

# MARKED LENGTH RIGIDITY FOR FUCHSIAN BUILDINGS

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ABSTRACT. We consider finite 2-complexes  $X$  that arise as quotients of Fuchsian buildings by subgroups of the combinatorial automorphism group, which we assume act freely and cocompactly. We show that locally CAT(-1) metrics on  $X$  which are piecewise hyperbolic, and satisfy a natural non-singularity condition at vertices are marked length spectrum rigid within certain classes of negatively curved, piecewise Riemannian metrics on  $X$ . As a key step in our proof, we show that the marked length spectrum function for such metrics determines the volume of  $X$ .

## 1. Introduction

One of the central results in hyperbolic geometry is Mostow's rigidity theorem, which states that for closed hyperbolic manifolds of dimension  $\geq 3$ , isomorphism of fundamental groups implies isometry. Moving away from the constant curvature case, one must impose some additional constraints on the isomorphism of fundamental groups if one hopes to conclude it is realized by an isometry. On any closed negatively curved manifold  $M$ , each free homotopy class of loops contains a unique geodesic representative. This gives a well-defined class function  $MLS : \pi_1(M) \rightarrow \mathbb{R}^+$ , called the *marked length spectrum function*. Given a pair of negatively curved manifolds  $M_0, M_1$ , we say they have *the same* marked length spectrum if there is an isomorphism  $\phi : \pi_1(M_0) \rightarrow \pi_1(M_1)$  with the property that  $MLS_1 \circ \phi = MLS_0$ . The marked length spectrum conjecture predicts that closed negatively curved manifolds with the same marked length spectrum must be isometric (and that the isomorphism of fundamental groups is induced by an isometry). In full generality, the conjecture is only known to hold for closed surfaces, which was independently established by Croke [Cr] and Otal [Ot1]. In the special case where one of the Riemannian metrics is locally symmetric, the conjecture was established by Hamenstädt [Ha] (see also Dal'bo and Kim [DK] for analogous results in the higher rank case).

Of course, it is possible to formulate the marked length spectrum conjecture for other classes of geodesic spaces – for example, compact locally CAT(-1) spaces. Still in the realm of surfaces, Hersonsky and Paulin [HP] extended the result to some singular metrics on surfaces, while Banković and Leininger [BL] and Constantine [Co] give extensions to the case of non-positively curved metrics. Moving away from the surface case, the conjecture was verified independently by Alperin and Bass [AB] and by Culler and Morgan [CM] in the special case of locally CAT(-1) spaces whose universal covers are metric trees. This was recently extended by the authors to the context of compact geodesic spaces of topological (Lebesgue) dimension one, see [CL].

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In this paper, we are interested in the marked length spectrum conjecture for compact quotients of *Fuchsian buildings*, a class of polygonal 2-complexes supporting locally CAT(-1) metrics. Fixing such a quotient  $X$ , we can then look at various families of locally negatively curved metrics on  $X$ . The metrics we consider are *piecewise Riemannian*: each polygon in the complex is equipped with a Riemannian metric with geodesic boundary edges. They are also assumed to be locally negatively curved, which means that the metrics satisfy Gromov’s “large link condition” at all the vertices. We consider three classes of such metrics: those whose curvatures are everywhere bounded above by -1, those whose curvature is everywhere hyperbolic, and those whose curvatures are everywhere within the interval  $[-1, 0)$ . The space of such metrics will be denoted  $\mathcal{M}_{\leq}(X)$ ,  $\mathcal{M}_{\equiv}(X)$ , and  $\mathcal{M}_{\geq}(X)$  respectively. Note that the family of piecewise hyperbolic metrics  $\mathcal{M}_{\equiv}(X)$  are precisely the metrics lying in the intersection  $\mathcal{M}_{\leq}(X) \cap \mathcal{M}_{\geq}(X)$ . Furthermore, all three of these classes of metrics lie within the space  $\mathcal{M}_{neg}(X)$ , consisting of all (locally) negatively curved, piecewise Riemannian metrics on  $X$ . Finally, if we impose some further regularity conditions on the vertices, we obtain subclasses of metrics  $\mathcal{M}_{\leq}^v(X)$ ,  $\mathcal{M}_{\equiv}^v(X)$ ,  $\mathcal{M}_{\geq}^v(X)$ , and  $\mathcal{M}_{neg}^v(X)$ . We refer our reader to Section 2 for further background on Fuchsian buildings, including precise definitions for these classes of metrics – let us just mention that, amongst these, the most “regular” metrics are those lying in the class  $\mathcal{M}_{\equiv}^v(X)$ , which forms an analogue of Teichmüller space for  $X$ .

**Main Theorem.** *Let  $X$  be a quotient of a Fuchsian building  $\tilde{X}$  by a subgroup  $\Gamma \leq \text{Aut}(\tilde{X})$  of the combinatorial automorphism group  $\text{Aut}(\tilde{X})$  which acts freely and cocompactly. Consider a pair of negatively curved metrics  $g_0, g_1$  on  $X$ , where  $g_0$  is in  $\mathcal{M}_{\equiv}^v(X)$ , and  $g_1$  is in  $\mathcal{M}_{\geq}^v(X)$ . Then  $(X, g_0)$  and  $(X, g_1)$  have the same marked length spectrum if and only if they are isometric.*

In the process of establishing the **Main Theorem**, we also obtain a number of auxiliary results which may be of some independent interest. Let us briefly mention a few of these. Throughout the rest of this section,  $X$  will denote a quotient of a Fuchsian building  $\tilde{X}$  by a subgroup  $\Gamma \leq \text{Aut}(\tilde{X})$  which acts freely and cocompactly.

The first step is to obtain marked length spectrum rigidity for pairs of metrics in  $\mathcal{M}_{\leq}^v(X)$ .

**Theorem 1** (MLS rigidity – special case). *Let  $g_0, g_1$  be any two metrics in  $\mathcal{M}_{\equiv}^v$  and  $\mathcal{M}_{\leq}(X)$  respectively. Then  $(X, g_0)$  and  $(X, g_1)$  have the same marked length spectrum if and only if they are isometric.*

This result is established in Section 3, and is based on an argument outlined to us by an anonymous referee. Next, we study the volume functional on the space of metrics. We note that the volume is constant on the subspace  $\mathcal{M}_{\equiv}^v(X)$ , and in Section 4, we show the following rigidity result:

**Theorem 2** (Minimizing the volume). *Let  $g_0$  be a metric in  $\mathcal{M}_{\equiv}^v(X)$ , and  $g_1$  an arbitrary metric in  $\mathcal{M}_{\geq}^v(X)$ . If  $\text{Vol}(X, g_1) \leq \text{Vol}(X, g_0)$ , then  $g_1$  must lie within  $\mathcal{M}_{\equiv}^v(X)$  (and the inequality is actually an equality).*

Finally, the last (and hardest) step in the proof is a general result relating the marked length spectrum and the volume. We show:

**Theorem 3** (MLS determines volume). *Let  $g_0, g_1$  be an arbitrary pair of metrics in  $\mathcal{M}_{neg}^v(X)$ . If  $\text{MLS}_0 \leq \text{MLS}_1$ , then  $\text{Vol}(X, g_0) \leq \text{Vol}(X, g_1)$ .*

The analogous result for negatively curved metrics on a closed surface is due to Croke and Dairbekov [CrD], who also established a version for conformal metrics on negatively curved manifolds (see also some related work by Fanai [Fa] and by Z. Sun [S]). Our proof of Theorem 3 roughly follows the approach in [CrD] – after setting up the preliminaries in Sections 6 and 7, we prove the theorem in Section 8.

Finally, using these three theorems, the proof of the **Main Theorem** is now straightforward.

*Proof of Main Theorem.* Let  $g_0$  be a metric in  $\mathcal{M}_{\neq}^v(X)$ , and  $g_1$  a metric in  $\mathcal{M}_{\neq}^v(X)$ . If  $MLS_0 \equiv MLS_1$ , then by Theorem 3, we see that  $Vol(g_1) = Vol(g_0)$ . So Theorem 2 forces  $g_1$  to lie in the space  $\mathcal{M}_{=}^v(X)$ . Since they have the same marked length spectrum, Theorem 1 now allows us to conclude that  $(X, g_0)$  is isometric to  $(X, g_1)$ , completing the proof.  $\square$

These results provide partial evidence towards the general marked length spectrum conjecture for these compact quotients of Fuchsian buildings, which we expect to hold for any pairs of metrics in  $\mathcal{M}_{neg}(X)$ . We should mention that rigidity theorems for such quotients  $X$  are often difficult to prove. For instance combinatorial (Mostow) rigidity was established by Xiangdong Xie [X] (building on previous work of Bourdon [Bou2]). Quasi-isometric rigidity was also established by Xie [X], generalizing earlier work of Bourdon and Pajot [BP]. Superrigidity with targets in the isometry group of  $\tilde{X}$  was established by Daskalopoulos, Mese, and Vdovina [DMV]. Finally, in the context of volume entropy, recent work of Ledrappier and Lim [LL] leave us uncertain as to which metrics in  $\mathcal{M}_{=}^v(X)$  minimize the volume growth entropy (they show that the “obvious” candidate for a minimizer is actually not a minimizer).

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## 2. BACKGROUND MATERIAL

**Fuchsian buildings.** We start by summarizing basic notation and conventions on Fuchsian buildings, which were first introduced by Bourdon [Bou3]. These are 2-dimensional polyhedral complexes which satisfy a number of axioms. First, one starts with a compact convex hyperbolic polygon  $R \subset \mathbb{H}^2$ , with each angle of the form  $\pi/m_i$  for some  $m_i$  associated to the vertex ( $m_i \in \mathbb{N}, m_i \geq 2$ ). Reflection in the geodesics extending the sides of  $R$  generate a Coxeter group  $W$ , and the orbit of  $R$  under  $W$  gives a tessellation of  $\mathbb{H}^2$ . Cyclically labeling the edges of  $R$  by the integers  $\{1\}, \dots, \{k\}$  (so that the vertex between the edges labelled  $i$  and  $i + 1$  has angle  $\pi/m_i$ ), one can apply the  $W$  action to obtain a  $W$ -invariant labeling of the tessellation of  $\mathbb{H}^2$ ; this edge labeled polyhedral 2-complex will be denoted  $A_R$ , and called the *model apartment*.

A polygonal 2-complex  $\tilde{X}$  is called a 2-dimensional hyperbolic building if it contains an edge labeling by the integers  $\{1, \dots, k\}$ , along with a distinguished collection of subcomplexes  $\mathcal{A}$  called the *apartments*. The individual polygons in  $\tilde{X}$  will be called *chambers*. The complex is required to have the following properties:

- each apartment  $A \in \mathcal{A}$  is isomorphic, as an edge labeled polygonal complex, to the model apartment  $A_R$ ,
- given any two chambers in  $\tilde{X}$ , one can find an apartment  $A \in \mathcal{A}$  which contains the two chambers, and
- given any two apartments  $A_1, A_2 \in \mathcal{A}$  that share a chamber, there is an isomorphism of labeled 2-complexes  $\varphi: A_1 \rightarrow A_2$  that fixes  $A_1 \cap A_2$ .

If in addition each edge labeled  $i$  has a fixed number  $q_i$  of incident polygons, then  $\tilde{X}$  is called a *Fuchsian building*. The group  $\text{Aut}(\tilde{X})$  will denote the group of combinatorial (label-preserving) automorphisms of the Fuchsian building  $\tilde{X}$ .

Throughout this paper we make the standing assumption that  $\tilde{X}$  is *thick*, i.e. that every edge is contained in at least three chambers. Thus, the overall geometry of the building  $\tilde{X}$  will involve an interplay between the geometry of the apartments, and the combinatorics of the branching along the edges.

Note that making each polygon in  $\tilde{X}$  isometric to  $R$  via the label-preserving map produces a CAT(-1) metric on  $\tilde{X}$ . However, a given polygonal 2-complex might have several metrizations as a Fuchsian building: these correspond to varying the hyperbolic metric on  $R$  while preserving the angles at the vertices. Any such variation induces a new CAT(-1) metric on  $\tilde{X}$ . The hyperbolic polygon  $R$  is called *normal* if it has an inscribed circle that touches all its sides – fixing the angles of a polygon to be  $\{\pi/m_1, \dots, \pi/m_k\}$ , there is a unique normal hyperbolic polygon with those given vertex angles. We will call the quantity  $\pi/m_i$  the *combinatorial angle* associated to the corresponding vertex. A Fuchsian building will be called *normal* if all metric angles are equal to the corresponding combinatorial angles and the metric on each chamber is normal. We can now state Xiangdong Xie’s version of Mostow rigidity for Fuchsian buildings (see [X]):

**Theorem 4** (Xie). *Let  $\tilde{X}_1, \tilde{X}_2$  be a pair of Fuchsian buildings, and let  $\Gamma_i \leq \text{Isom}(\tilde{X}_i)$  be a uniform lattice. Assume that we have an isomorphism  $\phi: \Gamma_1 \rightarrow \Gamma_2$ . Then there is a  $\phi$ -equivariant homeomorphism  $\Phi: \tilde{X}_1 \rightarrow \tilde{X}_2$ . Moreover, if both buildings are normal, then one can choose  $\Phi$  to be a  $\phi$ -equivariant isometry.*

Another notion that will reveal itself useful is the following: inside  $\tilde{X}$ , we have a collection of *walls*, which are defined as follows. First, recall that each apartment in the building is (combinatorially) modeled on a  $W$ -invariant polygonal tessellation of  $\mathbb{H}^2$ . The geodesics extending the various sides of the polygons give a  $W$ -invariant collection of geodesics in  $\mathbb{H}^2$ , which are also a collection of combinatorial paths in the tessellation. This gives a distinguished collection of combinatorial paths in the model apartment  $A_R$ . Via the identification of apartments  $A \in \mathcal{A}$  in  $\tilde{X}$  with the model apartment  $A_R$ , we obtain the notion of wall in an apartment of  $\tilde{X}$ . Note that every edge in  $\tilde{X}$  is contained in many different walls of  $\tilde{X}$ .

