Available online at www.sciencedirect.com
ScienceDirect

Expo. Math. 25 (2007) 21 – 35

EXPOSITIONES MATHEMATICAE

www.elsevier.de/exmath

A note on the characteristic classes of non-positively curved manifolds

Jean-François Lafont[a,*], Ranja Roy[b]

[a]*Department of Mathematics, The Ohio State University, Columbus, OH 43210-1174, USA*
[b]*Department of Mathematics, New York Institute of Technology, Old Westbury, NY 11568, USA*

Received 8 August 2005; received in revised form 16 February 2006

Abstract

In this expository note, we give a simple conceptual proof of the Hirzebruch proportionality principle for Pontrjagin numbers of non-positively curved locally symmetric spaces. We also establish (non)-vanishing results for Stiefel–Whitney and Pontrjagin numbers of (finite covers of) the Gromov–Thurston examples of compact negatively curved manifolds. A byproduct of our argument gives a constructive proof of a well-known result of Rohlin: every closed orientable 3-manifold bounds orientably. We mention some geometric corollaries: a lower bound for degrees of covers having tangential maps to the non-negatively curved duals and estimates for the complexity of some representations of certain uniform lattices.

MSC 2000: primary 57R2053C35; secondary 57T10

Keywords: Dual space; Tangential maps; Locally symmetric spaces; Characteristic class

1. Introduction

A well known result asserts that closed hyperbolic manifolds have zero Pontrjagin numbers. The standard argument for this consists of using the Hirzebruch proportionality principle (see Appendix 1 in Hirzebruch [1]) for Pontrjagin numbers, and to observe that the dual space (a sphere) has vanishing Pontrjagin numbers. This note originated in a desire to

* Corresponding author.
E-mail addresses: jlafont@math.ohio-state.edu (J.-F. Lafont), rroy@nyit.edu (R. Roy).

0723-0869/$ - see front matter © 2006 Elsevier GmbH. All rights reserved.
doi:10.1016/j.exmath.2006.04.001

give a simple, conceptual proof of this proportionality principle, which we do in Section 3. The main advantage of our approach lies in that the characteristic numbers are computed *via an actual map* between the non-positively curved locally symmetric spaces and their non-negatively curved duals.

Now recall that there is another well-known class of negatively curved closed manifolds arising from the Gromov–Thurston construction [2] (see also the older construction of Mostow–Siu [3]). These manifolds are ramified coverings of closed hyperbolic manifolds, where the ramification occurs over a totally geodesic, codimension two submanifold that is null-homologous. Note that the behavior of characteristic numbers under ramified coverings is unclear (though see the recent result of Izawa [4]). In Section 4, we show that the Gromov–Thurston manifolds always have a finite cover that bounds orientably. A byproduct of our argument also gives a very elementary (constructive!) proof of a result of Rohlin [5]: every orientable closed 3-manifold bounds orientably. Finally, in Section 5, we point out some corollaries of our main results. We conclude in Section 6 with some open questions.

2. Preliminaries

In this section, we briefly remind the reader of the basics of the theory of vector bundles: classifying spaces, characteristic classes, and characteristic numbers. We also include a brief discussion of the manifolds we will be interested in, namely locally symmetric spaces of non-compact type, as well as the Gromov–Thurston examples of manifolds of negative curvature.

Given a smooth manifold M, a *k-dimensional real vector bundle* over the manifold M is a space E, equipped with a map $\rho : E \to M$ with the property that each point pre-image $\rho^{-1}(x)$ is equipped with a real k-dimensional vector space structure. Furthermore, the vector space operations are required to vary smoothly from fiber to fiber, and locally E looks like a product with $\mathbb{R}^k$. One can think of the space E as a smooth family of vector spaces, parameterized by points in the base M. We say two vector bundles E_1, E_2 over M are *isomorphic* provided there is a diffeomorphism $\psi : E_1 \to E_2$ having the property that $\rho_2 \circ \psi = \rho_1$, and ψ restricts to a vector space isomorphism on each individual fiber.

The example which we will be working with is the tangent bundle TM to a smooth manifold M, where at each point $x \in M$, the corresponding fiber is the tangent space $T_x M$ to M through the point x. Recall that the tangent space to a point x is obtained by looking at all smooth curves $\gamma \subset M$ having the property that $\gamma(0) = x$, modulo the equivalence relation of having the same derivative in a fixed smooth chart containing the point x. Another example of a vector bundle comes from looking at the Grassmanian of k-planes in $\mathbb{R}^n$, denoted by Gr_k^n. Recall that points in Gr_k^n correspond bijectively to k-planes in $\mathbb{R}^n$, and hence there is a canonical vector bundle $E_k^n \to \mathrm{Gr}_k^n$, where the fiber over each point in Gr_k^n is precisely the corresponding k-plane in $\mathbb{R}^n$. Now note that given any smooth map $\phi : M_1 \to M_2$, and given a vector bundle $\rho : E \to M_2$, one can form the *pull-back bundle* $\phi^{-1}(E)$, defined to be the subset $(x, v) \in M_1 \times E$ satisfying $\phi(x) = \rho(v)$. There is an obvious map to M_1 given by projection on the first factor, and for each $x \in M_1$, the pre-image under this map is a copy of the fiber $\rho^{-1}(\phi(x))$. For example, given any smooth manifold M^k, one can find a smooth embedding i of M^k in a suitably large $\mathbb{R}^n$. This induces a natural map from $\bar{i} : M^k \to \mathrm{Gr}_k^n$, assigning to each point $x \in M^k$ the tangent space to $i(M)$ at $i(x)$; since the latter is a k-plane in $\mathbb{R}^n$, one can view it as a point in the Grassmanian Gr_k^n. In this situation, the pull-back of the canonical bundle $E_k^n \to \mathrm{Gr}_k^n$ under the map $\bar{i}$ yields a vector bundle $\bar{i}^{-1}(E_k^n)) \to M^k$, which is isomorphic to the tangent bundle TM^k. A map $f : M \to N$ between smooth manifolds is said to be *tangential* provided the pullback satisfies $f^{-1}(TN) = TM$, where TN, TM are the tangent bundles to N, M, respectively.

