same metric on G, seen as a subset of the vertex set in each case.)

The following criterion for relative quasiconvexity is well-known to experts and follows from a relative version of Milnor–Švarc obtained by Charney and Crisp in [8]. We include a proof for the convenience of the reader. Here, we say that an action is *discontinuous* if orbits are discrete and an *isotropy subgroup* is a subgroup with non-empty fixed set.

Lemma 2.2. *Let X be a hyperbolic length space. Let G be a finitely generated group admitting a discontinuous, cocompact, isometric action on X. Let $\mathcal{P}$ be a collection of subgroups of G consisting* *of exactly one representative from each conjugacy class of the maximal isotropy subgroups for this action. Assume that G is relatively hyperbolic with respect to $\mathcal{P}$. Let $Y \subseteq X$ be a quasiconvex subset. Let H be the stabilizer of Y in G. If H acts cocompactly on Y, then H is a relatively quasiconvex subgroup of $(G, \mathcal{P})$.*

Proof. Fix a basepoint $p \in Y$. It follows from the proof of [8, Theorem 5.1] that the orbit map $G \to X$, $g \mapsto g.p$ is a quasi-isometry, when G is equipped with the metric induced from $\widehat{\Gamma}$. Since every point of $\widehat{\Gamma}$ is at distance at most $\frac{1}{2}$ from G, this orbit map extends to a (λ, ε)-quasi-isometry $f : \widehat{\Gamma} \to X$ for some constants $\lambda \geqslant 1, \varepsilon > 0$. Now, the proof proceeds as in the absolute case.

Let δ be the hyperbolicity constant of X and K the quasiconvexity constant of Y. Now, pick $h \in H$ and a geodesic $\gamma : I \to \widehat{\Gamma}$ from 1_G to h. Then, $f(\gamma)$ is a (λ, ε)-quasi-geodesic in X from p to $f(h) = h.p$. Let α be a geodesic path in X from p to $f(h) = h.p$. By the Morse lemma (see, for instance, [7, Theorem III.H.1.7]) we get that $f(\gamma) \subseteq \mathcal{N}_A(\alpha)$ for some constant $A = A(\delta, \lambda, \varepsilon)$. Moreover since $p, h.p \in Y$ and Y is K-quasiconvex, we get also that $\alpha \subseteq \mathcal{N}_K(Y)$. Finally, since H acts cocompactly on Y, there is some $F > 0$ such that $Y \subseteq \mathcal{N}_F(f(H))$. Combining these statements we see that $f(\gamma) \subseteq \mathcal{N}_B(f(H))$ for some $B = B(\delta, K, \lambda, \varepsilon, F)$.

In particular, for each $x \in \gamma$ there exists some $h_x \in H$ such that $d_X(f(x), f(h_x)) \leqslant B$. But then, since f is a (λ, ε)-quasi-isometric embedding, we also get

$$d_{\widehat{\Gamma}}(x, h_x) \leqslant \lambda(d_X(f(x), f(h_x)) + \varepsilon) \leqslant \lambda(B + \varepsilon)$$

which shows that there exists a constant $C = C(\delta, K, \lambda, \varepsilon, F)$ such that $\gamma \subseteq \mathcal{N}_C(H)$, that is, H is C-quasiconvex in $\widehat{\Gamma}$. □

In particular, when Y is just the fixed point of a maximal parabolic subgroup one recovers the fact that each maximal parabolic subgroup is relatively quasiconvex.

Finally, we will need the following definitions. A subgroup $H \subseteq G$ is *full relatively quasiconvex* if it is relatively quasiconvex and for any $g \in G$ and for any peripheral subgroup $P \in \mathcal{P}$ the intersection $H \cap gPg^{-1}$ is either finite or of finite index in gPg^{-1}. For instance, since the collection of maximal parabolic subgroups is almost malnormal, a maximal parabolic subgroup is automatically a full relatively quasiconvex subgroup. A subgroup $H \subseteq G$ is *strongly relatively quasiconvex* if it is relatively quasiconvex and for any $g \in G$ and for any peripheral subgroup $P \in \mathcal{P}$ the intersection $H \cap gPg^{-1}$ is finite.

3 | RELATIVE STRICT HYPERBOLIZATION

Let K be a finite connected simplicial complex, and L a (not necessarily connected) subcomplex. The main example the reader should keep in mind is when K is a compact manifold with boundary L.

3.1 | The basic spaces

We consider the relative hyperbolization procedure of [12, §2], as well as its strict version, see [6]. See Figure 1 for a picture of the main steps involved in the construction. For an exposition of

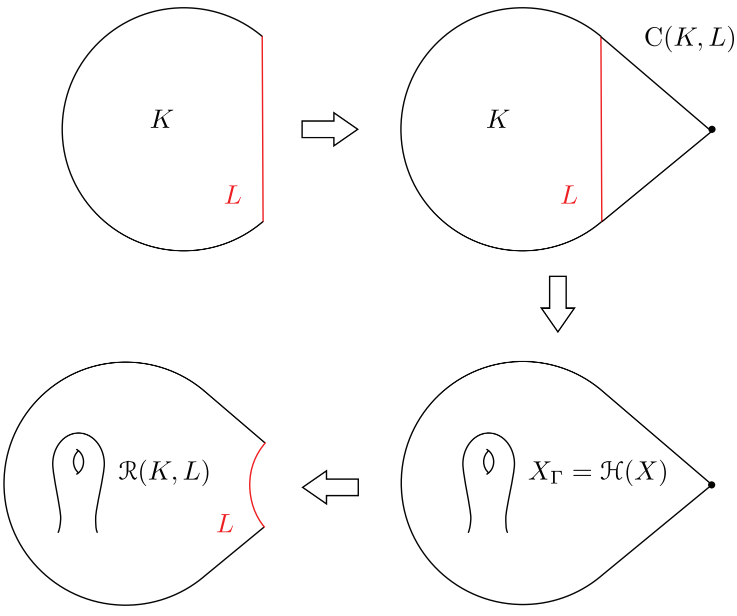

FIGURE 1 The relative strict hyperbolization procedure.

Gromov's cylinder construction and Charney–Davis strict hyperbolization, we refer the reader to [33], and references therein.

Let $\mathrm{C}(K, L)$ be the simplicial complex obtained by attaching a cone C_i over each connected component L_i of L. Denote by y_i the cone point. We will assume that $\mathrm{C}(K, L)$ is foldable, which can always be achieved by taking a barycentric subdivision of $\mathrm{C}(K, L)$.

Let $X = \mathcal{G}(\mathrm{C}(K, L))$ be the cubical complex obtained by applying Gromov's cylinder construction. Then, X is a non-positively curved and foldable cubical complex. Cone points arising from the cone points of $\mathrm{C}(K, L)$ are still denoted by y_i.

Let $X_\Gamma = \mathcal{H}(X) = \mathcal{H}(\mathcal{G}(\mathrm{C}(K, L)))$ be the piecewise hyperbolic complex obtained by applying the Charney–Davis strict hyperbolization from [9] to $X = \mathcal{G}(\mathrm{C}(K, L))$. Roughly speaking, this is obtained by replacing each cube of X with a certain hyperbolic manifold with corners $\square^n_\Gamma$. We still denote by y_i the points arising from the cone points. Links are essentially preserved, in the sense that $\mathrm{lk}(y_i, X_\Gamma)$ is isomorphic to a subdivision of $\mathrm{lk}(y_i, \mathrm{C}(K, L)) = L_i$.

Let $\mathcal{R}(K, L)$ be the space obtained from X_Γ by removing a small open ball around each y_i. This is the *relative strict hyperbolization* of K with respect to L. Note that L embeds (up to subdivision) as a subcomplex of $\mathcal{R}(K, L)$. The following are the main features of this procedure, see [6, 12].

(1) $\mathcal{R}(K, L)$ is aspherical if and only if each component of L is aspherical.
(2) If K is a PL manifold with boundary L, then $\mathcal{R}(K, L)$ is a PL manifold with boundary L. The same is true in the smooth category, if one works with smooth triangulations.
(3) Each component L_i is π_1-injective. If $\mathcal{P}$ denotes a set of representatives of the conjugacy classes of the subgroups $\pi_1(L_i)$, then Belegradek proved in [6] that $G = \pi_1(\mathcal{R}(K, L))$ is hyperbolic relative to $\mathcal{P}$. Note that since K is finite, $\mathcal{P}$ is finite and $P \in \mathcal{P}$ is finitely presented.

Remark 3.1. X_Γ is negatively curved, but in general the metric induced on $\mathcal{R}(K, L)$ is not even non-positively curved.

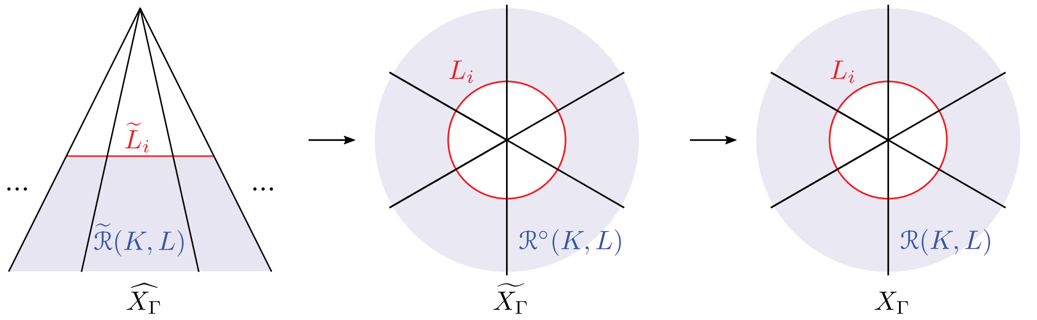

FIGURE 2 Some covering spaces, local pictures around a cone/branch point.

Remark 3.2. If K is compact and homogeneous, and $L = \partial K$, then $\pi_1(X_\Gamma)$ is hyperbolic and virtually compact special by [33]. However, $\mathcal{R}(K,L)$ is not π_1-injective in X_Γ, so $G = \pi_1(\mathcal{R}(K,L))$ is not naturally a subgroup of $\pi_1(X_\Gamma)$.

Remark 3.3 (A standing assumption). To use the techniques from [33] we need one additional assumption, namely that the pair (K,L) is such that $\mathrm{C}(K,L)$ is homogeneous and without boundary. This condition ensures that $\mathcal{G}(\mathrm{C}(K,L))$ is an *admissible* cubical complex in the sense of [33, §3]. We will assume this condition in this paper. In particular, $X = \mathcal{G}(\mathrm{C}(K,L))$ comes with a folding $f : X \to \square^n$.

Example 3.4. The motivating case is when K is a compact manifold with boundary L. More generally, K could be any homogeneous complex with boundary $L = \partial K$, or we could take K to be homogeneous and without boundary and L to be a homogeneous codimension-1 subcomplex without boundary.