**Structure of vertex links.** For a Fuchsian building, the combinatorial axioms force some additional structure on the links of vertices: these graphs must be (thick) *generalized  $m$ -gons* (see for instance [ABr, Prop. 4.9 and 4.44]). Work of Feit and Higman [FH] then implies that each  $m_i$  must lie in the set  $\{2, 3, 4, 6, 8\}$ .

Viewed as a combinatorial graph, a generalized  $m$ -gon has diameter  $m$  and girth  $2m$ . Moreover, taking the collection of cycles of length  $2m$  within the graph to be the set of apartments, such a graph has the structure of a (thick) spherical building (based on the action of the dihedral group  $D_{2m}$  of order  $2m$  acting on  $S^1$ ).

For instance, when  $m = 2$ , a generalized 2-gon is just a complete bipartite graph  $K_{p,q}$ . When  $m = 3$ , generalized 3-gons correspond to the incidence structure on finite projective planes (whose classification is a notorious open problem). When  $m > 3$ , examples are harder to find. An extensive discussion of generalized 4-gons can be found in the book [PT]. For generalized 6-gons and 8-gons, the only known examples arise from certain incidence structures associated to some of the finite groups of Lie type (see e.g. [vM]).

Note that, at a given vertex  $v$ , the edges incident to  $v$  always have one of two possible (consecutive) labels. On the level of the link, this means that  $lk(v)$  comes equipped with an induced 2-coloring of the vertices by the integers  $i, i + 1$ . Since all edges with a given label  $i$  have  $q_i$  incident chambers, this means that the vertices in  $lk(v)$  colored  $i, i + 1$  have degrees  $q_i, q_{i+1}$  respectively. In the case of generalized 2-gons, the vertex 2-coloring is the one defining the complete bipartite graph structure. For a generalized 3-gon, the identification of the graph with the incidence structure of a finite projective plane  $\mathcal{P}$  provides the 2-coloring: the colors determine whether a vertex in the graph corresponds to a point or to a line in  $\mathcal{P}$ .

Split the vertex set into  $\mathcal{V}_i, \mathcal{V}_{i+1}$ , the set of vertices with label  $i, i + 1$  respectively. From the bipartite nature of the graph, the number of edges in the graph satisfies  $|\mathcal{E}| = q_i |\mathcal{V}_i| = q_{i+1} |\mathcal{V}_{i+1}|$ . Given an edge  $e \in \mathcal{E}$ , we now count the number of apartments (i.e.  $2m$ -cycles) passing through  $e$ . In a generalized  $m$ -gon, any path of length  $m + 1$  is contained in a unique apartment (see, e.g. [We, Prop. 7.13]). Thus, to count the number of apartments through  $e$ , it is enough to count the number of ways to extend  $e$  to a path of length  $m + 1$ . The number of branches we can take at each vertex alternates between a  $q_i$  and a  $q_{i+1}$ . So if  $m$  is even, we obtain that the number of edges is  $N := q_i^{m/2} q_{i+1}^{m/2}$ .

If  $m = 3$  is odd, then we note that  $q_i = q_{i+1}$ . Indeed, opposite vertices in one of the 6-cycles have labels  $q_i$  and  $q_{i+1}$ . But for each vertex in  $lk(v)$  (which corresponds to an edge in the original building) the valence corresponds to the number of chambers which share that edge. Since the branching in the ambient building occurs along walls, for two opposite vertices in an apartment in the link, the valence must be the same. So in this case, the number of apartments through an edge is  $N := q_i^3 = q_{i+1}^3$ .

**Spaces of metrics.** Now consider a compact quotient  $X = \tilde{X}/\Gamma$  of a Fuchsian building, where  $\Gamma \subset \text{Aut}(\tilde{X})$  is a lattice in the group of combinatorial automorphisms of  $\tilde{X}$ . On the quotient space  $X$ , we will consider metrics which are *piecewise Riemannian*, i.e. whose restriction to each chamber of  $X$  is a Riemannian metric, such that all the sides of the chamber are geodesics. Moreover, we will restrict to metrics which are locally negatively curved – and thus will require the metrics on each chamber to have sectional curvature  $< 0$ . We will denote this class of metrics by  $\mathcal{M}_{neg}$ . If we instead require each chamber to be hyperbolic (i.e. to have curvature  $\equiv -1$ ), then we obtain the space  $\mathcal{M}_{\equiv}$ . Similarly, we can require each chamber to have curvature  $\leq -1$ , or curvature in the interval  $[-1, 0)$ . These give rise to the corresponding spaces  $\mathcal{M}_{\leq}$  or  $\mathcal{M}_{\geq}$  respectively. Clearly, we have a proper inclusion  $\mathcal{M}_{\leq} \cup \mathcal{M}_{\equiv} \cup \mathcal{M}_{\geq} \subset \mathcal{M}_{neg}$ , as well as the equality  $\mathcal{M}_{\equiv} = \mathcal{M}_{\leq} \cap \mathcal{M}_{\geq}$ . Notice that, for

all of these classes of metrics, the negative curvature property imposes some constraints on the metric near the vertices of  $X$ : they must always satisfy Gromov’s “large link condition” (see discussion below).

In order to obtain a true analogue of hyperbolic metrics on  $X$ , one needs to impose some additional regularity condition. To illustrate this, consider the case of piecewise hyperbolic metrics on ordinary surfaces. One can pullback a hyperbolic metric on a surface  $\Sigma_2$  of genus two via a degree two map  $\Sigma_4 \rightarrow \Sigma_2$  ramified over a pair of points. The resulting metric on the surface  $\Sigma_4$  of genus four is piecewise hyperbolic, but has two singular points with cone angle =  $4\pi$ , so in particular is *not* hyperbolic. By analogy, an analogue of a constant curvature metric on  $X$  should have “as few” singular points as possible.

Of course, the only possible singularities occur at the vertices of  $X$ . Given a vertex  $\tilde{v} \in \tilde{X}$ , one has several apartments passing through  $\tilde{v}$ , and one can restrict the metric to each of these apartments. The negative curvature condition implies that each of these apartments inherits a (possibly singular) negatively curved metric. This tells us that the sum of the angles around the vertex  $\tilde{v}$  in each apartment is  $\geq 2\pi$ . We say that the vertex  $\tilde{v}$  is *metrically non-singular* if, when restricted to each apartment through  $\tilde{v}$ , the sum of the angles at  $\tilde{v}$  is exactly  $2\pi$ . A metric has *non-singular vertices* if every vertex is metrically non-singular. We will denote the subspace of such metrics inside  $\mathcal{M}_{neg}$  by  $\mathcal{M}_{neg}^v$ . We can similarly define the subsets  $\mathcal{M}_{\leq}^v$ ,  $\mathcal{M}_{=}^v$  and  $\mathcal{M}_{\geq}^v$  inside the spaces  $\mathcal{M}_{\leq}$ ,  $\mathcal{M}_{=}$ ,  $\mathcal{M}_{\geq}$  respectively (the superscript  $v$  is intended to denote non-singular vertices).

When  $X$  is equipped with a piecewise Riemannian metric  $g$ , each vertex link  $lk(v)$  gets an induced metric  $d$ . Indeed, an edge in  $lk(v)$  corresponds to a chamber corner in  $X$ . Since the chamber  $C$  has a Riemannian metric with geodesic sides, the corner has an angle  $\theta$  measured in the  $g$  metric. The  $d$ -length of the corresponding edge is defined to be the angle  $\theta$ . With respect to this metric, the negative curvature condition at  $v$  translates to saying that every  $2m$ -cycle in the generalized  $m$ -gon  $lk(v)$  has total  $d$ -length  $\geq 2\pi$  (Gromov’s “large link” condition). The metric  $g \in \mathcal{M}_{neg}$  lies in the subclass  $\mathcal{M}_{neg}^v$  precisely if for every vertex link  $lk(v)$ , the metric  $d$ -length of *every*  $2m$ -cycle is *exactly*  $2\pi$ . Of course, a similar statement holds for  $\mathcal{M}_{\leq}^v, \mathcal{M}_{=}^v, \mathcal{M}_{\geq}^v$ . As we will see below (Corollary 6), the non-singularity condition on vertices imposes very strong constraints on the vertex angles – they will always equal the corresponding combinatorial angle.

### 3. MLS RIGIDITY FOR METRICS IN $\mathcal{M}_{=}^v$

This section is devoted to proving Theorem 1. The argument we present here was suggested to us by the anonymous referee. We start by reminding the reader of some metric properties of boundaries of CAT(-1) spaces. If  $(\tilde{X}, d)$  is any CAT(-1) space, with boundary at infinity  $\partial^\infty(\tilde{X}, d)$ , the *cross-ratio* is a function on 4-tuples  $(\xi, \xi', \eta, \eta')$  of distinct points in  $\partial^\infty(\tilde{X}, d)$ . It is defined by:

$$[\xi\xi'\eta\eta'] := \lim_{(a,a',b,b') \rightarrow (\xi,\xi',\eta,\eta')} \text{Exp} \left( \frac{1}{2} (d(a,b) + d(a',b') - d(a,b') - d(a',b)) \right)$$

and the 4-tuple  $(a, a', b, b')$  converges radially towards  $(\xi, \xi', \eta, \eta')$ . If  $\tilde{Y}$  is another CAT(-1) space, a topological embedding  $\Phi : \partial^\infty \tilde{Y} \rightarrow \partial^\infty \tilde{X}$  is a *Möbius map* if it respects the cross-ratio, i.e. for all 4-tuples of distinct points  $(\xi, \xi', \eta, \eta')$  in  $\partial^\infty Y$ ,

we have

$$[\Phi(\xi)\Phi(\xi')\Phi(\eta)\Phi(\eta')] = [\xi\xi'\eta\eta'].$$

Note that an isometric embedding of CAT(-1) spaces automatically induces a Möbius map between boundaries at infinity. As a consequence, for a totally geodesic subspace of a CAT(-1) space, the intrinsic cross-ratio (defined from within the subspace) coincides with the extrinsic cross-ratio (restriction of the cross-ratio from the ambient space).

*Proof of Theorem 1.* Lifting the metrics  $g_0, g_1$  to the universal cover, the identity map lifts to a quasi-isometry  $\Phi : (\tilde{X}, \tilde{g}_0) \rightarrow (\tilde{X}, \tilde{g}_1)$ . This induces a map between boundaries at infinity  $\partial^\infty \Phi : \partial^\infty(\tilde{X}, \tilde{g}_0) \rightarrow \partial^\infty(\tilde{X}, \tilde{g}_1)$ . Otal showed that, if an isomorphism of fundamental groups preserves the marked length spectrum, then the induced map on the boundaries at infinity is Möbius (see [Ot2] – the argument presented there is for negatively curved closed manifolds, but the proof extends verbatim to the CAT(-1) setting).

Now let  $\mathcal{A}$  be the collection of apartments in the building  $\tilde{X}$  (note that this is independent of the choice of metric on  $X$ ). Since  $g_0 \in \mathcal{M}_{\equiv}^v(X)$ , each apartment  $A \subset \tilde{X}$  inherits a piecewise hyperbolic metric, with no singular vertices. So each  $(A, \tilde{g}_0|_A)$  is a totally geodesic subspace of  $(\tilde{X}, \tilde{g}_0)$ , isometric to  $\mathbb{H}^2$ . The map  $\partial^\infty \Phi$  sends the circle corresponding to  $\partial^\infty(A, \tilde{g}_0|_A)$  to the circle in  $\partial^\infty(\tilde{X}, \tilde{g}_1)$  corresponding to the totally geodesic subspace  $(A, \tilde{g}_1|_A)$  (see [X]). Since the map  $\partial^\infty \Phi$  preserves the cross-ratio, work of Bourdon [Bou1] implies that there is an isometric embedding  $F_A : (A, \tilde{g}_0|_A) \rightarrow (\tilde{X}, \tilde{g}_1)$  which “fills-in” the boundary map. This isometry must have image  $(A, \tilde{g}_1|_A)$ , which hence must also be isometric to  $\mathbb{H}^2$ . Applying this to every apartment, we see that the metric  $g_1$ , which was originally assumed to be in  $\mathcal{M}_{\leq}(X)$ , must actually lie in the subspace  $\mathcal{M}_{\equiv}^v(X)$ .

Finally, we claim that there is an equivariant isometry between  $(\tilde{X}, \tilde{g}_0)$  and  $(\tilde{X}, \tilde{g}_1)$ . For each apartment  $A \in \mathcal{A}$ , we have an isometry  $F_A : (A, \tilde{g}_0|_A) \rightarrow (A, \tilde{g}_1|_A)$ . From Xiangdong Xie’s work, the boundary map  $\partial^\infty F_A \equiv \partial^\infty \Phi|_{\partial^\infty A}$  maps endpoints of walls to endpoints of walls (see [X, Lemma 3.11]), so the isometry  $F_A$  respects the tessellation of the apartment  $A$ , i.e. sends chambers in  $A$  isometrically onto chambers in  $A$ . But a priori, we might have two different apartments  $A, A'$  with the property that  $F_A$  and  $F_{A'}$  send a given chamber to two distinct chambers. So in order to build a global isometry from  $(\tilde{X}, \tilde{g}_0)$  to  $(\tilde{X}, \tilde{g}_1)$ , we still need to check that the collection of maps  $\{F_A\}_{A \in \mathcal{A}}$  are compatible.