Now the example described above is by no means exceptional. Indeed, a crucial fact in bundle theory is the existence of a *classifying space*, namely a space Gr_k^∞ (the Grassmanian of k-planes in $\mathbb{R}^\infty$), equipped with a canonically defined k-dimensional vector bundle $E_k^\infty \to \mathrm{Gr}_k^\infty$, having the property that isomorphism classes of k-dimensional vector bundles over a manifold M are in precise bijective correspondence with homotopy classes of maps from M to Gr_k^∞. The correspondence is given by associating to a map $f : M \to \mathrm{Gr}_k^\infty$ the pullback bundle $f^{-1}(E_k^\infty) \to M$. The wonderful consequence of this result is that *bundle theory reduces to homotopy theory*. One concrete application lies in the existence of *characteristic classes*: given an element α in the cohomology $H^i(\mathrm{Gr}_k^\infty; \Lambda)$ (where Λ is some coefficient ring), one can associate to any k-dimensional vector bundle over a manifold M the cohomology class $f^*(\alpha) \in H^i(M; \Lambda)$, where $f : M \to \mathrm{Gr}_k^\infty$ is the map classifying the given vector bundle over X. Such a cohomology class is called a characteristic class of the vector bundle, and gives an invariant of the vector bundle. One can now focus on the $\mathbb{Z}_2$-coefficients, in which case the cohomology ring $H^*(\mathrm{Gr}_k^\infty; \mathbb{Z}_2)$ is a free polynomial algebra over $\mathbb{Z}_2$, with one generator in each dimension (up to k). For a k-dimensional real vector bundle $E \to M$, the characteristic classes corresponding to the generators of $H^*(\mathrm{Gr}_k^\infty; \mathbb{Z}_2)$ are called the *Stiefel–Whitney classes* of the real vector bundle.

Now analogous to real vector bundles over a manifold M, one can consider k-dimensional *complex vector bundles* over M: one merely requires each of the fibers of the map $E \to M$ to have the structure of k-dimensional complex vector space. In this situation, the theory still pushes through: one has a classifying space (the space $\mathrm{Gr}_k^\infty(\mathbb{C})$ consisting of k-dimensional complex vector subspaces in $\mathbb{C}^\infty$), and hence one can define characteristic classes. In the complex situation, working with $\mathbb{Z}$-coefficients, one sees that the cohomology ring $H^*(\mathrm{Gr}_k^\infty(\mathbb{C}); \mathbb{Z})$ is a free polynomial ring over $\mathbb{Z}$, with one generator in every *even* dimension (up to $2k$). For a k-dimensional complex vector bundle $E \to M$, the characteristic classes corresponding to the generators of $H^*(\mathrm{Gr}_k^\infty(\mathbb{C}); \mathbb{Z})$ are called the *Chern classes* of the complex vector bundle.

Finally, let us briefly remind the reader of the definition of the *Pontrjagin classes* of a real vector bundle. If one starts out with a k-dimensional real vector bundle $E \to M$, we can construct a k-dimensional complex vector bundle $E_{\mathbb{C}} \to M$ by complexifying each fiber. Now the Chern classes of the complex vector bundle $c_i(E_{\mathbb{C}}) \in H^{2i}(M; \mathbb{Z})$ will be invariants of the original bundle $E \to M$. It is not too hard to see that the *odd* Chern classes satisfy $2 \cdot c_{2i+1}(E_{\mathbb{C}}) = 0 \in H^{4i+2}(M; \mathbb{Z})$, and hence are not too interesting (since they always have order two). Focusing on the *even* dimensional Chern classes, we define the Pontrjagin classes to be $p_i(E) = (-1)^i c_{2i}(E_{\mathbb{C}}) \in H^{4i}(M; \mathbb{Z})$ (the coefficient $(-1)^i$ is chosen to simplify certain formulas involving Pontrjagin classes).

Now for a compact smooth manifold M, the characteristic classes of M will be defined to be the corresponding characteristic classes of TM, the tangent bundle of M. We can now make sense of the Stiefel–Whitney and Pontrjagin classes of a manifold M. Note that for any manifold M, the top dimensional homology class $H_n(M;\mathbb{Z}_2)$ contains a single non-zero element $[M]$, called the *fundamental class* of the manifold M. Given a product of Stiefel–Whitney classes that lies in $H^n(M;\mathbb{Z}_2)$, we can define the corresponding *Stiefel–Whitney number* of M by evaluating the cohomology class on $[M]$; this yields an element in $\mathbb{Z}_2$. Likewise, if M is oriented, the orientation determines a generator $[M]$ for $H_n(M;\mathbb{Z})\cong\mathbb{Z}$, and given a product of Pontrjagin classes lying in $H^n(M;\mathbb{Z})$, one can define the corresponding *Pontrjagin number* of M by evaluating the cohomology class on $[M]$ (giving us an element in $\mathbb{Z}$). For more details on characteristic classes, we refer the reader to the classic text by Milnor–Stasheff [6].

