Primitive geodesic lengths and (almost) arithmetic progressions

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Abstract

In this article, we investigate when the set of primitive geodesic lengths on a Riemannian manifold have arbitrarily long arithmetic progressions. We prove that in the space of negatively curved metrics, a metric having such arithmetic progressions is quite rare. We introduce almost arithmetic progressions, a coarsification of arithmetic progressions, and prove that every negatively curved, closed Riemannian manifold has arbitrarily long almost arithmetic progressions in its primitive length spectrum. Concerning genuine arithmetic progressions, we prove that every noncompact arithmetic hyperbolic 2– or 3–manifold has arbitrarily long arithmetic progressions in its primitive length spectrum. We end with a conjectural characterization of arithmeticity in terms of arithmetic progressions in the primitive length spectrum. We also give an approach to a well known spectral ridigity problem based on the scarcity of manifolds with arithmetic progressions.

1 Introduction

Given a Riemannian manifold M, the associated geodesic length spectrum is an invariant of central importance. When the manifold M is closed and equipped with a negatively curved metric, there are several results that show primitive, closed geodesics on M play the role of primes in $\mathbf{Z}$ (or prime ideals in $\mathscr{O}_K$). Prime geodesic theorems like Huber [13], Margulis [18], and Sarnak [28] on growth rates of closed geodesics of length at most t are strong analogs of the prime number theorem (see, for instance, also [4], [22], [31], and [32]). Sunada's construction of length isospectral manifolds [33] was inspired by a similar construction of non-isomorphic number fields with identical Dedekind ζ–functions (see [20]). The Cebotarev density theorem has also been extended in various directions to lifting behavior of closed geodesics on finite covers (see [34]). There are a myriad of additional results, and this article continues to delve deeper into this important theme. Let us start by introducing some basic terminology:

Definition. Let (M,g) be a Riemannian orbifold, and $[g]$ a conjugacy class inside the orbifold fundamental group $\pi_1(M)$. We let $L_{[g]} \subset \mathbf{R}^+$ consist of the lengths of all closed orbifold geodesics in M which represent the conjugacy class $[g]$. The **length spectrum** of (M,g) is the multiset $\mathscr{L}(M,g)$ obtained by taking the union of all the sets $L_{[g]}$, where $[g]$ ranges over all conjugacy classes in M.

We say a conjugacy class $[g]$ is **primitive** if the element g is not a proper power of some other element (in particular g must have infinite order). The **primitive length spectrum** of (M,g) is the multiset $\mathscr{L}_p(M,g)$ obtained by taking the union of all the sets $L_{[g]}$, where $[g]$ ranges over all primitive conjugacy classes in M.

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1.1 Arithmetic progressions

Partially inspired by the analogy with primes, we are interested in understanding, for a closed Riemannian manifold (M,g), the structure of the primitive length spectrum $\mathscr{L}_p(M,g)$. Specifically, we would like to analyze whether or not the multiset of positive real numbers $\mathscr{L}_p(M,g)$ contains arbitrarily long arithmetic progressions.

Definition. We say that a multiset S contains a **k–term arithmetic progression** if it contains a sequence of numbers $x_1 < x_2 < \cdots < x_k$ with the property that, for some suitable a,b, we have $x_j = aj + b$.

We will say a (multi)-set S **has arithmetic progressions** if it contains k–term arithmetic progressions for all $k \geq 3$. We will say that a (multi)-set of positive numbers **has no arithmetic progressions** if it contains no 3–term arithmetic progressions (and hence, no k–term arithmetic progression with $k \geq 3$). Note that we do not allow for *constant* arithmetic progressions – so that multiplicity of entries in S are not detected by, and do not influence, our arithmetic progressions. Our first result indicates that generically, the primitive length spectrum of a negatively curved manifold has no arithmetic progression.

Theorem 1.1. *Let M be a closed, smooth manifold and let $\mathscr{M}(M)$ denote the space of all negatively curved Riemannian metrics on M, equipped with the Lipschitz topology. If $\mathscr{X}(M) \subseteq \mathscr{M}(M)$ is the set of negatively curved metrics g whose primitive length spectrum $\mathscr{L}_p(M,g)$ has no arithmetic progression, then $\mathscr{X}(M)$ is a dense G_δ set inside $\mathscr{M}(M)$.*

Recall that any two Riemannian metrics g,h on the manifold M are automatically bi-Lipschitz equivalent to each other. Let $1 \leq \lambda_0$ denote the infimum of the set of real numbers λ such that there exists a λ–bi-Lipschitz map

$$f_\lambda : (M,g) \longrightarrow (M,g').$$

The **Lipschitz distance** between g,g' is defined to be $\log(\lambda_0)$, and the **Lipschitz topology** on the space of metrics is the topology induced by this metric.

The key to establishing Theorem 1.1 lies in showing that any negatively curved metric can be slightly perturbed to have no arithmetic progression:

Theorem 1.2. *Let (M,g) be a negatively curved closed Riemannian manifold. Then for any $\varepsilon > 0$, there exists a new Riemannian metric $(M,\bar{g})$ with the property that:*

- *$(M,\bar{g})$ is negatively curved (hence $\bar{g} \in \mathscr{M}(M)$).*
- *For any $v \in TM$, we have the estimate*

$$(1-\varepsilon)\,||v||_g \leq ||v||_{\bar{g}} \leq ||v||_g\,.$$

- *The corresponding length spectrum $\mathscr{L}_p(M,\bar{g})$ has no arithmetic progression.*

In particular, the metric $\bar{g}$ lies in the subset $\mathscr{X}(M)$

The proof of Theorem 1.2 is fairly involved and will be carried out in Section 2. Let us deduce Theorem 1.1 from Theorem 1.2.