Given any two apartments  $A, A' \in \mathcal{A}$  with non-empty intersection  $A \cap A' = K$ , we want to check that the maps  $F_A$  and  $F_{A'}$  coincide on the set  $K$ . Let us first consider the case where  $K$  is a half-space, i.e. there is a single wall  $\gamma$  lying in  $A \cap A'$ , and  $K$  coincides with the subset of  $A$  (respectively  $A'$ ) lying to one side of  $\gamma$ . In this special case, it is easy to verify that  $F_A$  and  $F_{A'}$  restrict to the same map on  $K$ . Indeed, Bourdon constructs the map  $F_A$  as follows: given a point  $p \in K$  take any two geodesics  $\eta, \xi$  passing through  $p \in (A, \tilde{g}_0)$ , look at the corresponding pair of geodesics  $\eta', \xi'$  in  $(A, \tilde{g}_1)$  (obtained via the boundary map), and define  $F_A(p) := \eta' \cap \xi'$ . Bourdon argues that this intersection is non-empty, and independent of the choice of pairs of geodesics. The map  $F_{A'}$  is defined similarly. But now if  $p \in \text{Int}(K)$ , one can choose a pair of geodesics  $\eta, \xi \subset \text{Int}(K)$ . Since  $\text{Int}(K)$  is contained in both  $A, A'$ , this pair of geodesics can be used to see that

$F_A(p) = \eta' \cap \xi' = F_{A'}(p)$ . This shows that, if  $K = A \cap A'$  is a half-space, then  $F_A|_K \equiv F_{A'}|_K$ .

For the general case, we now assume that we have a pair of apartments  $A, A'$  with the property that  $A \cap A' = K$  contains a chamber, and let  $x$  be an interior point of this chamber. Then work of Hersensky and Paulin [HP, Lemma 2.10] produces a sequence of apartments  $\{A_i\}_{i \in \mathbb{N}}$  with the property that  $A_1 = A$ , each  $A_i \cap A_{i+1}$  is a half-space containing  $x$ , and the  $A_i$  converge to  $A'$  in the topology of uniform convergence on compacts. From the discussion in the previous paragraph, one concludes that  $F_A(x) = F_{A_i}(x)$  for all  $i \in \mathbb{N}$ , and from the uniform convergence, it is easy to deduce that  $F_{A'}(x) = \lim F_{A_i}(x) = F_A(x)$ . This verifies that the maps  $\{F_A\}_{A \in \mathcal{A}}$  all coincide on a full-measure set (the interior points to chambers), and hence patch together to give a global isometry  $F : (\tilde{X}, \tilde{g}_0) \rightarrow (\tilde{X}, \tilde{g}_1)$ . Equivariance of the isometry follows easily from the naturality of the construction, along with the geometric nature of the maps  $F_A$ . Descending to the compact quotient completes the proof of Theorem 1.  $\square$

**Remark.** The argument presented here relies crucially on Bourdon's result in [Bou1]. In the proof of the latter, the normalization of the spaces under consideration is important. The hyperbolic space mapping in must have curvature which matches the upper bound on the curvature in the target space. This is the key reason why the argument presented here does not immediately work in the setting of the **Main Theorem**, where the metric  $g_1$  is assumed to have piecewise curvature  $\geq -1$ .

#### 4. $\mathcal{M}_{\equiv}^v(X)$ MINIMIZES THE VOLUME

This section is dedicated to proving Theorem 2. For a vertex  $v$  in our building, let  $lk(v)$  denote the link of the vertex. Combinatorially, this link is a generalized  $m$ -gon, hence a 1-dimensional spherical building. The edges of the generalized  $m$ -gon correspond to the chamber angles at  $v$ , and so any piecewise Riemannian metric on the building induces a metric on the link:

$$d_i : E(lk(v)) \rightarrow \mathbb{R}^+.$$

For these metrics,  $Vol(-, d_i)$  is simply the sum of all edge lengths. We first argue that the vertex regularity hypothesis strongly constrains the angles.

**Lemma 5.** *Let  $\mathcal{G}$  be a thick generalized  $m$ -gon. Assume we have a metric  $d$  on  $\mathcal{G}$  with the property that every  $2m$ -cycle in  $\mathcal{G}$  has length exactly  $2\pi$ . Then every edge has length  $\pi/m$ .*

*Proof.* Consider a pair of vertices  $v, w$  in  $\mathcal{G}$  at combinatorial distance  $= m$ . Let  $\mathcal{P}$  denote the set of all paths of combinatorial length  $m$  joining  $v$  to  $w$ . Note that, since any two paths in  $\mathcal{P}$  have common endpoints at  $v, w$ , they cannot have any other vertices in common – for otherwise one would find a closed loop of length  $< 2m$ , which is impossible. The concatenation of any two paths in  $\mathcal{P}$  form a  $2m$ -cycle, so has length exactly  $2\pi$ . By the thickness hypothesis, there are at least three such paths, hence every path in  $\mathcal{P}$  has metric length  $= \pi$ . Applying this argument to all pairs of antipodal vertices in  $\mathcal{G}$ , we see that *every* path in  $\mathcal{G}$  of combinatorial length  $m$  has metric length  $= \pi$ .

Now let us return to our original pair  $v, w$ . Every edge emanating from  $v$  can be extended to a (unique) combinatorial path of length  $m$  terminating at  $w$  (and

likewise for edges emanating from  $w$ ). This gives a bijection between edges incident to  $v$  and edges incident to  $w$ . Let  $e_i^w$  denote the edge incident to  $w$  associated to the edge  $e_i^v$  incident to  $v$ . Choosing  $i \neq j$ , we have a  $2m$ -cycle obtained by concatenating the paths  $\mathbf{p}_i$  and  $\mathbf{p}_j$  of combinatorial length  $m$ , joining  $v$  to  $w$  and passing through  $e_i^v, e_j^w$ . Within this  $2m$ -cycle, we have a path of combinatorial length  $m-1$  which can be extended, at each endpoint, by  $e_i^v, e_j^w$  respectively. Since every path of combinatorial length  $m$  has metric length exactly  $\pi$ , we see that the edges  $e_i^v, e_j^w$  must have the same metric length. By the thickness hypothesis, we have  $\deg(v) = \deg(w) \geq 3$ , and it follows that every edge at the vertex  $v$  has exactly the same metric length.

Using the same argument at every vertex, and noting that  $\mathcal{G}$  is a connected graph, we see that every edge in  $\mathcal{G}$  has exactly the same metric length. Finally, from the fact that the  $2m$ -cycles have length  $= 2\pi$ , we see that this common length must be  $= \pi/m$ .  $\square$

Applying Lemma 5 to the links of each vertex in  $X$ , gives us:

**Corollary 6.** *If  $g \in \mathcal{M}_{neg}^v$ , then at every vertex  $v \in X$ , all the metric angles are equal to the combinatorial angles.*

Recall that the area of a hyperbolic (geodesic) polygon, by the Gauss-Bonnet formula, is completely determined by the number of sides and the angles at the vertices. So we also obtain:

**Corollary 7.** *The volume functional is constant on the space  $\mathcal{M}_{\leq}^v(X)$ .*

We are now ready to establish Theorem 2

*Proof of Theorem 2.* We will argue by contradiction. Assume we have a metric  $g_1 \in \mathcal{M}_{\geq}^v(X) \setminus \mathcal{M}_{\leq}^v(X)$  with the property that  $Vol(X, g_1) \leq Vol(X, g_0)$ . Applying the Gauss-Bonnet theorem to any chamber  $C$ , we obtain for either metric that

$$\int_C K_i dvol_i = -\pi(n-2) + \sum_{j=1}^n \theta_i^{(j)}$$

where  $n$  is the number of sides for any chamber,  $\theta_i^{(j)}$  are the interior angles of  $C$ , and  $K_i$  is the curvature function for the metric  $g_i$ . Denote by  $\mathcal{P}(X)$  the collection of chambers in  $X$ . For the whole space  $X$ , we have

$$(4.1) \quad \sum_{C \in \mathcal{P}(X)} \int_C K_i dvol_i = -|\mathcal{P}(X)|\pi(n-2) + \sum_{C \in \mathcal{P}(X)} \sum_{j=1}^n \theta_i^{(j)}.$$

Under the assumptions of the Theorem, we have

$$\begin{aligned} \sum_{C \in \mathcal{P}(X)} \int_C K_0 dvol_0 &= \sum_{C \in \mathcal{P}(X)} \int_C -1 dvol_0 \\ &= -Vol(X, g_0) \\ &\leq -Vol(X, g_1) \\ &= \sum_{C \in \mathcal{P}(X)} \int_C -1 dvol_1 \\ &< \sum_{C \in \mathcal{P}(X)} \int_C K_1 dvol_1. \end{aligned}$$

(The last inequality is strict, since from the assumption that  $g_1 \in \mathcal{M}_\geq^v(X) \setminus \mathcal{M}_\leq^v(X)$ , there must be at least one interior point on some chamber where the curvature  $K_1$  is greater than  $-1$ .) Since the quantity  $-|\mathcal{P}(X)|\pi(n-2)$  is independent of the choice of metric, applying equation (4.1) gives us

$$\sum_{C \in \mathcal{P}(X)} \sum_{j=1}^n \theta_0^{(j)} < \sum_{C \in \mathcal{P}(X)} \sum_{j=1}^n \theta_1^{(j)}.$$

But each of these two sums can be interpreted as  $\sum_v \text{Vol}(lk(v), d_i)$  for the respective metrics. Hence, there must be at least one vertex  $v$  whose  $d_0$ -volume is strictly smaller than its  $d_1$ -volume. But by Corollary 6, the vertex regularity hypothesis forces the volumes of the links to be equal, a contradiction. This completes the proof of the Theorem 2.  $\square$

## 5. GEODESIC FLOWS AND GEODESIC CURRENTS ON FUCHSIAN BUILDINGS

In this section, we set up the terminology needed for the proof of Theorem 3.

**Geodesic flow.** Let  $\tilde{X}$  be a hyperbolic building, equipped with a  $CAT(-\epsilon^2)$  metric for some  $\epsilon > 0$ , and  $X = \tilde{X}/\Gamma$  a compact quotient of  $\tilde{X}$  by a lattice  $\Gamma \leq \text{Aut}(X)$  in the group of combinatorial automorphisms, acting freely, isometrically, and co-compactly. We make the following definitions:

- Let  $G(\tilde{X})$  be the set of unit-speed parametrizations of geodesics in  $\tilde{X}$ . Since  $\tilde{X}$  is  $CAT(-\epsilon^2)$ ,  $G(\tilde{X}) = (\partial^\infty \tilde{X} \times \partial^\infty \tilde{X} \times \mathbb{R}) \setminus (\Delta \times \mathbb{R})$  where  $\Delta$  is the diagonal in  $\partial^\infty \tilde{X} \times \partial^\infty \tilde{X}$ . The quotient space  $G(X) := G(\tilde{X})/\Gamma$  by the naturally induced  $\Gamma$ -action is the space of unit-speed geodesic parametrizations on  $X = \tilde{X}/\Gamma$ .
- As in [BB, Section 3], let  $S'$  denote the set of all unit length vectors based at a point in  $X^{(1)} \setminus X^{(0)}$  (i.e. at an edge but not a vertex) and pointing into a chamber.  $S'C$  is the set pointing into a particular chamber  $C$ .  $S'_x C$  is the set pointing into  $C$  and based at  $x$ .  $S'_x = \cup S'_x C_i$  is the union over all chambers adjacent to  $x$ .
- For  $v \in S'C$ , let  $I(v) \in S'C$  to be the vector tangent to the geodesic segment through  $C$  generated by  $v$  and pointing the opposite direction. Let  $F(v) \subset S'$  be the set of all vectors based at the footpoint of  $I(v)$  which geodesically extend the segment defined by  $v$ . Let  $W$  be the set of all bi-infinite sequences  $(w_n)_{n \in \mathbb{Z}}$  such that  $w_{n+1} \in F(w_n)$  for all  $n$ .
- Let  $\sigma$  be the left shift on  $W$ .
- Let  $t_v$  be the length of the geodesic segment in  $C$  generated by  $v$ .

The geodesic flow on  $G(\tilde{X})$  is  $g_t(\gamma(s)) = \gamma(s+t)$ . It can also be realized by the suspension flow over  $\sigma : W \rightarrow W$  with suspension function  $g((w_n)) = t_{w_0}$ . Denote the suspension flow by  $f_t : W_g \rightarrow W_g$  where

$$W_g = \{((w_n), s) : 0 \leq s \leq g((w_n))\} / [((w_n), g((w_n))) \sim (\sigma((w_n)), 0)]$$

and  $f_t((w_n), s) = ((w_n), s+t)$ . An explicit conjugacy between the suspension flow and the geodesic flow on the space  $G'(X)$  of all geodesics which do not hit a vertex is as follows:  $h : G'(X) \rightarrow W_g$  by  $h(\gamma(t)) = ((w_n^\gamma), t^\gamma)$  where  $(w_n^\gamma)$  is the trajectory of  $\gamma$  through  $S'$  indexed so that  $w_0$  is  $\dot{\gamma}(-t^\gamma)$  for  $t^\gamma$  the smallest  $t \geq 0$  for which  $\dot{\gamma}(-t^\gamma)$  belongs to  $S'$ .

**Remark.** The spaces  $G(\tilde{X})$  and  $G(X)$  are independent of the choice of metric on  $X$ . In contrast, the spaces  $W$  and  $G'(X)$  do a priori depend on the choice of metric.