We now turn our attention to the manifolds whose characteristic classes we will be computing. The first types of manifolds we will be considering are *irreducible, closed, non-positively curved locally symmetric spaces*. These spaces are obtained by the following procedure: start with G a non-compact semi-simple Lie group having trivial center, and let K be a maximal compact subgroup (such a subgroup is unique up to conjugacy). One can consider the coset space G/K which has a natural smooth manifold structure. The tangent space at the coset eK containing the identity e can be identified with a subspace T of the Lie algebra $\mathfrak{g}$ of G. On the Lie algebra, the *Killing form* defines a quadratic form having the property that it is positive definite when restricted to T, and hence defines an inner product on the tangent space $T_{eK}(G/K)$ at the distinguished point eK. Using the left multiplication, one can push-forward this inner product, obtaining a canonical Riemannian metric on G/K. An explicit computation shows that the resulting Riemannian metric has non-positive sectional curvature. Now if $\Gamma\leqslant G$ is a torsion-free *uniform lattice* in G (i.e. a discrete, cocompact subgroup of G), we have a natural isometric Γ-action on G/K given by left multiplication. The spaces we are interested in arise as quotients $\Gamma\backslash G/K$: these are compact manifolds equipped with a Riemannian metric of non-positive curvature, and indeed are the main examples of such manifolds. We refer the reader to Helgason's book [7] for more information on these Riemannian manifolds.

The other well-known family of compact *negatively curved* Riemannian manifolds were constructed by Gromov–Thurston [2]. Their construction starts from an oriented n-dimensional *hyperbolic manifold*, that is to say, a manifold of the form $M=\Gamma\backslash G/K$, where $G=SO(n,1)$, $K=SO(n)$, and $\Gamma\leqslant SO(n,1)$ a torsion-free uniform lattice. Furthermore, one assumes that the manifold M^n contains a totally geodesic codimension two submanifold N^{n-2} (such a submanifold is itself a hyperbolic manifold), which is homologically trivial, i.e. $[N^{n-2}]=0\in H_{n-2}(M^n;\mathbb{Z})$. They then proceed to take an oriented cyclic ramified cover $\bar{M}$ of M^n, ramified over the submanifold N^{n-2} (see Section 4 for a discussion of such ramified covers). The cover $\bar{M}$ comes equipped with a map $p:\bar{M}\to M^n$ having the property that it is a diffeomorphism when restricted to the pre-image of N^{n-2}, and away from that submanifold, it is a cyclic covering. Note that there is a natural (*not* complete) Riemannian metric on $\bar{M}-p^{-1}(N^{n-2})$. Gromov–Thurston showed that this Riemannian metric can be "smoothed out" near the subset $p^{-1}(N^{n-2})$ to a Riemannian metric *which still has strict negative sectional curvature*. Their smoothing argument relies on the fact that the singular submanifold $p^{-1}(N^{n-2})$ has codimension two in $\bar{M}$. In fact, the "smoothing" question while preserving negative (or non-positive) curvature becomes much more delicate for higher codimension singular subsets, and is known to fail in certain cases (see Davis–Januszkiewicz[8]). Gromov–Thurston then proceeded to show that, provided the ramified covering has high enough degree, these give genuinely *new* examples of manifolds of strict negative curvature, i.e. that the manifolds $\bar{M}$ are topologically distinct from the locally symmetric examples we discussed earlier. In Section 4, we will exhibit a vanishing result for the Pontrjagin numbers of the Gromov–Thurston examples.

3. Characteristic numbers of locally symmetric spaces

Let us start by recalling the construction of the non-negatively curved dual space associated to any non-positively curved closed locally symmetric space. If G is a real Lie group, K its maximal compact subgroup, we let $G_C = G \otimes \mathbb{C}$ be the complexification of G and G_U the maximal compact subgroup of G_C. The factor spaces G/K and $M_U = G_U/K$ are called dual symmetric spaces [9]. By abuse of language, if Γ is a uniform lattice in G, we will still say that $M := \Gamma\backslash G/K$ and M_U are dual spaces. In [9], Okun showed that if M^n is a non-positively curved closed locally symmetric space, then there is a tangential map from some finite cover $\bar{M}^n$ to the dual symmetric space. We start by showing the following easy lemma.

Lemma 1. *Assume $f : M \longrightarrow N$ is a tangential map between two n-dimensional manifolds. Then*

- $p_I(M) = \pm \deg(f) \cdot p_I(N) \in \mathbb{Z}$,
- $sw_I(M) = \deg(f) \cdot sw_I(N) \in \mathbb{Z}/2\mathbb{Z}$,

where p_I, sw_I denote the Pontrjagin and Stiefel–Whitney numbers associated to a product of Pontrjagin or Stiefel–Whitney classes.

Proof. Since the map is tangential, the pullbacks of Pontrjagin classes (respectively, Stiefel –Whitney classes) of N yield the corresponding classes for M. If we denote by $\tau_I(N)$ a product of Pontrjagin classes, we have $f^*(\tau_I(N)) = \tau_I(M)$. Likewise, if $\sigma_I(N)$ denotes a product of Stiefel–Whitney classes, $f^*(\sigma_I(N)) = \sigma_I(M)$. Now we have that

$$\begin{aligned} p_I(M) &= \langle \tau_I(M), [M] \rangle = \langle f^*(\tau_I(N)), [M] \rangle \\ &= \pm \langle \tau_I(N), f_*([M]) \rangle = \pm \langle \tau_I(N), \deg(f) \cdot [N] \rangle \\ &= \pm \deg(f) \cdot \langle \tau_I(N), [N] \rangle = \pm \deg(f) \cdot p_I(N). \end{aligned}$$

And the argument for part (b) of the lemma is identical. □

Note that, from the discussion above, we have associated to any closed locally symmetric space M^n a diagram:

$$M^n \xleftarrow{f} \bar{M}^n \xrightarrow{t} M_U,$$

where $\bar{M}^n$ is a finite cover, M_U is the non-negatively curved dual, and the maps in the diagram are tangential. Since a covering map never has zero degree, Lemma 1 tells us that we can solve for the Pontrjagin numbers of M^n

$$p_I(M^n) = \frac{\deg(t)}{\deg(f)} \cdot p_I(M_U).$$

Of course, if we are trying to relate the vanishing/non-vanishing of Pontrjagin numbers of M^n with those of M_U, it is crucial to know when $\deg(t) \neq 0$. Conceivably if $\deg(t) = 0$, one could have non-zero Pontrjagin numbers for M_U, but with the corresponding Pontrjagin number for M^n equal to zero. That this does not occur is the content of the next Lemma:

