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Barycentric straightening and bounded cohomology

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Abstract. We study the barycentric straightening of simplices in higher rank irreducible symmetric spaces of non-compact type. We show that, for an n-dimensional symmetric space of rank $r \geq 2$ (excluding $\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$ and $\mathrm{SL}(4,\mathbb{R})/\mathrm{SO}(4)$), the p-Jacobian has uniformly bounded norm, provided $p \geq n-r+2$. As a consequence, for the corresponding non-compact, connected, semisimple real Lie group G, in degrees $p \geq n-r+2$, every degree p cohomology class has a bounded representative. This answers Dupont's problem in small codimension. We also give examples of symmetric spaces where the barycentrically straightened simplices of dimension $n-r$ have unbounded volume, showing that the range in which we obtain boundedness of the p-Jacobian is very close to optimal.

Keywords. Barycenter method, bounded cohomology, semisimple Lie group, Dupont's problem

1. Introduction

When studying the bounded cohomology of groups, an important theme is the comparison map from bounded cohomology to ordinary cohomology. In the context of non-compact, connected, semisimple Lie groups, Dupont raised the question of whether this comparison map is always surjective [10] (see also Monod's ICM address [17, Problem A′], and [4, Conjecture 18.1]). Properties of these Lie groups G are closely related to properties of the corresponding non-positively curved symmetric space $X = G/K$. Geometric methods on the space X can often be used to recover information about the Lie group G. This philosophy was used by Lafont and Schmidt [16] to show that the comparison map is surjective in degree $\dim(X)$. In the present paper, we extend this result to smaller degrees, and show:

Main Theorem. *Let $X = G/K$ be an n-dimensional irreducible symmetric space of non-compact type of rank $r = \mathrm{rank}(X) \geq 2$, excluding $\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$ and $\mathrm{SL}(4,\mathbb{R})/\mathrm{SO}(4)$, and Γ a cocompact torsion-free lattice in G. Then the comparison maps $\eta : H^*_{c,b}(G,\mathbb{R}) \to H^*_c(G,\mathbb{R})$ and $\eta' : H^*_b(\Gamma,\mathbb{R}) \to H^*(\Gamma,\mathbb{R})$ are both surjective in all degrees $* \geq n-r+2$.*

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The idea of the proof is similar to that in [16]. One defines a barycentric straightening of simplices in X, and uses it to construct bounded cocycles representing any given cohomology class. These cocycles are obtained by integrating a suitable differential form on various straightened simplices. Since the differential form has bounded norm, the key step is to show that the Jacobian of the straightened simplex is uniformly controlled (independent of the simplex or the point in it). Showing this later property requires some work, and is done in Sections 3 and 4 (following the general approach of Connell and Farb [6], [7]). The proof of the Main Theorem is then given in Section 5.

Remark. For various families of higher rank symmetric spaces, the dimension grows roughly quadratically in the rank. Our Main Theorem thus answers Dupont's question for continuous cohomology classes in degree close to the dimension of the symmetric space. Prior results on this problem include some work on the degree two case (Domic and Toledo [8], as well as Clerk and Ørsted [5]) as well as the top-degree case (Lafont and Schmidt [16]). In his seminal paper on the subject, Gromov showed that characteristic classes of flat bundles are bounded classes [13]. Using Gromov's result, Hartnick and Ott [14] were able to obtain complete answers for several specific classes of Lie groups (e.g. of Hermitian type, as well as some other cases).

The preprint [15] of Inkang Kim and Sungwoon Kim uses similar methods to obtain uniform control of the Jacobian in codimension one. Their paper also contains a wealth of other applications, which we have not pursued in the present paper. On the other hand, their results do not produce any new bounded cohomology classes (since in the higher rank case, the codimension one continuous cohomology always vanishes).

2. Preliminaries

2.1. Symmetric spaces of non-compact type

In this section, we give a quick review of some results on symmetric spaces of non-compact type; for more details, we refer the reader to Eberlein's book [11]. Let $X = G/K$ be a symmetric space of non-compact type, where G is semisimple and K is a maximal compact subgroup of G. Geometrically G can be identified with $\mathrm{Isom}_0(X)$, the connected component of the isometry group of X that contains the identity, and $K = \mathrm{Stab}_p(G)$ for some $p \in X$. Fixing a basepoint $p \in X$, we have a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ of the Lie algebra $\mathfrak{g}$ of G, where $\mathfrak{k}$ is the Lie algebra of K, and $\mathfrak{p}$ can be isometrically identified with T_pX using the Killing form. Let $\mathfrak{a} \subseteq \mathfrak{p}$ be a maximal abelian subalgebra of $\mathfrak{p}$. We can identify $\mathfrak{a}$ with the tangent space of a *flat* $\mathcal{F}$ at p—that is, an isometrically embedded Euclidean space $\mathbb{R}^r \subseteq X$, where r is the rank of X. Given any vector $v \in T_pX$, there exists a flat $\mathcal{F}$ that is tangent to v. We say v is *regular* if such a flat is unique, and *singular* otherwise.

Now let $v \in \mathfrak{p}$ be a regular vector. This direction defines a point $v(\infty)$ on the visual boundary ∂X of X. The group G acts on ∂X. The orbit set $Gv(\infty) = \partial_F X \subseteq \partial X$ is called a *Furstenberg boundary* of X. Since both G and K act transitively on $\partial_F X$, $\partial_F X$ is compact. In fact, a point stabilizer for the G-action on $\partial_F X$ is a minimal parabolic subgroup P, so we can also identify $\partial_F X$ with the quotient G/P. In the rest of this paper, we will use a specific realization of the Furstenberg boundary: the one given by choosing the regular vector v to point towards the barycenter of a Weyl chamber in the flat.

For each element α in the dual space $\mathfrak{a}^*$ of $\mathfrak{a}$, we define $\mathfrak{g}_\alpha = \{Y \in \mathfrak{g} \mid [A, Y] = \alpha(A)Y$ for all $A \in \mathfrak{a}\}$. We call α a *root* if $\mathfrak{g}_\alpha$ is non-trivial, and in such case we call $\mathfrak{g}_\alpha$ the *root space* of α. We denote by Λ the finite set of roots, and we have the root space decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Lambda} \mathfrak{g}_\alpha$$

where $\mathfrak{g}_0 = \{Y \in \mathfrak{g} \mid [A, Y] = 0$ for all $A \in \mathfrak{a}\}$, and the direct sum is orthogonal with respect to the canonical inner product on $\mathfrak{g}$.

Let θ be the Cartan involution at the point p. Then θ is an involution on $\mathfrak{g}$ which acts by I on $\mathfrak{k}$ and $-I$ on $\mathfrak{p}$, hence it preserves the Lie bracket. We can define $\mathfrak{k}_\alpha = (I + \theta)\mathfrak{g}_\alpha \subseteq \mathfrak{k}$ and $\mathfrak{p}_\alpha = (I - \theta)\mathfrak{g}_\alpha \subseteq \mathfrak{p}$, with the following properties:

Proposition 2.1 ([11, Proposition 2.14.2]).

(1) *$I + \theta : \mathfrak{g}_\alpha \to \mathfrak{k}_\alpha$ and $I - \theta : \mathfrak{g}_\alpha \to \mathfrak{p}_\alpha$ are linear isomorphisms. Hence* $\dim(\mathfrak{k}_\alpha) = \dim(\mathfrak{g}_\alpha) = \dim(\mathfrak{p}_\alpha)$.
(2) *$\mathfrak{k}_\alpha = \mathfrak{k}_{-\alpha}$ and $\mathfrak{p}_\alpha = \mathfrak{p}_{-\alpha}$ for all $\alpha \in \Lambda$, and $\mathfrak{k}_\alpha \oplus \mathfrak{p}_\alpha = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$.*
(3) *$\mathfrak{k} = \mathfrak{k}_0 \oplus \bigoplus_{\alpha \in \Lambda^+} \mathfrak{k}_\alpha$ and $\mathfrak{p} = \mathfrak{a} \oplus \bigoplus_{\alpha \in \Lambda^+} \mathfrak{p}_\alpha$, where $\mathfrak{k}_0 = \mathfrak{g}_0 \cap \mathfrak{k}$, and Λ^+ is the set of positive roots.*

Remark. Since $\mathfrak{p}_\alpha = (\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha}) \cap \mathfrak{p}$, the direct sum of $\mathfrak{p}$ in (3) of Proposition 2.1 is also orthogonal with respect to the canonical inner product on $\mathfrak{p}$.

We now analyze the adjoint action of $\mathfrak{k}$ on $\mathfrak{a}$. Let $u \in \mathfrak{k}_\alpha$ and $v \in \mathfrak{a}$, we can write u as $(I + \theta)w$ where $w \in \mathfrak{g}_\alpha$, hence

$$\begin{aligned}[u, v] &= [(I + \theta)w, v] = [w, v] + [\theta w, v] = -\alpha(v)w + \theta[w, -v] \\ &= -\alpha(v)w + \theta(\alpha(v)w) = -\alpha(v)(I - \theta)(w) = -\alpha(v)(I - \theta)(I + \theta)^{-1}u.\end{aligned}$$

This gives the following proposition.

Proposition 2.2. *Let α be a root. The adjoint action of $\mathfrak{k}_\alpha$ on $\mathfrak{a}$ is given by*

$$[u, v] = -\alpha(v)(I - \theta)(I + \theta)^{-1}u$$

for any $u \in \mathfrak{k}_\alpha$ and $v \in \mathfrak{a}$. In particular, $\mathfrak{k}_\alpha$ maps v into $\mathfrak{p}_\alpha$.

Assume $v \in \mathfrak{a} \subseteq T_x X$ is inside a fixed flat through x, and let K_v be the stabilizer of v in K. Then the space $K_v\mathfrak{a}$ is the tangent space of the union of all flats that goes through v. Equivalently, it is the union of all vectors that are parallel to v, hence it can be identified with $\mathfrak{a} \oplus \bigoplus_{\alpha \in \Lambda^+, \alpha(v)=0} \mathfrak{p}_\alpha$. In particular, if v is regular, then the space is just $\mathfrak{a}$. Moreover, if we denote by $\mathfrak{k}_v$ the Lie algebra of K_v, then $\mathfrak{k}_v = \{u \in \mathfrak{k} \mid [u, v] = 0\} = \mathfrak{k}_0 \oplus \bigoplus_{\alpha \in \Lambda^+, \alpha(v)=0} \mathfrak{k}_\alpha$.