Proof of Theorem 1.1. To begin, note that the second condition in Theorem 1.2 ensures that the identity map is a $(1-\varepsilon)^{-1}$–bi-Lipschitz map from (M,g) to $(M,\bar{g})$. Hence, by choosing ε small enough, we can arrange for the Lipschitz distance between g, $\bar{g}$ to be as small as we want. In particular, we can immediately conclude that $\mathscr{X}(M)$ is dense inside $\mathscr{M}(M)$.

Since M is compact, the set $[S^1,M]$ of free homotopy classes of loops in M is countable (it corresponds to conjugacy classes of elements in the finitely generated group $\pi_1(M)$). Let $\mathrm{Tri}(M)$ denote the set of ordered triples of distinct elements in $[S^1, M]$, which is still a countable set. Fix a triple $t := (\gamma_1, \gamma_2, \gamma_3) \in \mathrm{Tri}(M)$ of elements in $[S^1, M]$. For any $g \in \mathscr{M}(M)$, we can measure the length of the g–geodesic in the free homotopy class represented by each γ_i. This yields a continuous function

$$L_t : \mathscr{M}(M) \longrightarrow \mathbf{R}^3$$

when $\mathscr{M}(M)$ is equipped with the Lipschitz metric. Consider the subset $A \subset \mathbf{R}^3$ consisting of all points whose three coordinates form a 3–term arithmetic progression. Note that A is a closed subset in $\mathbf{R}^3$, as it is just the union of the three hyperplanes $x+y=2z$, $x+z=2y$, and $y+z=2x$. Since $\mathbf{R}^3 \setminus A$ is open, so is $L_t^{-1}\left(\mathbf{R}^3 \setminus A\right) \subset \mathscr{M}(M)$. However, we have by definition that

$$\mathscr{X}(M) = \bigcap_{t \in T(M)} L_t^{-1}\left(\mathbf{R}^3 \setminus A\right)$$

establishing that $\mathscr{X}(M)$ is a G_δ set. □

It is perhaps worth mentioning that our proof of Theorem 1.1 is actually quite general, and can be used to show that, for any continuous finitary relation on the reals, one can find a dense G_δ set of negatively curved metrics whose primitive length spectrum *avoids* the relation (see Remark 2.2). As a special case, one obtains a proof of a well-known folk result – that there is a dense G_δ set of negatively curved metrics whose primitive length spectrum is multiplicity free.

Now Theorem 1.1 tells us that, for negatively curved metrics, the property of having arithmetic progressions in the primitive length spectrum is quite rare. There are two different ways to interpret this result:

(1) Arithmetic progressions are the wrong structures to look for in the primitive length spectrum.

(2) Negatively curved metrics whose primitive length spectrum have arithmetic progressions should be very special.

The rest of our results attempt to explore these two viewpoints.

1.2 Almost arithmetic progressions

Let us start with the first point of view (1). Since the property of having arbitrarily long arithmetic progressions is easily lost under small perturbations of the metric (e.g. our Theorem 1.2), we next consider a coarsification of this notion.

Definition. A finite sequence $x_1 < \cdots < x_k$ is a k–term **ε–almost arithmetic progression** ($k \geq 2$, $\varepsilon > 0$) provided we have

$$\left| \frac{x_i - x_{i-1}}{x_j - x_{j-1}} - 1 \right| < \varepsilon$$

for all $i, j \in \{2, \ldots, k\}$.

Definition. A multiset of real numbers $S \subset \mathbf{R}$ is said to have **almost arithmetic progressions** if, for every $\varepsilon > 0$ and $k \in \mathbf{N}$, the set S contains a k–term ε–almost arithmetic progression.

We provide a large class of examples of Riemannian manifolds (M, g) whose primitive length spectra $\mathscr{L}_p(M, g)$ have almost arithmetic progressions.

Theorem 1.3. *If (M, g) is a closed Riemannian manifold with strictly negative sectional curvature, then $\mathscr{L}_p(M, g)$ has almost arithmetic progressions.*

We will give two different proofs of Theorem 1.3 in Section 3. The first proof is geometric/dynamical, and uses the fact that the geodesic flow on the unit tangent bundle, being Anosov, satisfies the specification property. The second proof actually shows a more general result. Specifically, any set of real numbers that is asymptotically "dense enough" will contain almost arithmetic progressions. Theorem 1.3 is then obtained from an application of Margulis' [18] work on the growth rate of the primitive geodesics. The second approach is based on the spirit of Szemerédi's Theorem [35] (or more broadly the spirit of the Erdös–Turan conjecture) that large sets should have arithmetic progressions.