3.2 | Some useful covering spaces

Our goal is to construct a nice action of $G = \pi_1(\mathcal{R}(K,L))$ on a CAT(0) cubical complex. We follow [33] and construct a cubical complex dual to a certain collection of subspaces (mirrors) defined on a suitable cover of X_Γ. Some adaptations are needed to work relatively to L. The idea is to follow the treatment in [12, §2]; see Figure 2.

Let $\widetilde{X_\Gamma}$ be the universal cover of X_Γ. Note that the link of each lift of a cone point y_i is naturally identified with a subdivision of L_i.

Let $\mathcal{R}^\circ(K,L)$ be the lift of $\mathcal{R}(K,L)$ to $\widetilde{X_\Gamma}$. This is a covering space of $\mathcal{R}(K,L)$, but it is in general not simply connected. Indeed, there is an injection $\pi_1(L_i) \to \pi_1(\mathcal{R}^\circ(K,L))$, because there is a retraction $\mathcal{R}^\circ(K,L) \to \mathrm{lk}(y_i, \widetilde{X_\Gamma}) \cong L_i$ for each cone point y_i (see [12, Lemma 2.2]).

Let $\widehat{X_\Gamma}$ be the *branched universal cover* of X_Γ, that is, the space obtained by puncturing $\widetilde{X_\Gamma}$ at all the cone points, taking the universal cover, and then taking the metric completion. The ideal points in the completion are isolated points, covering the cone points, and we call them *branch points*. The space $\widehat{X_\Gamma}$ is a piecewise hyperbolic complex, which is locally isometric to $\widetilde{X_\Gamma}$, except at the branch points. The link of a branch point above a cone point y_i is isomorphic to a subdivision of the universal cover $\widetilde{L}_i$ of L_i. Since $\widetilde{X_\Gamma}$ is CAT(−1), we have that L_i is CAT(1), and therefore $\widetilde{L}_i$ is CAT(1). It follows that $\widehat{X_\Gamma}$ is a CAT(−1) piecewise hyperbolic complex; see [7, §II.5] for details.

Finally, let $\widetilde{\mathcal{R}}(K,L)$ be the lift of $\mathcal{R}^\circ(K,L)$ to $\widehat{X_\Gamma}$. Note that $\widetilde{L}_i$ is simply connected and each branch point has a simply connected neighborhood. By Seifert–Van Kampen we get that $\widetilde{\mathcal{R}}(K,L)$ is simply connected, so $\widetilde{\mathcal{R}}(K,L)$ is the universal cover of $\mathcal{R}(K,L)$.

Remark 3.5. The space $\widehat{X_\Gamma}$ is homeomorphic to the space obtained from the universal cover $\widetilde{\mathcal{R}}(K,L)$ by coning off each copy of the universal cover $\widetilde{L}_i$ of L_i. Moreover, there is a natural continuous map $\hat{\pi} : \widehat{X_\Gamma} \to \widetilde{X_\Gamma}$. This is a covering map in the complement of the branch points.

As a result of the above construction, we obtain a *folding map*

$$\hat{f} : \widehat{X_\Gamma} \xrightarrow{\hat{\pi}} \widetilde{X_\Gamma} \xrightarrow{\pi} X_\Gamma \xrightarrow{g_X} X \xrightarrow{f} \square^n.$$

As in [33], we define a *mirror* in $\widehat{X_\Gamma}, \widetilde{X_\Gamma}$ and X_Γ to be a connected component of the full preimage of a codimension-1 face of $\square^n$ via this folding map. The collection of mirrors induces a stratification of $\widehat{X_\Gamma}$, and we can define a *k-cell* to be the closure of a connected component of the full preimage of an open k-face of $\square^n$. In particular, a 0-cell is a vertex, a 1-cell is an edge, and an n-cell is called a *tile*, and is isometric to the universal cover $\widetilde{\square^n_\Gamma}$ of $\square^n_\Gamma$ (recall this is the hyperbolic manifold with corners constructed in [9] to define the strict hyperbolization procedure). This provides a *stratification* and a *tiling* of $\widehat{X_\Gamma}$, analogous to the ones obtained for $\widetilde{X_\Gamma}$ in [33].

Any mirror going through a branch point might have some non-locally finite behavior, but the collection of mirrors itself is locally finite: at most $n = \dim X$ mirrors go through each point of $\widehat{X_\Gamma}$.

Remark 3.6 (The case of L with simply connected components). The link of a cone point of X_Γ is homeomorphic to the corresponding component of L. In particular, if each component of L is simply connected, then each cone point has a simply connected link and a simply connected neighborhood (namely, the cone over the link). It follows from the Seifert–Van Kampen theorem that puncturing X_Γ at the (finitely many) cone points does not change the fundamental group, that is, the inclusion $\mathcal{R}(K,L) \subseteq X_\Gamma$ induces an isomorphism on fundamental groups. In particular, the group $G = \pi_1(\mathcal{R}(K,L)) = \pi_1(X_\Gamma)$ is hyperbolic and virtually compact special by [33].

For a motivating example, let K be a manifold whose boundary $L = \partial K$ has simply connected components. In [46], this set up was considered to construct manifolds with hyperbolic fundamental group that do not admit any real projective or flat conformal structures, in any dimension at least 5.

3.3 | The action of G on $\widehat{X_\Gamma}$

Recall from Section 3.1 that if $\mathcal{P}$ is a set of representatives of the conjugacy classes of the subgroups $\pi_1(L_i)$, where L_i is a connected component of L, then the group $G = \pi_1(\mathcal{R}(K,L))$ is hyperbolic relative to $\mathcal{P}$. Moreover, G naturally acts on $\widetilde{\mathcal{R}}(K,L)$ by deck transformations. As noted in Remark 3.5 this space embeds in $\widehat{X_\Gamma}$, the complement being given by the cones over the various copies of the universal cover $\widetilde{L}_i$ of L_i. Note that the action permutes these subspaces $\widetilde{L}_i$, and each maximal parabolic subgroup stabilizes one of them.

We extend the action of G to the entire $\widehat{X_\Gamma}$ by defining it on the cones in the obvious way. The resulting action is cocompact and by isometries, but not proper, because each maximal parabolic fixes a branch point. To address the failure of properness, we now study the cell stabilizers for the action of G on $\widehat{X_\Gamma}$.

Lemma 3.7. *Let σ be a cell of $\widehat{X_\Gamma}$.*

(1) *If σ is a branch point, then* $\mathrm{Stab}_G(\sigma)$ *is a maximal parabolic subgroup.*
(2) *If σ is not a branch point, then* $\mathrm{Stab}_G(\sigma)$ *is a hyperbolic and virtually compact special group.*

Proof. Since (1) follows directly from the construction of the space $\widehat{X_\Gamma}$, we just prove (2). Following the second part of the proof of [33, Lemma 5.13], we have that $\widehat{X_\Gamma}$ folds to $\square^n_\Gamma$, and cell stabilizers map isomorphically to quasiconvex subgroups of $\Gamma_X = \pi_1(\square^n_\Gamma)$. This group is hyperbolic and virtually compact special (see [33, Lemma 5.12]), so cell stabilizers are hyperbolic and virtually compact special too (see [33, Lemma 5.10]). □

Lemma 3.8. *Let σ be an n-cell of $\widehat{X_\Gamma}$ which is not a branch point, and let $\hat{y}$ be a branch point of $\widehat{X_\Gamma}$. Then,* $\mathrm{Stab}_G(\sigma) \cap \mathrm{Stab}_G(\hat{y}) = 1$.

Proof. Let $g \in \mathrm{Stab}_G(\sigma) \cap \mathrm{Stab}_G(\hat{y})$. We distinguish two cases, but in each case we construct a fixed point for g in $\widehat{X_\Gamma}$ which is not a branch point. Since the action of G is free away from branch points, this forces $g = 1$.

Case 1: $\hat{y} \notin \sigma$. Since σ is closed and convex in the CAT(-1) space $\widehat{X_\Gamma}$, we can consider the nearest point projection $\pi_\sigma(\hat{y})$ of $\hat{y}$ to σ. Note that π_σ is g-equivariant, because $g \in \mathrm{Stab}_G(\sigma)$. Hence, g fixes $\pi_\sigma(\hat{y})$.

Case 2: $\hat{y} \in \sigma$. Let $p_1, \dots, p_m$ be the 0-cells of σ which are adjacent to $\hat{y}$ and of minimal distance from $\hat{y}$ ($m \leqslant n$). Then, g permutes this collection of points. Let z be their barycenter inside σ. Then, g fixes z. □

Finally, we show that stabilizers are full relatively quasiconvex subgroups of $(G, \mathcal{P})$; see Section 2 for definitions.

Lemma 3.9. *Let σ be a cell of $\widehat{X_\Gamma}$. Then,* $\mathrm{Stab}_G(\sigma)$ *is a full relatively quasiconvex subgroup of $(G, \mathcal{P})$.*

Proof. By (1) in Lemma 3.7, if σ is a branch point, then $\mathrm{Stab}_G(\sigma)$ is a maximal parabolic subgroup of $(G, \mathcal{P})$ and the statement follows. So, let us assume that σ is not a branch point. Notice that the action of G on $\widehat{X_\Gamma}$ fits in the setting of Lemma 2.2. Indeed, it is an action by deck transformations in the complement of the branch points. Moreover, σ is convex in $\widehat{X_\Gamma}$, and $\mathrm{Stab}_G(\sigma)$ acts on it cocompactly (the quotient is a face of $\square^n_\Gamma$, which is a compact manifold with boundary). Therefore, $\mathrm{Stab}_G(\sigma)$ is a relatively quasiconvex subgroup of $(G, \mathcal{P})$. Finally, let $P \in \mathcal{P}$ be any peripheral subgroup and $g \in G$. Then, $gPg^{-1} = \mathrm{Stab}_G(\hat{y})$ for some branch point $\hat{y}$, so we conclude by Lemma 3.8. □

Remark 3.10. The above proof actually shows that if a cell stabilizer is not maximal parabolic then it is strongly relatively quasiconvex.