2.2. *Patterson–Sullivan measures*

Let $X = G/K$ be a symmetric space of non-compact type, and Γ be a cocompact lattice in G. Albuquerque [1] generalized the construction of Patterson–Sullivan to higher rank symmetric spaces. He showed that to each $x \in X$, we can assign a probability measure $\mu(x)$ that is G-equivariant and is fully supported on the Furstenberg boundary $\partial_F X$. Moreover, for $x, y \in X$ and $\theta \in \partial_F X$, the Radon–Nikodym derivative is given by

$$\frac{d\mu(x)}{d\mu(y)}(\theta) = e^{hB(x,y,\theta)}$$

where h is the volume entropy of X/Γ, and $B(x, y, \theta)$ is the Busemann function on X. Recall that, in a non-positively curved space X, the *Busemann function* B is defined by

$$B(x, y, \theta) = \lim_{t\to\infty} (d_X(y, \gamma_\theta(t)) - t)$$

where γ_θ is the unique geodesic ray from x to θ. Fixing a basepoint O in X, we shorten $B(O, y, \theta)$ to just $B(y, \theta)$. Notice that for fixed $\theta \in \partial_F X$ the Busemann function is convex on X, and by integrating on $\partial_F X$, we obtain, for any probability measure ν that is fully supported on the Furstenberg boundary $\partial_F X$, a strictly convex function

$$x \mapsto \int_{\partial_F X} B(x, \theta)\, d\nu(\theta)$$

(see [6, Proposition 3.1] for a proof of this last statement).

Hence we can define the barycenter $\mathrm{bar}(\nu)$ of ν to be the unique point in X where the function attains its minimum. It is clear that this definition is independent of the choice of basepoint O.

2.3. *Barycenter method*

In this section, we discuss the barycentric straightening introduced by Lafont and Schmidt [16] (based on the barycenter method originally developed by Besson, Courtois, and Gallot [3]). Let $X = G/K$ be a symmetric space of non-compact type, and Γ be a cocompact lattice in G. We denote by Δ_s^k the standard spherical k-simplex in the Euclidean space, that is,

$$\Delta_s^k = \left\{(a_1, \ldots, a_{k+1}) \;\middle|\; a_i \geq 0, \sum_{i=1}^{k+1} a_i^2 = 1\right\} \subseteq \mathbb{R}^{k+1},$$

with the induced Riemannian metric from $\mathbb{R}^{k+1}$, and with ordered vertices $(e_1, \ldots, e_{k+1})$. Given any singular k-simplex $f : \Delta_s^k \to X$, with ordered vertices $V = (x_1, \ldots, x_{k+1}) = (f(e_1), \ldots, f(e_{k+1}))$, we define the *k-straightened simplex*

$$\mathrm{st}_k(f) : \Delta_s^k \to X, \qquad \mathrm{st}_k(f)(a_1, \ldots, a_{k+1}) := \mathrm{bar}\Big(\sum_{i=1}^{k+1} a_i^2 \mu(x_i)\Big),$$

where $\mu(x_i)$ is the Patterson–Sullivan measure at x_i. We notice that $\mathrm{st}_k(f)$ is determined by the (ordered) vertex set V, and we denote $\mathrm{st}_k(f)(\delta)$ by $\mathrm{st}_V(\delta)$, for $\delta \in \Delta_s^k$.

Observe that the map $\mathrm{st}_k(f)$ is C^1, since one can view it as the restriction of the C^1-map $\mathrm{st}_n(f)$ to a k-dimensional subspace (see e.g. [16, Property (3)]). For any $\delta = \sum_{i=1}^{k+1} a_i e_i \in \Delta_s^k$, $\mathrm{st}_k(f)(\delta)$ is defined to be the unique point where the function

$$x \mapsto \int_{\partial_F X} B(x,\theta)\, d\Big(\sum_{i=1}^{k+1} a_i^2 \mu(x_i)\Big)(\theta)$$

is minimized. Hence, by differentiating at that point, we get the 1-form equation

$$\int_{\partial_F X} dB_{(\mathrm{st}_V(\delta),\theta)}(\cdot)\, d\Big(\sum_{i=1}^{k+1} a_i^2 \mu(x_i)\Big)(\theta) \equiv 0,$$

which holds identically on the tangent space $T_{\mathrm{st}_V(\delta)}X$. Differentiating in a direction $u \in T_\delta \Delta_s^k$ in the source, one obtains the 2-form equation

$$\begin{aligned}\sum_{i=1}^{k+1} 2a_i \langle u, e_i\rangle_\delta & \int_{\partial_F X} dB_{(\mathrm{st}_V(\delta),\theta)}(v)\, d(\mu(x_i))(\theta) \\ &+ \int_{\partial_F X} DdB_{(\mathrm{st}_V(\delta),\theta)}(D_\delta(\mathrm{st}_V)(u), v)\, d\Big(\sum_{i=1}^{k+1} a_i^2 \mu(x_i)\Big)(\theta) \equiv 0, \qquad (2.1)\end{aligned}$$

which holds for every $u \in T_\delta \Delta_s^k$ and $v \in T_{\mathrm{st}_V(\delta)}X$. Now we define two positive semi-definite quadratic forms Q_1 and Q_2 on $T_{\mathrm{st}_V(\delta)}X$:

$$\begin{aligned} Q_1(v,v) &= \int_{\partial_F X} dB^2_{(\mathrm{st}_V(\delta),\theta)}(v)\, d\Big(\sum_{i=1}^{k+1} a_i^2 \mu(x_i)\Big)(\theta), \\ Q_2(v,v) &= \int_{\partial_F X} DdB_{(\mathrm{st}_V(\delta),\theta)}(v,v)\, d\Big(\sum_{i=1}^{k+1} a_i^2 \mu(x_i)\Big)(\theta). \end{aligned}$$

In fact, Q_2 is positive definite since $\sum_{i=1}^{k+1} a_i^2 \mu(x_i)$ is fully supported on $\partial_F X$ (see [6, Section 4]). From (2.1) we obtain, for $u \in T_\delta \Delta_s^k$ a unit vector and $v \in T_{\mathrm{st}_V(\delta)}X$ arbitrary,

$$\begin{aligned} |Q_2(D_\delta(\mathrm{st}_V)(u), v)| &= \Big| -\sum_{i=1}^{k+1} 2a_i \langle u, e_i\rangle_\delta \int_{\partial_F X} dB_{(\mathrm{st}_V(\delta),\theta)}(v)\, d(\mu(x_i))(\theta)\Big| \\ &\le \Big(\sum_{i=1}^{k+1} \langle u, e_i\rangle_\delta^2\Big)^{1/2} \Big(\sum_{i=1}^{k+1} 4a_i^2 \Big(\int_{\partial_F X} dB_{(\mathrm{st}_V(\delta),\theta)}(v)\, d(\mu(x_i))(\theta)\Big)^2\Big)^{1/2} \\ &\le 2\Big(\sum_{i=1}^{k+1} a_i^2 \int_{\partial_F X} dB^2_{(\mathrm{st}_V(\delta),\theta)}(v)\, d(\mu(x_i))(\theta) \int_{\partial_F X} 1\, d(\mu(x_i))\Big)^{1/2} \\ &= 2Q_1(v,v)^{1/2} \qquad (2.2) \end{aligned}$$

via two applications of the Cauchy–Schwarz inequality.

We restrict these two quadratic forms to the subspace $S = \mathrm{Im}(D_\delta(\mathrm{st}_V)) \subseteq T_{\mathrm{st}_V(\delta)}X$, and denote the corresponding k-dimensional endomorphisms by H_δ and K_δ, that is,

$$Q_1(v, v) = \langle H_\delta(v), v\rangle_{\mathrm{st}_V(\delta)}, \qquad Q_2(v, v) = \langle K_\delta(v), v\rangle_{\mathrm{st}_V(\delta)},$$

for all $v \in S$.

For points $\delta \in \Delta_s^k$ where st_V is non-degenerate, we now pick orthonormal bases $\{u_1, \dots, u_k\}$ in $T_\delta\Delta_s^k$ and $\{v_1, \dots, v_k\}$ in $S \subseteq T_{\mathrm{st}_V(\delta)}X$. We choose these so that $\{v_i\}_{i=1}^k$ are eigenvectors of H_δ, and $\{u_1, \dots, u_k\}$ is the resulting basis obtained by applying the orthonormalization process to the collection $\{(K_\delta \circ D_\delta(\mathrm{st}_V))^{-1}(v_i)\}_{i=1}^k$ of pullback vectors. So we obtain

$$\det(Q_2|_S) \cdot |\mathrm{Jac}_\delta(\mathrm{st}_V)| = |\det(K_\delta) \cdot \mathrm{Jac}_\delta(\mathrm{st}_V)| = |\det(\langle K_\delta \circ D_\delta(\mathrm{st}_V)(u_i), v_j\rangle)|.$$

By the choice of bases, the matrix $(\langle K_\delta \circ D_\delta(\mathrm{st}_V)(u_i), v_j\rangle)$ is upper triangular, so we have

$$\begin{aligned}|\det(\langle K_\delta \circ D_\delta(\mathrm{st}_V)(u_i), v_j\rangle)| &= \Big|\prod_{i=1}^k \langle K_\delta \circ D_\delta(\mathrm{st}_V)(u_i), v_i\rangle\Big| \\ &\le \prod_{i=1}^k 2\langle H_\delta(v_i), v_i\rangle^{1/2} = 2^k \det(H_\delta)^{1/2} = 2^k \det(Q_1|_S)^{1/2}\end{aligned}$$

where the middle inequality is obtained via (2.2). Hence

$$|\mathrm{Jac}_\delta(\mathrm{st}_V)| \le 2^k \cdot \frac{\det(Q_1|_S)^{1/2}}{\det(Q_2|_S)}.$$

We summarize the above discussion in the following proposition.

Proposition 2.3. *Let Q_1, Q_2 be the positive semi-definite quadratic forms defined as above (note Q_2 is actually positive definite). Assume there exists a constant C, only depending on X, such that*

$$\frac{\det(Q_1|_S)^{1/2}}{\det(Q_2|_S)} \le C$$

for any k-dimensional subspace $S \subseteq T_{\mathrm{st}_V(\delta)}X$. Then the quantity $|\mathrm{Jac}(\mathrm{st}_V)(\delta)|$ is universally bounded, independently of the choice of the $(k+1)$-tuple of points $V \subset X$, and of the point $\delta \in \Delta_s^k$.

3. Jacobian estimate

Let $X = G/K$ be an irreducible symmetric space of non-compact type. We fix an arbitrary point $x \in X$ and identify T_xX with $\mathfrak{p}$. Let μ be a probability measure that is fully supported on the Furstenberg boundary $\partial_F X$. Using the same notation as in Section 2.3, we define a positive semi-definite quadratic form Q_1 and a positive definite quadratic form Q_2 on T_xX by

$$Q_1(v,v) = \int_{\partial_F X} dB^2_{(x,\theta)}(v)\, d\mu(\theta), \qquad Q_2(v,v) = \int_{\partial_F X} DdB_{(x,\theta)}(v,v)\, d\mu(\theta),$$

for $v \in T_xX$. We will follow the techniques of Connell and Farb [6], [7], and show the following theorem.