1.3 Arithmetic manifolds and progressions

Now we move to viewpoint (2) – a manifold whose primitive length spectrum has arithmetic progressions should be special. We show that several arithmetic manifolds have primitive length spectra that have arithmetic progressions. In the moduli space of constant (-1)–curvature metrics on a closed surface, the arithmetic structures make up a finite set. One reason to believe that such manifolds would be singled out by this condition is, vaguely, that one expects solutions to extremal problems on surfaces to be arithmetic. For example, the Hurwitz surfaces, which maximize the size of the isometry group as a function of the genus, are always arithmetic; it is a consequence of the Riemann–Hurwitz formula that such surfaces are covers of the $(2,3,7)$–orbifold and consequently are arithmetic.

Note that a 3–term arithmetic progression $x<y<z$ is a solution to the equation $x+z=2y$, and similarly, a k–term arithmetic progression can be described as a solution to a set of linear equations in k variables. Given a "generic" discrete subset of $\mathbf{R}^+$, one would not expect to find any solutions to this linear equation within the set, and hence would expect no arithmetic progressions. Requiring the primitive length spectrum to have arithmetic progressions forces it to contain infinitely many solutions to a linear system that generically has none. Of course, constant (-1)–curvature is already a rather special class of negatively curved metrics. Even within this special class of metrics, a 3–term progression in the length spectrum is still a non-trivial condition on the space of (-1)–curvature metrics. Our first result in this direction is a sample of a more general result.

Theorem 1.4. *Let X_n be the arithmetic, locally symmetric orbifold associated to the lattice* $\mathrm{PSL}(n,\mathbf{Z})$*. Then $\mathscr{L}_p(X_n)$ has arithmetic progressions.*

In Section 4, we prove Theorem 1.4, as well as various much stronger (but more technical) results, see e.g. Corollary 4.9. For instance, every primitive length occurs in arithmetic progressions (see Section 4.3 for a precise definition of this notion). We also establish arbitrarily long arithmetic progressions for other arithmetic manifolds, see for instance Corollary 4.8 and Corollary 4.15. The strongest statement our techniques yield, in our present language, is the following:

Theorem 1.5. *If (M,g) is a noncompact, arithmetic hyperbolic* 2*– or* 3*–manifold, then $\mathscr{L}_p(M,g)$ has arithmetic progressions. In fact, given any $\ell \in \mathscr{L}_p(M,g)$ and $k \in \mathbf{N}$, we can find an integer multiple of ℓ which occurs as the initial entry in a k–term arithmetic progression.*

The noncompactness condition helps avoid some technical difficulties that can be overcome with additional work. In particular, we believe that all arithmetic manifolds contain arithmetic progressions.

This suggests an approach to proving the primitive length spectrum determines a locally symmetric metric either locally or globally in the space of Riemannian metrics. This would require an upgrade of Theorem 1.1. We also provide a conjectural characterization of arithmeticity, and discuss a few related conjectural characterizations of arithmeticity in Section 5.

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2 Arithmetic progressions are non-generic

In this section, we provide a proof of Theorem 1.2. Starting with a negatively curved closed Riemannian manifold (M,g), we want to construct a perturbation $\overline{g}$ of the metric so that the primitive length spectrum $\mathscr{L}_p(M,\overline{g})$ contains no arithmetic progressions. The basic idea of the proof is to enumerate the geodesics in (M,g) according to their length. One then goes through the geodesics in order, and each time we see a geodesic whose length forms the third term of an arithmetic progression, we perturb the metric along the geodesic to destroy the corresponding 3–term arithmetic progression. The perturbations are chosen to have smaller and smaller support and amplitude, so that they converge to a limiting Riemannian metric. The limiting metric will then have no arithmetic progressions. We now proceed to make this heuristic precise.

2.1 Perturbing to kill a single arithmetic progression

Given a negatively curved Riemannian manifold (M,g), we will always fix an indexing of the set of primitive geodesic loops $\{\gamma_1,\gamma_2,\ldots\}$ according to the lengths, i.e. for all $i<j$, we have $\ell(\gamma_i)\leq\ell(\gamma_j)$. We now can establish the basic building block for our metric perturbations.

Proposition 2.1. *Let (M,g) be a negatively curved closed Riemannian manifold, γ_k a primitive geodesic in (M,g) of length $\ell(\gamma_k)=L$, and $\varepsilon>0$ a given constant. Then one can construct a negatively curved Riemannian metric $(M,\overline{g})$ satisfying the following properties:*

(a) For any vector $v\in TM$, we have

$$(1-\varepsilon)\,||v||_g\leq||v||_{\overline{g}}\leq||v||_g\,.$$

Moreover, all derivatives of the metric $\overline{g}$ are also ε–close to the corresponding derivatives of the metric g.

(b) For an appropriate point p, the metric $\overline{g}$ coincides with g on the complement of the ε–ball centered at p.

Given a loop η, we denote by $\overline{\eta}$ the unique $\overline{g}$–geodesic loop freely homotopic to η, and ℓ (or $\overline{\ell}$) denotes the g–length (or $\overline{g}$–length) of any curve in M. Then the lengths of geodesics change as follows:

(c) We have $L-\varepsilon\leq\overline{\ell}(\overline{\gamma_k})<L$.

(d) If $i\neq k$ with $\ell(\gamma_i)\leq L$ then $\overline{\ell}(\overline{\gamma_i})=\ell(\gamma_i)$.