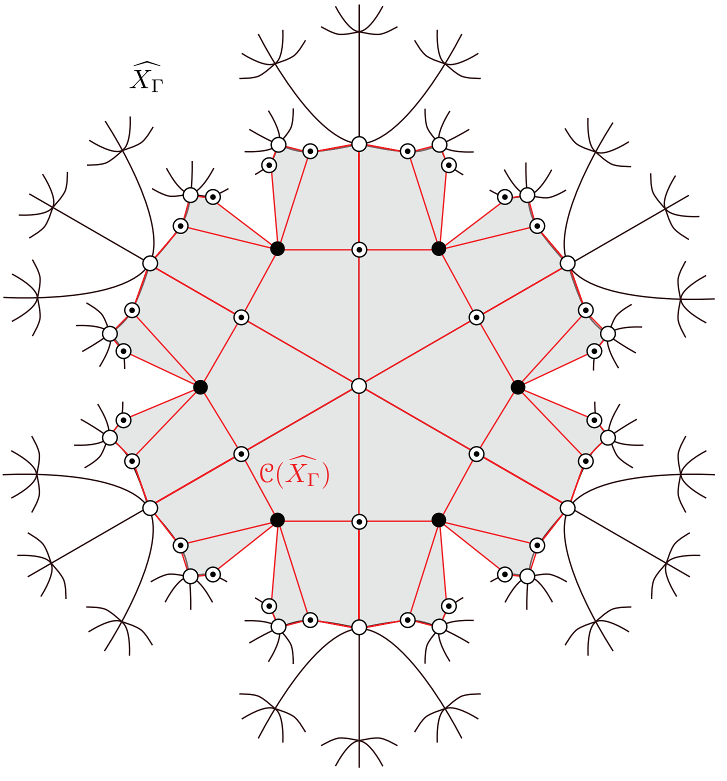

FIGURE 3 The dual cubical complex $C(\widehat{X_\Gamma})$ superimposed on the stratification of $\widehat{X_\Gamma}$. Key: ○, ⊙, and ● denote a vertex of height 0, 1, and 2.

4 | THE DUAL CUBICAL COMPLEX

Following [33] we construct a cubical complex that is dual to the stratification induced by the collection of mirrors. This is defined as follows; see Figure 3.

- vertices are k-cells in the stratification of $\widehat{X_\Gamma}$,
- edges represent codimension-1 inclusions,
- higher-dimensional cubes are glued in whenever their 1-skeleton appears.

The resulting cubical complex is denoted by $C(\widehat{X_\Gamma})$. It comes with a *height* function on the 0-skeleton, recording the dimension of the dual cell. The action of $G = \pi_1(\mathcal{R}(K, L))$ on $\widehat{X_\Gamma}$ extends to an action on $C(\widehat{X_\Gamma})$ by cubical isometries. Since K is finite, the action is cocompact. However, it is not proper. Note that running the same construction on $\widetilde{X_\Gamma}$ leads to the CAT(0) cubical complex $C(\widetilde{X_\Gamma})$ studied in [33].

Remark 4.1 (The case of L with simply connected components, continued). An argument analogous to the one in Remark 3.6 shows that if each component of L is simply connected, then puncturing $\widetilde{X_\Gamma}$ at its cone points does not change the fundamental group. Therefore, $\widehat{X_\Gamma} = \widetilde{X_\Gamma}$, and it follows that $C(\widehat{X_\Gamma}) = C(\widetilde{X_\Gamma})$, which is known to be CAT(0) by [33].

Vertices of height at least 2 in $C(\widehat{X_\Gamma})$ do not admit compact neighborhoods. The vertex dual to a branch point will be called a *branch vertex* of $C(\widehat{X_\Gamma})$. Branch vertices have height 0 and are the only vertices of height less than 2 that do not have compact neighborhoods. Let $\mathcal{BV}$ denote the collection of branch vertices of $C(\widehat{X_\Gamma})$. Analogously, let $\mathcal{CV}$ denote the collection of *cone vertices* of $C(\widetilde{X_\Gamma})$, that is, the vertices of $C(\widetilde{X_\Gamma})$ that arise from cone points of X. The branched covering map $\hat{\pi} : \widehat{X_\Gamma} \to \widetilde{X_\Gamma}$ from Section 3.2 induces a cubical branched covering map $C(\widehat{X_\Gamma}) \to C(\widetilde{X_\Gamma})$, see Section 4.1 for details.

We introduce certain subcomplexes on $C(\widehat{X_\Gamma})$, following [33], where the analogous subcomplexes were defined for $C(\widetilde{X_\Gamma})$. For each tile τ of $\widehat{X_\Gamma}$, we define the *dual tile* $C(\tau)$ of $C(\widehat{X_\Gamma})$ to be the full subcomplex consisting of vertices dual to cells of τ. Note that if v is the vertex dual to τ, then $C(\tau)$ is the cubical 1-neighborhood of v. Similarly, for each mirror M of $\widehat{X_\Gamma}$, we define the *dual mirror* $C(M)$ of $C(\widehat{X_\Gamma})$ to be the full subcomplex consisting of vertices dual to cells of M.

4.1 | $C(\widehat{X_\Gamma})$ is non-positively curved

Thanks to Gromov's link condition, it is enough to check that links of vertices of $C(\widehat{X_\Gamma})$ are flag simplicial complexes. Since the definition of adjacency in the dual cubical complex $C(\widehat{X_\Gamma})$ is given in terms of codimension-1 inclusion of cells in the stratification of $\widehat{X_\Gamma}$, the combinatorics of the link of a cell in $\widehat{X_\Gamma}$ completely determine the combinatorics of the link of the dual vertex in $C(\widehat{X_\Gamma})$. Note that the branched covering map $\hat{\pi} : \widehat{X_\Gamma} \to \widetilde{X_\Gamma}$ respects the stratifications of these spaces, in the sense that it sends cells of $\widehat{X_\Gamma}$ to cells of $\widetilde{X_\Gamma}$, preserving inclusions. Hence, it induces combinatorial maps on the dual cube complexes.

First, let $v \in C(\widehat{X_\Gamma})$ be a vertex that is not a branch vertex, and let $\sigma \subseteq \widehat{X_\Gamma}$ be its dual cell. Note that $\hat{\pi} : \widehat{X_\Gamma} \to \widetilde{X_\Gamma}$ is a covering map in the complement of branch points. In particular, it induces an isomorphism $\mathrm{lk}(\sigma, \widehat{X_\Gamma}) \to \mathrm{lk}(\hat{\pi}(\sigma), \widetilde{X_\Gamma})$, and therefore an isomorphism $\mathrm{lk}(v, C(\widehat{X_\Gamma})) \to \mathrm{lk}(w, C(\widetilde{X_\Gamma}))$, where w is the vertex of $C(\widetilde{X_\Gamma})$ dual to the cell $\hat{\pi}(\sigma)$. The latter link is known to be flag by [33, Proposition 4.10 (3)].

Let us now consider the link of a branch vertex of $\widehat{X_\Gamma}$. Recall from Section 3.2 that the link of a branch vertex of $\widehat{X_\Gamma}$ is isomorphic to a subdivision of $\widetilde{L}_i$ for some connected component L_i of L. The argument is similar to that in [33, Proposition 4.10 (2)] (note that a branch vertex has height 0). The main difference is that the map

$$\widehat{g_X} : \widehat{X_\Gamma} \xrightarrow{\hat{\pi}} \widetilde{X_\Gamma} \xrightarrow{\pi} X_\Gamma \xrightarrow{g_X} X$$

does not induce an isomorphism on links at the branch points. This is addressed by the following lemma, which is the analog of [33, Lemma 3.17].

Lemma 4.2. *Let $\hat{y} \in \widehat{X_\Gamma}$ be a branch point, and let $y = \widehat{g_X}(\hat{y}) \in X$ be the corresponding cone point. Then, the following hold.*

(1) *$\widehat{g_X}$ induces the universal covering map $\mathrm{lk}(\hat{y}, \widehat{X_\Gamma}) \to \mathrm{lk}(y, X)$.*
(2) *$\mathrm{lk}(\hat{y}, \widehat{X_\Gamma})$ is a piecewise spherical simplicial complex with vertices given by the edges containing $\hat{y}$, and in which $m+1$ vertices span an m-simplex if and only if the corresponding edges are contained in a $(m+1)$-cell.*

Proof. The maps π and g_X induce isomorphisms on the link of any vertex. Recall from Remark 3.5 that $\widehat{X_\Gamma}$ can be obtained from $\widetilde{\mathcal{R}}(K, L)$ by coning off each copy of the universal cover $\widetilde{L_i}$ of the components L_i of L. In particular, the map $\hat{\pi}$ induces the universal covering map on the links of the branch points. Therefore, we obtain (1). To prove (2) just argue as in [33, Lemma 3.17 (3)] with $k = 0$. □

Lemma 4.3. *Let $v \in C(\widehat{X_\Gamma})$ be a branch vertex. Then,* $\operatorname{lk}(v, C(\widehat{X_\Gamma}))$ *is a flag simplicial complex, isomorphic to* $\operatorname{lk}(\hat{y}, \widehat{X_\Gamma})$.

Proof. Let $\hat{y} \in \widehat{X_\Gamma}$ be the branch point dual to v, and let $y = \widehat{g_X}(\hat{y}) \in X$ be the corresponding cone point. Any vertex w_i of $\operatorname{lk}(v, C(\widehat{X_\Gamma}))$ corresponds to a vertex v_i of $C(\widehat{X_\Gamma})$ adjacent to v, and therefore to an edge e_i of $\widehat{X_\Gamma}$ that contains $\hat{y}$. Note that if w_i, w_j are two adjacent vertices in $\operatorname{lk}(v, C(\widehat{X_\Gamma}))$, then there is a vertex v_{ij} such that v, v_i, v_j, v_{ij} span a square in $C(\widehat{X_\Gamma})$. Since v has height 0, necessarily v_{ij} has height 2, hence it is dual to a 2-cell of $\widehat{X_\Gamma}$ containing the edges e_i, e_j dual to v_i, v_j. In particular, e_i, e_j are adjacent in $\operatorname{lk}(\hat{y}, \widehat{X_\Gamma})$. This shows that $\operatorname{lk}(v, C(\widehat{X_\Gamma}))$ and $\operatorname{lk}(\hat{y}, \widehat{X_\Gamma})$ have the same 1-skeleton.

By Lemma 4.2, $\operatorname{lk}(\hat{y}, \widehat{X_\Gamma})$ identifies with the universal cover of $\operatorname{lk}(y, X)$. Since X is non-positively curved, $\operatorname{lk}(y, X)$ is flag, so $\operatorname{lk}(\hat{y}, \widehat{X_\Gamma})$ is flag too. Therefore, the lemma is proved if we show that $\operatorname{lk}(v, C(\widehat{X_\Gamma}))$ is flag.