Lemma 2. *If t has degree zero, then the Pontrjagin numbers $p_I(M_U)$ are all equal to zero.*

Proof. We start by noting that Okun ([9, Corollary 6.5]) showed that if G_U and K have equal rank, then t has non-zero degree. Hence if $\deg(t) = 0$, we must have $\mathrm{rk}(G_U) > \mathrm{rk}(K)$. Recall that the *toral rank* of a compact manifold N, denoted by $\mathrm{trk}(N)$, is the largest dimension of a torus that has a smooth, rationally-free action on N (where an action is rationally-free provided all point stabilizers are finite). Now Allday–Halperin [10] have shown that $\mathrm{trk}(G_U/K) = \mathrm{rk}(G_U) - \mathrm{rk}(K)$, hence if $\deg(t) = 0$, we have that $\mathrm{trk}(M_U) > 0$. But Conner–Raymond [11] have shown that if N is a compact manifold with $\mathrm{trk}(N) > 0$, then all the Pontrjagin numbers of N are equal to zero. Applying their result to M_U completes the proof. □

For completeness, we point out that by a result of Papadima [12], for the homogenous space $M_U = G_U/K$, we have that the toral rank of M_U is zero if and only if the Euler characteristic of M_U is non-zero. Hence to verify that the map t has non-zero degree, it is sufficient to verify that the Euler characteristic of M_U is non-zero. Combining the previous two Lemmas, we obtain the immediate:

Theorem A (*Hirzebruch proportionality principle*). *Let M^n be a non-positively curved closed locally symmetric space, and let M_U be the non-negatively curved dual. Then $p_I(M^n) \neq 0$ if and only if $p_I(M_U) \neq 0$. Furthermore, the ratio of these Pontrjagin numbers is a constant that depends solely on M^n.*

We refer to Helgason [7] for the classification of the irreducible non-positively curved symmetric spaces, as well as for the notation used in our discussion. Amongst the classical families, we have

Corollary 1. *Let M^n be a closed irreducible locally symmetric space, and assume that M^n is locally modelled on one of the following:*

(1) $SL(n, \mathbb{R})/SO(n)$;
(2) $SU^*(2n)/Sp(n)$;
(3) $SO_0(p, q)/SO(p) \times SO(q)$ *where p and q are both odd*;
(4) *an irreducible globally symmetric spaces of Type IV, see pp.* 515–516 *in* [7].

Then M^n has all Pontrjagin numbers equal to zero.

Proof of Corollary 1. An explicit computation shows that amongst the non-positively curved symmetric spaces, those mentioned in Corollary 1, are precisely the ones having $\mathrm{rk}(G_U) > \mathrm{rk}(K)$, and hence from the discussion above the corresponding M^n have all Pontrjagin numbers equal to zero.

Remark. The remaining families of non-positively curved locally symmetric spaces could conceivably have non-vanishing Pontrjagin numbers. Since the procedure for calculating the Pontrjagin numbers of the non-negatively curved duals is well established (see Borel–Hirzebruch [13]), and in view of Theorem A, one could in principle find out which of these spaces actually have a non-vanishing Pontrjagin number. As this procedure is primarily combinatorial in nature, we leave the precise computations to the interested reader, and content ourselves with computing them for the *negatively* curved locally symmetric spaces. In the process, we also discuss the exceptional locally symmetric space $F_{4(-20)}/\mathrm{Spin}(9)$ giving rise to Cayley hyperbolic manifolds.

Corollary 2. *Let M^n be a compact orientable manifold, and assume that one of the following holds*:

(1) *M^n is real hyperbolic*;
(2) *M^n is complex hyperbolic, and $n = 4k + 2$*;
(3) *M^n is quaternionic hyperbolic, and $n = 8k + 4$.*

Then M^n has a finite cover that bounds. In the first two cases, there is a finite cover that bounds orientably (and hence M^n has all Pontrjagin numbers equal to zero).

Corollary 3. *Let M^n be a compact orientable manifold, and assume that one of the following holds*:

(1) *M^n is Cayley hyperbolic (so $n = 16$)*;
(2) *M^n is complex hyperbolic, and $n = 4k$*;
(3) *M^n is quaternionic hyperbolic of dimension at least* 8.

Then M^n has some non-zero Pontrjagin numbers, and hence no finite cover can bound orientably. Furthermore, in the case (2), *we have that* all *Pontrjagin numbers are non-zero.*

Since the arguments are closely related, we simultaneously prove both corollaries.

Proof of Corollaries 2 and 3. We note that for the negatively curved symmetric spaces, the duals are easy to compute. Indeed we have that:

- the dual to real hyperbolic space is the sphere,
- the dual to complex hyperbolic space is complex projective space,
- the dual to quaternionic hyperbolic space is quaternionic projective space,
- the dual to Cayley hyperbolic space is the Cayley projective plane.

Since the characteristic classes of the duals are well known, we can apply Lemmas 1 and 2 in each case to obtain information on the negatively curved locally symmetric spaces. $\bar{M}^n$ will always denote the finite cover that supports a tangential map to the positively curved dual. The various cases are:

M^n is real hyperbolic: Since the sphere bounds orientably, all its characteristic numbers (both Stiefel–Whitney and Pontrjagin) are zero. Applying Lemma 1, we see that all the characteristic numbers of $\bar{M}^n$ are zero. By a result of Wall [14], this is equivalent to $\bar{M}^n$ bounding orientably, giving (1) of Corollary 2.

M^{2n} is complex hyperbolic: Then its dual space is the complex projective space $\mathbb{C}P^n$, which is a $2n$-dimensional real manifold. We now have two cases:

(A) If $n = 2k$, then the Pontrjagin numbers are all non-zero [6, p. 185], hence from Theorem A, the same holds for M^{2n}.