Theorem 3.1. *Let X be an irreducible symmetric space of non-compact type excluding* $\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$ *and* $\mathrm{SL}(4,\mathbb{R})/\mathrm{SO}(4)$*, and let $r = \mathrm{rank}(X) \geq 2$. If $n = \dim(X)$, then there exists a constant C, only depending on X, such that*

$$\frac{\det(Q_1|_S)^{1/2}}{\det(Q_2|_S)} \leq C$$

for any subspace $S \subseteq T_xX$ with $n - r + 2 \leq \dim(S) \leq n$.

In view of Proposition 2.3, this implies that the barycentrically straightened simplices of dimension $\geq n - r + 2$ have uniformly controlled Jacobians. The reader whose primary interest is bounded cohomology, and who is willing to take Theorem 3.1 on faith, can skip ahead to Section 5 for the proof of the Main Theorem.

The rest of this section will be devoted to the proof of Theorem 3.1. In Section 3.1, we explain some simplifications of the quadratic forms, allowing us to give geometric interpretations for the quantities involved in Theorem 3.1. In Section 3.2, we formulate the "weak eigenvalue matching" Theorem 3.3 (which will be established in Section 4). Finally, in Section 3.3, we will deduce Theorem 3.1 from Theorem 3.3.

3.1. Simplifying quadratic forms

Following [6, Section 4.3], we fix a flat $\mathcal{F}$ going through x, and denote the tangent space by $\mathfrak{a}$, so $\dim(\mathfrak{a}) = r$ is the rank of X. By abuse of notation, we identify $\mathfrak{a}$ with $\mathcal{F}$. Choose an orthonormal basis $\{e_i\}$ in T_xX such that $\{e_1, \dots, e_r\}$ spans $\mathcal{F}$, and assume e_1 is regular so that $e_1(\infty) \in \partial_F X$. Then Q_1, Q_2 can be expressed in the following matrix forms:

$$Q_1 = \int_{\partial_F X} O_\theta \begin{pmatrix} 1 & 0 \\ 0 & 0^{(n-1)} \end{pmatrix} O_\theta^*\, d\mu(\theta), \qquad Q_2 = \int_{\partial_F X} O_\theta \begin{pmatrix} 0^{(r)} & 0 \\ 0 & D_\lambda^{(n-r)} \end{pmatrix} O_\theta^*\, d\mu(\theta),$$

where $D_\lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_{(n-r)})$, and O_θ is the orthogonal matrix corresponding to the unique element in K that sends e_1 to $v_{(x,\theta)}$, the direction at x pointing towards θ. Moreover, there exists a constant $c > 0$ that only depends on X, so that $\lambda_i \geq c$ for $1 \leq i \leq n - r$. For more details, we refer the reader to the original [6].

Denote by $\bar{Q}_2$ the quadratic form given by

$$\bar{Q}_2 = \int_{\partial_F X} O_\theta \begin{pmatrix} 0^{(r)} & 0 \\ 0 & I^{(n-r)} \end{pmatrix} O_\theta^*\, d\mu(\theta).$$

Then the difference $Q_2 - c\bar{Q}_2$ is positive semi-definite, hence $\det(Q_2|_S) \geq \det(c\bar{Q}_2|_S)$. So in order to show Theorem 3.1, it suffices to assume Q_2 has the matrix form

$$\int_{\partial_F X} O_\theta \begin{pmatrix} 0^{(r)} & 0 \\ 0 & I^{(n-r)} \end{pmatrix} O_\theta^* \, d\mu(\theta).$$

Given any $v \in T_x X$, we have the following geometric estimates on the value of the quadratic form:

$$\begin{aligned} Q_1(v, v) &= \int_{\partial_F X} v^t O_\theta \begin{pmatrix} 1 & 0 \\ 0 & 0^{(n-1)} \end{pmatrix} O_\theta^* v \; d\mu(\theta) = \int_{\partial_F X} \langle O_\theta^* v, e_1 \rangle^2 \, d\mu(\theta) \\ &\leq \int_{\partial_F X} \sum_{i=1}^{r} \langle O_\theta^* v, e_i \rangle^2 \, d\mu(\theta) = \int_{\partial_F X} \sin^2(\angle(O_\theta^* v, \mathcal{F}^\perp)) \, d\mu(\theta). \end{aligned} \tag{3.1}$$

Roughly speaking, $Q_1(v, v)$ is bounded above by the weighted average of the time the K-orbit spends away from $\mathcal{F}^\perp$. Similarly we can estimate

$$\begin{aligned} Q_2(v, v) &= \int_{\partial_F X} v^t O_\theta \begin{pmatrix} 0^{(r)} & 0 \\ 0 & I^{(n-r)} \end{pmatrix} O_\theta^* v \; d\mu(\theta) = \int_{\partial_F X} \sum_{i=r+1}^{n} \langle O_\theta^* v, e_i \rangle^2 \, d\mu(\theta) \\ &= \int_{\partial_F X} \sin^2(\angle(O_\theta^* v, \mathcal{F})) \, d\mu(\theta). \end{aligned} \tag{3.2}$$

So again, $Q_2(v, v)$ roughly measures the weighted average of the time the K-orbit spends away from $\mathcal{F}$.

3.2. Eigenvalue matching

In their original paper, Connell and Farb showed an eigenvalue matching theorem [6, Theorem 4.4], in order to get the Jacobian estimate in top dimension. For the small eigenvalues of Q_2 (there are at most r of them), they want to find twice as many comparatively small eigenvalues of Q_1. Then by taking the product of those eigenvalues, they obtain a uniform upper bound on the ratio $\det(Q_1)^{1/2}/\det(Q_2)$, which yields an upper bound on the Jacobian. However, as was pointed out by Inkang Kim and Sungwoon Kim, there was a mistake in the proof. Connell and Farb fixed the gap by showing a weak eigenvalue matching theorem [7, Theorem 0.1], which was sufficient to imply the Jacobian inequality.

We generalize this method and show that in fact we can find $r - 2$ additional small eigenvalues of Q_1 that are bounded by a universal constant times the smallest eigenvalue of Q_2. This allows for the Jacobian inequality to be maintained when we pass down to a subspace of codimension at most $r - 2$. We now state our version of the weak eigenvalue matching theorem.

Definition 3.2. We call a set $\{w_1, \ldots, w_k\}$ of unit vectors a *δ-orthonormal k-frame* if $\langle w_i, w_j \rangle < \delta$ for all $1 \leq i < j \leq k$.

Theorem 3.3 (Weak eigenvalue matching). *Let X be an irreducible symmetric space of non-compact type such that $r = \text{rank}(X) \geq 2$, excluding $\text{SL}(3, \mathbb{R})/\text{SO}(3)$ and $\text{SL}(4, \mathbb{R})/\text{SO}(4)$. There exist constants C', C, δ, only depending on X, such that the following holds. Given any $\epsilon < \delta$, and any orthonormal k-frame $\{v_1, \dots, v_k\}$ in $T_x X$ with $k \leq r$, whose span V satisfies $\angle(V, \mathcal{F}) \leq \epsilon$, there is a $C'\epsilon$-orthonormal $(2k + r - 2)$-frame given by vectors $\{v_1', v_1'', \dots, v_1^{(r)}, v_2', v_2'', \dots, v_k', v_k''\}$ such that for $i = 1, \dots, k$ and $j = 1, \dots, r$, we have*

$$\angle(hv_i', \mathcal{F}^\perp) \leq C\angle(hv_i, \mathcal{F}), \qquad \angle(hv_i'', \mathcal{F}^\perp) \leq C\angle(hv_i, \mathcal{F}),$$
$$\angle(hv_1^{(j)}, \mathcal{F}^\perp) \leq C\angle(hv_1, \mathcal{F}),$$

for all $h \in K$, where hv is the linear action of $h \in K$ on $v \in T_x X \simeq \mathfrak{p}$.

The proof of Theorem 3.3 is postponed to Section 4.

3.3. *Proof of Theorem 3.1*

In this section, we will prove Theorem 3.1 using Theorem 3.3. For the proof, we will need the following three elementary results from linear algebra.

Lemma 3.4. *Let Q be a positive definite quadratic form on some Euclidean space V of dimension n, with eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$. Let $W \subseteq V$ be a subspace of codimension l, and let $\mu_1 \leq \cdots \leq \mu_{n-l}$ be the eigenvalues of Q restricted to W. Then $\lambda_i \leq \mu_i \leq \lambda_{i+l}$ for $i = 1, \dots, n - l$.*

Proof. Assume that, on the contrary, $\mu_i > \lambda_{i+l}$ for some i. Take the subspace $W_0 \subseteq W$ spanned by the eigenvectors corresponding to $\mu_i, \mu_{i+1}, \dots, \mu_{n-l}$; clearly $\dim(W_0) = n-l-i+1$. So for any non-zero vector $v \in W_0$, we have $Q(v, v) \geq \mu_i \|v\|^2 > \lambda_{i+l}\|v\|^2$. However, if we denote by $V_0 \subseteq V$ the $(i + l)$-dimensional subspace spanned by the eigenvectors corresponding to $\lambda_1, \dots, \lambda_{i+l}$, we have $Q(v, v) \leq \lambda_{i+l}\|v\|^2$ for any $v \in V_0$. Now $\dim(W_0 \cap V_0) \geq \dim(W_0) + \dim(V_0) - \dim(V) = 1$ implies $W_0 \cap V_0$ is non-trivial, so we obtain a contradiction. This establishes $\mu_i \leq \lambda_{i+l}$. A similar argument shows $\lambda_i \leq \mu_i$. □

Lemma 3.5. *Let Q be a positive definite quadratic form on some Euclidean space V of dimension n, with eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$. If $\{v_1, \dots, v_n\}$ is any orthonormal frame in V, ordered so that $Q(v_1, v_1) \leq \cdots \leq Q(v_n, v_n)$, then $Q(v_i, v_i) \geq \lambda_i/n$ for $i = 1, \dots, n$.*

Proof. We use induction on the dimension of V. The statement is clear when $n = 1$, so assume it holds for $\dim(V) = n - 1$. Now if $\dim(V) = n$, we restrict the quadratic form Q to the $(n - 1)$-dimensional subspace W spanned by $v_1, \dots, v_{n-1}$, and denote the eigenvalues of $Q|_W$ by $\mu_1 \leq \cdots \leq \mu_{n-1}$. By the induction hypothesis and Lemma 3.4, we obtain

$$Q(v_i, v_i) \geq \frac{\mu_i}{n-1} \geq \frac{\lambda_i}{n-1} \geq \frac{\lambda_i}{n}$$

for $1 \leq i \leq n-1$. Finally, for the last vector, we have

$$Q(v_n, v_n) \geq \frac{Q(v_1, v_1) + \cdots + Q(v_n, v_n)}{n} = \frac{tr(Q)}{n} = \frac{\lambda_1 + \cdots + \lambda_n}{n} \geq \frac{\lambda_n}{n}. \qquad \square$$