**Liouville measure.** We also want an analogue of Liouville measure. We use the one constructed in [BB]. On  $S'$  define  $\mu$  by

$$d\mu(v) = \cos \theta(v) d\lambda_x(v) dx$$

where  $\theta(v)$  is the angle between  $v$  and the normal to the edge it is based at,  $\lambda_x$  is the Lebesgue measure on  $S'_x$  and  $dx$  is the volume on the edge. This measure is invariant under  $I$  by an argument well known from billiard dynamics (see e.g. [CFS]).

Consider  $W$  as the state space for a Markov chain with transition probabilities

$$p(v, w) = \begin{cases} \frac{1}{|F(v)|} & \text{if } w \in F(v) \\ 0 & \text{else.} \end{cases}$$

Ballmann and Brin prove that  $\mu$  is a stationary measure for this Markov chain ([BB] Prop 3.3) and hence  $\mu$  induces a shift invariant measure  $\mu^*$  on the shift space  $W$ . Under the suspension flow on  $W_g$ ,  $\mu^* \times dt$  is invariant. Using the conjugacy  $h$ , pull back this measure to  $G'(X) \subset G(X)$  and denote the resulting geodesic flow-invariant measure induced on  $G(X)$  by  $L$  (or  $L_g$  if we want to specify the underlying metric  $g$ ). As Ballmann and Brin remark,  $\mu \times dt$  is the Liouville measure on the interior of each chamber  $C$ , so  $L$  is a natural choice as a Liouville measure analogue on  $G(X)$ .

We close this section with a quick remark about geodesics along walls. In  $\mathcal{M}_*^v$ , every wall is geodesic and every geodesic tangent to an edge remains in  $X^{(1)}$ , by Corollary 6. Without the non-singular vertices condition (i.e. in  $\mathcal{M}_{neg}$ ) it is possible that a geodesic might spend some time along a wall, leave it, then return to it or another wall later. This sort of behavior introduces some technical issues in the argument of section 8, which we will note there. For now, we make the following observation:

**Proposition 8.** *Let  $g$  be a metric in  $\mathcal{M}_{neg}$ . Let  $T$  be the set of geodesics which are tangent to a wall at some point. Then  $L_g(T) = 0$ .*

*Proof.* By a standing assumption, each edge in  $X$  is geodesic. Thus, any geodesic which is tangent to a wall at some point will hit a vertex. These geodesics are omitted in the construction of  $L_g$ , and hence form a zero measure set when we think of  $L_g$  as a measure on all of  $G(X)$ .  $\square$

### Geodesic currents.

**Definition 9.** Let  $\mathcal{G}(\tilde{X}_i)$  denote the space of (un-parametrized and un-oriented) geodesics in  $(\tilde{X}, \tilde{g}_i)$ . A *geodesic current* on  $X = \tilde{X}/\Gamma$  is a positive Radon measure on  $\mathcal{G}(\tilde{X}_i)$  which is  $\Gamma$ -invariant and cofinite (recall that a Radon measure is a Borel measure which is both inner regular and locally finite).

**Example 10.** The following are geodesic currents on a compact Fuchsian building quotient  $X$  which will play a role in our later proofs:

- Any geodesic flow-invariant Radon measure on  $G(\tilde{X})/\Gamma$  induces a geodesic current on  $X$ , so  $L_g$  induces the Liouville current, also denoted  $L_g$ .

- For any primitive closed geodesic  $\alpha$  in  $X$ , the Dirac mass  $\mu_\alpha$  supported on the  $\Gamma$ -orbit of  $\tilde{\alpha}$  is a geodesic current.
- For a non-primitive closed geodesic  $\beta = \alpha^n$ , one can scale the current  $\mu_\alpha$  associated to its primitive representative  $\alpha$  by the multiplicity  $n$ , i.e. define  $\mu_\beta := n \cdot \mu_\alpha$ .

Let  $\mathcal{C}(X)$  denote the space of geodesic currents. We equip it with the weak-\* topology, under which it is complete (see, e.g. Prop. 2 of [Bon1]).

**Proposition 11.** *Let  $\mathcal{C} \subset \mathcal{C}(X/\Gamma)$  be the set of currents which are supported on a single closed geodesic. (I.e., it consists of all positive multiples of the  $\mu_\alpha$  described above.) Then  $\mathcal{C}$  is dense in  $\mathcal{C}(X/\Gamma)$ .*

*Proof.* In [Bon2, Theorem 7], Bonahon establishes the analogous property for geodesic currents on  $\delta$ -hyperbolic groups, with a proof given in [Bon2, Section 3]. Bonahon's argument makes use of the Cayley graph  $\text{Cay}(G)$  of  $G$ , but only relies on negative curvature properties of the Cayley graph – the group structure plays no role in the proof. A careful reading of the arguments shows that it applies verbatim in our setting.  $\square$

Since  $X_0$  and  $X_1$  are compact locally  $CAT(-\epsilon^2)$  spaces with the same fundamental group, there is a  $\Gamma$ -equivariant homeomorphism  $\partial^\infty \tilde{X}_0 \rightarrow \partial^\infty \tilde{X}_1$  induced by the identification of  $\partial^\infty \tilde{X}_i$  with  $\partial^\infty \Gamma$ . Since the space of geodesics is given by  $\mathcal{G}(\tilde{X}_i) = [(\partial^\infty \tilde{X}_i \times \partial^\infty \tilde{X}_i) \setminus \Delta] / [(\gamma_1, \gamma_2) \sim (\gamma_2, \gamma_1)]$ , there is an induced homeomorphic orbit equivalence  $\varphi : \mathcal{G}(\tilde{X}_0)/\Gamma \rightarrow \mathcal{G}(\tilde{X}_1)/\Gamma$ . Thus,  $\Gamma$ -invariant Radon measures on  $\mathcal{G}(\tilde{X}_1)$  of cofinite mass can be pulled back via  $\varphi$  to obtain  $\Gamma$ -invariant Radon measures on  $\mathcal{G}(\tilde{X}_0)$  of cofinite mass. Similarly, currents can be pushed forward via  $\varphi$ . We fix the following notation:

**Definition 12.** For a current  $\alpha \in \mathcal{C}(X_0)$ , the push-forward of  $\alpha$  under  $\varphi$  is  $\varphi_*\alpha$ , defined by  $\varphi_*(A) = \alpha(\varphi^{-1}(A))$ . For a current  $\beta \in \mathcal{C}(X_1)$ , the pull-back of  $\beta$  under  $\varphi$  is  $\varphi^*\beta$ , defined by  $\varphi^*(B) = \beta(\varphi(B))$ .

The following is a straightforward consequence of the  $\Gamma$ -equivariance of  $\varphi$ .

**Lemma 13.** *Let  $[\gamma]$  be any element of  $\Gamma = \pi_1(X)$ , and let  $\gamma_i \in \mathcal{C}(X_i)$  be the current supported on the closed  $g_i$ -geodesic representing  $\gamma$ . Then  $\varphi_*\gamma_0 = \gamma_1$  and  $\varphi^*\gamma_1 = \gamma_0$ .*

If we equip  $\mathcal{C}(X_i)$  with the weak-\* topology, then the following is also an easy consequence of the definitions:

**Lemma 14.** *If  $\alpha_n \rightarrow \alpha$  in  $\mathcal{C}(X_0)$ , then  $\varphi_*\alpha_n \rightarrow \varphi_*\alpha$  in  $\mathcal{C}(X_1)$ . If  $\beta_n \rightarrow \beta$  in  $\mathcal{C}(X_1)$ , then  $\varphi^*\beta_n \rightarrow \varphi^*\beta$  in  $\mathcal{C}(X_0)$ .*

The Corollary below will be used for a technical step in the proof of Theorem 3:

**Lemma 15.** *Let  $\partial^\infty f : \partial^\infty(\tilde{X}, \tilde{g}_0) \rightarrow \partial^\infty(\tilde{X}, \tilde{g}_1)$  denote the  $\Gamma$ -equivariant homeomorphism induced by identifying  $\partial^\infty \tilde{X}_i$  with  $\partial^\infty \Gamma$ , and  $\partial^\infty \Phi : \partial^\infty(\tilde{X}, \tilde{g}_0) \rightarrow \partial^\infty(X, \tilde{g}_1)$  the map induced by the lift of the identity map  $X \rightarrow X$ . Then  $\partial^\infty f = \partial^\infty \Phi$ .*

*Proof.* Fix a fundamental domain  $\mathcal{F}$  for the  $\Gamma$ -action on  $(\tilde{X}, \tilde{g}_0)$ . Any geodesic ray  $\eta$  in  $(\tilde{X}, \tilde{g}_0)$  can be encoded by a sequence  $(\gamma_i \cdot \mathcal{F}_i)$  through which it passes; the sequence  $(\gamma_i)$  approaches the point in  $\partial^\infty \Gamma$  corresponding to  $\eta(+\infty)$ . The sequence  $(\gamma_i \cdot \Phi(\mathcal{F}))$  approaches  $\partial^\infty \Phi(\eta(+\infty))$ . Likewise,  $\partial^\infty f(\eta(+\infty))$  is the point in  $\partial^\infty(\tilde{X}, \tilde{g}_1)$  corresponding to  $(\gamma_i)$ , namely the point which  $(\gamma_i \cdot \Phi(\mathcal{F}))$  approaches. Thus  $\partial^\infty f(\eta(+\infty)) = \partial^\infty \Phi(\eta(+\infty))$ .  $\square$

**Corollary 16.** *The map  $\varphi : \mathcal{G}(\tilde{X}_0)/\Gamma \rightarrow \mathcal{G}(\tilde{X}_1)/\Gamma$  sends walls to walls.*

*Proof.* As noted in the proof of Theorem 1,  $\partial^\infty \Phi$  maps endpoints of walls to endpoints of walls (see [X, Lemma 3.11]). By the previous Lemma, so does  $\partial^\infty f$ . Since  $\partial^\infty f$  defines  $\varphi$ ,  $\varphi$  sends the geodesic connecting the endpoints at infinity of some wall in  $(\tilde{X}, \tilde{g}_0)$  to the endpoints at infinity of the same wall in  $(\tilde{X}, \tilde{g}_1)$ . Our vertex regularity condition, together with Corollary 6 implies that these geodesics are the walls themselves.  $\square$

## 6. INTERSECTION PAIRING ON FUCHSIAN BUILDINGS

The key tool in the proof of Theorem 3, as in Otal's original work on MLS rigidity and Croke and Dairbekov's work on MLS and volume is the intersection pairing for geodesic currents. This is a finite, bilinear pairing on the space of currents, which recovers the intersection number for geodesics when the currents in question are Dirac measures on closed geodesics, and can also recover lengths of closed geodesics and the total volume of the space. For surfaces it is defined by

$$i(\mu, \lambda) = (\mu \times \lambda)(DG(X))$$

where  $DG(X)$  is the set of all transversally intersecting pairs of geodesics on  $X$ .

The main problem in extending this tool to the building case is the fact that  $DG(X)$  is not naturally defined for buildings. For a surface, transverse intersection of geodesics is detected by linking of their endpoints in the circle  $\partial^\infty \tilde{S}$ . This is no longer the case for buildings, and one can imagine a pair of geodesics which intersect in  $(\tilde{X}, g_0)$  but whose images under  $\varphi$  do not intersect in  $(\tilde{X}, g_1)$  due to the branching of the building.

Therefore, we must introduce an 'adjusted' version of both  $DG(X)$  and the intersection pairing and prove that it retains enough of the necessary properties of  $i(-, -)$  for our purposes. We begin by examining to what extent we can detect intersections of geodesics in a building. Before doing so we make the following definition:

**Definition 17.** We say that two geodesics  $\gamma_1$  and  $\gamma_2$  *intersect transversally* if there exists an apartment  $A$  such that the restrictions of  $\gamma_1$  and  $\gamma_2$  to  $A$  intersect transversally in their interiors. Note in particular that each  $\gamma_i \cap A$  is a non-degenerate segment. We say that they *intersect in a point* if  $\gamma_1 \cap \gamma_2$  is a single point.

Note that two geodesics intersecting in a point in the interior of a chamber automatically intersect transversally. If  $\gamma_1$  and  $\gamma_2$  intersect transversally, then there is a well-defined angle between them, up to the symmetry about  $\frac{\pi}{2}$ .

**A lemma on links.** Let  $I_p$  denote the interval graph with  $p$  segments. Let  $S_q^1$  denote the circle graph with  $q$  segments.

**Lemma 18.** *Let  $G$  be a thick, generalized  $m$ -gon. Then any embedding of  $I_p$  into  $G$  for  $p \leq 2m$  can be extended to an embedding of  $S_q^1$  into  $G$  for some  $q \geq p$ .*

*Proof.* If  $p \leq m + 1$ , then the embedding of  $I_p$  lies in an apartment in  $G$ ; the apartment itself is an extension of  $I_p$  to an embedding of  $S_{2m}^1$ .

Assume then that  $p > m + 1$ . Let  $v_0$  and  $v_p$  be the endpoints of the embedding of  $I_p$  in  $G$ . Let  $v_m$  be the vertex at distance  $m$  from  $v_0$  along this embedding. Let

$a$  denote the segment of the embedding from  $v_0$  to  $v_m$ . Let  $b$  denote the segment of the embedding from  $v_m$  to  $v_p$ ; it has length  $p - m$  with  $m \geq p - m > 1$ .