(B) If $n=2k+1$, then $\mathbb{C}P^n$ bounds orientably [6, p. 186]. Arguing as in the real hyperbolic case, we see that $\bar{M}^n$ bounds orientably.

This gives us (2) of Corollaries 2 and 3.

M^{4n} is quaternionic hyperbolic: Then its dual space is the quaternionic projective space $\mathbb{O}P^n$, which is a $4n$-dimensional real manifold. We again have two cases:

(A) If $n = 2k + 1$, then $\mathbb{O}P^n$ bounds, and hence has vanishing Stiefel–Whitney numbers. By Lemma 1, the same holds for $\bar{M}^{2n}$, giving (3) of Corollary 2.

(B) In general, the total Pontrjagin class of $\mathbb{O}P^n$ is given by $(1 + u)^{2n+2}(1 + 4u)^{-1}$, where $u \in H^4(\mathbb{O}P^n)$ is a generator for the truncated polynomial ring $H^*(\mathbb{O}P^n)$. Since the coefficient of u in the power series expansion equals $2n - 2$, we see that the Pontrjagin number $p_1^n(M_U)$ is equal to $(2n - 2)^n$. So provided $n \geqslant 2$, we can apply Theorem A to obtain (3) of Corollary 3.

M^{16} is Cayley hyperbolic: Then its dual space is the Cayley projective plane $\mathrm{Cay}P^2$. The Cayley plane has two non-vanishing Pontrjagin numbers: $p_2^2[\mathrm{Cay}P^2]=36$ and $p_4[\mathrm{Cay}P^2]=39$ (see Borel–Hirzebruch [13, pp. 535–536]). Applying Theorem A, we get that $\bar{M}^{16}$ has non-vanishing Pontrjagin numbers. This deals with case (1) of Corollary 3, and hence completes the proof of the corollaries. □

Remark. We note that information on the Stiefel–Whitney numbers of the rank one locally symmetric spaces is much harder to obtain. Indeed, anytime the degree of one of the two maps f, t is even, there is a potential loss of information.

Corollary 4. *If M^n is a manifold supporting a metric of constant sectional curvature, then all of its Pontrjagin numbers are zero.*

Proof. The case of constant negative curvature has been dealt with above. In the remaining two cases, M^n has a finite cover that bounds orientably (either a sphere, or a torus, depending on curvature). The corollary follows. □

Remark. Recall that Farrell–Jones have constructed exotic smooth structures on certain closed hyperbolic manifolds, and have shown that these manifolds support Riemannian metrics of negative curvature [15]. There results were subsequently extended by various authors to providing exotic smooth structures on a variety of different locally symmetric spaces, see for instance [16–20]. Observe that while the Pontrjagin classes are smooth invariants, the *rational* Pontrjagin classes are topological invariants, by a celebrated result of Novikov [21]. Since the Pontrjagin numbers of a manifold only depend on the rational Pontrjagin classes (i.e. the torsion part of the Pontrjagin classes do not influence the Pontrjagin numbers), the discussion in Corollaries 2 and 3 gives us vanishing (or non-vanishing) results for the Pontrjagin numbers of these exotic manifolds as well.

4. Characteristic numbers of the Gromov–Thurstonexamples

Definition. Let X be an oriented differentiable manifold (with or without boundary) on which the cyclic group $\mathbb{Z}_k$ acts semifreely by orientation-preserving diffeomorphisms with fixed set a codimension two submanifold Y (an action is semifree if every point is either fixed, or has trivial stabilizer). Denote the quotient space by $X' := X/\mathbb{Z}_k$, and the canonical projection map by π: $X \to X'$. Let Y be the fixed set of the action on X, and note that $\pi : Y \to Y'$ is a diffeomorphism. Observe that X' is a manifold. We say that X is an *oriented cyclic ramified cover* of X', of order k, ramified over Y'. If Y' bounds a smooth embedded codimension one submanifold in X', we say that the ramified covering is *nice*.

Remark. If a ramified covering is nice, then it is particularly easy to describe it. Indeed, let $N \subset X'$ be the codimension one embedded submanifold satisfying $\partial N = Y'$. Then the pre-image of N in the ramified cover X will consist of multiple (embedded) copies of N which all coincide along their boundary (which will equal Y). Cutting X open along the pre-images of N will yield k homeomorphic copies of $X' - N$. Now consider the space with boundary the double DN of N, obtained by cutting open X' along N. Then X is obtained by taking k copies of this space, $X_1, \dots X_k$, and for each space, cyclically gluing ∂X_i^+ to ∂X_{i+1}^-, where $\partial X_i^\pm$ denotes the two copies of N in $\partial X_i = DN$.

Proposition. *Assume that M^n bounds, and that $p : \bar{M}^n \to M^n$ is an oriented cyclic ramified cover of M^n (ramified over N^{n-2}). If the covering is nice, then $\bar{M}^n$ also bounds. If M^n bounds orientably, then so does $\bar{M}^n$.*

Proof. Let $M^n = \partial L^{n+1}$, and note that since the ramified covering is nice, there exists a smoothly embedded $K_0^{n-1} \subset M^n$ satisfying $\partial K_0^{n-1} = N^{n-2}$. Since M^n is collarable in L^{n+1}, there is a manifold $K^{n-1} \subseteq L^{n+1}$ of dimension $n-1$ with the properties:

- $K^{n-1} \cap \partial L^{n+1} = N^{n-2} = \partial K_0^{n-1}$,
- K^{n-1} and K_0^{n-1} are cobordant in L^{n+1},
- the cobordism W^n is an embedded submanifold satisfying $W^n \cap M^n = K_0^{n-1}$.