Lemma 3.6. *Let Q be a positive definite quadratic form on some Euclidean space V of dimension n. If $\{v_1, \ldots, v_k\}$ is any τ-orthonormal k-frame for τ sufficiently small (only depending on n), ordered so that $Q(v_1, v_1) \leq \cdots \leq Q(v_k, v_k)$, then there is an orthonormal k-frame $\{u_1, \ldots, u_k\}$ such that $Q(u_i, u_i) \leq 2Q(v_i, v_i)$.*

Proof. The Gram–Schmidt process applied to $\{v_1, \ldots, v_k\}$ yields an orthonormal k-frame $\{u_1, \ldots, u_k\}$. Notice $\{v_1, \ldots, v_k\}$ is τ-orthonormal, so $u_i = v_i + O(\tau)v_1 + \cdots + O(\tau)v_i$, where $O(\tau)$ denotes a number that has universally bounded ratio to τ (with the bound only depending on n). This implies

$$Q(u_i, u_i) = Q(v_i, v_i) + O(\tau) \sum_{1 \leq s \leq t \leq i} Q(v_s, v_t).$$

Since $|Q(v_s, v_t)| \leq \sqrt{Q(v_s, v_s)Q(v_t, v_t)} \leq Q(v_i, v_i)$, we obtain

$$Q(u_i, u_i) \leq Q(v_i, v_i) + O(\tau)Q(v_i, v_i) \leq 2Q(v_i, v_i)$$

for τ sufficiently small. $\square$

Proof of Theorem 3.1. As was shown in [6, Section 4.4], for any fixed $\epsilon_0 \leq 1/(r+1)$ there are at most r eigenvalues of Q_2 that are smaller than ϵ_0 (we will choose ϵ_0 in the course of the proof). By Lemma 3.4 the same is true for $Q_2|_S$. We arrange these small eigenvalues in the order $L_1 \leq \cdots \leq L_k$, where $k \leq r$. Observe that if no such eigenvalue exists, then by Lemma 3.4, $\det(Q_2|_S)$ is uniformly bounded below, and the conclusion holds (since the eigenvalues of $Q_1|_S$ are all ≤ 1). So we will henceforth assume $k \geq 1$. We denote the corresponding unit eigenvectors by $v_1, \ldots, v_k$ (so that v_i has eigenvalue L_i). Although $V = \mathrm{span}\{v_1, \ldots, v_k\}$ might not have small angle with $\mathcal{F}$, it is shown in [7, Section 3] that there is a $k_0 \in K$ such that $\angle(k_0 v_i, \mathcal{F}) \leq 2\epsilon_0^{1/4}$ for each i.

Let ϵ be a constant so small that $\epsilon < \delta$, where δ is from Theorem 3.3, and also $\tau := C'\epsilon$ satisfies the condition of Lemma 3.6 (where C' is obtained from Theorem 3.3). Hence the choice of ϵ only depends on X. We now make a choice of ϵ_0 such that $2\epsilon_0^{1/4} < \epsilon$, and hence $\angle(k_0 V, \mathcal{F}) < \epsilon$. (Note again the choice of ϵ_0 only depends on X.)

Apply Theorem 3.3 to the frame $\{k_0 v_1, \ldots, k_0 v_k\}$, and translate the resulting $C'\epsilon$-orthonormal frame by k_0^{-1}. This gives us a $C'\epsilon$-orthonormal $(2k + r - 2)$-frame $\{v_1', v_1'', \ldots, v_1^{(r)}, v_2', v_2'', \ldots, v_k', v_k''\}$, such that for $i = 1, \ldots, k$ and $j = 1, \ldots, r$, we have

$$\angle(hv_i', \mathcal{F}^{\perp}) \leq C\angle(hv_i, \mathcal{F}), \qquad \angle(hv_i'', \mathcal{F}^{\perp}) \leq C\angle(hv_i, \mathcal{F}),$$
$$\angle(hv_1^{(j)}, \mathcal{F}^{\perp}) \leq C\angle(hv_1, \mathcal{F}),$$

for all $h \in K$ (note that we have absorbed the k_0-translation into h).

Note that $\angle(hv_i', \mathcal{F}^\perp) \le C\angle(hv_i, \mathcal{F})$ implies $\sin^2(\angle(hv_i', \mathcal{F}^\perp)) \le C_0 \sin^2(\angle(hv_i, \mathcal{F}))$ for some C_0 depending on C. For convenience, we still use C for this new constant. Hence,

$$\begin{aligned} Q_1(v_i', v_i') &\le \int_{\partial_F X} \sin^2(\angle(O_\theta^* v_i', \mathcal{F}^\perp))\, d\mu(\theta) \\ &\le C \int_{\partial_F X} \sin^2(\angle(O_\theta^* v_i, \mathcal{F}))\, d\mu(\theta) = C Q_2(v_i, v_i) = C L_i. \end{aligned}$$

An identical estimate gives us $Q_1(v_i'', v_i'') \le CL_i$, and $Q_1(v_1^{(j)}, v_1^{(j)}) \le CL_1$.

We rearrange the $C'\epsilon$-orthonormal $(2k+r-2)$-frame as $\{u_1', u_1'', \dots, u_1^{(r)}, u_2', u_2'', \dots, u_k', u_k''\}$ so that it has increasing order when Q_1 is applied. Then the inequalities still hold for this new frame:

$$Q_1(u_i', u_i') \le CL_i, \quad Q_1(u_i'', u_i'') \le CL_i, \quad Q_1(u_1^{(j)}, u_1^{(j)}) \le CL_1.$$

Since the choice of ϵ makes $C'\epsilon$ satisfy the condition of Lemma 3.6, we apply the lemma to this $C'\epsilon$-orthonormal frame. This gives us an orthonormal $(2k+r-2)$-frame $\{\overline{u_1'}, \overline{u_1''}, \dots, \overline{u_1^{(r)}}, \overline{u_2'}, \overline{u_2''}, \dots, \overline{u_k'}, \overline{u_k''}\}$, such that

$$\begin{aligned} Q_1(\overline{u_i'}, \overline{u_i'}) &\le 2Q_1(u_i', u_i') \le 2CL_i, \\ Q_1(\overline{u_i''}, \overline{u_i''}) &\le 2Q_1(u_i'', u_i'') \le 2CL_i, \\ Q_1(\overline{u_1^{(j)}}, \overline{u_1^{(j)}}) &\le 2Q_1(u_1^{(j)}, u_1^{(j)}) \le 2CL_1. \end{aligned}$$

Again, we can rearrange the orthonormal basis to have increasing order when applying Q_1, and it is easy to check that, for the resulting rearranged orthonormal basis, the same inequalities still hold.

We denote the first $2k+r-2$ eigenvalues of Q_1 by $\lambda_1' \le \lambda_1'' \le \cdots \le \lambda_1^{(r)} \le \lambda_2' \le \lambda_2'' \le \cdots \le \lambda_k' \le \lambda_k''$, and the first $2k$ eigenvalues of $Q_1|_S$ by $\mu_1' \le \mu_1'' \le \cdots \le \mu_k' \le \mu_k''$. Applying Lemma 3.5, we have

$$\begin{aligned} \lambda_i' &\le nQ_1(\overline{u_i'}, \overline{u_i'}) \le 2nCL_i, \\ \lambda_i'' &\le nQ_1(\overline{u_i''}, \overline{u_i''}) \le 2nCL_i, \\ \lambda_1^{(j)} &\le nQ_1(\overline{u_1^{(j)}}, \overline{u_1^{(j)}}) \le 2nCL_1, \end{aligned}$$

for $1 \le i \le k$ and $1 \le j \le l$.

Notice $\dim(S) \ge n-r+2$. We apply Lemma 3.4 to obtain

$$\begin{aligned} &\mu_1' \le \lambda_1^{(r-1)} \le 2nCL_1, \qquad \mu_1'' \le \lambda_1^{(r)} \le 2nCL_1, \\ &\mu_i' \le \lambda_i' \le 2nCL_i, \qquad\qquad \mu_i'' \le \lambda_i'' \le 2nCL_i, \end{aligned}$$

for $2 \le i \le k$. The eigenvalues of $Q_1|_S$ are bounded above by 1, and $L_1, \dots, L_k$ are the only eigenvalues of $Q_2|_S$ that are below ϵ_0 (and recall that the choice of ϵ_0 only depends on X). Therefore,

$$\det(Q_1|_S) \le \prod_{i=1}^{k} \mu_i' \mu_i'' \le \prod_{i=1}^{k} (2nCL_i)^2 \le (2nC)^{2k} \left[\frac{\det(Q_2|_S)}{\epsilon_0^{\dim(S)-k}} \right]^2 \le \overline{C} \det(Q_2|_S)^2$$

where $\overline{C}$ only depends on X. This completes the proof of Theorem 3.1. □

4. Reduction to a combinatorial problem

In this section, we will prove the "weak eigenvalue matching" Theorem 3.3, which was introduced in Section 3.2. The approach is to follow [7], and reduce the theorem to a combinatorial problem. Then we apply Hall's Marriage Theorem to solve it.

4.1. Hall's Marriage Theorem

We recall Hall's classical Marriage Theorem; later on we will apply a slightly stronger version (Corollary 4.3 below) in the proof of Lemma 4.5.

Theorem 4.1 (Hall's Marriage Theorem). *Suppose we have a set of m different species $A = \{a_1, \dots, a_m\}$, and a set of n different planets $B = \{b_1, \dots, b_n\}$. Let $\phi : A \to \mathcal{P}(B)$ be a map which sends a species to the set of all planets suitable for its survival. Then we can assign to each species a different planet to survive if and only if for any subset $A_0 \subseteq A$, we have the cardinality inequality $|\phi(A_0)| \ge |A_0|$.*

Corollary 4.2. *Under the assumption of Theorem 4.1, we can assign to each species two different planets if and only if for any subset $A_0 \subseteq A$, we have $|\phi(A_0)| \ge 2|A_0|$.*

Proof. If there exists such an assignment, the cardinality condition holds obviously. On the other hand, assume the cardinality condition; we want to show there is an assignment. We make an identical copy of each species and form the set $A' = \{a_1', \dots, a_m'\}$. We apply Hall's Marriage Theorem to the set $A \cup A'$ relative to B. Then for every i, a_i and a_i' each have its own planet, and that means there are two planets for the original species a_i.