The length  $m$  segment in the embedding from  $v_0$  to  $v_m$  can be extended to an apartment  $A_1$  by connecting  $v_m$  and  $v_0$  by a length  $m$  segment  $c$ . Since the  $m$ -gon is thick, we may assume that the segment of  $c$  leaving  $v_n$  does not agree with the segment of  $b$  leaving  $v_m$ . Let  $c'$  be the initial segment of  $c$  of length  $m+1-(p-m)$  so that the concatenation of  $b$  with  $c'$  is a path of length  $m+1$ . Note that  $b \cap c = \{v_m\}$ , else a loop of length  $< 2m$  exists, which is not allowed. As any path of length  $m+1$  can be extended to an apartment, there must exist a segment  $d$  of length  $m-1$  connecting  $v_p$  to the end of  $c'$  and forming apartment  $A_2$ . (See Figure 1.)

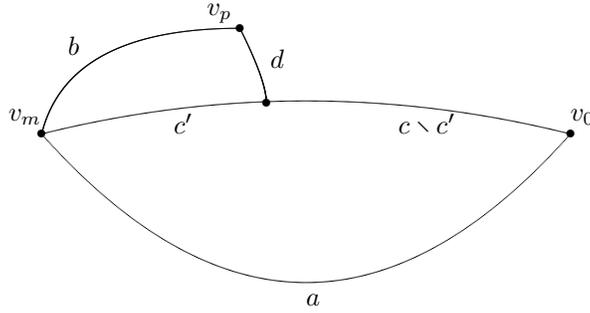


FIGURE 1. Segments in the proof of Lemma 18.

Note that if  $d \subset (c \setminus c') \cup a$  but  $d$  is not contained in  $c \setminus c'$  (which is ruled out as  $b \cap c = \emptyset$ ), then apartments  $A_1$  and  $A_2$  agree along a path of length  $m+1$ , which would imply they are the same apartment, and contradict our choice of the segment  $c$  as branching away from  $b$  at  $v_m$ .

Next we note that  $d \cap a = \emptyset$ , for if not, let  $d' \subset d$  be the segment of  $d$  between its final intersection with  $a$  and its subsequent intersection with  $c$ . The segments  $a$  and  $c$  form an embedded circle of length  $2m$ ;  $d'$  is a chord of length  $< m$  connecting two points on this circle. But then there must be a loop of length  $< 2m$  present in this configuration, which is a contradiction. Finally, since  $d \cap a = \emptyset$ , the concatenation of  $a, b, d$  and  $c \setminus c'$  (canceling any portion of  $d$  that lies along  $c \setminus c'$ ) provides the desired embedded circle extending  $I_p$ . □

**Intersections with walls and edges.** Since  $\varphi$  sends walls to walls via the combinatorial isomorphism (see Corollary 16), if  $w_1 \cap w_2 \neq \emptyset$ , then  $\varphi(w_1) \cap \varphi(w_2) \neq \emptyset$ . A more interesting result is the following:

**Proposition 19.** *Let  $\gamma$  be any geodesic and  $w$  a wall. Suppose that  $\gamma$  and  $w$  intersect transversally. Then  $\varphi(\gamma)$  and  $\varphi(w)$  intersect.*

Observe that, a geodesic intersecting a wall in the interior of an edge automatically intersects the wall transversely. Thus the proposition immediately implies

**Corollary 20.** *If the geodesic  $\gamma$  and wall  $w$  intersect in a non-vertex point, then  $\varphi(\gamma)$  and  $\varphi(w)$  must also intersect.*

**Definition 21.** A *totally geodesic strip* is any union of chambers of  $\tilde{X}$  which is convex, on which the induced metric is locally CAT(-1), and which is homeomorphic to  $[0, 1] \times \mathbb{R}$ . A *totally geodesic plane* is similarly defined, but is homeomorphic to  $\mathbb{R}^2$ .

The CAT(-1) condition means that, around any vertex in the interior of a totally geodesic strip  $S$ , the total angle is  $\geq 2\pi$ . If a vertex  $v$  lies on  $\partial S$ , i.e. the image of  $\{0, 1\} \times \mathbb{R}$  under the homeomorphism, then the image of  $S$  in  $lk(v)$  consists of an embedded sequence of some number of consecutive edges (i.e. an embedded combinatorial path in  $lk(v)$ ). Let  $l(v)$  denote this number of edges. If  $lk(v)$  is a generalized  $m_v$ -gon, then convexity at  $v$  is equivalent to  $l(v) \leq m_v$ . Convexity of the strip then reduces to this condition at each boundary vertex.

The proof of Proposition 19 relies on the following technical lemma. In it, we show that a convex totally geodesic strip can be thickened to another convex totally geodesic strip, subject to a certain constraint on how that thickening occurs near a particular edge of  $\partial S$ .

**Lemma 22.** *Let  $S$  be a convex totally geodesic strip. Fix an edge  $e$  in one component  $B$  of  $\partial S$  with endpoints  $v_0$  and  $v_1$ . Let  $\hat{S}$  be the union of  $S$  with a sequence of  $t \leq m_{v_1}$  chambers  $C_1, \dots, C_t$ , with the property that (i)  $C_1 \cap B = e$ , (ii) for  $i \geq 2$ ,  $C_i \cap B = \{v_1\}$ , and (iii) each  $C_i \cap C_{i+1}$  is an edge. (See Figure 2.) Then there exists a convex totally geodesic strip  $S'$  containing  $\hat{S}$  such that  $B \cap \partial S' = \emptyset$ .*

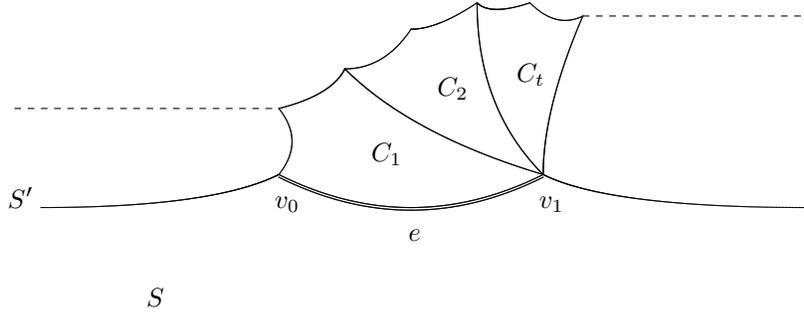


FIGURE 2.  $\hat{S}$ , extended to  $S'$  with the new boundary of  $S'$  in dashed lines, as in Lemma 22.

*Proof.* We will prove this Lemma under two cases, first the case where the chambers have at least four sides, and then the case that they are triangles. The latter case requires some slight additions to the argument.

**Case 1: Chambers have at least four sides.**

Enumerate the vertices in  $B$  in order as  $\{v_i\}_{i \in \mathbb{Z}}$ , agreeing with the specification of  $v_0$  and  $v_1$ . Convexity of the strip guarantees that this is possible.

Consider the image of  $\hat{S}$  in  $lk(v_1)$ . It is an interval of  $\leq 2m_{v_1}$  edges, since at most  $m_{v_1}$  edges are contributed by  $S$  and an additional  $t \leq m_{v_1}$  edges come from

the chambers added to  $S$  to form  $\hat{S}$ . By Lemma 18, this interval can be extended to an embedded  $S^1$  in  $lk(v)$ . Glue chambers to  $\hat{S}$  adjacent to  $v_1$  according to this  $S^1$ ; call the result  $\hat{S}_1$ .

Now consider  $v_2$ . Of the chambers added to  $\hat{S}$  in forming  $\hat{S}_1$ , exactly one is adjacent to  $v_2$ , specifically the added chamber corresponding to the edge in  $S^1 \subset lk(v_1)$  which completes the  $S^1$ . None of the other added chambers can be adjacent to  $v_2$  because chambers are convex, and a second chamber attached to  $v_1$  and  $v_2$  not along the edge between them would provide a second geodesic path between the two vertices, a contradiction. Thus, the image of  $\hat{S}_1$  in  $lk(v_2)$  is an interval of  $\leq m_{v_2} + 1 < 2m_{v_2}$  edges. As in the previous step, apply Lemma 18 and glue new chambers accordingly to obtain  $\hat{S}_2$ . Via the same argument as above, the image of  $\hat{S}_2$  in  $lk(v_3)$  is an interval of  $\leq m_{v_3} + 1 < 2m_{v_3}$  edges.

Continue in this fashion, gluing new chambers to vertices  $v_i$  with  $i \rightarrow \infty$ . Then run the same process on  $v_i$  with  $i \rightarrow -\infty$ . Call the resulting union of chambers  $S'$ . By construction,  $B \cap \partial S' = \emptyset$ . It also is clear from the construction that  $S'$  is locally CAT(-1), since its image in the link of any interior vertex  $v$  is a circle with  $\geq 2m_v$  edges. That it is homeomorphic to  $[0, 1] \times \mathbb{R}$  will follow once we check the convexity condition on the vertices of  $\partial S'$ , since convexity implies the vertices along the boundary components can be ordered linearly as noted above, and hence that there are only two boundary components.

Suppose that  $C_1$  and  $C_2$  are chambers attached to  $v_i$  and  $v_j$  with  $|i - j| \geq 2$ . We claim that  $C_1 \cap C_2 = \emptyset$ . Indeed, from  $|i - j| \geq 2$  it follows that there exists a chamber  $C_3$  incident to an edge  $e$  along  $\partial S$ , and lying between  $C_1$  and  $C_2$ . Consider the two edges  $e_1, e_2$  of  $C_3$  which are adjacent to  $e$ , and form the sets  $W_i$  consisting of all walls passing through  $e_i$ . Since the chamber  $C_3$  has at least four sides, we have  $e_1 \cap e_2 = \emptyset$ . The chamber  $C_3$  is metrically a convex hyperbolic polygon, so it contains a geodesic segment  $\eta$  which is orthogonal to both  $e_1$  and  $e_2$ . This implies  $W_1 \cap W_2 = \emptyset$ . Finally, we note that each  $W_i$  separates  $\tilde{X}$  into two connected components. By construction, we have that  $C_1$  and  $C_2$  lie in different connected components of (each)  $\tilde{X} \setminus W_i$ , and so indeed,  $C_1 \cap C_2 = \emptyset$ .

Now that we know  $C_1$  and  $C_2$  do not intersect, we see that a vertex  $v \in \partial S'$  can belong to at most two chambers attached to  $v_i$ , at most two attached to  $v_{i+1}$  and no chambers attached to other vertices in  $\partial S$ . The only case in which the bound of two is achieved for both vertices is when the chamber is a triangle, which will be dealt with in Case 2. In fact, for chambers with at least four edges, if two chambers adjacent to  $v_i$  meet at  $v$ , then no chamber adjacent to  $v_{i+1}$ , besides, perhaps, one of these two chambers, contains  $v$ . Thus, for all vertices in  $\partial S'$ ,  $l(v) \leq 2$ , and convexity holds. This finishes the proof of Case 1.

### Case 2: The chambers are triangles.

We repeat the argument above, but need to adjust slightly to ensure convexity of  $S'$ . The arguments in Case 1 apply except for the argument proving  $C_1 \cap C_2 = \emptyset$ . A very similar argument works except for the case where  $|i - j| = 2$ . Then there is a triangle  $T$  bounded by two edges along  $w$  and one edge each from  $C_1$  and  $C_2$ . The pair of edges between  $v_i$  and  $v_j$  along  $w$  must form a geodesic segment, otherwise an argument similar to Case 1 works again, extending each of these edges into a collection of walls  $W_i$ . Furthermore, if the combinatorial angle at the vertex  $v_k$  between these edges is not  $\pi/2$ , then there are two distinct edges from  $v_k$  which

can be extended to collections  $W_i$ , allowing Case 1's argument again. Note that by Feit-Higman ([FH]) the combinatorial angles of the chamber are  $(\pi/2, \pi/3, \pi/8)$  or  $(\pi/2, \pi/k, \pi/l)$  with  $k, l \in \{4, 6, 8\}$  but not both = 4.

Combining this with the arguments of the last paragraph of Case 1, we see that in the triangle case, we may have vertices in  $\partial S'$  with  $l(v) = 3$  or 4, the latter only in the case that  $|i - j| = 2$  and the combinatorial angle at  $v_k$  is  $\pi/2$ . The only problems for convexity at such a vertex arise when the combinatorial angle there is  $\pi/2$  or  $\pi/3$ . Figure 3 shows the only two non-convex possibilities.

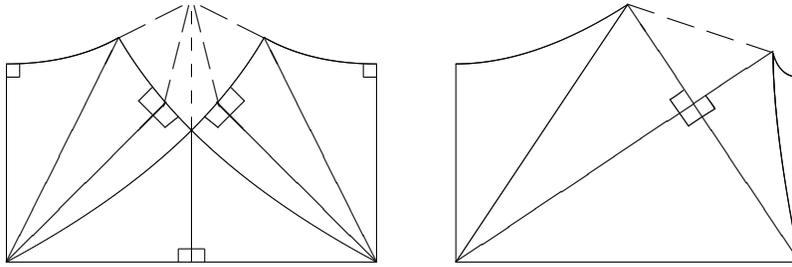


FIGURE 3. Filling in non-convex regions in Lemma 22.

In each of these possible configurations we see in the dashed additions to the figure, that a few extra chambers can be added (using Lemma 18, or the fact that any  $m_v + 1$  adjacent chambers around a vertex can be completed into a circuit of length  $2m_v$ ) to remove the non-convexity at the problematic vertex. This proves Case 2.

□

*Proof of Proposition 19.* We may assume that  $\gamma$  is not a wall, else the statement follows from the fact that  $\varphi$  agrees with the combinatorial isomorphism along walls.

Let  $\mathcal{C}$  be the union of a minimal set of chambers which covers  $\gamma$ , which includes the chamber(s) witnessing the transversal intersection of  $\gamma$  and  $w$ , and such that any chamber in the collection has at least two edges which belong to another chamber in the collection.