(e) If $\ell(\gamma_i)>L$, then $\overline{\ell}(\overline{\gamma_i})>L$.

Proof. Consider the geodesic γ_k whose length we want to slightly decrease, along with the finite collection

$$\mathscr{S}:=\{\gamma_i\,:\,i\neq k,\ell(\gamma_i)\leq L\}$$

of closed geodesics whose lengths should be left unchanged. Note that any $\gamma_i\in\mathscr{S}$ is distinct from γ_k, hence $\gamma_i\cap\gamma_k$ is a finite set of points. Now choose $p\in\gamma_k$ which does not lie on any of the $\gamma_i\in\mathscr{S}$, and let δ be smaller than the distance from p to all of the $\gamma_i\in\mathscr{S}$, smaller than $\varepsilon/2$, and smaller than the injectivity radius of (M,g). We will modify the metric g within the g–metric ball $B(p;\delta)$ centered at p of radius δ. This will immediately ensure that property (b) is satisfied. Since the g–geodesics $\gamma_i\in\mathscr{S}$ lie in the complement of $B(p;\delta)$, they will remain $\overline{g}$–geodesics. This verifies property (d).

Next, we consider the set of g–geodesics whose lengths are greater than L. Since the length spectrum of a closed negatively curved Riemannian manifold is discrete, there is a $\delta'>0$ with the property that for any γ_i with $\ell(\gamma_i)>L$, we actually have

$$(1-\delta')\ell(\gamma_i)>L.$$

By shrinking δ' if need be, we can also assume that $\delta' < \varepsilon$. We will modify the metric on $B(p;\delta)$ so that, for any $v \in TB(p;\delta)$, we have

$$(1-\delta')||v||_g \leq ||v||_{\overline{g}} \leq ||v||_g. \tag{1}$$

Since $\delta' < \varepsilon$, the first statement in property (a) will follow. Moreover, if γ is any closed g–geodesic, and $\overline{\gamma}$ is the $\overline{g}$–geodesic freely homotopic to γ, then we have the inequalities:

$$\begin{aligned} \overline{\ell}(\overline{\gamma}) &= \int_{S^1} \left|\left|\overline{\gamma}'(t)\right|\right|_{\overline{g}} dt & (2)\\ &\geq (1-\delta') \int_{S^1} \left|\left|\overline{\gamma}'(t)\right|\right|_{g} dt & (3)\\ &= (1-\delta')\ell(\overline{\gamma}) & (4)\\ &\geq (1-\delta')\ell(\gamma) & (5) \end{aligned}$$

Inequality (3) follows by applying (1) point-wise, while inequality (5) comes from the fact that γ is the g–geodesic freely homotopic to the loop $\overline{\gamma}$. By the choice of δ', we conclude that

$$\overline{\ell}(\overline{\gamma_i}) \geq (1-\delta')\ell(\gamma_i) > L,$$

verifying property (e).

So to complete the proof, we are left with explaining how to modify the metric on $B(p;\delta)$ in order to ensure both property (a) (in particular, equation (1)) and property (c). We start by choosing a very small $\delta'' < \delta/2$, which is also smaller than the normal injectivity radius of γ_k. We will focus on an exponential normal δ''–neighborhood of the geodesic γ_k near the point p (we can reparametrize so that $\gamma_k(0) = p$). Choose an orthonormal basis $\{e_1, \dots e_n\}$ at the point $\gamma_k(0)$, with $e_1 = \gamma'(0)$, and parallel transport along γ to obtain an orthonormal family of vector fields $E_1, \dots E_n$ along γ. The vector fields $E_2, \dots, E_n$ provides us with a diffeomorphism between the normal bundle $N\gamma_k$ of $\gamma_k|_{(-\delta'',\delta'')}$ and $(-\delta'',\delta'') \times \mathbf{R}^{n-1}$. Let $D \subset \mathbf{R}^{n-1}$ denote the open ball of radius δ'', and using the exponential map, we obtain a neighborhood N of the point p which is diffeomorphic to $(-\delta'',\delta'') \times D$. We use this identification to parametrize N via pairs $(t,z) \in (-\delta'',\delta'') \times D$.

Next, observe that this neighborhood N comes equipped with a natural foliation, given by the individual slices $\{t\} \times D$. This is a smooth foliation by smooth codimension one submanifolds, and assigning to each point $q \in N$ the unit normal vector (in the positive t–direction) to the leaf through q, we obtain a smooth vector field V defined on N. We can (locally) integrate this vector field near any point $q = (t_0, z_0) \in N$ to obtain a well-defined function $\tau \colon N \to \mathbf{R}$, defined in a neighborhood of q (with initial condition given by $\tau \equiv 0$ on the leaf through q). Observe that, along the geodesic γ_k, we have that $\tau(t,0) = t$, but that in general, $\tau(t,z)$ might not equal t. In this (local) parametrization near any point $q \in N$, our g–metric takes the form

$$g = d\tau^2 + h_t, \tag{6}$$

where h_t is a Riemannian metric on the leaf $\{t\} \times D$. We now change this metric on N.