Let $w_0, \dots, w_p$ pairwise adjacent vertices in $\operatorname{lk}(v, C(\widehat{X_\Gamma}))$, and let $v_0, \dots, v_p$ be the corresponding vertices of $C(\widehat{X_\Gamma})$. Let $e_0, \dots, e_p$ be the edges in $\widehat{X_\Gamma}$ that correspond to $v_0, \dots, v_p$. These edges contain $\hat{y}$, and are pairwise adjacent in $\operatorname{lk}(\hat{y}, \widehat{X_\Gamma})$. Since $\operatorname{lk}(\hat{y}, \widehat{X_\Gamma})$ is flag, there is a cell μ of $\widehat{X_\Gamma}$ containing $e_0, \dots, e_p$. The collection of cells that contain $\hat{y}$ and are contained in μ give rise to a cube in $C(\widehat{X_\Gamma})$ that contains $v_0, \dots, v_p$. As a result, $w_0, \dots, w_p$ span a simplex in $\operatorname{lk}(v, C(\widehat{X_\Gamma}))$. □

We note that a completely analogous argument shows that the link of a vertex in $\widetilde{X_\Gamma}$ and the link of its dual vertex in $C(\widetilde{X_\Gamma})$ are isomorphic. Also note that it follows from the above discussion that the complex $C(\widehat{X_\Gamma})$ is a branched covering of the complex $C(\widetilde{X_\Gamma})$, branching over the set $\mathcal{BV}$ of branched vertices.

4.2 | $C(\widehat{X_\Gamma})$ is simply-connected

In this section, we show that the dual cubical complex is simply connected. Recall that we are working under the standing assumption in Remark 3.3. We follow the approach in [33], which is based on the following two observations about the combinatorial geometry of the dual cubical complex $C(\widetilde{X_\Gamma})$.

(DT) An edge–loop entirely contained in a dual tile is nullhomotopic.
(DM) If an edge–loop is not entirely contained in a dual tile, then it can be cut along dual mirrors and decomposed into a product of edge–loops, each of which is contained in a dual tile.

The step (DT) carries over verbatim from [33, §4.2], because the arguments there are completely local, in the sense that they only depend on the geometry and combinatorics of a single tile, and tiles of $\widehat{X_\Gamma}$ are isomorphic to those of $\widetilde{X_\Gamma}$.

For the step (DM), we will check that all the arguments from [33] carry over to the current setting, because they do not rely in any essential way on the local finiteness of the spaces involved.

Indeed, each point of $\widehat{X_\Gamma}$ is contained in at most finitely many mirrors, and each finite edge–path of $C(\widehat{X_\Gamma})$ intersects only finitely many dual mirrors (each of them only finitely many times).

The following statement provides one of the main properties of mirrors. It is a direct consequence of foldability as in [33, Proposition 3.14].

Proposition 4.4. *Each mirror of $\widehat{X_\Gamma}$ is a closed connected convex subspace of $\widehat{X_\Gamma}$.*

Next, we turn to the separation properties of the collection of mirrors. Following [33, §3.6], for each $i = 1, \dots, n$, let $\widehat{\mathcal{M}}_i$ be the collection of mirrors of $\widehat{X_\Gamma}$ that fold to one of the two parallel ith faces of $\square^n = [0,1]^n$, that is, $\{x_i = 0\}$ and $\{x_i = 1\}$. Notice that by construction any two elements of $\widehat{\mathcal{M}}_i$ are disjoint, and even have disjoint ε-neighborhoods for ε sufficiently small (because Γ is cocompact). Let $\widehat{C}_i$ be the collection of connected components of $\widehat{X_\Gamma} \setminus \cup_{M \in \widehat{\mathcal{M}}_i} M$. For each mirror $M \in \widehat{\mathcal{M}}_i$ and for each component $C \in \widehat{C}_i$, consider the following *equidistant space*, obtained by pushing the mirror M into the component C.

$$E^\varepsilon_{M,C} = \{x \in C \mid d(x, M) = \varepsilon\}.$$

Note that the local geometry of a component $C \in \widehat{C}_i$ in a neighborhood of a mirror $M \in \widehat{\mathcal{M}}_i$ is not sensitive to the fact that the stratification of $\widehat{X_\Gamma}$ is not locally finite around a branch point. More precisely, if $\mathcal{M}_i$ and C_i denote the analogous collections of mirrors and complementary components in $\widetilde{X_\Gamma}$, then we have that the map $\hat{\pi} : \widehat{X_\Gamma} \to \widetilde{X_\Gamma}$ maps each component $C \in \widehat{C}_i$ locally isometrically to a component $\hat{\pi}(C) \in C_i$ in $\widetilde{X_\Gamma}$. So, we obtain the following analog of [33, Lemma 3.27].

Lemma 4.5. *For $\varepsilon > 0$ small enough there is $k \in (-1, 0)$ such that the metric induced on $E^\varepsilon_{M,C}$ is* CAT(k)*.*

As a consequence, we obtain the following analog of [33, Proposition 3.29].

Proposition 4.6. *$\widehat{X_\Gamma}$ admits the structure of a graph of spaces, with underlying graph a connected tree.*

The construction and the proof are the same as in the case of $\widetilde{X_\Gamma}$. The only difference is that in the case of $\widehat{X_\Gamma}$ the tree is not locally finite: a mirror that goes through a branch point intersects the closure of possibly infinitely many complementary components, so the corresponding vertex has possibly infinitely many neighbors. However, this property is not needed. The relevant property of this tree is that it has no boundary, and this is still true in our setting, as we now explain.

The arguments from [33, §3.7] are local in nature: they deal with a cell σ on a mirror M and a *framing* for it, that is a choice of two tiles τ_1, τ_2 such that $\sigma \subseteq \tau_1 \cap \tau_2 \subseteq M$. Once again, the map $\hat{\pi} : \widehat{X_\Gamma} \to \widetilde{X_\Gamma}$ preserves the structure of framings, and this can be used to obtain the following statement, which is analogous to [33, Proposition 3.37].

Proposition 4.7. *Each $M \in \widehat{\mathcal{M}}_i$ separates $\widehat{X_\Gamma}$. More precisely, let $M \in \widehat{\mathcal{M}}_i$ be a mirror, let $\sigma \subseteq M$ be a k-cell, and let $\{\tau_1, \tau_2\}$ be a framing for σ. Then, τ_1, τ_2 are contained in the closure of two distinct connected components of $\widehat{X_\Gamma} \setminus M$.*

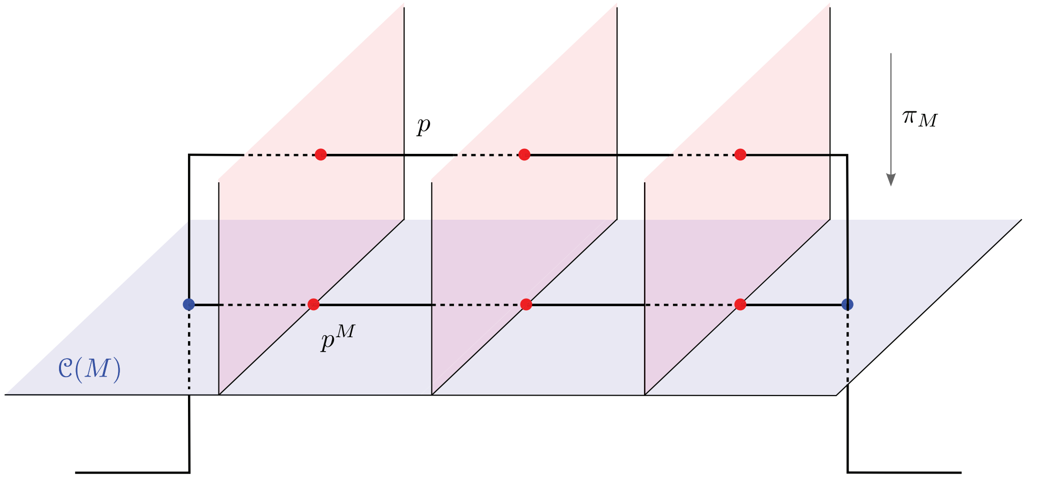

FIGURE 4 A minimal bridge p supported by M, and its projection to $\mathcal{C}(M)$.

Arguing as in [33, §4.3] it follows that each dual mirror $\mathcal{C}(M)$ separates the dual complex $\mathcal{C}(\widehat{X_\Gamma})$. When an edge–loop is not entirely contained in a tile, we want to decompose it into subpaths by cutting it along dual mirrors. The following definitions are taken from [33, §4.3.2]. Let $p = (v_0, \ldots, v_s)$ be an edge–path in $\mathcal{C}(\widehat{X_\Gamma})$, and let $\sigma_0, \ldots, \sigma_s$ be the cells of $\widehat{X_\Gamma}$ dual to its vertices. We say that p is a *bridge* if there exists a mirror M of $\widehat{X_\Gamma}$ such that $v_0, v_s \in \mathcal{C}(M)$, but $p \nsubseteq \mathcal{C}(M)$. In other words, $\sigma_0, \sigma_s \subseteq M$ but some of the other cells $\sigma_1, \ldots, \sigma_{s-1}$ are not contained in M. In this case, we say that p is *supported* by M. We say that p is a *minimal bridge* if none of its subpaths is a bridge.

The arguments in [33, §4.3–4] do not use the local finiteness of $\widetilde{X_\Gamma}$, so they carry over to $\widehat{X_\Gamma}$. Indeed, they just rely on the orthogonality properties of the collection of mirrors and tiles, such as the fact that either two mirrors are disjoint, or they intersect orthogonally and the projection of one to the other is contained in their intersection; see Lemma [33, Lemma 4.22]. This is enough to turn the nearest point projection $\pi_M : \widehat{X_\Gamma} \to M$ to a mirror M into a length-decreasing projection from a certain neighborhood of $\mathcal{C}(M)$ in $\mathcal{C}(\widehat{X_\Gamma})$ onto $\mathcal{C}(M)$, see [33, Lemma 4.24] and the discussion after it. This neighborhood is the one consisting of all the dual tiles that intersect $\mathcal{C}(M)$. Moreover, if a minimal bridge p is supported by M, then p remains inside this neighborhood of $\mathcal{C}(M)$; see Lemma 4.25 in [33]. It follows that we can project p to $\mathcal{C}(M)$ and obtain a shorter path; see Figure 4. This is stated in the following lemma, which is the core of the step (DM), and is the analog of [33, Lemma 4.26].

Lemma 4.8. *Let p be a minimal bridge supported on a mirror M. Then, there exists an edge–path $p^M \subseteq \mathcal{C}(M)$, such that p^M has the same endpoints as p and $\ell(p^M) \leqslant \ell(p) - 2$.*

We are now ready to prove the main result of this section.

Theorem 4.9. *The complex $\mathcal{C}(\widehat{X_\Gamma})$ is a connected* CAT(0) *cubical complex.*

Proof. By construction, the complex $\mathcal{C}(\widehat{X_\Gamma})$ is a path-connected cubical complex. Moreover, the link of any vertex is a flag simplicial complex (see Lemma 4.3 for the link of a branch vertex, and [33, Proposition 4.10] for the other ones). So, $\mathcal{C}(\widehat{X_\Gamma})$ is non-positively curved by Gromov's link condition.