Let  $\mathbf{C}$  be the 1-skeleton of  $\mathcal{C}$ . The boundary of  $\mathcal{C}$  consists of two embedded copies of  $\mathbb{R}$  with marked points at the vertices. At each marked vertex the number  $l(v)$  indicates the combinatorial length of the image of  $\mathcal{C}$  in  $lk(v)$ . We have  $l(v) \leq m_v + 1$  for all  $v$  since the geodesic  $\gamma$  lies in some apartment containing  $v$  and can only intersect each wall through  $v$  once (see Figure 4).

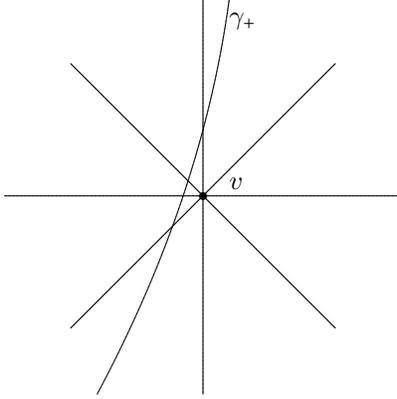


FIGURE 4. A geodesic passing near a vertex.

We now want to use Lemma 22 to extend  $\mathcal{C}$  to a totally geodesic containing  $\gamma$  and  $w$ .

First, consider  $w$  as a convex totally geodesic strip of zero width. Let  $v_0$  and  $v_1$  be the vertices in  $\mathcal{C} \cap w$  nearest  $\gamma \cap w$ . There are at most  $l(v_1)$  chambers in  $\mathcal{C}$  adjacent to  $v_1$ , but since  $\gamma$  crosses  $w$ , we may restrict our attention to chambers containing a ray  $\gamma_+$ , of which at most  $l(v_1) - 1 \leq m_{v_1}$  are adjacent to  $v_1$  and one is adjacent to  $v_0$  (up to interchanging the two vertices). We can then apply Lemma 22 to extend  $w$  on this side to a wider convex totally geodesic strip.

(This argument needs to be slightly adjusted if  $\gamma$  and  $w$  intersect at a vertex, so that the chambers witnessing their transversal intersection – which are in  $\mathcal{C}$  – form a full circuit around the vertex. In this case, declare this vertex to be  $v_1$  and consider those chambers to be the first step in building the strip.)

In doing so, we make sure to fill using chambers in  $\mathcal{C}$  whenever possible, so that the new strip covers as much of  $\gamma_+$  as possible and  $\gamma_+$  exits the new strip through its boundary. This will always be possible, because those chambers in  $\mathcal{C}$  adjacent to the strip built so far, together with the strip itself, always have image in the link of any boundary vertex with length  $\leq 2m_v$ , allowing the application of Lemma 18.

Continue to apply Lemma 22 inductively. At each stage  $\hat{S}$  consists of the strip produced so far, as well as any chambers in  $\mathcal{C}$  adjacent to that strip on the  $\gamma_+$  side. There are always at most  $l(v) - 1 \leq m_v$  chambers from  $\mathcal{C}$  added to  $S$ , since at least one chamber adjacent to  $v$  has already been included in  $S$ . The induction produces a totally geodesic half-space containing  $\gamma_+$  and  $w$ . Note that it need not be a half apartment – there may be singular vertices with total angle  $> 2\pi$ .

Run the same argument on the other side of  $w$  to cover the other ray  $\gamma_-$  and  $w$ . The fact that  $\gamma$  and  $w$  intersect transversally guarantees that the two half-spaces so produced together form a totally geodesic plane, call it  $S^*$ . Suppose it is not embedded. Then there are two distinct geodesics from  $\gamma \cap w$  to a single point  $x$  – a point which is encountered twice in the process of constructing  $S^*$ . But this contradicts the fact that  $\tilde{X}$  is CAT(-1).

Take any geodesic segment in  $S^*$  emanating from  $\gamma \cap w$ . It is clear from the construction of  $S^*$  that such a segment can be continued indefinitely, since it certainly



inductively build a totally geodesic plane  $S^*$  inside  $\tilde{X}$  which contains both  $\gamma$  and  $w'$ . From the linking of the endpoints of  $\gamma, w'$  in the boundary at infinity  $\partial^\infty S^* \cong S^1$  of the plane, it again follows that the corresponding  $\varphi(\gamma)$  and  $\varphi(w')$  must intersect. This completes the proof.  $\square$

This proposition has the following consequence.

**Proposition 24.** *Let  $\gamma$  be a geodesic in  $(\tilde{X}, g_0)$  and  $e$  an edge of a wall  $w$  such that  $\gamma$  and  $w$  intersect transversally and  $\gamma \cap e \neq \emptyset$ . Let  $\varphi(e)$  be the edge in  $(\tilde{X}, g_1)$  identified by the combinatorial isomorphism. Then  $\varphi(\gamma) \cap \varphi(e) \neq \emptyset$ .*

*Proof.* Find, using Propositions 19 or 23, a set of walls  $\{w_i\}$  intersecting  $\gamma$  with  $\cap_i w_i = e$ . Then, using those propositions,  $\varphi(\gamma)$  and  $\varphi(w_i)$  intersect, for all  $i$ . But  $\cap_i \varphi(w_i) = \varphi(e)$ , and from the fact that geodesics in negatively curved spaces intersect at most once it is fairly easy to see that  $\varphi(\gamma)$  intersects  $\varphi(e)$ , as desired.  $\square$

**Corollary 25.** *If a geodesic  $\gamma$  passes through a vertex  $v$ , then  $\varphi(\gamma)$  passes through  $\varphi(v)$ .*

*Proof.* Pick any apartment  $A$  containing  $\gamma$  and let  $\{e_1, \dots, e_{2n}\}$  be the collection of edges in  $A$  containing the vertex  $v$ .  $\gamma$  passes through each of these so by Proposition 24,  $\varphi(\gamma)$  hits all  $\varphi(e_i)$ . But  $\varphi(v)$  is the intersection of all of these, so  $\varphi(\gamma)$  intersects  $\varphi(v)$ .  $\square$

This implies a stronger version of Proposition 19:

**Proposition 26.** *If the geodesic  $\gamma$  and the wall  $w$  intersect, then so do  $\varphi(\gamma)$  and  $\varphi(w)$ .*

*Proof.* The only case not covered by Proposition 19 is when the intersection of  $\gamma$  and  $w$  is a vertex. This is handled by Corollary 25.  $\square$

**Good position.** Having analyzed the possible intersections of a geodesic with a wall, we now move to the more general situation: intersections of two arbitrary geodesics. We first identify a situation where it is easy to ensure that the image geodesics still intersect.

**Definition 27.** Let  $\gamma_1$  and  $\gamma_2$  be two  $g_0$ -geodesics in  $\tilde{X}$ . We say that the pair  $(\gamma_1, \gamma_2)$  is in *good position* if there is some chamber  $C$  in  $\tilde{X}$  through which both  $\gamma_i$  pass, and four distinct edges of  $C$  such that  $\gamma_1$  intersects the boundary of  $C$  in edges  $e_1$  and  $e_3$ ,  $\gamma_2$  intersects the boundary of  $C$  in edges  $e_2$  and  $e_4$ , and the cyclic order of the edges around the boundary of  $C$  is  $e_1, e_2, e_3, e_4$ . (See Figure 6).

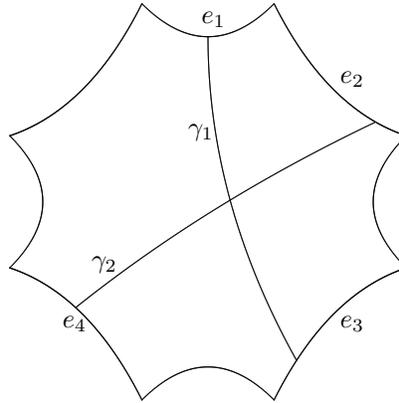


FIGURE 6. Geodesics in good position.

It is clear that two geodesics in good position must intersect in a point. That yields the following:

**Proposition 28.** *Let  $\gamma_1$  and  $\gamma_2$  be  $g_0$ -geodesics in good position. Then  $\varphi(\gamma_1)$  and  $\varphi(\gamma_2)$  intersect in a point. If  $\gamma_1$  and  $\gamma_2$  intersect in the interior of  $C$  (and hence intersect transversally), then  $\varphi(\gamma_1)$  and  $\varphi(\gamma_2)$  intersect transversally.*

*Proof.* Using Proposition 24,  $\varphi(\gamma_1)$  and  $\varphi(\gamma_2)$  must be in good position, and hence must intersect in a point. The only cases where this intersection could fail to be transversal are if the intersection occurs at a vertex of  $C$ . But using Corollary 25 we see that this case does not occur for  $\varphi(\gamma_1)$  and  $\varphi(\gamma_2)$  if  $\gamma_1 \cap \gamma_2$  is not a vertex.  $\square$

**Intersections of general geodesics.** Proposition 28 takes care of a large class, but not all pairs of intersecting  $g_0$ -geodesics. The behavior of transversally intersecting geodesics which are not in good position is the focus of this section.

We remarked above in the proof of Proposition 28 that the intersections detectable by good position are necessarily transversal if we avoid vertices. Throughout this section and below, we will restrict our attention to pairs of geodesics which intersect in non-vertex points. We note that the geodesics in such pairs are drawn from a  $\varphi$ -invariant (Corollary 25), full measure set for any Liouville current.

**Definition 29.** Let  $\gamma_1$  and  $\gamma_2$  be a pair of  $g_0$  geodesics intersecting transversally in an interior point of a chamber  $C$ . A *window* for the pair  $(\gamma_1, \gamma_2)$  is any union of chambers  $C_i$  such that each  $C_i$  intersects  $\gamma_1$  and  $\gamma_2$  in the interior of  $C_i$ . (See Figure 7.)

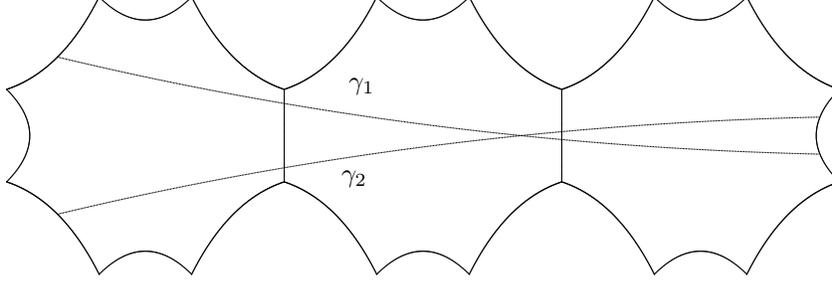


FIGURE 7. A window for  $\gamma_1$  and  $\gamma_2$ . Note that it is not a good window (Defn 30) because the geodesics exit on the right through the same edge.

Note that if  $\gamma_1$  and  $\gamma_2$  are in good position,  $C$  itself is the only window for  $(\gamma_1, \gamma_2)$ . In fact, for any window, both  $\gamma_1$  and  $\gamma_2$  must cross all the edges shared by more than one chamber in the window. Note that a window is a convex, two-dimensional subset of the building.

**Definition 30.** Say  $\gamma_1 \cap \gamma_2$  is a non-vertex point. A window for  $(\gamma_1, \gamma_2)$  is a *good window* for the pair  $(\gamma_1, \gamma_2)$  if  $\gamma_1$  intersects its boundary in the edges  $e_1, e_3$ ,  $\gamma_2$  intersects its boundary in the edges  $e_2, e_4$ , and the cyclic ordering of these (distinct) edges around the boundary of the window is  $e_1, e_2, e_3, e_4$ .

We will denote by  $\varphi(W)$  the image in  $(\tilde{X}, g_1)$  of a window in  $(\tilde{X}, g_0)$  under the combinatorial isomorphism between the buildings.

**Lemma 31.** *If  $W$  is a good window for  $(\gamma_1, \gamma_2)$ , then  $\varphi(\gamma_1)$  and  $\varphi(\gamma_2)$  intersect transversally in the good window  $\varphi(W)$ .*

*Proof.* Same as Proposition 28.  $\square$

For a geodesic  $\gamma$  passing through a chamber  $C$ , let  $\gamma|_C$  denote the geodesic segment  $\gamma \cap C$ .

**Definition 32.** A geodesic segment  $\gamma'$  is an *extension* of  $\gamma|_C$  if  $\gamma' \cap C = \gamma|_C$ .

**Definition 33.** Suppose  $\gamma_1$  and  $\gamma_2$  are  $g_0$ -geodesics intersecting in a non-vertex point of a chamber  $C$ . Let  $\mathcal{GW}(\gamma_1, \gamma_2)$  be the set of all  $W$  where  $W$  is a good window for  $(\gamma'_1, \gamma'_2)$  and where  $\gamma'_i$  is an extension of  $\gamma_i|_C$ .

**Lemma 34.** *The pair  $(\gamma_1, \gamma_2)$  is already in good position if and only if  $\mathcal{GW}(\gamma_1, \gamma_2) = \{C\}$ .*

*Proof.* Clear.  $\square$

**Lemma 35.** *If  $\gamma_1$  and  $\gamma_2$  intersect in  $C$ , then every apartment through  $C$  contains a good window for extensions of  $(\gamma_1, \gamma_2)$ , and  $\mathcal{GW}(\gamma_1, \gamma_2)$  is finite.*

*Proof.* Let  $A$  be an apartment through  $C$  and  $\gamma'_i$  the extensions of  $\gamma_i|_C$  to  $A$ . Since they intersect transversely and the curvature is negative, they cannot stay at a bounded distance from one another, and thus the sequence of edges  $\gamma'_1$  and  $\gamma'_2$  pass

through must differ eventually, in both directions. Thus they lie in a good window  $W \subset A$ .