Pick a monotone smooth function

$$f \colon [0,\delta''] \longrightarrow [1-\delta', 1],$$

which is identically 1 in a neighborhood of δ'', and is identically $1-\delta'$ in a neighborhood of 0. Recall that we had the freedom of choosing δ' as small as we want. By further shrinking δ' if need be, we can also arrange for the smooth function f to have all order derivatives very close to 0. There is a continuous function $r \colon N \to [0,\delta'')$ given by sending a point to its distance from the geodesic γ_k. We define a new metric in the neighborhood N which is given in local coordinates by:

$$\overline{g} = f(r)f(t)d\tau^2 + h_t \tag{7}$$

where r denotes the distance to the geodesic γ_k (i.e. the distance to the origin in the D parameter).

Let us briefly describe in words this new metric. We are shrinking our original metric g in the directions given by the τ parameter. In a small neighborhood of the point p, the τ parameter vector (which coincides with γ_k'

along γ_k) is shrunk by a factor of $1-\delta'$. As you move away from p in the t and r directions, the τ parameter vector is shrunk by a smaller and smaller amount (f gets closer to 1), until you are far enough, at which point the metric coincides with the g–metric.

By the choice of δ'', this neighborhood N is entirely contained in $B(p;\delta)$, hence our new metric $\overline{g}$ coincides with the original one outside of $B(p,\delta)$. The fact that equation (1) holds is easy to see. Specifically, at any point $x=(t,z)\in N$ we can decompose any given tangent vector $\vec{v}\in T_xM$ as

$$\vec{v}=v_\tau\frac{d}{d\tau}+\vec{v}_z,$$

with $v_\tau\in\mathbf{R}$ and $\vec{v}_z\in T_{t,z}\big(\{t\}\times D\big)$. In terms of these, we have that the original g–length of $\vec{v}$ is given by

$$||\vec{v}||_g^2=v_\tau^2+||\vec{v}_z||_{h_t}^2,$$

while the new $\overline{g}$–length of $\vec{v}$ is given by

$$||\vec{v}||_{\overline{g}}^2=f(t)f(r)v_\tau^2+||\vec{v}_z||_{h_t}^2.$$

Now the fact that the function f takes values in the interval $[1-\delta',1]$ yields equation (1) (which as discussed earlier, gives the first statement in property (a)).

Before continuing, we remark that the curvature operator can be expressed as a continuous function of the Riemannian metric and its derivatives. The metrics $\overline{g}$ and g only differ on N, where they are given by equations (6) and (7) respectively. However, the function f was chosen to have all derivatives very close to 0. It follows that the metrics $\overline{g}$ and g are close, as are all their derivatives (giving the second statement in property (a)). Hence their curvature operators (as well as their sectional curvatures) will correspondingly be close. Since g is negatively curved, and M is compact, by choosing the parameters small enough, we can also ensure that $\overline{g}$ is negatively curved.

Lastly, we have to verify property (c), which states that the $\overline{g}$–length of the geodesic $\overline{\gamma_k}$ in the free homotopy class of the curve γ_k has length strictly smaller than L but no smaller than $L-\varepsilon$. For the strict upper bound, we merely observe that

$$\overline{\ell}(\overline{\gamma_k})\leq\overline{\ell}(\gamma_k)<\ell(\gamma_k)=L$$

The second inequality follows from equation (1), along with the fact that, in the vicinity of the point p, the tangent vectors γ_k' have $\overline{g}$–length equal to $1-\delta'$ which is strictly smaller than their g–length of 1. This establishes the upper bound in property (c). The lower bound follows immediately from property (a), using the same chain of inequalities appearing in Equations (2) - (5). This completes the verification of property (c), and hence concludes the proof of Proposition 2.1.

□

2.2 Perturbations with no arithmetic progressions

Finally, we have the necessary ingredients to prove Theorem 1.2.

Proof of Theorem 1.2. Given our negatively curved closed Riemannian manifold (M,g), we will inductively construct a sequence of negatively curved Riemannian metrics g_i, starting with $g_0=g$. We will denote by $\gamma_k^{(i)}$ the k^{th} shortest primitive geodesic in the g_i–metric. To alleviate notation, let us denote by $\mathscr{L}_i$ the primitive length spectrum of (M,g_i), which we think of as a non-decreasing function

$$\mathscr{L}_i\colon\mathbf{N}\longrightarrow\mathbf{R}^+.$$

In particular, $\mathscr{L}_i(k)=\ell\big(\gamma_k^{(i)}\big)$, the length of $\gamma_k^{(i)}$ in the g_i-metric.

We will be given an arbitrary sequence $\{\varepsilon_n\}_{n\in\mathbf{N}}$ satisfying $\lim\varepsilon_n=0$. For each $n\in\mathbf{N}$, the sequence of metrics g_i will then be chosen to satisfy the following properties:

1. For all $i \geq n$, the functions $\mathscr{L}_i$ coincide on $\{1,\dots,n\}$.
2. Each subset $\mathscr{L}_n(\{1,\dots,n\}) \subset \mathbf{R}^+$ contains no 3–term arithmetic progressions.
3. Each $g_{n+1} \equiv g_n$ on the complement of a closed set B_n, where each B_n is a (contractible) metric ball in the g–metric of radius strictly smaller than ε_n, and the sets B_n are pairwise disjoint.
4. On the balls B_n, we have that for all vectors $v \in TB_n$,

   $$(1-\varepsilon_n)\,||v||_{g_n} \leq ||v||_{g_{n+1}} \leq ||v||_{g_n}$$

   Moreover, for each $n \in \mathbf{N}$, all derivatives of the metric g_{n+1} are close to the corresponding derivatives of the metric g_n.
5. For each $i > n$, we have that

   $$\gamma_i^{(n)} \setminus \bigcup_{j=1}^{n} B_j \neq \emptyset.$$

6. The sectional curvatures of the metrics g_n are uniformly bounded away from zero, and uniformly bounded below.

Assertion: There is a sequence of metrics g_n ($n \in \mathbf{N}$) satisfying properties (1)–(6).