To conclude, we need to show that $\mathcal{C}(\widehat{X_\Gamma})$ is simply connected. As in the proof of [33, Theorem 4.29], we argue that edge–loops are nullhomotopic by induction on their length. Let p be an edge–loop in $\mathcal{C}(\widehat{X_\Gamma})$. If p does not cross any mirror, then p stays in a tile and is therefore nullhomotopic. So let us assume that p crosses a mirror. Then, it must cross it an even number of time. Each pair of crossings determines a decomposition of p into two bridges. Make a choice of a minimal bridge, and use the projection from Lemma 4.8 to introduce a shortcut along the supporting mirror, which allows us to write p as the product of two shorter edge–loops. Iterating this process decomposes p into a product of loops that are entirely contained in a dual tile and are therefore nullhomotopic. □

4.3 | The action of G on $\mathcal{C}(\widehat{X_\Gamma})$

We now turn to the study of cube stabilizers. The idea is to follow the approach in [33], and relate the cube stabilizers for the action of $G = \pi_1(\mathcal{R}(K,L))$ on $\mathcal{C}(\widehat{X_\Gamma})$ to the cell stabilizers for the action of G on $\widehat{X_\Gamma}$. Recall that the action of G on $\widetilde{\mathcal{R}}(K,L)$ by deck transformations extends to an action on $\widehat{X_\Gamma}$ in which each maximal parabolic subgroup identifies with the stabilizer of a branch point. The following is the leading observation, which follows directly from the definition of the action of G on $\mathcal{C}(\widehat{X_\Gamma})$.

Lemma 4.10. *The stabilizer of a vertex in $\mathcal{C}(\widehat{X_\Gamma})$ coincides with the stabilizer of its dual cell in $\widehat{X_\Gamma}$. In particular, the stabilizer of a branch vertex is a maximal parabolic subgroup.*

To deal with higher dimensional cubes of $\mathcal{C}(\widehat{X_\Gamma})$, we observe the following. By invariance of the height function, the stabilizer of a cube C is always contained in the stabilizer of its vertex of minimal height. If this minimal vertex is not a branch vertex, then they are actually equal, as established by the following result. It is obtained as in [33, Lemma 5.4], where there are no branch vertices.

Lemma 4.11. *Let C be a cube in $\mathcal{C}(\widetilde{X_\Gamma})$. If the vertex of minimal height v of C is not a branch vertex, then* $\mathrm{Stab}_G(C) = \mathrm{Stab}_G(v)$.

We now consider the case in which C contains a branch vertex. (Note that if C contains a branch vertex v, then v is necessarily the vertex of minimal height.)

Lemma 4.12. *Let C be a cube in $\mathcal{C}(\widetilde{X_\Gamma})$ such that the vertex of minimal height v of C is a branch vertex.*

(1) *If $C = v$, then* $\mathrm{Stab}_G(C)$ *is a maximal parabolic subgroup.*
(2) *If $C \neq v$, then* $\mathrm{Stab}_G(C) = 1$.

Proof. The proof of (1) just follows from Lemma 4.10 and (1) in Lemma 3.7. So, let us prove (2). Let $g \in \mathrm{Stab}_G(C)$. Since the height function is invariant, g must fix v, by uniqueness of the vertex of minimal height. So, $g \in \mathrm{Stab}_G(v)$ too. If σ is the cell dual to C and $\hat{y}$ is the branch point dual to

v, then it follows from Lemma 4.10 that $g \in \mathrm{Stab}_G(C) \cap \mathrm{Stab}_G(v) = \mathrm{Stab}_G(\sigma) \cap \mathrm{Stab}_G(\hat{y})$. But this intersection is trivial by Lemma 3.8. □

We are now ready to collect the ideas, and prove that the action of G on $C(\widehat{X_\Gamma})$ looks like a relatively geometric action in the sense of [17, Definition 1.1] or [24, Definition 1.9], but with some larger stabilizers "away from the parabolics". Recall from Section 3 that $\mathcal{P}$ is a set of representatives of the conjugacy classes of the subgroups $\pi_1(L_i)$, where L_i is a connected component of L.

Theorem A. *The action of the relatively hyperbolic group $G = \pi_1(\mathcal{R}(K, L))$ on the* CAT(0) *cubical complex $C(\widehat{X_\Gamma})$ satisfies the following properties.*

1. *$G \backslash C(\widehat{X_\Gamma})$ is compact.*
2. *Each $P \in \mathcal{P}$ acts elliptically on $C(\widehat{X_\Gamma})$.*
3. *For each cube C of $C(\widehat{X_\Gamma})$,* $\mathrm{Stab}_G(C)$ *is either maximal parabolic, or else is full relatively quasi-convex, hyperbolic, and virtually compact special.*

Proof. First of all, $C(\widehat{X_\Gamma})$ is CAT(0) by Theorem 4.9. The action of G on it is cocompact because K is finite. Each $P \in \mathcal{P}$ fixes a branch point in $\widehat{X_\Gamma}$, and therefore it fixes the dual vertex in $C(\widehat{X_\Gamma})$. This proves the first two statements, so let us prove the third one. Let $C \subseteq C(\widehat{X_\Gamma})$ be a cube. There are three cases to consider (recall that if a cube contains a branch vertex, then that vertex is the vertex of minimal height).

First, if C is a branch vertex, then $\mathrm{Stab}_G(C)$ is a maximal parabolic subgroup of G by (1) in Lemma 4.12. Next, consider the case that C is not a branch vertex but its vertex of minimal height v is a branch vertex. By (2) in Lemma 4.12 we have that $\mathrm{Stab}_G(C) = 1$. Finally, suppose that C is not a branch vertex and its vertex of minimal height v is not a branch vertex. In this case, $\mathrm{Stab}_G(C) = \mathrm{Stab}_G(v)$ by Lemma 4.11. Moreover, by Lemma 4.10 this also equals $\mathrm{Stab}_G(\sigma)$, where σ is the cell of $\widehat{X_\Gamma}$ dual to v. By (2) in Lemma 3.7 we know that this is a hyperbolic and virtually compact special group. Moreover, by Lemma 3.9 we also know that it is a full relatively quasiconvex subgroup of $(G, \mathcal{P})$. This concludes the proof. □

Remark 4.13. It follows from Remark 3.10 that if a cube stabilizer is not a maximal parabolic subgroup then it is strongly relatively quasiconvex.

Theorem B. *Let $G = \pi_1(\mathcal{R}(K, L))$ and $\mathcal{P}$ be as above. Then, the following hold.*

1. *If each $P \in \mathcal{P}$ is residually finite, then G is residually finite and each $P \in \mathcal{P}$ is separable.*
2. *If each $P \in \mathcal{P}$ is hyperbolic and virtually compact special, then G is hyperbolic and virtually compact special.*

Proof. By Theorem A, the pair $(G, \mathcal{P})$ satisfies the conditions in Theorem A.4 from the Appendix. Since the trivial subgroup and the peripheral subgroups are full relatively quasiconvex in $(G, \mathcal{P})$, the statement in (1) follows.

To prove (2), we argue as follows. First of all, if a group is hyperbolic relative to a hyperbolic subgroup then it is itself hyperbolic, see [42, Corollary 2.41]. So, we have that G is hyperbolic. Consider the action of G on the CAT(0) cubical complex $C(\widehat{X_\Gamma})$ and let H be a non-trivial cube stabilizer. We claim that H is quasiconvex in G and virtually compact special. Indeed, by Theorem A we have two cases. If H is a maximal parabolic, then it is quasiconvex by [30, Corollary 8.2], and virtually

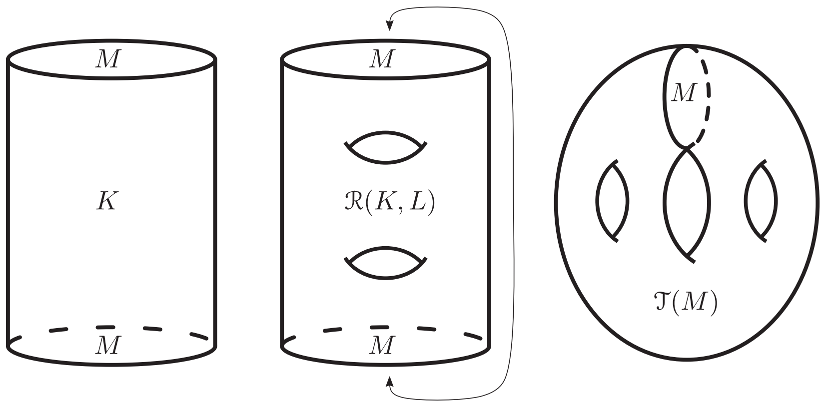

FIGURE 5 The construction of the hyperbolized mapping torus $\mathcal{T}(M)$.

compact special by assumption. Otherwise, H is virtually compact special, and strongly relatively quasiconvex in $(G, \mathcal{P})$; see Remark 4.13. By [42, Theorem 1.9] H is quasi-isometrically embedded in G, hence H is quasiconvex, since G is hyperbolic. It follows that the action of G on $\mathcal{C}(\widehat{X_\Gamma})$ satisfies the conditions in [25, Theorem D], so G is virtually compact special. $\square$

5 | APPLICATIONS TO MANIFOLDS

We collect some applications to the study of aspherical manifolds in Section 5.1, cobordism of manifolds in Section 5.2, and (non-)triangulability of manifolds in Section 5.3.

5.1 | Aspherical manifolds with residually finite fundamental groups

The purpose of this section is to obtain examples of closed aspherical manifolds whose fundamental groups have interesting algebraic properties. In Theorem 5.3, we construct closed aspherical manifolds in each dimension $n \geqslant 6$, whose fundamental group is residually finite. These examples can be chosen to be Riemannian and non-positively curved, and even negatively curved for $n \geqslant 9$; see Remark 5.5. These examples are new, in the sense that they are not homotopy equivalent to manifolds for which residual finiteness of the fundamental group was previously known by other methods. Finally, in Theorem 5.8 we construct negatively curved Riemannian manifolds of dimension $n \geqslant 5$, which are not locally symmetric and whose fundamental groups are virtually compact special.

All of the examples constructed in this section are based on a procedure that associates to a closed triangulable manifold M another closed triangulable manifold $\mathcal{T}(M)$, which contains M as a codimension-1 submanifold. We call $\mathcal{T}(M)$ the *hyperbolized mapping torus* of M. The construction of $\mathcal{T}(M)$ is as follows, see Figure 5. Let K_0 be a triangulation of M. Extend K_0 to a triangulation K_1 of $M \times [0, 1]$. Glue two copies of K_1 via the identity on $M \times \{0\}$ to obtain a triangulation K of $M \times [-1, 1]$. Denote by L the boundary of K; it consists of two components, each homeomorphic to M. Notice that $j : K \to K, (m, t) \mapsto (m, -t)$ is a simplicial involution exchanging the two components of L. Now, apply relative strict hyperbolization to the pair (K, L) to obtain a compact manifold with boundary $\mathcal{R}(K, L)$ whose boundary $\partial\mathcal{R}(K, L)$ is homeomorphic to $M \times \{\pm 1\}$.