Indeed, homotoping K_0^{n-1} (relative $\partial K_0^{n-1} = N^{n-2}$) into a collared neighborhood of M^n in L^{n+1} gives both K^{n-1}, and the manifold W^n (the image of the homotopy, which we can assume to have no self-intersections). Now note that $K^{n-1} \subseteq L^{n+1}$ is a codimension two submanifold which bounds W^n. Hence we can take the i-ramified covering of L^{n+1} over K^{n-1} (see the remark preceding this Proposition). But note that on $\partial L^{n+1} = M^n$, this ramified covering yields $\bar{M}^n$. Hence if $\bar{L}^{n+1}$ is the covering, we have $\partial \bar{L}^{n+1} = \bar{M}^n$. Finally, we note that if L^{n+1} is orientable, then so is the ramified covering $\bar{L}^{n+1}$. □

Corollary 5 (*Rohlin's Theorem*). *Let M be a closed, orientable, 3-dimensional manifold. Then M bounds orientably.*

Proof. It is a well known result (due independently to Hilden [22] and Montesinos [23]) that every closed orientable 3-manifold is a ramified covering of the 3-dimensional sphere S^3 along a knot. Since every knot in S^3 bounds a compact embedded surface (a Seifert surface for the knot), this ramified cover is nice. Since S^3 bounds orientably, the proposition gives us the claim. □

Remark. Corollary 5 was first established by Rohlin [5]. It also follows easily from the subsequent results of Thom and Wall: the Pontrjagin numbers are automatically zero, since M is 3-dimensional. As for the Stiefel–Whitney numbers, there are only three of them to consider: $s_1^3, s_1^2 s_2$, and s_3. Note that since M is orientable, $s_1 = 0$, so the first two numbers vanish. As for s_3, it is just the mod 2 reduction of the Euler characteristic, which has to be zero as we are in odd dimension. Applying Wall's theorem [14], we get that M must bound orientably. The advantage of our approach is that the bounding manifold can be seen *explicitly*, and we avoid appealing to the sophisticated results of Thom and Wall.

Theorem B. *Let N be a Gromov–Thurston non-positively curved manifold. Then N has a finite cover that bounds orientably* (*and hence all Pontrjagin numbers of N are zero*).

Proof of Theorem B. Let M be a real hyperbolic manifold and N be a Gromov–Thurston non-positively curved manifold obtained as a ramified covering of M. From Corollary 2, M has a finite cover $\bar{M}$ that bounds orientably. We claim that there is a space $\bar{N}$ yielding the commutative diagram:

$$\begin{array}{ccc} \bar{N} & \xrightarrow{\bar{\psi}} & \bar{M} \\ {\scriptstyle \bar{\phi}}\downarrow & & \downarrow{\scriptstyle \phi} \\ N & \xrightarrow[\psi]{} & M \end{array}$$

where $\bar{\phi}$ is a covering map and $\bar{\psi}$ is a ramified covering (and ψ is the original ramified covering, ϕ the original covering).

In order to see this, we make the following general observation: assume that X^{n-2} is a smooth embedded codimension two submanifold in Y^n, and let $W \subset Y^n$ be a closed tubular neighborhood of X^{n-2}. Note that W is a $\mathbb{D}^2$-bundle over X^{n-2}, and hence that ∂W is an S^1-bundle over X^{n-2}. Now let $Y' \subset Y^n$ be the manifold with boundary obtained by removing the interior of W from Y^n, and assume that $\bar{Y}' \to Y'$ is a covering map. Then we have:

(1) the covering map $f : \bar{Y}' \to Y'$ extends to a covering $\bar{f} : \bar{Y} \to Y$ if and only if, for each fiber F of the bundle $S^1 \to \partial W \to X^{n-2}$, we have that $f^{-1}(F)$ consists of $\deg(f)$ disjoint copies of S^1.

(2) the covering map $f : \bar{Y}' \to Y'$ extends to a *ramified* covering $\bar{f} : \bar{Y} \to Y$ of degree $\deg(f)$ over X^{n-2} if and only if, for each fiber F of the bundle $S^1 \to \partial W \to X^{n-2}$, we have that $f^{-1}(F)$ is connected.

Indeed, one direction of the implications is immediate, since a covering (respectively, a ramified covering over X^{n-2}) exhibits precisely the aforementioned behavior on the boundary of a regular neighborhood. Conversely, assume that we have a covering map $f : \bar{Y}' \to Y'$ satisfying one of the above properties. Then note that the pre-image $f^{-1}(\partial W)$ naturally inherits a smooth foliation with S^1 leaves. Now consider the space $\bar{W}$ obtained by smoothly gluing in $\mathbb{D}^2$'s along their boundary to the leaves. Observe that this can be done, since the foliation on $f^{-1}(\partial W)$ is the lift of a fibration, and hence is locally a product. Finally, form the space $\bar{Y}$ by gluing $\bar{Y}'$ with $\bar{W}$ along their common boundary.

Now in case (1) above, we immediately get that the covering map f extends to a covering map $\bar{f}$, by simply extending linearly along each $\mathbb{D}^2$. In case (2), we again extend linearly, but this time also extend the action of $\mathbb{Z}_{\deg(f)}$ (by deck transformations) from each S^1 to each $\mathbb{D}^2$. Note that this gives a smooth $\mathbb{Z}_{\deg(f)}$ action on $\bar{Y}$, whose fixed point set maps diffeomorphically to the original X^{n-2}.

Now in the setting we have, proceed as follows: if K^{n-2} is the codimension two submanifold of M^n that is being ramified over, then let W be a closed tubular neighborhood of K, W_0 it's interior. Note that ψ is an actual *covering*, when restricted to the preimage of $M - W_0$ (as we are throwing away a neighborhood of the set where the ramification occurs). Consider the commutative diagram:

$$\begin{array}{ccc} M' & \longrightarrow & \phi^{-1}(M - W_0) \\ \downarrow & & \downarrow \phi \\ \psi^{-1}(M - W_0) & \xrightarrow[\psi]{} & M - W_0 \end{array}$$

where M' is the pullback of the covering maps. By commutativity of the diagram, we see that the covering $M' \to \phi^{-1}(M - W_0)$ satisfies (2) from our discussion above, while the covering $M' \to \psi^{-1}(M - W_0)$ satisfies (1) from the discussion above. In particular, extending M' as above, we obtain a space $\bar{N}$ which is simultaneously a ramified covering of $\bar{M}$, and an actual covering of N, as desired.