Let us for the time being assume the **Assertion**, and explain how to deduce Theorem 1.2. The **Assertion** provides us with a sequence of negatively curved Riemannian metrics on the manifold M. By choosing a sequence $\{\varepsilon_n\}_{n\in\mathbf{N}}$ which decays to zero fast enough, it is easy to verify (using (3) and (4)) that these metrics converge uniformly to a limiting Riemannian metric g_∞ on M. Moreover, this metric is negatively curved (see (6)), and has the property that $\mathscr{L}_p(M,g_\infty)$ has no arithmetic progression. To see that there are no arithmetic progressions, we just need the following claim:

Claim: For any given free homotopy class of loops, one can choose a sufficiently large n so that, in the g_n–metric, we have that the geodesic $\gamma_k^{(n)}$ in the given free homotopy class satisfies $k \leq n$.

Let us for the moment assume this **Claim** and show that $\mathscr{L}_p(M,g_\infty)$ has no arithmetic progression. Given three free homotopy classes of loops, the claim implies that for sufficiently large n, we have that the three corresponding g_n–geodesics $\gamma_i^{(n)},\gamma_j^{(n)},\gamma_k^{(n)}$ satisfy $i,j,k \leq n$. Then property (2) ensures that the three real numbers $\mathscr{L}_n(i),\mathscr{L}_n(j),\mathscr{L}_n(k)$ do not form a 3–term arithmetic progression. Property (1) ensures that this property still holds for all metrics g_m, where $m \geq n$, and hence holds for the limiting metric g_∞. We conclude that $\mathscr{L}_p(M,g_\infty)$ has no arithmetic progression.

*Proof of **Claim***. To verify the **Claim**, we proceed via contradiction. Let L denote the g_0–length of the g_0–geodesic in the given free homotopy class. For each $n \in \mathbf{N}$, we have that there are at least n primitive g_n–geodesics whose g_n length is no larger than the g_n–length of the g_n–geodesic in the given free homotopy class. On the other hand, from properties (3) and (4), we know that the g_n–length of the g_n–geodesic in the given free homotopy class is no longer than L (each successive g_i can only shorten the length of minimal representatives). Property (3) and (5) ensures that these g_n–geodesics are also g_∞–geodesics. This implies that for the g_∞–metric on M, we have infinitely many geometrically distinct primitive geodesics whose lengths are uniformly bounded above by L. However, this is in direct conflict with the fact that $\mathscr{L}_p(M,g_\infty)$ is a discrete multiset in $\mathbf{R}$ (since g_∞ has strictly negative curvature). Having derived a contradiction, we can conclude the validity of the **Claim**. □

So to complete the proof, we are left with constructing the sequence of metrics postulated in the **Assertion**.

*Proof of **Assertion***. By induction, let us assume that g_n is given, and let us construct g_{n+1}. We consider the set $\mathscr{L}_n(\{1,\dots,n+1\}) \subset \mathbf{R}^+$, and check whether or not it contains any arithmetic progression. If it does not, we set $g_{n+1} \equiv g_n$, $B_{n+1} = \emptyset$, and we are done. If it does contain an arithmetic progression, then from the induction hypothesis we know that it is necessarily a 3–term arithmetic progression with last term given by $\mathscr{L}_n(n+1)$, the length of the g_n–geodesic $\gamma^{(n)}_{n+1}$.

From property (5), the complement

$$\gamma^{(n)}_{n+1} \setminus \bigcup_{j=1}^{n} B_j$$

is a non-empty set and can be viewed as a collection of open subgeodesics of $\gamma^{(n)}_{n+1}$. As each of the sets

$$\gamma^{(n)}_{n+1} \cap \gamma^{(n)}_i$$

is finite, we can choose a point p on

$$\gamma^{(n)}_{n+1} \setminus \bigcup_{j=1}^{n} B_j$$

which does not lie on any of the geodesics $\gamma^{(n)}_i$ for $i \leq n$. We choose a parameter $\varepsilon' < \varepsilon_n$, small enough so that the ε'–ball centered at p is disjoint from

$$\left(\bigcup_{j=1}^{n} B_j\right) \cup \left(\bigcup_{j=1}^{n} \gamma^{(n)}_j\right).$$

Note that, in view of property (3), on the complement of

$$\bigcup_{j=1}^{n} B_j,$$

we have that

$$g_n \equiv g_{n-1} \equiv \cdots \equiv g_0.$$

In particular, for ε' small, the metric ball centered at p will be independent of the metric used. Shrinking ε' further if need be, we can apply Proposition 2.1 (with a parameter $\varepsilon < \varepsilon'$ to be determined below), obtaining a metric g_{n+1} which differs from g_n solely in the ε'–ball centered at p. We define B_{n+1} to be the ε'–ball centered at p, and now proceed to verify properties (1)–(6) for the resulting metric.