(Alternatively, one could apply strict hyperbolization to the simplicial suspension of K_0, and then remove sufficiently small open neighborhoods of the cone points.) Finally, let $\mathcal{T}(M)$ be the closed manifold obtained by gluing the two boundary components of $\mathcal{R}(K,L)$ together. By construction, $\mathcal{T}(M)$ contains a π_1-injective codimension-1 submanifold homeomorphic to M arising from the two boundary components of $\mathcal{R}(K,L)$ that have been glued together. We keep referring to this submanifold as M.

Lemma 5.1. *If $\pi_1(M)$ is residually finite, then $\pi_1(\mathcal{T}(M))$ is residually finite.*

Proof. First of all, note that $\pi_1(\mathcal{T}(M))$ splits as a nice HNN extension. Indeed, the involution $j : K \to K$ defined above induces an automorphism j_* of $G = \pi_1(\mathcal{R}(K,L))$ that exchanges the subgroups H_1, H_2 corresponding to the two boundary components of $\mathcal{R}(K,L)$. So, $\pi_1(\mathcal{T}(M))$ can be presented as the HNN extension

$$\langle t, G \mid tht^{-1} = j_*(h), h \in H_1 \rangle.$$

The problem of residual finiteness for HNN extensions of this type was considered by Baumslag–Tretkoff in [3, Lemma 4.4]. More precisely, they proved that an HNN extension induced by an automorphism of G as above is residually finite as soon as G and H_1 satisfy the following two properties: (1) G is finitely generated and residually finite; (2) for any $x_1, \dots, x_n \in G \setminus H_1$ there is a normal subgroup N of finite index in G such that $x_i H_1 \cap N = \emptyset$ for all $i = 1, \dots, n$. We will now verify that these two properties are satisfied in our case.

To check (1), we argue as follows. We know that G is finitely generated and hyperbolic relative to $\mathcal{P} = \{H_1, H_2\}$. Notice that H_1, H_2 are both isomorphic to $\pi_1(M)$, which is assumed to be residually finite. So, by Theorem B we have that G is residually finite.

Next, note that (2) is satisfied as soon as H_1 is separable in G. To see this, let $x_1, \dots, x_n \in G \setminus H_1$. If H_1 is separable in G, then for each $i = 1, \dots, n$ we can find a normal subgroup of finite index $N_i \leqslant G$ such that $H_1 \leqslant N_i$ and $x_i \notin N_i$. Let $N = \cap N_i$. Then, N is still normal and of finite index in G. Moreover, $H_1 \leqslant N$, and thus $x_i H_1 \subseteq x_i N$. Since $x_i \notin N$, it follows that $x_i N \cap N = \emptyset$, hence $x_i H_1 \cap N = \emptyset$ for all $i = 1, \dots, n$, as desired. To conclude, note that our H_1 is indeed separable in G thanks to Theorem B and the assumption that $\pi_1(M)$ is residually finite. □

Remark 5.2 (Smoothness). When M is smooth, one can ensure that $\mathcal{T}(M)$ is smooth too, by applying strict hyperbolization with a sufficiently large hyperbolizing cell to a smooth triangulation K of $M \times [-1,1]$ as in [41]. Moreover, if M admits a non-positively (resp. negatively) curved Riemannian metric, then $\mathcal{T}(M)$ admits a Riemannian metric of non-positive (resp. negative) curvature in which M embeds totally geodesically; see [39] for details. The sectional curvatures of $\mathcal{T}(M)$ can be pinched arbitrarily close to -1 along planes that are sufficiently far away from the tangent bundle to M. While smoothness is not essential in the following discussion, we will work in this Riemannian setting for convenience.

We now use the above construction to obtain the desired examples. Let Λ be a torsion-free uniform lattice in $\mathrm{SL}(3,\mathbb{R})$ and let $M = \Lambda \backslash \mathrm{SL}(3,\mathbb{R}) / \mathrm{SO}(3,\mathbb{R})$ be the corresponding locally symmetric space. Then, M is a closed non-positively curved Riemannian manifold of dimension 5, whose fundamental group $\pi_1(M) = \Lambda$ is residually finite and has property (T). Let $N_5 = M$ and recursively define $N_n = \mathcal{T}(N_{n-1})$.

Theorem 5.3. *For all $n \geqslant 6$, N_n is a closed Riemannian n-manifold of non-positive curvature with residually finite fundamental group and not homotopy equivalent to any of the following:*

(1) *a locally symmetric space,*
(2) *a Gromov–Thurston manifold,*
(3) *a strictly hyperbolized manifold,*
(4) *a closed Kähler manifold.*

Here, a *Gromov–Thurston manifold* is one of the examples constructed in [22], and a *strictly hyperbolized manifold* is the result of applying Charney–Davis strict hyperbolization or Ontaneda's Riemannian hyperbolization to any closed triangulable manifold. Moreover, recall that all the examples of negatively curved manifolds constructed by Mostow–Siu in [38], Deraux in [15], or Stover–Toledo in [48, 49] are Kähler manifolds.

Proof. First of all, by construction N_n is a closed non-positively curved Riemannian manifold of dimension n that contains N_{n-1} as a totally geodesic codimension-1 submanifold; see Remark 5.2. In particular, since $N_5 = M$, we have that $\pi_1(N_n)$ contains a subgroup isomorphic to $\Lambda = \pi_1(M)$. Moreover, $\pi_1(N_n)$ is residually finite by Lemma 5.1.

By construction $\pi_1(N_n)$ splits as an HNN extension, hence it acts on the associated Bass–Serre tree without a global fixed point. In particular, it does not have property (FA), hence it does not have property (T), see [51]. It follows that $\pi_1(N_n)$ cannot be a lattice in a Lie group of higher rank, nor in the isometry group of a quaternionic hyperbolic space or the Cayley hyperbolic plane, because all of these Lie groups have property (T). To prove (1) we need to deal with the possibility that $\pi_1(N_n)$ is a lattice in the isometry group of a real or complex hyperbolic space. If this were the case, we would obtain a free isometric action of $\Lambda \subseteq \pi_1(N_n)$ on a real or complex hyperbolic space. But this would be impossible: since Λ has property (T), any isometric action on a real or complex hyperbolic space must have a fix point by Corollary 23 in [13]. This proves (1).

To prove (2) and (3) we argue as follows. Suppose by contradiction that N_n is homotopy equivalent to a Gromov–Thurston manifold or a strictly hyperbolized manifold. Then, $\pi_1(N_n)$ acts geometrically on a finite-dimensional CAT(0) cubical complex by [20] or [33], respectively. In particular, we obtain a proper cubical action of Λ on a CAT(0) cube complex. Since Λ has property (T), this is in contradiction with Theorem B in [40].

Finally, we prove (4). Suppose by contradiction that N_n is homotopy equivalent to a closed Kähler manifold X. Since N_n is non-positively curved, N_n and X are actually homeomorphic by [2]. Then, $N_n \setminus N_{n-1}$ is homeomorphic to an open subset of X. But by definition of N_n as $\mathcal{T}(N_{n-1})$, we have that $N_n \setminus N_{n-1}$ is the interior of a manifold obtained by relative strict hyperbolization on $N_{n-1} \times [-1, 1]$, so we reached a contradiction with [6, Theorem 10.5]. □

Remark 5.4. The crucial property of M that we have used above is that M is a closed aspherical manifold whose fundamental group is residually finite and contains a subgroup Λ with property (T). Notably, these properties are preserved when passing from M to its hyperbolized mapping torus $\mathcal{T}(M)$. So, given any such M one can recursively define a sequence of manifolds as in Theorem 5.3 and certain other prescribed features, as in Remark 5.5.

Remark 5.5 (Negatively curved examples). Let Λ' be a torsion-free uniform lattice in the isometry group of the quaternionic hyperbolic plane $\mathrm{Isom}(\mathbb{H}^2_{\mathbb{H}}) = \mathrm{Sp}(1,2)$ and let $M' = \mathbb{H}^2_{\mathbb{H}}/\Lambda'$ be the associated eight-dimensional manifold. Now, let $N'_8 = M'$ and recursively define $N'_n = \mathcal{T}(N'_{n-1})$.

Then for all $n \geqslant 9$, the manifold N'_n satisfies the conclusion of Theorem 5.3, with the added feature of negative curvature; see Remark 5.2. In particular, $\pi_1(N'_n)$ is a residually finite hyperbolic group.

Remark 5.6 (Property (T) vs. Haagerup property). Note that the above constructions provide examples of one-ended CAT(0) and hyperbolic groups that do not have property (T), as they split as HNN extensions, and do not have the Haagerup property, as they contain an infinite subgroup with (T). On the other hand, all these groups have relative property (T) with respect to that subgroup; see [4, §1.4].

The negatively curved examples from Remark 5.5 do not have cubulated fundamental group, because they are constructed starting with a manifold M whose fundamental group has property (T). On the other hand, if one starts with a manifold M whose fundamental group is cubulated (e.g., a real hyperbolic manifold defined by a uniform real hyperbolic lattice of simple type; see [28]), then one can obtain negatively curved examples whose fundamental groups are cubulated too. This is based on the following lemma, which is the analog of Lemma 5.1.

Lemma 5.7. *If M be a negatively curved Riemannian manifold and $\pi_1(M)$ is virtually compact special, then $\mathcal{T}(M)$ is a negatively curved Riemannian manifold and $\pi_1(\mathcal{T}(M))$ is virtually compact special.*

Proof. As in the proof of Lemma 5.1, $G = \pi_1(\mathcal{R}(K, L))$ is hyperbolic relative to $\mathcal{P} = \{H_1, H_2\}$ and $\pi_1(\mathcal{T}(M))$ can be represented as the HNN extension of G with respect to H_1. Note that $H_i \cong \pi_1(M)$ is hyperbolic and virtually compact special, so by Theorem B we have that G is hyperbolic and virtually compact special. By Remark 5.2 we know that $\mathcal{T}(M)$ admits a negatively curved Riemannian metric in which M embeds as a totally geodesic submanifold. In particular, $\pi_1(\mathcal{T}(M))$ is hyperbolic and H_1 is quasiconvex. Then by [52, Theorem 13.1] we have that $\pi_1(\mathcal{T}(M))$ is virtually compact special. □

This can be used to construct examples of Riemannian manifolds of negative curvature with cubulated fundamental group which are not real hyperbolic. Note that manifolds with the same properties can also be obtained directly via strict hyperbolization (see [33, 41]). We suspect that such manifolds are different from the examples in Theorem 5.8, but we are not aware of invariants that can distinguish the two classes.