Further, the distance between  $\gamma'_1$  and  $\gamma'_2$  parametrized by arc length must diverge at least linearly, and so we can bound the diameter of the good window containing them. Since there are only finitely many restrictions of apartments to a given metric ball containing  $C$ , there are only finitely many good windows for the pair  $(\gamma_1, \gamma_2)$ .  $\square$

**Definition 36.** Let  $DG_0(\tilde{X})$  be the set of  $g_0$ -geodesic pairs  $(\gamma_1, \gamma_2)$  intersecting transversally in  $\tilde{X}$  in a non-vertex point. Let

$$\mathcal{GW}(\tilde{X}) = \bigcup_{DG_0(\tilde{X})} \mathcal{GW}(\gamma_1, \gamma_2).$$

**Lemma 37.**  $\mathcal{GW}(\tilde{X})$  is countable.

*Proof.* We prove the Lemma for  $\mathcal{GW}(C)$ , the set of good windows for pairs intersecting transversally in  $C$ , then use the fact that there are countably many chambers in  $\tilde{X}$ .

There is a countable partition of  $(0, \pi/2]$  into intervals  $I_n$  (independent of choice of chamber or intersection point) such that if  $\gamma_1$  and  $\gamma_2$  intersect at angle  $\theta \in I_n$  then any good window for  $(\gamma_1, \gamma_2)$  contains at most  $n$  chambers. As there are finitely many windows containing  $C$  and containing  $n$  chambers,  $\mathcal{GW}(C)$  is a countable union of finite sets, proving the statement.  $\square$

**Definition 38.** Enumerate the windows in  $\mathcal{GW}(\tilde{X})$  as  $\{W_i\}$ . Then let

$$\widehat{DG}_0(\tilde{X}) = \bigcup_i \widehat{DG}_0(W_i)$$

where  $\widehat{DG}_0(W_i) = \{(\gamma_1, \gamma_2) : W_i \text{ is a good window for } (\gamma_1, \gamma_2) \text{ and } \gamma_1 \cap \gamma_2 \neq \text{a vertex}\}$ .

The following Lemma is easy to check.

**Lemma 39.** For any pair  $(\gamma_1, \gamma_2)$  of  $g_0$ -geodesics intersecting transversally in  $\tilde{X}$ , there is at least one  $(\gamma'_1, \gamma'_2) \in \widehat{DG}_0(\tilde{X})$  such that  $\gamma'_i$  is an extension of  $\gamma_i|_C$ .

**$\widehat{DG}(X)$  and the intersection pairing.** We now show that  $\widehat{DG}_0(\tilde{X})$  descends to a well-defined,  $\varphi$ -invariant collection of intersecting geodesic pairs on  $X$ , and define our adjusted intersection pairing.

**Lemma 40.** Let  $\varphi$  send  $g_0$ -geodesics to  $g_1$ -geodesics as defined previously. Then

$$\varphi(\widehat{DG}_0(\tilde{X})) = \widehat{DG}_1(\tilde{X})$$

where  $\varphi((\gamma_1, \gamma_2)) = (\varphi(\gamma_1), \varphi(\gamma_2))$ .

*Proof.* If  $(\gamma_1, \gamma_2) \in \widehat{DG}_0(\tilde{X})$  with  $W$  a good window for the pair, then by Lemma 31,  $\varphi(W)$  is a good window for  $(\varphi(\gamma_1), \varphi(\gamma_2))$ . Therefore  $\varphi(\gamma_1)$  and  $\varphi(\gamma_2)$  intersect in some chamber belonging to  $\varphi(W)$  (and not in a vertex, by Corollary 25). Then  $(\varphi(\gamma_1), \varphi(\gamma_2)) \in \widehat{DG}_1(\tilde{X})$ . Thus  $\varphi(\widehat{DG}_0(\tilde{X})) \subseteq \widehat{DG}_1(\tilde{X})$ . Reversing the roles of  $g_0$  and  $g_1$  gives the other inclusion and completes the proof.  $\square$

Given this lemma, we may without ambiguity write  $\widehat{DG}(\tilde{X})$  and suppress the dependence on the metric, unless we specifically want to view a pair of geodesics in terms of one metric or the other.

**Lemma 41.** *Under the obvious action,  $\widehat{DG}(\tilde{X})$  is  $\Gamma$ -invariant and  $\varphi: \widehat{DG}_0(\tilde{X}) \rightarrow \widehat{DG}_1(\tilde{X})$  is  $\Gamma$ -equivariant.*

*Proof.* This is clear, as  $\Gamma$  acts by isometries, which also preserve the combinatorial structure of the building.  $\square$

The first part of Lemma 41 justifies this definition:

**Definition 42.** Let  $\widehat{DG}(X) := \widehat{DG}(\tilde{X})/\Gamma$ .

Lemma 41 then has the following immediate consequence:

**Lemma 43.** *The map  $\varphi$  descends to a map  $\varphi: \widehat{DG}_0(X) \rightarrow \widehat{DG}_1(X)$ .*

**Lemma 44.** *Let  $\{U_i\}$  denote the countable collection of  $\Gamma$ -orbits in the collection  $\{W_i\}$ , and for each  $U_i = \Gamma \cdot W_i$ , set  $\widehat{DG}(U_i) := \Gamma \cdot \widehat{DG}(W_i)$ . Then we have a disjoint union*

$$\widehat{DG}(X) = \bigcup_i \widehat{DG}(U_i).$$

*Proof.* This is again easy to check as  $\Gamma$  acts by isometries and preserves the combinatorial structure of the building.  $\square$

We now define the ‘adjusted’ intersection number. Let  $e$  be an edge in  $X$ . Recall that  $q(e)$  is the integer such that there are  $q(e)$  chambers containing the edge  $e$ . As the building is thick,  $q(e) \geq 3$ .

**Definition 45.** Let  $W$  be a window and let  $\{e_1, \dots, e_n\}$  be the set of edges shared by consecutive pairs of chambers in  $W$ . The *weight* of  $W$  is defined as

$$\varpi(W) = \prod_{i=1}^n (q(e_i) - 1).$$

**Definition 46.** Let  $\mu$  and  $\lambda$  be geodesic currents on  $X$ . We define their *adjusted intersection number* as

$$\hat{i}(\mu, \lambda) = \sum_i \varpi(U_i) (\mu \times \lambda)(\widehat{DG}(U_i)).$$

**Intersection number calculations.** To justify this adjusted notion of intersection number, we want to calculate the following three standard types of pairings:

- $\hat{i}(\alpha, \beta)$  where  $\alpha$  and  $\beta$  are the geodesic currents defined by Dirac unit masses supported on the closed geodesics  $\alpha$  and  $\beta$ .
- $\hat{i}(L_g, \beta)$  where  $L_g$  is the Liouville current.
- $\hat{i}(L_g, L_g)$ .

**Proposition 47.** *When  $\alpha, \beta$  are a pair of geodesic currents supported on closed geodesics, we have*

$$\hat{i}(\alpha, \beta) = \sum_j \mathfrak{p}_j$$

where  $j$  enumerates the intersections of the geodesics  $\alpha$  and  $\beta$  in  $X$  and  $\mathfrak{p}_j = \varpi(U_j)$  where  $U_j$  is the good window witnessing the  $j^{\text{th}}$  intersection of these geodesics if such a window exists, and  $\mathfrak{p}_j = 0$  if no good window witnessing the  $j^{\text{th}}$  intersection exists (or if the intersection occurs at a vertex).

**Corollary 48.** *If all intersections between  $\alpha$  and  $\beta$  are in good position and not at vertices, then  $\hat{i}(\alpha, \beta)$  is the number of intersections.*

*Proof of Cor 48.* In this case, all  $\mathfrak{p}_j = 1$ . □

**Remark.** In the case of Corollary 48, we recover the intersection number of the two closed geodesics, exactly as in the surface case.

*Proof of Prop 47.* By our definition of the intersection pairing, all contributions to  $\hat{i}(\alpha, \beta)$  come from measuring  $\widehat{DG}(U_i)$ . Since the currents  $\alpha$  and  $\beta$  are supported on the single geodesics  $\alpha$  and  $\beta$ , the contribution from  $\widehat{DG}(U_i)$  is non-zero if and only if  $U_i$  is a good window for  $(\alpha, \beta)$ . Thus, there is a contribution of  $\varpi(U_i)$  to  $\hat{i}(\alpha, \beta)$  precisely for each intersection witnessed by the good window  $U_i$ , and those intersections which are not witnessed by any good window contribute nothing. □

**Proposition 49.** *For any closed geodesic which is not tangent to a wall,*

$$\hat{i}(L_g, \beta) = 2 \cdot \text{Length}_g(\beta).$$

*Proof.* As in Proposition 47, the only contributions to  $\hat{i}(L_g, \beta)$  will come from windows  $U_i$  which are good for  $(\gamma, \beta)$  for some geodesic  $\gamma$ . For such windows,  $(L_g \times \beta)(\widehat{DG}(U_i)) = L_g(A_i)$  where  $A_i$  consists of the geodesics  $\gamma$  which pair with  $\beta$  so that  $U_i$  is a good window for  $(\gamma, \beta)$ . From the definition of  $L_g$  we have that

$$L_g(A_i) = \frac{1}{\varpi(U_i)} \int_{A_i} \cos \theta d\theta dp$$

where the integral is over the intersections of the geodesics in  $A_i$  with  $\beta$ , and  $\theta$  and  $p$  are the coordinates for these intersections with respect to the metric  $g$ . Therefore,

$$\varpi(U_i)(L_g \times \beta)(\widehat{DG}(U_i)) = \int_{A_i} \cos \theta d\theta dp.$$

Since every vector with basepoint on  $\beta$  and making a nonzero angle with  $\beta$  generates exactly one geodesic  $\gamma$  such that  $(\gamma, \beta)$  lie in a good window  $U_i$ , summing the expression above over all  $i$  gives

$$\hat{i}(L_g, \beta) = \int_{t \in \beta, \theta \in (-\frac{\pi}{2}, \frac{\pi}{2})} \cos \theta d\theta dt = 2 \cdot \text{Length}_g(\beta).$$

□

**Remark.** This result also reproduces the analogous result for surfaces. For geodesics along walls, one obtains  $\hat{i}_g(L_g, \beta) = K \cdot \text{Length}_g(\beta)$ , where  $K$  is the number of chambers adjacent to any edge in the wall.

**Proposition 50.**

$$\hat{i}(L_g, L_g) = \frac{1}{4\pi} \text{Vol}_g(X).$$

*Proof.* We compute chamber by chamber, noting that

$$\text{Vol}_g(X) = \sum_C \text{Vol}_g(C).$$

If we denote by  $S^1C$  the unit tangent vectors to  $C$  based at edge points and pointing into  $C$ , then

$$Vol_g(C) = \frac{1}{2\pi} \int_{v \in S^1 C} l_g(v) d\mu$$

where  $d\mu = \cos \theta d\theta dp$  is a measure on  $S^1 C$  and  $l_g(v)$  is the length of the geodesic segment in  $C$  generated by  $v$ . Call this segment  $\gamma_v$ .

Now we also have

$$l_g(v) = \frac{1}{2} \int_{A_v} 1 d\mu$$

where  $d\mu = \cos \theta d\theta dp$  is the same measure but for tangent vectors along  $\gamma_v$ , and  $A_v$  is the set of geodesics intersecting  $\gamma_v$  transversally in  $C$ .

Consider the pairs  $(\gamma_v, \eta)$  with  $\eta$  a geodesic intersecting  $\gamma_v$  transversally in  $C$ . Not all of these are in good position, so we replace them with

$$\mathcal{G}_i(v) = \{(\gamma'_v, \eta) : U_i \text{ is a good window for } (\gamma'_v, \eta) \text{ where } \gamma'_v \text{ is an extension of } \gamma_v\}.$$

Then, just as in the proof of Proposition 49,

$$l_g(v) = \frac{1}{2} \sum_i \int_{\mathcal{G}_i(v)} \varpi(U_i) \frac{d\delta_{\gamma'_v}}{\varpi(U_i)} dL_g(\eta)$$

where  $\delta_{\gamma'_v}$  is the Dirac mass on  $\gamma'_v$ . We must divide by the factor  $\varpi(U_i)$  to correct for the fact that there are multiple extensions  $\gamma'_v$  of  $\gamma_v$ . Weighting by  $\frac{1}{\varpi(U_i)}$  precisely cancels out this error.

We now have

$$Vol_g(C) = \frac{1}{4\pi} \int_{v \in S^1 C} \sum_i \int_{\mathcal{G}_i(v)} \varpi(U_i) \frac{d\delta_{\gamma'_v}}{\varpi(U_i)} dL_g(\eta) d\mu.$$

From the description of  $L_g$ ,  $d\delta_{\gamma'_v} d\mu = \varpi(U_i) dL_g$ , so

$$Vol_g(C) = \frac{1}{4\pi} \sum_i \int_{\mathcal{G}_i(C)} \varpi(U_i) dL_g(\eta) dL_g(\gamma')$$

where the sets  $\mathcal{G}_i(C)$  are defined by

$$\mathcal{G}_i(C) = \{(\gamma', \eta) : U_i \text{ is a good window for } (\gamma', \eta) \text{ where } \gamma' \text{ intersects } \eta \text{ transversally in } C\}.$$

Summing this over all  $C$ , we have

$$Vol_g(X) = \frac{1}{4\pi} \sum_i \varpi(U_i) (L_g \times L_g)(\widehat{DG}(U_i))$$

which is just

$$Vol_g(X) = \frac{1}{4\pi} \hat{i}(L_g, L_g),$$

completing the proof.  $\square$

**Remark.** Again, this reproduces the analogous result for surfaces.

Another key property of the intersection pairing is finiteness, some cases of which are proven above. We will also need

**Proposition 51.**

$$\hat{i}(L_{g_0}, L_{g_1}) < \infty.$$

*Proof.* Let  $G(U_i)$  be the set of all geodesics which appear as one of the two geodesics in any element of  $\widehat{DG}(U_i)$  (recall such elements are pairs of geodesics). Fix, for each good window  $U_i$ , a chamber  $C$  contained in  $U_i$  and let  $A_i$  be the set of geodesics whose restriction to that chamber agrees with the restriction of a geodesic in  $G(U_i)$ . As noted above,  $L_g(A_i) = \varpi(U_i)L_g(G(U_i))$ . Denote by  $|U_i|$  the number of chambers in  $U_i$ .