To see why the cardinality condition holds, we choose an arbitrary subset $H \cup K' \subseteq A \cup A'$ where $H \subseteq A$ and $K' \subseteq A'$. Let K be the corresponding identical copy of K' in A. We have $\phi(H \cup K') = \phi(H \cup K) \ge 2|H \cup K| \ge |H| + |K| = |H \cup K'|$. □

Corollary 4.3. *Suppose we have a set $V = \{v_1, \dots, v_r\}$ of vectors, and for each v_i, the selectable set is denoted by $B_i \subseteq B$. If for any subset $V_0 = \{v_{i_1}, \dots, v_{i_k}\} \subseteq V$, we have $|B_{i_1} \cup \cdots \cup B_{i_k}| \ge 2k + r - 2$, then we can pick $3r - 2$ distinct elements $\{b_1', \dots, b_1^{(r)}, b_i', b_i''$ $(2 \le i \le r)\}$ in B such that $b_1', \dots, b_1^{(r)} \in B_1$ and $b_i', b_i'' \in B_i$.*

Proof. First we choose V_0 to be the singleton set $\{v_1\}$. By hypothesis, $|B_1| \ge r > r - 2$, hence we are able to choose $r - 2$ elements $b_1^{(3)}, \dots, b_1^{(r)}$ for v_1. Next we can easily check the cardinality condition and apply Corollary 4.2 to the set V with respect to $B \setminus \{b_1^{(3)}, \dots, b_1^{(r)}\}$ to obtain the pairs $\{b_i', b_i''\}$ (for each $1 \le i \le r$). □

4.2. Angle inequality

Throughout this section, we will work exclusively with *unit vectors* in $T_xX \simeq \mathfrak{p}$. We embed the point stabilizer K_x into $\mathrm{Isom}(T_xX) \simeq O(n)$, and endow it with the induced metric. This gives rise to a norm on K, defined by $\|k\| = \max_{v \in T_xX} \angle(v, kv)$ for $k \in K$. We denote the Lie algebra of $K_x \simeq K$ by $\mathfrak{k}$, which has root space decomposition $\mathfrak{k} = \mathfrak{k}_0 \oplus \bigoplus_{\alpha \in \Lambda^+} \mathfrak{k}_\alpha$. For each small element $k \in K$, the action on a vector v can be approximated by the Lie algebra action, that is, if $k = \exp(u)$ is small, then $\|[u, v]\| \approx \|kv - v\| \sim \angle(v, kv)$, where we write $A \sim B$ if A/B and B/A are both universally bounded. By abuse of notation, we do not distinguish between $\|k\|$ and $\|u\|$ inside a very small neighborhood $\mathcal{U}$ of 0 inside $\mathfrak{k}$. Although $\|\cdot\|$ is not linear on $\mathcal{U}$, it is linear up to a universal constant, that is, $\|tu\| \sim t\|u\|$ for all $u \in \mathcal{U}$ and t such that $tu \in \mathcal{U}$. We now show the following lemmas.

Lemma 4.4 (compare [7, Lemma 1.1]). *Let $X = G/K$ be a rank $r \geq 2$ irreducible symmetric space of non-compact type, and fix a flat $\mathcal{F} \subseteq T_xX$ at x. Then for any small $\rho > 0$, there is a constant $C(\rho)$ with the following property. If $v \in \mathcal{F}$ is arbitrary, and $v^* \in \mathcal{F}$ is a maximally singular vector in the ρ-neighborhood of v (in the sense that the dimension of K_{v^*} is as large as possible), then*

$$\angle(hu, \mathcal{F}^\perp) \leq C\angle(hv, \mathcal{F})$$

for any $h \in K$ and $u \in (K_{v^*}\mathcal{F})^\perp \simeq \bigoplus_{\alpha \in \Lambda^+,\, \alpha(v^*) \neq 0} \mathfrak{p}_\alpha$, where Λ^+ is the set of all positive roots. Moreover,

$$\angle(hu, \mathcal{F}^\perp) \leq C\angle(hk_0v, K_{v^*}\mathcal{F})$$

for any $h \in K$, $u \in (K_{v^*}\mathcal{F})^\perp$, and $k_0 \in K_{v^*}$.

Proof. We only need to verify the inequality when $\angle(hv, \mathcal{F})$ is small. Notice that for any $v \in \mathcal{F}$, and any small element $w \in \mathfrak{k}_\alpha = (I + \theta)\mathfrak{g}_\alpha = (I + \theta)(I - \theta)^{-1}\mathfrak{p}_\alpha$, the Lie algebra action (see Proposition 2.2) has norm

$$\|[w, v]\| = \|-\alpha(v) \cdot (I - \theta)(I + \theta)^{-1} w\| \sim |\alpha(v)| \cdot \|w\|. \tag{4.1}$$

This is due to the fact that $(I + \theta)(I - \theta)^{-1}$ is a linear isomorphism between $\mathfrak{k}_\alpha$ and $\mathfrak{p}_\alpha$ (see Proposition 2.1), and when restricted to $\mathfrak{k}_\alpha \cap \mathcal{U}$, it preserves the norms up to a uniform multiplicative constant.

Infinitesimally speaking, for $h = \exp(w)$, we have $hv - v = [w, v]$, so the estimate on the Lie algebra action tells us about the infinitesimal growth of $\|hv - v\|$. We also see that since $[w, v] \in \mathfrak{p}_\alpha$, h moves the vector v in the direction $\mathfrak{p}_\alpha$ (which we recall is orthogonal to the flat $\mathcal{F}$, see Proposition 2.1). Now v^* is a maximally singular vector in the ρ-neighborhood of the unit vector v, so once ρ is small enough, if α is any root with $\alpha(v^*) \neq 0$, then $\alpha(v)$ will be uniformly bounded away from zero (depending only on the choice of ρ). This shows that if a root α satisfies $\alpha(v^*) \neq 0$, then $\angle(hv, \mathcal{F}) \sim \|h\|$ for all $h \in \exp(\mathfrak{k}_\alpha \cap \mathcal{U})$.

Now we move to analyzing the general case $h = \exp(w)$, where $w \in \mathfrak{k}$ is arbitrary. If $\angle(hv, \mathcal{F})$ is small, then the components of hv on each $\mathfrak{p}_\alpha$ must be small. From the discussion above, this implies that the component of w in each $\mathfrak{k}_\alpha|_{\alpha(v^*)\neq 0}$ is small, i.e. w almost lies in $\mathfrak{k}_{v^*} = \mathfrak{k}_0 \oplus \bigoplus_{\alpha(v^*)=0} \mathfrak{k}_\alpha$. Since h almost lies in K_{v^*}, there exists $h_0 \in K_{v^*}$ such that $h_0^{-1}h$ is close to the identity. We write $h = h_0 h_1$, where $h_1 = \exp(w_1) \in \exp(\mathfrak{k}_{v^*}^{\perp}) = \exp(\bigoplus_{\alpha(v^*)\neq 0} \mathfrak{k}_\alpha)$, and observe that the analysis in the previous paragraph applies to h_1. Now observe that, infinitesimally, $h_1 v - v = [u_1, v] \in \bigoplus_{\alpha(v^*)\neq 0} \mathfrak{p}_\alpha$, so h_1 moves v in a direction lying in $\bigoplus_{\alpha(v^*)\neq 0} \mathfrak{p}_\alpha$. On the other hand, infinitesimally, K_{v^*} moves the entire flat $\mathcal{F}$ in the directions $\bigoplus_{\alpha(v^*)=0} \mathfrak{p}_\alpha$ (corresponding to the action of its Lie algebra $\mathfrak{k}_{v_*}$). But these two directions are orthogonal, which means that $h_1 v$ leaves orthogonally not just $\mathcal{F}$, but actually the entire orbit $K_{v^*}\mathcal{F}$. This allows us to estimate

$$\angle(hv, \mathcal{F}) = \angle(h_1 v, h_0^{-1}\mathcal{F}) \geq \angle(h_1 v, K_{v^*}\mathcal{F}) \sim \|h_1\|, \tag{4.2}$$

where at the last step, we use the fact that h_1 moves v orthogonally off the K_{v^*}-orbit of $\mathcal{F}$. On the other hand, we are assuming that the vector u lies in $(K_{v^*}\mathcal{F})^{\perp}$, hence also in $h_0^{-1}\mathcal{F}^{\perp}$. Therefore

$$\angle(hu, \mathcal{F}^{\perp}) = \angle(h_1 u, h_0^{-1}\mathcal{F}^{\perp}) \leq \angle(h_1 u, u) \leq \|h_1\|. \tag{4.3}$$

Combining (4.2) and (4.3) gives us the first inequality of the conclusion.

Similarly, $\angle(hk_0 v, K_{v^*}\mathcal{F})$ being small also implies that the component of h on each $\mathfrak{k}_\alpha|_{\alpha(v^*)\neq 0}$ is small. So by writing $h = h_0 h_1$ in the same manner, we get $\angle(hk_0 v, K_{v^*}\mathcal{F}) = \angle(h_1 k_0 v, K_{v^*}\mathcal{F}) = \angle(k_0^{-1} h_1 k_0 v, K_{v^*}\mathcal{F})$. Notice that K_{v^*} conjugates $\mathfrak{k}_{v^*}^{\perp}$ to itself, so $k_0^{-1} h_1 k_0$ is in $\exp(\mathfrak{k}_{v^*}^{\perp})$. In view of (4.1) and the fact that $k_0^{-1} h_1 k_0 v$ leaves $K_{v^*}\mathcal{F}$ orthogonally, we obtain $\angle(k_0^{-1} h_1 k_0 v, K_{v^*}\mathcal{F}) \sim \|k_0^{-1} h_1 k_0\| = \|h_1\|$. Combining this estimate with (4.3) gives the second inequality. □

Lemma 4.5. *Let $X = G/K$ be a rank $r \geq 2$ irreducible symmetric space of non-compact type excluding* $\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$ *and* $\mathrm{SL}(4,\mathbb{R})/\mathrm{SO}(4)$*, and fix a flat $\mathcal{F} \subseteq T_x X$. Then there exists a constant $C > 0$, only depending on X, such that for any $1/2$-orthonormal r-frame $\{v_1, \ldots, v_r\}$ in $\mathcal{F}$, there is an orthonormal $(3r-2)$-frame $\{v_1', v_1'', \ldots, v_1^{(r)}, v_i', v_i''\ (2 \leq i \leq r)\}$ in $\mathcal{F}^{\perp}$ such that*

$$\angle(hv_i', \mathcal{F}^{\perp}) \leq C\angle(hv_i, \mathcal{F}), \qquad \angle(hv_i'', \mathcal{F}^{\perp}) \leq C\angle(hv_i, \mathcal{F}),$$
$$\angle(hv_1^{(j)}, \mathcal{F}^{\perp}) \leq C\angle(hv_1, \mathcal{F}),$$

for all $h \in K$, $i = 2, \ldots, r$ and $j = 1, \ldots, r$.