Theorem 5.8. *The hyperbolized mapping torus construction $M \mapsto \mathcal{T}(M)$ can provide for all $n \geqslant 5$ a closed Riemannian n-manifold N''_n such that:*

(1) *N''_n has negative sectional curvature.*
(2) *$\pi_1(N''_n)$ is virtually compact special.*
(3) *N''_n is not homotopy equivalent to a locally symmetric space or Kähler manifold.*

Proof. Let M be a negatively curved Riemannian 4-manifold with virtually compact special fundamental group that is not a real hyperbolic manifold, such as a Gromov–Thurston manifold or a strictly hyperbolized manifold. These are known to have virtually special fundamental group by [20] and [33], respectively. Let $N''_4 = M$ and for $n \geqslant 5$ let $N''_n = \mathcal{T}(N_{n-1})$. By Lemma 5.7, we have that N''_n is a negatively curved Riemannian manifold and that $\pi_1(N''_n)$ is virtually compact special, which proves (1) and (2).

To prove (3) we argue as follows. As in Theorem 5.3, we have that $\pi_1(N''_n)$ does not have property (T) and N''_n is not Kähler. So, we only need to check that N''_n cannot be a real hyperbolic manifold. By contradiction, suppose that $\pi_1(N''_n)$ is isomorphic to a real hyperbolic lattice $\Gamma \subseteq \mathrm{Isom}(\mathbb{H}^n)$. Since N''_{n-1} is a totally geodesic submanifold of N''_n, its fundamental group identifies with a quasiconvex subgroup $H \subseteq \Gamma$.

By construction, N''_n admits an involution that fixes N''_{n-1} pointwise. This provides an outer automorphism ϕ of Γ that preserves H. By Mostow rigidity, ϕ is realized by an involution $g \in \mathrm{Isom}(\mathbb{H}^n)$. Since g fixes a totally geodesic $\mathbb{H}^k \subseteq \mathbb{H}^n$ for some $k \leqslant n-1$, the fixed set $\mathrm{Fix}_\infty(g)$ for the induced action of g on $\partial_\infty \mathbb{H}^n$ is a round sphere $S^{k-1} = \partial_\infty \mathbb{H}^k$. The Gromov boundary of $H = \pi_1(N''_{n-1})$ is S^{n-2} because N''_{n-1} is a negatively curved Riemannian $(n-1)$-manifold.

The limit set $\Lambda(H) = S^{n-2}$ of H in $\partial_\infty \mathbb{H}^n$ is a priori just a topologically embedded sphere (not necessarily a round one). However, since ϕ preserves H, we have $\Lambda(H) = S^{n-2} \subseteq \mathrm{Fix}_\infty(g) = S^{k-1}$. Therefore, we must have $\Lambda(H) = \mathrm{Fix}_\infty(g) = S^{n-2}$. This implies that $\pi_1(N''_{n-1}) = H$ acts geometrically on the totally geodesic $\mathbb{H}^{n-1}$ stabilized by g, and therefore that N''_{n-1} admits a real hyperbolic structure too. One can repeat this argument all the way down to M and obtain a real hyperbolic structure on M, which provides a contradiction. □

5.2 | Cobordism of manifolds

Some of the most classical applications of hyperbolization procedures are to cobordism of manifolds. For instance, any closed triangulable manifold M is cobordant to $\mathcal{H}(M)$, which is a closed aspherical manifold with negative curvature and virtually compact special fundamental group; see [9, 33]. Moreover, if M is smooth, then $\mathcal{H}(M)$ can be chosen to be smooth and have pinched negative sectional curvatures; see [41]. The following statements provide more properties about the fundamental group of the cobordism. It applies in particular when M_1 is any closed triangulable manifold and $M_2 = \mathcal{H}(M_1)$ is its strict hyperbolization.

Corollary 5.9. *Let M_1, M_2 be two closed triangulable manifolds with residually finite fundamental group. Suppose there is a triangulable cobordism between them. Then, there exists a compact Riemannian manifold with boundary W such that*

(1) *W is a cobordism between M_1 and M_2*
(2) *$\pi_1(W)$ is relatively hyperbolic relative to $\{\pi_1(M_1), \pi_1(M_2)\}$*
(3) *$\pi_1(W)$ is residually finite and $\pi_1(M_i)$ is separable.*

Proof. Let W_0 be a triangulable cobordism between M_1 and M_2, and let $W = \mathcal{R}(W_0, M_1 \cup M_2)$. By construction $\partial W \cong M_1 \cup M_2$ and its fundamental group is relatively hyperbolic with respect to the fundamental groups of the boundary components; see [6]. The statement follows from (1) in Theorem B. □

Remark 5.10 (Variations). If M_1 and M_2 are smooth and smoothly cobordant, then W can be taken to be smooth by [41]. If $\pi_1(M_i)$ is hyperbolic and virtually compact special, then by (2) in Theorem B we have that $\pi_1(W)$ is also hyperbolic and virtually compact special.

We also propose the following more geometric application. Let N be a non-compact complete Riemannian manifold N of finite volume. A closed Riemannian manifold M *bounds* N

geometrically if there is a bounded set $B \subset N$ such that $N \setminus B$ is diffeomorphic to $M \times [0, +\infty)$. As shown by Long and Reid in [35], there are closed flat manifolds that do not bound geometrically a real hyperbolic manifold. On the other hand, Ontaneda proved that every closed flat manifold bounds geometrically a manifold of pinched negative curvature in [41]. The following statement provides information about its fundamental group.

Corollary 5.11. *Let $\varepsilon > 0$ and let M be a closed flat manifold. Then, there exists a complete Riemannian manifold of finite volume N such that*

(1) *N has sectional curvatures in $[-1-\varepsilon, -1]$.*
(2) *M bounds N geometrically.*
(3) *$\pi_1(N)$ is relatively hyperbolic relative to $\pi_1(M)$.*
(4) *$\pi_1(N)$ is residually finite and $\pi_1(M)$ is separable.*

Proof. The first two statement are due to Ontaneda, see [41, Corollary 7]. They are based on the fact that a closed flat manifold bounds smoothly a smooth compact manifold (see [29]), so that hyperbolization can be applied. The third one is due to Belegradek; see [6]. For the last statement, note that a flat manifold is virtually a torus by Bieberbach's theorem. In particular, $\pi_1(M)$ is virtually abelian hence residually finite. Then, we conclude again by (1) in Theorem B. □

As in [41], one can get a similar statement for an almost flat manifold. In this case, one needs to additionally assume that the manifold bounds smoothly and has residually finite fundamental group.

5.3 | Aspherical manifolds that cannot be triangulated

Manolescu showed in [36] that for each $n \geqslant 5$ there is a closed topological n-manifold that cannot be triangulated, that is, is not homeomorphic to a simplicial complex. Davis et al. showed in [10] that for $n \geqslant 6$ one can take such a manifold to be aspherical and to have hyperbolic fundamental group. We now show that the fundamental groups of these manifolds are cubulated.

Theorem 5.12. *For each $n \geqslant 6$, there is a closed aspherical n-manifold N^n that cannot be triangulated and whose fundamental group is hyperbolic and virtually compact special.*

Proof. The manifolds are the ones constructed in [§3, pp. 800–801] [10], by applying suitable strict hyperbolization procedures to a construction by Galewski and Stern in [18]. We consider the case $n = 6$. Higher dimensional examples are obtained in a similar way by taking product with tori of the initial building pieces. The manifold N^6 is a closed aspherical 6-manifold obtained by gluing two suitable manifolds with boundary P_1 and P_2 along their boundary, so $\pi_1(N^6)$ splits as the associated amalgamated free product.

The manifold with boundary P_1 is obtained via strict hyperbolization (i.e., $P_1 = \mathcal{H}(P')$ for a certain triangulable manifold with boundary P') and P_2 is obtained via relative strict hyperbolization (i.e., $P_2 = \mathcal{R}(W, \partial P_1)$ for a certain triangulable manifold W with boundary $\partial W = \partial P_1$). A priori, the former has hyperbolic fundamental group and the latter has relatively hyperbolic fundamental group. However, since $\partial P_1 = \partial\mathcal{H}(P') = \mathcal{H}(\partial P')$ is a closed manifold, we have that $\pi_1(\partial P_1)$ is a hyperbolic group. As observed in [10], it follows that $\pi_1(N^6)$ is hyperbolic. Moreover, we also have that $\pi_1(\partial P_1)$ is quasiconvex in $\pi_1(N^6)$, since it is quasiconvex in both factors; see [32, 44]. Thanks to [52, Theorem 13.1], we are left to show that $\pi_1(P_1)$ and $\pi_1(P_2)$ are virtually compact special.

Since $\partial P_1 = \mathcal{H}(\partial P')$ is the strict hyperbolization of a closed manifold, it follows from [33] that $\pi_1(\partial P_1)$ is virtually compact special. Then, $\pi_1(P_2)$ is virtually compact special by (2) in Theorem B. Finally, let us consider the manifold P'' obtained by doubling P' along its boundary. The fundamental group of its strict hyperbolization $\mathcal{H}(P'')$ is hyperbolic, and also virtually compact special by [33]. Since P' is a subcomplex of P'' (with respect to any triangulation), the inclusion $P_1 = \mathcal{H}(P') \hookrightarrow \mathcal{H}(P'')$ is a local isometry. It follows that $\pi_1(P_1)$ is a quasiconvex subgroup of $\pi_1(\mathcal{H}(P''))$, and therefore it is virtually compact special too. □

Remark 5.13 (Virtual triangulability). For a topological manifold M of dimension at least 5 the Kirby–Siebenmann class $\Delta(M) \in H^4(M, \mathbb{Z}_2)$ is an obstruction to the existence of a PL-structure on M, in the sense that M admits a PL-structure if and only if $\Delta(M) = 0$, see [19, Theorem 5]. Similarly, in dimension at least 6 there is a class $\delta(\Delta(M)) \in H^5(M, \ker(\mu))$ such that M admits a triangulation if and only if $\delta(\Delta(M)) = 0$. Here, $\mu : \Theta^3 \to \mathbb{Z}_2$ is the Rokhlin homomorphism for the homology cobordism group Θ^3, and $\delta : H^4(M, \mathbb{Z}_2) \to H^5(M, \ker(\mu))$ is the connecting homomorphism associated with the short exact sequence

$$0 \to \ker(\mu) \to \Theta^3 \xrightarrow{\mu} \mathbb{Z}_2 \to 0.$$

It turns out that if $Sq^1(\Delta(M)) \neq 0$ then $\delta(\Delta(M)) \neq 0$, where $Sq^1 : H^4(M, \mathbb{Z}_2) \to H^4(M, \mathbb{Z}_2)$ is the first Steenrod square.