Property (1): We need to check that the resulting length function $\mathscr{L}_{n+1}$ satisfies

$$\mathscr{L}_{n+1}(i) = \mathscr{L}_n(i)$$

when $i \leq n$. However, this equality follows from statement (d) in Proposition 2.1.

Property (2): In view of property (1), we have an equality of sets

$$\mathscr{L}_{n+1}(\{1,\ldots,n\}) = \mathscr{L}_n(\{1,\ldots,n\}).$$

By the inductive hypothesis, we know that there is no 3–term arithmetic progression in this subset. Since the set $\mathscr{L}_{n+1}(\{1,\ldots,n\})$ is finite, there are only finitely many real numbers which can occur as the 3rd term in a 3–term arithmetic progression whose first two terms lie in $\mathscr{L}_{n+1}(\{1,\ldots,n\})$; let T denote this finite set of real numbers, and observe that by hypothesis, $L := \mathscr{L}_n(n+1) \in T$. Since T is finite, we can choose $\varepsilon < \varepsilon'$ small enough so that we also have

$$[L-\varepsilon, L) \cap T = \emptyset.$$

Then it follows from statements (c) and (e) in our Proposition 2.1 that

$$L - \varepsilon \leq \mathscr{L}_{n+1}(n+1) < L$$

and hence $\mathscr{L}_{n+1}(n+1) \notin T$. Since $\mathscr{L}_{n+1}(n+1)$ cannot be the third term of an arithmetic progression, we conclude that the set $\mathscr{L}_{n+1}(\{1,\ldots,n+1\})$ contains no 3–term arithmetic progressions, verifying property (2).

Property (3): This follows immediately from our choice of $\varepsilon' < \varepsilon_n$ and point p, and property (b) in Proposition 2.1.

Property (4): This follows from the corresponding property (a) in Proposition 2.1 (recall that $\varepsilon < \varepsilon_n$).

Property (5): This follows readily from property (3), which implies that the individual B_j are the path connected components of the set

$$\bigcup_{j=1}^{n} B_j.$$

So if the closed geodesic $\gamma_i^{(n)}$ was entirely contained in

$$\bigcup_{j=1}^{n} B_j,$$

it would have to be contained entirely inside a single B_j. However, such a containment is impossible, as $\gamma_i^{(n)}$ is homotopically non-trivial in M, while each B_j is a contractible subspace of M.

Property (6): This is a consequence of property (4), as the curvature operator varies continuously with respect to changes in the metric and its derivatives. By choosing the sequence $\{\varepsilon_n\}_{n\in\mathbf{N}}$ to decay to zero fast enough, we can ensure that the change in sectional curvatures between successive g_n–metrics is slow enough to be uniformly bounded above and below by a pair of negative constants.

This completes the inductive construction required to verify the **Assertion**. □

Having verified the **Assertion**, our proof of Theorem 1.2 is complete. □

Remark. Let $\mathscr{R}$ be an r–ary relation ($r \geq 2$) on the reals $\mathbf{R}$, having the property that if $(x_1, x_2, \ldots, x_r)$ in $\mathscr{R}$, then

$$x_1 \leq x_2 \leq \cdots \leq x_r.$$

Assume the relation $\mathscr{R}$ also has the property that, given any $x_1 \leq x_2 \leq \cdots \leq x_{r-1}$, the set

$$\{z \,:\, (x_1, \ldots, x_{r-1}, z) \in \mathscr{R}\}$$

is *finite*. Then the reader can easily see that the proof given above for Theorem 1.1 also shows that there is a dense set of negatively curved metrics g with the property that the primitive length spectrum $\mathscr{L}_p(M,g)$ contains no r–tuple satisfying the relation $\mathscr{R}$. In the special case where there exists a continuous function $F\colon \mathbf{R}^r \to \mathbf{R}$ with the property that $(x_1, \ldots, x_r)$ is in $\mathscr{R}$ if and only if

$$x_1 \leq x_2 \leq \cdots \leq x_r$$

satisfies

$$F(x_1, \ldots, x_r) = 0,$$

one also has that this dense set of negatively curved metrics is a G_δ set (in the Lipschitz topology).

Our Theorem 1.1 corresponds to the 3–ary relation given by zeroes of the linear equation

$$F(x,y,z) = x - 2y + z.$$

For another example, consider the 2–ary relation corresponding to the zeroes of the linear equation

$$F(x,y) = x - y.$$

In this setting, we recover a folk result – that there is a dense G_δ set of negatively curved metrics on M which have no multiplicities in the primitive length spectrum. This result is undoubtedly well-known to experts, although there does not seem to be a proof in the literature.

3 Almost arithmetic progressions are generic

In this section, we give two proofs that almost arithmetic progressions can always be found in the primitive length spectrum of negatively curved Riemannian manifolds.