Proof. Once we have chosen a parameter ρ, we will denote by v_i^* a maximally singular vector in $\mathcal{F}$ that is ρ-close to v_i, and we will let $Q_i = (K_{v_i^*}\mathcal{F})^{\perp} \simeq \bigoplus_{\alpha\in\Lambda^+,\, \alpha(v_i^*)\neq 0} \mathfrak{p}_\alpha$. We now fix a ρ small enough that, for every $1/2$-orthonormal r-frame $\{v_1, \ldots, v_r\} \subset \mathcal{F}$, the corresponding $\{v_i^*\}_{i=1}^r$ are distinct. For each v_i, the vectors in Q_i are the possible choice of vectors that satisfy the angle inequality provided by Lemma 4.4. So it suffices to find r vectors in Q_1, and two vectors in each Q_i $(i \neq 1)$, such that the chosen $3r - 2$ vectors form an orthonormal frame.

Now for each root α, we pick an orthonormal frame $\{b_{\alpha_i}\}$ on $\mathfrak{p}_\alpha$, and we collect them into the set $B := \{b_i\}_{i=1}^{n-r}$, which forms an orthonormal frame on $\mathcal{F}^\perp$. We will pick the $(3r-2)$-frame from the vectors in B. For instance, the vector v_1 has selectable set $B_1 := Q_1 \cap B$, from which we want to choose r elements, while for $i = 2, \dots, r$, the vector v_i has selectable set $B_i := Q_i \cap B$, from which we want to choose two elements. Most importantly, the $3r-2$ chosen vectors have to be distinct from each other. This is a purely combinatorial problem, and can be solved by using Hall's Marriage Theorem. In view of Corollary 4.3, we only need to check the cardinality condition. We notice the selectable set of v_i is B_i, which spans Q_i, so $|B_i| = \dim(Q_i)$. The next lemma will estimate the dimension of the Q_i, and thus will complete the proof of Lemma 4.5. □

Lemma 4.6. *Let $X = G/K$ be a rank $r \geq 2$ irreducible symmetric space of non-compact type, excluding* $\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$ *and* $\mathrm{SL}(4,\mathbb{R})/\mathrm{SO}(4)$*, and fix a flat $\mathcal{F}$. Assume $\{v_1^*, \dots, v_r^*\}$ spans $\mathcal{F}$, and let $Q_i = K_{v_i^*}\mathcal{F}$. Then for any subcollection $\{v_{i_1}^*, \dots, v_{i_k}^*\}$, we have*

$$\dim(Q_{i_1} + \cdots + Q_{i_k}) \geq 2k + r - 2.$$

Proof. Since $Q_i = (K_{v_i^*}\mathcal{F})^\perp \simeq \bigoplus_{\alpha\in\Lambda^+,\, \alpha(v_i^*)\neq 0} \mathfrak{p}_\alpha$, we obtain $Q_{i_1} + \cdots + Q_{i_k} = \bigoplus_{\alpha\in\Lambda^+,\, \alpha(V)\neq 0} \mathfrak{p}_\alpha$, where $V = \mathrm{span}\{v_{i_1}^*, \dots, v_{i_k}^*\}$. We can estimate

$$\begin{aligned}\dim(Q_{i_1} + \cdots + Q_{i_k}) &= \sum_{\alpha\in\Lambda^+,\, \alpha(V)\neq 0} \dim(\mathfrak{p}_\alpha) \\ &\geq |\{\alpha \in \Lambda^+ \mid \alpha(V) \neq 0\}| = \tfrac{1}{2}(|\Lambda| - |\{\alpha \in \Lambda \mid H_\alpha \in V^\perp\}|),\end{aligned}$$

where $V^\perp$ is the orthogonal complement of V in $\mathcal{F}$, and H_α is the vector in $\mathcal{F}$ that represents α.

Now we denote $t_i = \frac{1}{2}\max_{U\subseteq\mathcal{F},\, \dim(U)=i} |\{\alpha \in \Lambda \mid H_\alpha \in U\}|$, the number of positive roots in the maximally rooted i-dimensional subspace. We use the following result that appears in [6, proof of Lemma 5.2]. For completeness, we also repeat their proof here.

Claim 4.7 ([6, Lemma 5.2]). *$t_i - t_{i-1} \geq i$ for $1 \leq i \leq r-1$.*

Proof. This is proved by induction on i. For $i = 1$, the inequality holds since $t_0 = 0$ and $t_1 = 1$. Assuming $t_{i-1} - t_{i-2} \geq i-1$ holds, we let V_{i-1} be an $(i-1)$-dimensional maximally rooted subspace. By definition, the number of roots that lie in V_{i-1} is $2t_{i-1}$. There exists a root α such that H_α does not lie in V_{i-1}, and does not lie in its orthogonal complement (by irreducibility of the root system). So $H_\alpha^\perp \cap V_{i-1} := Z$ is a codimension one subspace in V_{i-1}. By the induction hypothesis, there are at least $i-1$ pairs of root vectors that lie in $V_{i-1} - Z$; call them $\pm H_{\alpha_1}, \dots, \pm H_{\alpha_{i-1}}$. Hence by properties of root systems [11, Proposition 2.9.3], either $\pm(H_\alpha + H_{\alpha_l})$ or $\pm(H_\alpha - H_{\alpha_l})$ is a pair of root vectors, for each $1 \leq l \leq i-1$. Along with $\pm H_\alpha$, these pairs of vectors lie in $(V_{i-1} \oplus \langle H_\alpha\rangle) - V_{i-1}$. We have now found $2i$ root vectors in the i-dimensional subspace $V_{i-1} \oplus \langle H_\alpha\rangle$, which do not lie in the maximally rooted subspace V_{i-1}. This shows $t_i - t_{i-1} \geq i$. □

Finally, we can estimate $\dim(Q_{i_1} + \cdots + Q_{i_k}) \geq \frac{1}{2}(|\Lambda| - |\{\alpha \in \Lambda \mid H_\alpha \in V^\perp\}|) \geq t_r - t_{r-k}$. By the Claim, a telescoping sum gives us $t_r - t_{r-k} \geq r + (r-1) + \cdots + (r-k+1) = k(2r-k+1)/2$, whence $\dim(Q_{i_1} + \cdots + Q_{i_k}) \geq k(2r-k+1)/2$. When $r \geq 4$ or $k < r = 3$ or $k < r = 2$, it is easy to check that $k(2r-k+1)/2 \geq 2k+r-2$. This leaves the case when $r = k = 3$ or $r = k = 2$. When $r = k = 3$, we can instead estimate $\dim(Q_1 + Q_2 + Q_3) = \dim(\mathcal{F}^\perp) = n - 3 \geq 7 = 2k + r - 2$ provided $n \geq 10$, which only excludes the rank three symmetric space $\mathrm{SL}(4, \mathbb{R})/\mathrm{SO}(4)$. A similar analysis when $r = k = 2$ only excludes the rank two symmetric space $\mathrm{SL}(3, \mathbb{R})/\mathrm{SO}(3)$. This completes the proof of Lemma 4.6, and hence of Lemma 4.5. □

Remark. In the rank two case, both Theorems 3.3 and 3.1 only give statements about degree n. Our Main Theorem then only gives surjectivity of comparison maps in top degree, which agrees with the result of [16], and the corresponding Jacobian estimate is consistent with [6], [7].

4.3. Proof of Theorem 3.3

We assume $k = r$ without loss of generality since otherwise we can always extend the k-frame to an r-frame that has small angle to $\mathcal{F}$. Our first step is to move the frame so that it lies in $\mathcal{F}$, while controlling the angles between the resulting vectors (so that we can apply Lemma 4.5). This is done by first moving the vectors to the respective $K_{v_i^*}\mathcal{F}$, and then moving to $\mathcal{F}$.

As in the proof of Lemma 4.4, $\angle(v_i, \mathcal{F})$ being small implies that the components of v_i on each $\mathfrak{p}_\alpha$ is small. The K-orbit of v_i intersects $\mathcal{F}$ finitely many times (exactly once in each Weyl chamber), and if each of these intersections is ρ-close to a maximally singular vector, choose v_i^* to be one closest to v_i. The element in K moving v_i to $\mathcal{F}$ will almost lie in $K_{v_i^*}$ (by an argument similar to the one in Lemma 4.4). By decomposing this element as a product $\hat{k}_i k_i$, we obtain a small k_i which sends v_i to $K_{v_i^*}\mathcal{F}$ (and $\hat{k}_i \in K_{v_i^*}$). If $k_i^{-1} = \exp(u_i)$, we have $u_i \in \bigoplus_{\alpha \in \Lambda^+,\, \alpha(v^*) \neq 0} \mathfrak{k}_\alpha$.

We now estimate the norm $\|k_i\|$. From the identification of norms in a small neighborhood of the identity, we have $\|k_i\| = \|u_i\|$. Since $\hat{k}_i$ is an element in $K_{v_i^*}$ that sends $k_i v_i$ to $\mathcal{F}$, an argument similar to the proof of the second inequality in Lemma 4.4 gives

$$\angle(v_i, K_{v_i^*}\mathcal{F}) = \angle((\hat{k}_i k_i^{-1} \hat{k}_i^{-1})(\hat{k}_i k_i v_i), K_{v_i^*}\mathcal{F}) \sim_\rho \|\hat{k}_i k_i^{-1} \hat{k}_i^{-1}\| = \|k_i\|$$

(where the constant will depend on the choice of ρ). On the other hand, since $\mathcal{F} \subset K_{v_i^*}\mathcal{F}$, we obtain $\angle(v_i, K_{v_i^*}\mathcal{F}) \leq \angle(v_i, \mathcal{F})$. But by hypothesis, $\angle(v_i, \mathcal{F}) < \epsilon$. Putting all this together, we see that, for each fixed ρ, there exists a constant C', only depending on X, such that each $\|k_i\|$ is bounded above by $\frac{1}{2}C'\epsilon$. In particular, any $\{k_i\}_{i=1}^r$ perturbation of an orthonormal frame gives rise to a $C'\epsilon$-orthonormal frame, and hence the collection $\{k_1 v_1, \ldots, k_r v_r\}$ forms a $C'\epsilon$-orthonormal frame.

Next, since $\hat{k}_i$ is in $K_{v_i^*}$, it leaves v_i^* fixed. From the triangle inequality we obtain

$$\angle(\hat{k}_i k_i v_i, k_i v_i) \leq 2\angle(k_i v_i, v_i^*) < 2\rho.$$

It follows that the collection $\{\hat{k}_1 k_1 v_1, \dots, \hat{k}_r k_r v_r\} \subset \mathcal{F}$ is obtained from the $C'\epsilon$-orthonormal frame $\{k_1 v_1, \dots, k_r v_r\}$ by rotating each vector by an angle of at most 2ρ, hence forms a $(C'\epsilon + 4\rho)$-orthonormal basis in $\mathcal{F}$. In particular, once ρ and δ are chosen small enough, it gives us a $1/2$-orthonormal basis inside $\mathcal{F}$.