Finally, we note that the ramified covering $\bar{\psi} : \bar{N} \to \bar{M}$ is nice. Indeed, in the Gromov–Thurston construction, the ramified covering $\psi : N \to M$ is nice, so we have that $K^{n-2} = \partial L^{n-1}$ for a smooth, embedded codimension one manifold with boundary. But we have that the map $\bar{\psi}$ is ramified over $\phi^{-1}(K^{n-2})$, which clearly bounds the smooth, embedded codimension one submanifold $\phi^{-1}(L^{n-1})$. This confirms that $\bar{\psi}$ is nice, and since $\bar{M}$ bounds orientably, applying the Proposition, we see that $\bar{N}$ bounds orientably as well. This completes the proof of Theorem B. □

Remark. A related (unpublished) result is contained in the thesis of Ardanza–Trevijano Moras [24], and asserts that for the Gromov–Thurston ramified coverings, the individual Pontrjagin classes vanish. We note that while our approach does not give vanishing of individual *classes*, it does give vanishing of the Stiefel–Whitney numbers on a finite cover (which does not follow from the approach in [24]).

5. Geometric applications

As is well known, characteristic numbers provide obstructions to a wide range of topological problems. To mention but a few, if M^n has a non-zero Pontrjagin number, then

(1) no finite cover of M^n bounds orientably.
(2) M^n has no orientation reversing self-diffeomorphism.
(3) M^n does not support an almost quaternionic structure [25].

From Corollary 3, we immediately get these properties for the rank one locally symmetric manifolds that are either complex hyperbolic (with $n = 4k$), quaternionic hyperbolic with $n \geqslant 8$ or Cayley hyperbolic.

Our next application involves estimating the size of the cover that supports a tangential map to the dual space

Corollary 6. *Let M^{4n} be a compact orientable manifold which is locally symmetric. For each partition $I = i_1, i_2, \ldots, i_r$ of n, let $p_I(M^{4n})$ (respectively, $p_I(M_U)$) denote the Ith Pontrjagin number of M^{4n} (respectively, of the dual M_U). Note that if $p_I(M_U) \neq 0$, then we also have that $p_I(M^{4n}) \neq 0$ (from Lemma 2). Define*

$$\mu(M^{4n}) = \mathrm{LCM}_I\{\mathrm{LCM}(p_I(M^{4n}), p_I(M_U))/p_I(M^{4n})\},$$

where LCM *denotes least common multiple, and the outer* LCM *is over all partitions I of n for which $p_I(M^{4n}) \neq 0$. If $\bar{M}^{4n} \longrightarrow M^{4n}$ is a degree d cover having a tangential map $\bar{M}^{4n} \longrightarrow M_U$, then $\mu(M^{4n})$ divides d.*

Proof. Let r be the degree of the tangential map $\bar{M}^{4n} \longrightarrow M_U$. Then for each I, we have that $d \cdot p_I(M^{4n}) = r \cdot p_I(M_U)$. This implies that $d \cdot p_I(M^{4n})$ is a multiple of $\mathrm{LCM}(p_I(M^{4n}), p_I(M_U))$. Hence for each I, we see that d is a multiple of $\mathrm{LCM}(p_I(M^{4n}), p_I(M_U))/p_I(M^{4n})$. This forces d to be a multiple of their least common multiple. Therefore d is a multiple of $\mu(M^{4n})$. □

Remark. The argument for the last corollary applies equally well to give an identical estimate for the degree of the tangential map from $\bar{M}^n$ to M_U. Part of our interest in the covering map (rather than the tangential map), stems from the following

Corollary 7. *Let G/K be an non-positively curved, irreducible, symmetric space, and assume the dimension of G/K is divisible by 4. Let Γ be a torsion free subgroup of G, and denote by $\Gamma\backslash G/K =: M^{4n}$ the associated locally symmetric space. Consider the flat principal bundle $G/K \times_\Gamma G \longrightarrow M^{4n}$, and extend its structure group to the group G_C. The bundle naturally defines a homomorphism $\rho : \Gamma \longrightarrow G_C \subset GL(k, \mathbb{C})$ (for some suitable k). Let $A \subseteq \mathbb{C}$ be any subring of $\mathbb{C}$, finitely generated, with the property that $\rho(\Gamma) \subseteq GL(k, A)$, and let m_1, m_2 be any pair of maximal ideals in A with the property that the finite fields A/m_1 and A/m_2 have distinct characteristics. Then $\mu(M^{4n})$ divides the cardinality of the finite group $GL(2k+1, A/m_1) \times GL(2k+1, A/m_2)$.*

Proof. Given such a subring and a pair of maximal ideals, Deligne and Sullivan [26] exhibit a finite cover $\bar{M}^{4n}$ of M^{4n} having the property that:

(1) the pullback bundle to $\bar{M}^{4n}$ is trivial,
(2) the degree of the cover divides $|GL(2k+1, A/m_1) \times GL(2k+1, A/m_1)|$.

But Okun shows ([9], Proof of Theorem 5.1), that there is a tangential map from $\bar{M}^{4n}$ to M_U, hence applying Corollary 6 completes our proof. □

Remark. The previous corollary tells us that, in some sense, the *complexity* of the representation $\Gamma \to G_C \subset GL(k, \mathbb{C})$ can be estimated from below in terms of the Pontrjagin numbers of the quotient $\Gamma\backslash G/K$.