We thus have

$$\begin{aligned} \hat{i}(L_{g_0}, L_{g_1}) &= \sum_i \varpi(U_i)(L_{g_0} \times L_{g_1})(\widehat{DG}(U_i)) \\ &\leq \sum_i \varpi(U_i)L_{g_0}(G(U_i))L_{g_1}(G(U_i)) \\ &= \sum_i \frac{1}{\varpi(U_i)}L_{g_0}(A_i)L_{g_1}(A_i) \\ &= \sum_k \sum_{|U_i|=k} \frac{1}{\varpi(U_i)}L_{g_0}(A_i)L_{g_1}(A_i) \end{aligned}$$

Since geodesics diverge exponentially fast, it is easy to see that there is an exponentially decaying function  $f(k)$  such that  $L_{g_i}(A_i) \leq f(k)$  where  $|U_i| = k$  for all  $i$ . We then continue:

$$\begin{aligned} &\leq \sum_k f(k)^2 \sum_{|U_i|=k} \frac{1}{\varpi(U_i)} \\ &\leq \sum_k f(k)^2 \sum_{C \subset X/\Gamma} \sum_{|U_i|=k, U_i \supset C} \frac{1}{\varpi(U_i)}. \end{aligned}$$

It is easy to check that

$$\sum_{|U_i|=k, U_i \supset C} \frac{1}{\varpi(U_i)} \leq 1$$

so we have in total that

$$\hat{i}(L_{g_0}, L_{g_1}) \leq \#C \sum_k f(k)^2.$$

Since  $f(k)$  decays to zero exponentially fast, the sum converges, finishing the proof.  $\square$

We will also need the following lemma. The invariance under  $\varphi$  of  $\hat{i}(-, -)$  it expresses is the reason we had to re-define the intersection pairing.

**Lemma 52.** *If  $\varphi : G_0(X) \rightarrow G_1(X)$  is the identification of  $g_0$ - and  $g_1$ -geodesics and  $\mu, \lambda$  are two currents on  $G_0(X)$ , then*

$$\hat{i}(\varphi_*\mu, \varphi_*\lambda) = \hat{i}(\mu, \lambda).$$

*Proof.*  $\hat{i}(\varphi_*\mu, \varphi_*\lambda) = \sum_i \varpi(U_i)(\varphi_*\mu \times \varphi_*\lambda)(\widehat{DG}(U_i))$ . But  $\widehat{DG}(U_i)$  is invariant under  $\varphi$ , as is the combinatorial factor  $\varpi(U_i)$ . The result follows.  $\square$

7. CONTINUITY OF  $\hat{i}(-, -)$ 

The final, and crucial, property of the intersection pairing is some form of continuity. In the setting of geodesic currents on compact surfaces, the intersection pairing  $i(-, -)$  is continuous in both entries. In the Fuchsian building setting, given the difficulties involving transverse intersection and the new definition we adopted, it is not clear whether  $\hat{i}(-, -)$  is always continuous. Nevertheless, we can establish the following weaker result, which is sufficient for our purposes:

**Proposition 53.** *Let  $g_0, g_1$  be any metrics in  $\mathcal{M}_{neg}^v$ . Say  $\alpha_n$  are supported on closed geodesics such that  $\alpha_n \rightarrow L_{g_0}$ . Then*

$$\hat{i}(\alpha_n, L_{g_1}) \rightarrow \hat{i}(L_{g_0}, L_{g_1}).$$

*Proof.* By definition,

$$\hat{i}(\alpha_n, L_{g_1}) = \sum_i \varpi(U_i)(\alpha_n \times L_{g_1})(\widehat{DG}(U_i)).$$

Note that each  $\widehat{DG}(U_i)$ , as a subset of  $\mathcal{G}(X) \times \mathcal{G}(X)$ , is compact. So the fact that  $\alpha_n \times L_{g_1}$  weak-\* converges to  $L_{g_0} \times L_{g_1}$  implies that for all  $i$ ,

$$(\alpha_n \times L_{g_1})(\widehat{DG}(U_i)) \rightarrow (L_{g_0} \times L_{g_1})(\widehat{DG}(U_i)).$$

To deal with the (infinite) sum in the expression for  $\hat{i}(-, -)$ , we will prove the following claim:

**Claim:** Given  $\epsilon > 0$ , there exists some  $I_1$  such that for all  $n$ ,

$$\sum_{i>I_1} \varpi(U_i)(\alpha_n \times L_{g_1})(\widehat{DG}(U_i)) < \epsilon.$$

Once this claim is established, the proof of the proposition is as follows. Let  $\epsilon > 0$  be given. Let  $I_1$  be as provided by the Claim. Further, since  $\hat{i}(L_{g_0}, L_{g_1}) < \infty$  (Proposition 51), there exists some  $I_2$  such that

$$\sum_{i>I_2} \varpi(U_i)(L_{g_0} \times L_{g_1})(\widehat{DG}(U_i)) < \epsilon,$$

since the sum on  $i$  converges. Let  $I = \max\{I_1, I_2\}$ . Pick  $N$  so large that  $n > N$  implies that

$$\left| \sum_{i=1}^I \varpi(U_i)(\alpha_n \times L_{g_1})(\widehat{DG}(U_i)) - \sum_{i=1}^I \varpi(U_i)(L_{g_0} \times L_{g_1})(\widehat{DG}(U_i)) \right| < \epsilon,$$

using the weak-\* convergence of  $\alpha_n$  to  $L_{g_0}$  as noted above. Putting these three inequalities together, we get that for  $n > N$ ,

$$\left| \hat{i}(\alpha_n, L_{g_1}) - \hat{i}(\alpha_n, L_{g_0}) \right| < 3\epsilon,$$

finishing the proof.

*Proof of Claim:* Let  $G(U_i)$  be the set of all geodesics involved in a pair for which  $U_i$  is a good window. Note that  $G(U_i)$  consists of all geodesics which traverse all the chambers making up  $U_i$  consecutively. Fix any chamber  $C$  in  $U_i$ . Let  $A_i$  be the set of  $g_1$ -geodesics which agree with the restriction to  $C$  of the geodesics in  $G(U_i)$ . From the description of  $L_{g_1}$ , we see that  $L_{g_1}(G(U_i)) = \frac{1}{\varpi(U_i)} L_{g_1}(A_i)$ .

Now as the number of chambers in  $U_i$  increases, we see that the restriction of the geodesics in  $G(U_i)$  to  $C$  form a smaller and smaller set. In fact, since the curvature is bounded strictly away from zero and below zero, and geodesics in negative curvature diverge exponentially fast, it must be the case that  $L_{g_1}(A_i)$  tends to zero exponentially fast as the number of chambers in  $U_i$  increases (not necessarily as  $i$  increases). Let  $f(k)$  be a function decaying to zero exponentially fast so that if  $U_i$  has  $k$  chambers,  $L_{g_1}(U_i) < f(k)$ .

Note that  $L_{g_0}(G(X))$  is by definition finite, and as  $\alpha_n \rightarrow L_{g_0}$ , we see that  $\alpha_n(G(U_i)) \leq \alpha_n(G(X))$  is bounded, say by  $K$ .

Recall that  $\alpha_n$  is a Dirac measure supported on a closed geodesic;  $\alpha_n(G(U_i))$  is zero if the closed geodesic  $\alpha_n$  does not belong to  $G(U_i)$ . For any window with  $k$  chambers, the value of  $\alpha_n(G(U_i))$  is (proportional to) the number of times  $\alpha_n$  runs through  $U_i$ . Putting an arbitrary orientation on the closed geodesic  $\alpha_n$ , we see that for each time  $\alpha_n$  runs through  $U_i$ , there is a unique extension  $U'_i$  of  $U_i$  by one chamber in the forward direction of  $\alpha_n$  so that that  $\alpha_n$  runs through the extended window. From this, by induction, we see that

$$(7.1) \quad \sum_{U_i \text{ of size } 1} \alpha_n(G(U_i)) = \sum_{U_i \text{ of size a fixed } k > 1} \alpha_n(G(U_i)).$$

Combining all this, we get

$$\begin{aligned} \sum_i \varpi(U_i)(\alpha_n \times L_{g_1})(\widehat{DG}(U_i)) &\leq \sum_i \varpi(U_i) \alpha_n(G(U_i)) L_{g_1}(G(U_i)) \\ &= \sum_i \alpha_n(G(U_i)) L_{g_1}(A_i) \\ &= \sum_k \sum_{U_i \text{ containing } k \text{ chambers}} \alpha_n(G(U_i)) L_{g_1}(A_i) \end{aligned}$$

Letting  $A_i$  be the segments in the first chamber  $C$  of  $U_i$  corresponding to  $G(U_i)$  and using equation (7.1),

$$= \sum_k \sum_C \alpha_n(G(C)) L_{g_1}(A_i).$$

Note that  $k$  indexes the number of chambers involved in  $U_i$  inducing  $A_i$ . Using our bounding function  $f(k)$ ,

$$\begin{aligned} &< \sum_k \sum_C \alpha_n(G(C)) f(k). \\ &= \sum_k f(k) \sum_C \alpha_n(G(C)). \\ &\leq \sum_k f(k) (\# \text{chambers}) \alpha_n(G(X)) \\ &\leq \sum_k f(k) (\# \text{chambers}) K < \infty \end{aligned}$$

Since the sum converges, there must exist some  $N_\epsilon$  such that

$$\sum_{k > N_\epsilon} f(k) (\# \text{chambers}) K < \epsilon.$$

Letting  $I_i$  be the number of windows in  $X$  of length no more than  $N_\epsilon$ , we see that

$$\sum_{i>I_1} \varpi(U_i)(\alpha_n \times L_{g_1})(\widehat{DG}(U_i)) < \sum_{k>N_\epsilon} f(k)(\#\text{chambers})K < \epsilon$$

with  $I_1$  independent of  $n$  as desired. □

□

## 8. MARKED LENGTH SPECTRUM AND THE VOLUME FUNCTION

To prove Theorem 3, we move from the length inequality to the volume inequality via the intersection pairing. Our proof follows that in [CrD].

*Proof of Theorem 3.* For any closed geodesic  $\alpha_n$  not along a wall (using Corollary 16),

$$\hat{i}(\alpha_n, L_{g_0}) = 2 \cdot \text{Length}_{g_0}(\alpha_n) \leq 2 \cdot \text{Length}_{g_1}(\alpha_n) = \hat{i}(\alpha_n, L_{g_1}).$$

By Proposition 11 there is a sequence  $\alpha_n \rightarrow L_{g_0}$ . Since  $L_{g_0}$  gives zero measure to geodesics along walls, we can assume that no  $\alpha_n$  is along a wall. Applying Proposition 53 twice,

$$\hat{i}(L_{g_0}, L_{g_0}) \leq \hat{i}(L_{g_0}, L_{g_1}).$$

Similarly, for any closed geodesic  $\beta_n$  not along a wall,

$$\hat{i}(L_{g_0}, \beta_n) = 2 \cdot \text{Length}_{g_0}(\beta_n) \leq 2 \cdot \text{Length}_{g_1}(\beta_n) = \hat{i}(L_{g_1}, \beta_n).$$

Letting  $\beta_n \rightarrow L_{g_1}$  and applying the same continuity result,

$$\hat{i}(L_{g_0}, L_{g_1}) \leq \hat{i}(L_{g_1}, L_{g_1}).$$

Combining these inequalities, we have

$$\frac{1}{4\pi} \text{Vol}_{g_0}(X) = \hat{i}(L_{g_0}, L_{g_0}) \leq \hat{i}(L_{g_0}, L_{g_1}) \leq \hat{i}(L_{g_1}, L_{g_1}) = \frac{1}{4\pi} \text{Vol}_{g_1}(X)$$

as desired. □

**Remark.** We expect that Theorem 3 holds for any pair of metrics in  $\mathcal{M}_\leq(X)$ , without the non-singular vertices condition. Our arguments would prove this if one could show that, for a sequence  $\alpha_n \rightarrow L_g$ , the corresponding  $g'$ -geodesics spend proportion  $p_n \rightarrow 0$  of the time in the 1-skeleton of  $(X, g')$ . Without such a result, we can only ensure that  $\hat{i}(\alpha_n, L_g) \geq 2 \cdot \text{Length}_g(\alpha_n)$  (see the Remark after Proposition 49). This breaks the arguments for  $\hat{i}(L_{g_0}, L_{g_0}) \leq \hat{i}(L_{g_0}, L_{g_1})$  and  $\hat{i}(L_{g_0}, L_{g_1}) \leq \hat{i}(L_{g_1}, L_{g_1})$  above. This is the only place in our argument for Theorem 3 where the non-singular vertices assumption is used.

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