Applying Lemma 4.5 to the $1/2$-orthonormal frame $\{\hat{k}_1 k_1 v_1, \dots, \hat{k}_r k_r v_r\} \subset \mathcal{F}$ gives us an orthonormal $(3r-2)$-frame $\{v_1', \dots, v_1^{(r)}, v_i', v_i''\ (2 \le i \le r)\}$ such that the angle inequalities hold. Now by the second inequality of Lemma 4.4, we have

$$\angle(hv_i', \mathcal{F}^\perp) \le C\angle(hk_i v_i, K_{v_i^*}\mathcal{F}) \le C\angle(hk_i v_i, \mathcal{F}),$$
$$\angle(hv_i'', \mathcal{F}^\perp) \le C\angle(hk_i v_i, K_{v_i^*}\mathcal{F}) \le C\angle(hk_i v_i, \mathcal{F}),$$
$$\angle(hv_1^{(j)}, \mathcal{F}^\perp) \le C\angle(hk_1 v_1, K_{v_1^*}\mathcal{F}) \le C\angle(hk_1 v_1, \mathcal{F}),$$

for $2 \le i \le r$, $1 \le j \le r$ and any $h \in K$. Finally, we translate each of the vectors v_i', v_i'' by k_i^{-1}, and each $v_1^{(j)}$ by k_1^{-1}, producing a $C'\epsilon$-orthonormal $(3r-2)$-frame that satisfies the inequalities in Theorem 3.3, hence completing the proof.

5. Surjectivity of the comparison map in bounded cohomology

In this section, we provide some background on cohomology (see Section 5.1), establish the Main Theorem (Section 5.2), establish some limitations on our technique of proof (Section 5.3), and work out a detailed class of examples (Section 5.4).

5.1. Bounded cohomology

Let $X = G/K$ be a symmetric space of non-compact type, and Γ be a cocompact lattice in G. We recall the definition of group cohomology, working with $\mathbb{R}$ coefficients (so that we can relate these to the de Rham cohomology). Let $C^n(\Gamma, \mathbb{R}) = \{f : \Gamma^n \to \mathbb{R}\}$ be the space of n-cochains. Then the coboundary map $d : C^n(\Gamma, \mathbb{R}) \to C^{n+1}(\Gamma, \mathbb{R})$ is defined by

$$\begin{aligned} df(\gamma_1, \dots, \gamma_{n+1}) = f(\gamma_2, \dots, \gamma_{n+1}) &\\ + \sum_{i=1}^{n} (-1)^i f(\gamma_1, \dots, \gamma_{i-1}, \gamma_i\gamma_{i+1}, \gamma_{i+2}, \dots, \gamma_{n+1}) &\\ + (-1)^{n+1} f(\gamma_1, \dots, \gamma_n). & \end{aligned}$$

The homology of this chain complex is $H^*(\Gamma, \mathbb{R})$, the group cohomology of Γ with $\mathbb{R}$ coefficients. Moreover, if we restrict the cochains above to *bounded* functions, we obtain the space of bounded n-cochains, $C_b^n(\Gamma, \mathbb{R}) = \{f : \Gamma^n \to \mathbb{R} \mid f \text{ is bounded}\}$, and the corresponding bounded cohomology $H_b^*(\Gamma, \mathbb{R})$ of Γ. The inclusion of the bounded cochains into the ordinary cochains induces the comparison map $H_b^*(\Gamma, \mathbb{R}) \to H^*(\Gamma, \mathbb{R})$.

Similarly, we can define the (bounded) continuous cohomology of G, by taking the space of continuous n-cochains, $C_c^n(G, \mathbb{R}) = \{f : G^n \to \mathbb{R} \mid f \text{ is continuous}\}$, or the space of bounded continuous cochains, $C_{c,b}^n(G, \mathbb{R}) = \{f : G^n \to \mathbb{R} \mid f \text{ is continuous and bounded}\}$. With the same coboundary maps as above, this gives two new chain complexes, whose homology will be denoted by $H_c^*(G,\mathbb{R})$ and $H_{c,b}^*(G,\mathbb{R})$ respectively. Again, one has a naturally induced comparison map $H_{c,b}^*(G,\mathbb{R}) \to H_c^*(G,\mathbb{R})$.

Now let $M = X/\Gamma$ be the closed locally symmetric space covered by X. Note that M is a $K(\Gamma,1)$, so

$$H_{\mathrm{dR}}^*(M,\mathbb{R}) \simeq H_{\mathrm{sing}}^*(M,\mathbb{R}) \simeq H^*(\Gamma,\mathbb{R}).$$

The isomorphism between the de Rham cohomology and group cohomology is explicitly given by

$$\phi : H_{\mathrm{dR}}^k(M,\mathbb{R}) \to H^k(\Gamma,\mathbb{R}), \qquad \omega \mapsto f_\omega,$$

where $f_\omega(\gamma_1,\dots,\gamma_k) = \int_{\Delta(\gamma_1,\dots,\gamma_k)} \widetilde{\omega}$. Here, $\widetilde{\omega}$ is a lift of ω to X, and $\Delta(\gamma_1,\dots,\gamma_k)$ is any natural C^1 k-filling with ordered vertices $\{x, \gamma_1 x, (\gamma_1\gamma_2)x, \dots, (\gamma_1\cdots\gamma_k)x\}$ for some fixed basepoint $x \in X$ (for instance, one can choose $\Delta(\gamma_1,\dots,\gamma_k)$ to be the geodesic coning simplex, see Dupont [9]). Alternatively, we can use the barycentric straightened C^1 simplex $\mathrm{st}(\Delta(\gamma_1,\dots,\gamma_k))$ (which we defined in Section 2.3). That is, if we define $\overline{f_\omega}(\gamma_1,\dots,\gamma_k) = \int_{\mathrm{st}(\Delta(\gamma_1,\dots,\gamma_k))} \widetilde{\omega}$, then $\overline{f_\omega}$ represents the same cohomology class as f_ω. This is due to the fact that the barycentric straightening is Γ-equivariant (see [16, Section 3.2]). We call $\overline{f_\omega}$ the *barycentrically straightened cocycle*.

On the other hand, there is a theorem of van Est [18] which gives the isomorphism between the relative Lie algebra cohomology $H^*(\mathfrak{g},\mathfrak{k},\mathbb{R})$ and the continuous bounded cohomology $H_c^*(G,\mathbb{R})$. A class in $H^k(\mathfrak{g},\mathfrak{k},\mathbb{R})$ can be expressed by an alternating k-form φ on $\mathfrak{g}/\mathfrak{k} \simeq T_x X$. By left translation, it gives a closed C^∞ k-form $\widetilde{\varphi}$ on $X = G/K$. In [9], this isomorphism is explicitly given by

$$\phi : H^k(\mathfrak{g},\mathfrak{k},\mathbb{R}) \to H_c^k(G,\mathbb{R}), \qquad \varphi \mapsto f_\varphi,$$

where $f_\varphi(g_1,\dots,g_k) = \int_{\Delta(g_1,\dots,g_k)} \widetilde{\varphi}$, and $\Delta(g_1,\dots,g_k)$ is the geodesic simplex with ordered vertices consisting of $\{x, g_1 x, (g_1 g_2)x, \dots, (g_1\cdots g_k)x\}$ for some fixed basepoint $x \in X$. Again, we can replace $\Delta(g_1,\dots,g_k)$ by the barycentric straightened C^1 simplex $\mathrm{st}(\Delta(g_1,\dots,g_k))$, and the resulting barycentrically straightened function $\overline{f_\varphi}(g_1,\dots,g_k) = \int_{\mathrm{st}(\Delta(g_1,\dots,g_k))} \widetilde{\varphi}$ is in the same cohomology class as f_φ.

5.2. *Proof of the Main Theorem*

In this section, we use Theorem 3.1 to establish the Main Theorem. We need to show that both comparison maps η and η' are surjective. Let us start with η. We use the van Est isomorphism (see Section 5.1) to identify $H_c^*(G,\mathbb{R})$ with $H^*(\mathfrak{g},\mathfrak{k},\mathbb{R})$. For any class $[f_\varphi] \in H_c^k(G,\mathbb{R})$ where $f_\varphi(g_1,\dots,g_k) = \int_{\Delta(g_1,\dots,g_k)} \widetilde{\varphi}$, we instead choose the barycentrically straightened representative $\overline{f_\varphi}$. Then for any $(g_1,\dots,g_k) \in G^k$,

$$|\overline{f_\varphi}(g_1,\dots,g_k)| = \left| \int_{\mathrm{st}(\Delta(g_1,\dots,g_k))} \widetilde{\varphi} \right| \le \left| \int_{\Delta_s^k} \mathrm{st}_V^* \widetilde{\varphi} \right| \le \int_{\Delta_s^k} |\mathrm{Jac}(\mathrm{st}_V)| \cdot \|\widetilde{\varphi}\| \, d\mu_0 \qquad (5.1)$$

where $d\mu_0$ is the standard volume form of Δ_s^k. But from Proposition 2.3 and Theorem 3.1, the expression $|\mathrm{Jac}(\mathrm{st}_V)|$ is uniformly bounded above by a constant (independent of the choice of the vertices V and the point $\delta \in \Delta_s^k$), while the form $\widetilde{\varphi}$ is invariant under the G-action, hence bounded in norm. It follows that the last expression above is less than some constant C that depends only on the choice of the alternating form φ. We have thus produced, for each class $[f_\varphi]$ in $H_c^k(G, \mathbb{R})$, a bounded representative $\overline{f_\varphi}$. So the comparison map η is surjective. The argument for surjectivity of η' is virtually identical, using the explicit isomorphism between $H^k(\Gamma, \mathbb{R})$ and $H_{\mathrm{dR}}^k(M, \mathbb{R})$ discussed in Section 5.1. For any class $[f_\omega] \in H^k(\Gamma, \mathbb{R})$, we choose the barycentrically straightened representative $\overline{f_\omega}$. The differential form $\widetilde{\omega}$ has bounded norm, as it is the Γ-invariant lift of the smooth differential form ω on the compact manifold M. So again, the estimate in (5.1) shows the representative $\overline{f_\omega}$ is bounded, completing the proof.

5.3. Obstruction to straightening methods

In this section, we give a general obstruction to the straightening method that is applied in Section 5.2. In the next section, we will use this to give some concrete examples showing that the conclusion of Theorem 3.1 *is not true* when $\dim(S) \leq n - r$. Throughout this section, we let $X = G/K$ be an n-dimensional symmetric space of non-compact type, and we give the following definitions.

Definition 5.1. Let $C^0(\Delta^k, X)$ be the set of singular k-simplices in X, where Δ^k is assumed to be equipped with a fixed Riemannian metric. Assume that we are given a collection of maps $\mathrm{st}_k : C^0(\Delta^k, X) \to C^0(\Delta^k, X)$. We say that this collection forms a *straightening* if:

(a) the maps induce a chain map, that is, they commute with the boundary operators,
(b) st_n is C^1-smooth, that is, the image of st_n lies in $C^1(\Delta^n, X)$.