6. Some open questions

There remain a few interesting questions along the line of inquiry we are considering. For starters, Okun has provided sufficient conditions for establishing non-zero degree of the tangential map he constructs. One can ask the:

Question: Are there examples where Okun's tangential map has zero degree? In particular, if one has a locally symmetric space modelled on $SL(n, \mathbb{R})/SO(n)$, does the tangential map to the dual $SU(n)/SO(n)$ have non-zero degree?

Question: Is there an analogous construction of a tangential map in the case where M is a non-compact, finite volume, locally symmetric space?

Of course, the interest in the special case of $SL(n, \mathbb{R})/SO(n)$ is due to the "universality" of this example: every other locally symmetric space of non-positive curvature isometrically embedds in a space modelled on $SL(n, \mathbb{R})/SO(n)$. Now note that while the relationship between the cohomologies of M^n and M_U (with real coefficients) is well understood (and has been much studied) since the work of Matsushima [27], virtually nothing is known about the relationship between the cohomologies with other coefficients. One can ask:

Question: If $t : M^n \to M_U$ is the tangential map, what can one say about the induced map $t^* : H^*(M_U, \mathbb{Z}_p) \to H^*(M^n, \mathbb{Z}_p)$?

In particular, the case where $p = 2$ would be of some particular interest, as the Stiefel –Whitney classes lie in these cohomology groups. Finally, we point out that there are other classes of non-positively curved Riemannian manifolds, arising from Schroeder's cusp closing construction ([28,29]), doubling constructions, and related techniques.

Question: Compute the characteristic classes and/or the characteristic numbers for the remaining known examples of non-positively curved Riemannian manifolds.

Acknowledgments

The second author would like to thank the first for the exchange of ideas that made this collaboration possible. The authors would also like to thank Professor F. T. Farrell for his suggestions during the course of this work, and to thank F. Raymond for some helpful e-mails.

References

[1] Hirzebruch F. Topological methods in algebraic geometry. Berlin: Springer-Verlag; 1995.
[2] Gromov M, Thurston B. Pinching constants for hyperbolic manifolds. Invent Math 1987;89:1–12.
[3] Mostow GD, Siu YT. A compact Kahler manifold of negative curvature not covered by the ball. Ann Math 1980;112:321–60.
[4] Izawa T. Chern number formula for ramified coverings. J Math Soc Japan 2000;52:1–15.
[5] Rohlin. Three dimensional manifold is the boundary of a four-dimensional one. Doklady Acad Nauk SSSR (N.S.) 1951;81:355–7.
[6] Milnor JW, Stasheff JD. Characteristic classes. Ann Math Stud 1974: 76.
[7] Helgason S. Differential geometry, Lie groups, and symmetric spaces. New York: Academic Press; 1978.
[8] Davis M, Januszkiewicz T. Hyperbolization of polyhedra. J Differential Geom 1991;34:347–88.
[9] Okun B. Nonzero tangential maps between dual symmetric spaces. Algebr Geom Topol 2001;1:709–18.
[10] Allday C, Halperin S. Lie group actions on spaces of finite rank. Quart J Math 1978;29(2):63–76.
[11] Conner P, Raymond F. Injective operations of the toral group. Topology 1971;10:283–96.
[12] Papadima S. Discrete symmetry, toral symmetry and the Euler characteristic of manifolds. Proc Amer Math Soc 1988;103:612–4.
[13] Borel A, Hirzebruch F. Characteristic classes and homogenous spaces I. Amer J Math 1958;80:458–538.
[14] Wall CTC. Determination of the cobordism ring. Ann Math 1960;72(2):292–311.
[15] Farrell FT, Jones LE. Negatively curved manifolds with exotic smooth structures. J Amer Math Soc 1989;2: 899–908.
[16] Farrell FT, Jones LE. Complex hyperbolic manifolds and exotic smooth structures. Invent Math 1994;117: 57–74.
[17] Ontaneda P. Hyperbolic manifolds with negatively curved exotic triangulations in dimension six. J Differential Geom 1994;40:7–22.
[18] Farrell FT, Jones LE. Hyperbolic manifolds with negatively curved exotic triangulations in dimensions greater than five. J Differential Geom 1998;48:319–22.
[19] Aravinda CS, Farrell FT. Exotic structures and quaternionic hyperbolic manifolds. In: Algebraic groups and arithmetic. Mumbai: Tata Institute of Fundamental Research Studies on Mathematics; 2001.
[20] Aravinda CS, Farrell FT. Exotic negatively curved structures on Cayley hyperbolic manifolds. J Differential Geom 2003;63:41–62.
[21] Novikov SP. Rational Pontrjagin classes. Homeomorphism and homotopy type of closed manifolds. I. Izv Akad Nauk SSSR Ser Mat 1965;29:1373–88.
[22] Hilden HM. Three-fold branched coverings of S^3. Amer J Math 1976;98:989–97.
[23] Montesinos JM. Three-manifolds as 3-fold branched covers of S^3. Quart J Math 1976;27(2):85–94.
[24] Ardanza-Trevijano Moras SI. Exotic smooth structures on negatively curved manifolds that are not the homotopy type of a locally symmetric space. Ph.D. Thesis, S.U.N.Y. Binghamton, 2000.
[25] Sommese AJ. Quaternionic manifolds. Math Ann 1974/75;212:191–214.

[26] Deligne P, Sullivan D. Fibrés vectoriels complexes à groupe structural discret. CR Acad Sci Paris Ser A–B 1975;281:1081–3.
[27] Matsushima Y. On Betti numbers of compact, locally symmetric Riemannian manifolds. Osaka Math J 1962;14:1–20.
[28] Schroeder V. A cusp closing theorem. Proc Amer Math Soc 1989;106:797–802.
[29] Hummel C, Schroeder V. Cusp closing in rank one symmetric spaces. Invent Math 1996;123:283–307.