The manifold N^n from Theorem 5.12 has residually finite fundamental group, hence it admits a lot of finite covers. While N^n is not triangulable, one can ask if it is *virtually triangulable*, that is, if it admits a finite cover that is triangulable. We note that a triangulable cover must have even degree. Indeed, if $\pi : \widehat{N}^n_d \to N^n$ is a cover of odd degree d, then the induced map on cohomology $\pi^* : H^k(N^n, \mathbb{Z}_2) \to H^k(\widehat{N}^n_d, \mathbb{Z}_2)$ is injective. It is shown in [10] that $Sq^1(\Delta(N^n)) \neq 0$. By naturality of the Kirby–Siebenmann class and the Steenrod square, it follows that $Sq^1(\Delta(\widehat{N}^n_d)) \neq 0$, so $\widehat{N}^n_d$ does not admit a triangulation.

On the other hand, for a cover $\pi : \widehat{N}^n_d \to N^n$ of even degree d, the map π^* can have non-trivial kernel, so it is not clear whether either $\Delta(\widehat{N}^n_d)$ or $\delta(\Delta(\widehat{N}^n_d))$ vanishes. The existence of a triangulable cover would provide an example of an action of a finite group of even order d that cannot be made simplicial even up to changing the triangulation. Notice that in dimension 5 this happens already for $d = 2$: the Galewski–Stern 5-manifold from [18] is non-triangulable and non-orientable, but all orientable closed 5-manifolds are triangulable; see [47].

This problem about virtual vanishing of certain $\mathbb{Z}_2$-cohomology classes is reminiscent of the following question about vector bundles: given a *flat* vector bundle (i.e., a vector bundle with discrete structure group) over a compact polyhedron B, is there a finite cover of B on which the bundle becomes trivial? Triviality of a bundle over B with discrete structure group contained in $\mathrm{SL}(n, \mathbb{R})$ is obstructed by a class in $H^2(\pi_1(B), \mathbb{Z}_2)$, when $n \geqslant 3$. Millson constructed in [37] examples of flat real bundles for which this class remains non-trivial in all finite covers, that is, flat real bundles that are not virtually trivial. The analogous problem in the complex case is different: Deligne and Sullivan showed in [14] that all flat complex vector bundles are virtually trivial.

APPENDIX A: A CRITERION FOR RESIDUAL FINITENESS BY DANIEL GROVES AND JASON MANNING

In this Appendix, we explain why groups satisfying the conclusions of Theorem A are residually finite whenever the peripheral subgroups are. In fact, we prove the somewhat stronger statement that such groups are separable on full quasi-convex subgroups. Our proof relies on various group theoretic *Dehn filling* results so we begin by recalling the relevant definitions.

Definition A.1. Let $(G, \mathcal{P})$ be relatively hyperbolic. A *Dehn filling* of $(G, \mathcal{P})$ is a quotient $\overline{G}$ of G so that $\ker(G \to \overline{G})$ is generated by the union of a collection $\mathcal{N} = \{N_P \mid P \in \mathcal{P}\}$ where each $N_P \lhd P$. We may also write $\overline{G}$ as $G(\mathcal{N})$ if we want to keep track of the kernel. The filling is *peripherally finite* if $[P : N_P] < \infty$ for all $P \in \mathcal{P}$. If $\mathcal{H}$ is a family of subgroups of G, and we have $N_P < H^g$ whenever $H^g \cap P$ is infinite, then the filling is called an *$\mathcal{H}$–filling*.

Most Dehn filling results require the assumption that the filling is "sufficiently long." To be precise, we have the following.

Definition A.2. A statement F *holds for sufficiently long fillings* of $(G, \mathcal{P})$ if there is a finite set $S \subset G \setminus \{1\}$ so that the statement F holds for $G(\mathcal{N})$ whenever $\mathcal{N}$ satisfies $\bigcup \mathcal{N} \cap S = \emptyset$.

If G is a property of fillings, we say that F *holds for sufficiently long* G *fillings* if "F or not G" holds for sufficiently long fillings. For example G could be the property of being peripherally finite, or of being an $\mathcal{H}$–filling (or both).

Notice that a conjunction of *finitely many* statements which hold for sufficiently long fillings also holds for sufficiently long fillings.

Recall that a relatively quasiconvex subgroup $H \subseteq G$ is *full* if for every $g \in G$ and $P \in \mathcal{P}$ the intersection $H^g \cap P$ is either finite or of finite index in P. We will also need the following definition of Osin in the course of the proof.

DefinitionA.3 ([42, Definition 1.8]). A relatively quasi-convex subgroup $H \leqslant G$ is *strongly relatively quasi-convex* if for every $g \in G$ and $P \in \mathcal{P}$ the intersection $H^g \cap P$ is finite.

Theorem A.4. *Suppose that $(G, \mathcal{P})$ is a relatively hyperbolic pair with each element of $\mathcal{P}$ residually finite. Suppose also that G admits an isometric and cubical action on a* CAT(0) *cube complex X so that*

(1) *$G \backslash X$ is compact;*
(2) *Each $P \in \mathcal{P}$ acts elliptically on X; and*
(3) *For each cube $\sigma \in X$, $\mathrm{Stab}_G(\sigma)$ is either conjugate to an element of $\mathcal{P}$, or else is full relatively quasi-convex, hyperbolic, and virtually compact special.*

Then, every full relatively quasi-convex subgroup of G is separable. In particular, G is residually finite.

Proof. We first argue that it is enough to prove the theorem replacing (3) with the stronger hypothesis:

(3') Every cube stabilizer is either maximal parabolic or strongly quasi-convex and virtually compact special.

Let $\mathcal{P}' \subseteq \mathcal{P}$ be obtained by omitting those elements of $\mathcal{P}$ which are both hyperbolic and virtually compact special. Every full relatively quasi-convex subgroup of $(G, \mathcal{P}')$ is full relatively quasi-convex in $(G, \mathcal{P})$, so if we prove the theorem for $(G, \mathcal{P}')$ we will have proved it also for $(G, \mathcal{P})$. Replacing $\mathcal{P}$ by $\mathcal{P}'$ does not change hypotheses (1) or (2). Suppose H is a cube stabilizer which is not maximal parabolic. With respect to $\mathcal{P}$ it is therefore full relatively quasi-convex, hyperbolic, and virtually compact special. By [30, Theorem 9.1], H is hyperbolic relative to a collection $\mathcal{D}_H$ of finite index subgroups of conjugates of some parabolics $\mathcal{P}_H \subset \mathcal{P}$ (which may occur with multiplicity). Each of these subgroups is undistorted in H [42, Lemma 5.4]. In particular, each $D \in \mathcal{D}_H$ is hyperbolic and (using [26]) virtually special. It follows that each $P \in \mathcal{P}_H$ is hyperbolic and virtually special, so $\mathcal{P}_H \subset \mathcal{P} \setminus \mathcal{P}'$. In particular, H is strongly relatively quasi-convex with respect to $\mathcal{P}'$. Since H was an arbitrary non-maximal-parabolic cube stabilizer the action of $(G, \mathcal{P}')$ on X satisfies the stronger condition (3').

We therefore suppose all cube stabilizers are maximal parabolic or strongly quasi-convex and virtually compact special. Let $\mathcal{H}$ be a collection of conjugacy representatives of stabilizers of cubes in X which are not maximal parabolic.

Let L be a full relatively quasi-convex subgroup of $(G, \mathcal{P})$, and suppose that $g \in G \setminus L$.

Let $\sigma_1, \dots, \sigma_k$ be representatives of G-orbits of cubes in X. For each i, let Q_i be the finite-index subgroup of $\mathrm{Stab}(\sigma_i)$ which fixes σ_i pointwise, and let $\mathcal{Q} = \{Q_1, \dots Q_k\}$, and $\mathcal{Q}' = \mathcal{Q} \cup \{L\}$. Note that each element of $\mathcal{Q}'$ is full relatively quasi-convex.

Claim. A sufficiently long peripherally-finite $\mathcal{Q}'$-filling $\pi : G \to \overline{G} = G/K$ will have the following properties:

(1) $\overline{G}$ is hyperbolic;
(2) $K \backslash X$ is a CAT(0) cube complex;
(3) For each $H \in \mathcal{H}$ the map $\pi|_H$ is injective on H, and the image $\overline{H}$ is quasi-convex in $\overline{G}$;
(4) $\pi(g) \notin \pi(L)$. □

Proof of Claim. The fundamental theorem of relatively hyperbolic Dehn filling [43] says that for sufficiently long fillings $\overline{G}$ is hyperbolic relative to the images of the elements of $\mathcal{P}$. When these images are finite, this implies that $\overline{G}$ is hyperbolic.

The second item follows from [25, Corollary 6.6].

The third item follows from [23, Propositions 4.5 and 4.6] (the induced filling of each H has trivial filling kernels, because H is strongly quasi-convex, so does not intersect maximal parabolics except in finite groups, which can be avoided for sufficiently long fillings).

The fourth item follows from [23, Proposition 4.7].

That there exist fillings of the sort in the claim follows from the assumption that elements of $\mathcal{P}$ are residually finite. Indeed, for each $P \in \mathcal{P}$ there are finitely many infinite intersections $P \cap Q^g$ for $Q \in \mathcal{Q}$, $g \in G$; each such $P \cap Q^g$ is finite index in P. Let $\dot{P}$ be the normal core of the intersection of these $P \cap Q^g$. For any collection of finite index $\{N'_P \lhd P\}$, the collection $\{N_P = N'_P \cap \dot{P}\}$ determines a peripherally finite $\mathcal{Q}'$-filling. Since each P is residually finite, we may choose the N'_P to avoid any given finite set S.

Now, $\overline{G}$ acts cocompactly on $\overline{X} = K \backslash X$, with stabilizers which are either finite or quasi-convex and virtually special, and hence by [25, Theorem D] $\overline{G}$ is virtually special. The image $\pi(L)$ of L in $\overline{G}$ is quasi-convex, and $\pi(g) \notin \pi(L)$, so since quasi-convex subgroups of hyperbolic virtually compact special groups are separable by [27, Theorem 1.3], $\pi(g)$ can be separated from $\pi(L)$ in a finite quotient of $\overline{G}$. This is a finite quotient of G separating g from L.

Since L and $g \in G \setminus L$ were arbitrary, all full relatively quasi-convex subgroups of G are separable. Since $\{1\}$ is a full relatively quasi-convex subgroup of G, G is residually finite. □

Remark A.5. By using versions of the Malnormal special quotient theorem on the non-parabolic cell stabilizers, one can weaken Hypothesis (3) on these subgroups to, for example, be relatively hyperbolic with respect to the peripheral structure induced from $(G, \mathcal{P})$, and admitting a (weakly) relatively geometric action on a CAT(0) cube complex.

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