For a subgroup $H \leq G$, we say that the straightening is *H-equivariant* if the maps st_k all commute with the H-action.

Since X is simply connected, property (a) of Definition 5.1 implies that the chain map st_* is actually chain homotopic to the identity. Also, property (b) implies that the image of any straightened k-simplex is C^1-smooth, i.e. $\mathrm{Im}(\mathrm{st}_k) \subset C^1(\Delta^k, X)$. The barycentric straightening introduced in Section 2.3 is a G-equivariant straightening. As we saw in Section 5.2, obtaining a uniform control on the Jacobian of the straightened k-simplices immediately implies a surjectivity result for the comparison map from bounded cohomology to ordinary cohomology. This motivates the following:

Definition 5.2. We say the straightening is *k-bounded* if there exists a constant $C > 0$, depending only on X and the chosen Riemannian metric on Δ^k, with the following property. For any k-dimensional singular simplex $f \in C^0(\Delta^k, X)$ and the corresponding straightened simplex $\mathrm{st}_k(f) : \Delta^k \to X$,

$$|\mathrm{Jac}(\mathrm{st}_k(f))(\delta)| \leq C$$

where $\delta \in \Delta^k$ is arbitrary (and the Jacobian is computed relative to the fixed Riemannian metric on Δ^k).

Our Theorem 3.1 and Proposition 2.3 then tell us that when $r = \mathbb{R}\text{-rank}(G) \geq 2$ (excluding the two cases $SL(3,\mathbb{R})/SO(3)$ and $SL(4,\mathbb{R})/SO(4)$), our barycentric straightening is k-bounded for all $k \geq n - r + 2$. One can wonder whether this range can be improved. In order to obtain obstructions, we recall [16, Theorem 2.4]. Upon restriction to the case of locally symmetric spaces of non-compact type, the theorem says:

Theorem 5.3 ([16, Theorem 2.4]). *Let M be an n-dimensional locally symmetric space of non-compact type, with universal cover X, and Γ be the fundamental group of M. If X admits an n-bounded, Γ-equivariant straightening, then the simplicial volume of M is positive.*

Corollary 5.4. *If X splits off an isometric $\mathbb{R}$-factor, then X does not admit an n-bounded, G-equivariant straightening.*

Proof. Let $X \simeq X_0 \times \mathbb{R}$ for some symmetric space X_0. If X admits an n-bounded, G-equivariant straightening, then consider a closed manifold $M \simeq M_0 \times S^1$, where $\widetilde{M_0} \simeq X_0$. According to Theorem 5.3, the simplicial volume $\|M\|$ is positive. But on the other hand $\|M\| = \|M_0 \times S^1\| \leq C \cdot \|M_0\| \cdot \|S^1\| = 0$. This contradiction completes the proof. □

We will use subspaces satisfying the assumption of Corollary 5.4 to obstruct bounded straightenings.

Definition 5.5. For X a symmetric space of non-compact type, we define the *splitting rank* of X, denoted $\text{srk}(X)$, to be the maximal dimension of a totally geodesic submanifold $Y \subset X$ which splits off an isometric $\mathbb{R}$-factor.

For irreducible symmetric spaces of non-compact type, computations of the splitting rank can be found in a recent paper by the second author [19] (see also Berndt and Olmos [2] for some related work).

Theorem 5.6. *If $k = \text{srk}(X)$, then X does not admit any k-bounded, G-equivariant straightening.*

Proof. Assume, contrary to the claim, that $X = G/K$ admits a k-bounded, G-equivariant straightening st_i, and let $Y \subset X$ be a k-dimensional totally geodesic subspace which splits isometrically as $Y' \times \mathbb{R}$. Denote by $p : X \to Y$ the orthogonal projection from X to Y, and note that the composition $p \circ \text{st}_*$ is a straightening on Y, which we denote by $\overline{\text{st}_*}$. Notice that Y is also a symmetric space and can be identified with G_0/K_0 for some $G_0 < G$ and $K_0 < K$. Then the straightening $\overline{\text{st}_*}$ is certainly G_0-equivariant. We claim it is also k-bounded. This is because the projection map p is volume-decreasing, hence

$$|\text{Jac}(\overline{\text{st}_k}(f))| = |\text{Jac}(p(\text{st}_k(f)))| \leq |\text{Jac}(\text{st}_k(f))| \leq C$$

for any $f \in C^0(\Delta^k, X)$. Thus, Y admits a G_0-equivariant, k-bounded straightening, contrary to Corollary 5.4. □

Remark. In view of Proposition 2.3 and the arguments in Section 5.2, we can view Theorem 5.6 as *obstructing* the bounded ratio Theorem 3.1. Specifically, if $k = \mathrm{srk}(X)$, then Theorem 5.6 tells us that one has a sequence $f_i : \Delta_s^k \to X$ with the Jacobian of $\mathrm{st}_k(f_i)$ unbounded. From the definition of st_k, this means one has a sequence $V_i = \{v_0^{(i)}, \dots v_k^{(i)}\} \subset X$ of $(k+1)$-tuples of points (the vertices of the singular simplices f_i), and a sequence of points $\delta_i = (a_0^{(i)}, \dots, a_k^{(i)})$ inside the spherical simplex $\Delta_s^k \subset \mathbb{R}^{k+1}$, satisfying the following property. If one looks at the corresponding sequence of points

$$p_i := (\mathrm{st}_k(f_i))(\delta_i) = \mathrm{bar}\Big(\sum_{j=0}^{k} a_j^{(i)} \mu(v_j^{(i)})\Big),$$

one has a sequence of k-dimensional subspace $S_i \subset T_{p_i}X$ (given by the tangent spaces $D(\mathrm{st}_{V_i})(T_{\delta_i}\Delta_s^k)$ to the straightened simplex $\mathrm{st}_k(f_i)$ at the point p_i), and the sequence of ratios $\det(Q_1|_{S_i})^{1/2}/\det(Q_2|_{S_i})$ tends to infinity. It is not too hard to see that, for each $k' \leq k$, one can find a k'-dimensional subspace $\bar{S}_i \subset S_i$ such that the sequence of determinants for the quadratic forms restricted to the $\bar{S}_i$ must also tend to infinity. Thus the conclusion of the bounded ratio Theorem 3.1 fails whenever $k' \leq \mathrm{srk}(X)$.

5.4. *The case of* $\mathrm{SL}(m, \mathbb{R})$

We conclude with a detailed discussion of the special case of the Lie group $G = \mathrm{SL}(m, \mathbb{R})$, $m \geq 5$. Its continuous cohomology has been computed (see e.g. [12, p. 299]) and can be described as follows. If $m = 2k$ is even, then $H_c^*(\mathrm{SL}(2k, \mathbb{R}))$ is an exterior algebra on k generators in degrees $5, 9, \dots, 4k-3, 2k$. If $m = 2k+1$ is even, then $H_c^*(\mathrm{SL}(2k+1, \mathbb{R}))$ is an exterior algebra on k generators in degrees $5, 9, \dots, 4k+1$.

The associated symmetric space is $X = \mathrm{SL}(m, \mathbb{R})/\mathrm{SO}(m)$, and we have

$$n = \dim(X) = \dim(\mathrm{SL}(m, \mathbb{R})) - \dim(\mathrm{SO}(m)) = (m^2-1) - \tfrac{1}{2}m(m-1) = \binom{m+1}{2} - 1,$$

while the rank of the symmetric space is clearly $r = m - 1$. Thus, our Main Theorem tells us that, for these Lie groups, the comparison map

$$H_{c,b}^*(\mathrm{SL}(m, \mathbb{R})) \to H_c^*(\mathrm{SL}(m, \mathbb{R}))$$

is surjective within the range of degrees $* \geq \binom{m+1}{2} - m + 2$.

Observe that the exterior product of all the generators of $H_c^*(\mathrm{SL}(m, \mathbb{R}))$ yields the generator for the top-dimensional cohomology, which lies in degree $\binom{m+1}{2} - 1$. Dropping off the 5-dimensional generator in the exterior product yields a non-trivial class in degree $\binom{m+1}{2} - 6$. Comparing with the surjectivity range in our Main Theorem, we see that the first interesting example occurs in the case of $\mathrm{SL}(8, \mathbb{R})$, where our results imply that $H_{c,b}^{30}(\mathrm{SL}(8, \mathbb{R})) \neq 0$ (as well as $H_{c,b}^{35}(\mathrm{SL}(8, \mathbb{R})) \neq 0$, which was previously known). Of course, as m increases, our method provides more and more non-trivial bounded cohomology classes. For example, once we reach $\mathrm{SL}(12, \mathbb{R})$, we get new non-trivial bounded cohomology classes in $H_{c,b}^{68}(\mathrm{SL}(12, \mathbb{R}))$ and $H_{c,b}^{72}(\mathrm{SL}(12, \mathbb{R}))$.

Finally, let us consider Theorem 5.6 in the special case of $X = \mathrm{SL}(m, \mathbb{R})/\mathrm{SO}(m)$. Choose a maximally singular direction in the symmetric space X, and let X_0 be the set of geodesics that are parallel to that direction. Without loss of generality, we can take $X_0 = G_0/K_0$, where

$$G_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & a \end{pmatrix} \,\middle|\, \det(A) \cdot a = 1,\ a > 0 \right\} \quad \text{and} \quad K_0 = \mathrm{SO}(m) \cap G_0.$$

Moreover, X_0 clearly splits off an isometric $\mathbb{R}$-factor, and can be isometrically identified with $\mathrm{SL}(m-1, \mathbb{R})/\mathrm{SO}(m-1) \times \mathbb{R}$. This is the maximal-dimensional subspace of $\mathrm{SL}(m, \mathbb{R})$ that splits off an isometric $\mathbb{R}$-factor (see [2, Table 3]), and the splitting rank is just $\dim(X_0) = \binom{m}{2}$. So in this special case, Theorem 5.6 tells us that our method for obtaining bounded cohomology classes *fails* once we reach degrees $\leq \binom{m}{2}$. Comparing this to the range where our method works, we see that, in the special case where $G = \mathrm{SL}(m, \mathbb{R})$, the only degree which remains unclear is $\binom{m}{2} + 1$. This example shows our Main Theorem is very close to being optimal.

6. Concluding remarks

As we have seen, the technique used in our Main Theorem seems close to optimal, at least when restricted to the Lie groups $\mathrm{SL}(m, \mathbb{R})$. Nevertheless, the authors believe that for other families of symmetric spaces, there are likely to be improvements on the range of dimensions in which a barycentric straightening is bounded.

We also note that it might still be possible to bypass the limitations provided by the splitting rank. Indeed, the splitting rank arguments show that the barycentric straightening is not k-bounded when $k = \mathrm{srk}(X)$. But the barycentric straightening might still be k'-bounded for some $k' < \mathrm{srk}(X)$ (even though the bounded ratio Theorem 3.1 must fail for k'-dimensional subspaces).

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