3.1 Almost arithmetic progression - the dynamical argument

The first approach relies on the dynamics of the geodesic flow. Recall that closed geodesics in M correspond to periodic orbits of the geodesic flow ϕ defined on the unit tangent bundle T^1M. In the case where M is a closed negatively curved Riemannian manifold, it is well known that the geodesic flow is Anosov (see for instance [12, Section 17.6]). Our result is then a direct consequence of the following:

Proposition 3.1. *Let X be a closed manifold supporting an Anosov flow ϕ. Then for any $\varepsilon > 0$ and natural number $k \geq 3$, there exists a k–term ε–almost arithmetic progression $\tau_1 < \ldots < \tau_k$ and corresponding periodic points $z_1, \ldots, z_k$ in X with the property that each z_i has minimal period τ_i.*

Before establishing this result, we recall that the Anosov flow on X has the **specification property** (see [12, Section 18.3] for a thorough discussion of this notion). This means that, given any $\delta > 0$, there exists a real number $d > 0$ with the following property. Given the following specification data:

- any two intervals $[0, b_1]$ and $[b_1 + d, b_2]$ in $\mathbf{R}$ (here b_1, b_2 are arbitrary positive real numbers satisfying $b_1 + d < b_2$),
- a map
$$P\colon [0,b_1] \cup [b_1+d, b_2] \longrightarrow X$$
such that $\phi^{t_2-t_1}\big(P(t_1)\big) = P(t_2)$ holds whenever $t_1, t_2 \in [0, b_1]$ and whenever $t_1, t_2 \in [a_2, b_2]$ (so that P restricted to each of the two intervals defines a pair of ϕ–orbits),

one can find a periodic point x, of period s, having the property that for all $t \in [0,b_1] \cup [b_1+d, b_2]$ $d\big(\phi^t(x), P(t)\big) < \delta$ (so the periodic orbit δ–shadows the two given pairs of orbits). Moreover, the period s satisfies $\big|s - (b_2 + d)\big| < \delta$ (though s might not be the minimal period of the point x). We now use this specification property to establish the proposition.

Proof. We start by choosing a pair of distinct periodic orbits $\mathscr{O}_1$, $\mathscr{O}_2$ for the flow ϕ, with minimal periods A, B respectively (existence of distinct periodic orbits is a consequence of the Anosov property). Since the closed orbits are distinct, there is a δ with the property that the δ–neighborhoods of the two orbits are disjoint. Corresponding to this δ, we let $d > 0$ be the real number provided by the specification property. We fix a pair of points $p_i \in \mathscr{O}_i$, and now explain how to produce some new periodic points.

Given an $n \in \mathbf{N}$, we consider the two intervals $[0, A]$ and $[A+d, nB+A+d]$ in $\mathbf{R}$. We define a map

$$P\colon [0,A] \cup [A+d, nB+A+d] \longrightarrow X$$

by setting

$$P(t) = \begin{cases} \phi^t(p_1) & t \in [0,A] \\ \phi^{t-A-d}(p_2) & t \in [A+d, nB+A+d]. \end{cases}$$

From the specification property, one can find a periodic point $x_n \in X$, an s_n with $\phi^{s_n}(x_n) = x_n$ and

$$\big|s_n - (nB + A + 2d)\big| < \delta,$$

such that $d\big(\phi^t(x_n), P(t)\big) < \delta$ holds for all t in $[0,A] \cup [A+d, nB+A+d]$.

We now claim that, whenever $n > (A+2d+\delta)/B$, s_n is the minimal period of the point x_n. Indeed, under this hypothesis, the subinterval $[A+d, nB+A+d]$ is at least half the length of the period s_n. So if s_n were not minimal, one could find $t_1 \in [0,A]$ and $t_2 \in [A+d, nB+A+d]$ with the property that $y := \phi^{t_1}(x) = \phi^{t_2}(x)$. However, the shadowing property implies that

$$d\big(y, P(t_i)\big) = d\big(\phi^{t_i}(x), P(t_i)\big) < \delta,$$

which tells us that y lies in the δ–neighborhood of both sets $\mathscr{O}_1 = P\big([0,A]\big)$ and $\mathscr{O}_2 = P\big([A+d, nB+A+d]\big)$. This containment plainly contradicts the choice of δ. We conclude that s_n is indeed the minimal period of the point x_n.

Now that we have found a sequence $\{x_n\}$ of periodic points, with minimal periods $\{s_n\}$ (when n is sufficiently large), it is easy to find a k–term ε–almost arithmetic progression. First, pick the integer N to satisfy the inequality

$$N > \max\left\{\frac{4\delta+2\delta\varepsilon}{B\varepsilon}, \frac{A+2d+\delta}{B}\right\}$$

set $z_i := x_{iN}$, and $\tau_i := s_{iN}$. We claim that the real numbers $\tau_1, \ldots, \tau_k$ forms the desired almost arithmetic progression. Indeed, the condition

$$N > \frac{A+2d+\delta}{B}$$

ensures that τ_i is the minimal period of the corresponding x_i. We also have, from the specification property, that each τ_i satisfies the inequality

$$|\tau_i - (iNB + A + 2d)| < \delta$$

and an elementary calculation now shows that the ratio of any successive differences satisfies

$$1-\varepsilon < 1 - \frac{4\delta}{NB+2\delta} < \left|\frac{\tau_{i+1}-\tau_i}{\tau_{j+1}-\tau_j}\right| < 1 + \frac{4\delta}{NB-2\delta} < 1+\varepsilon$$

where the outer inequalities follow from

$$N > \frac{4\delta+2\delta\varepsilon}{B\varepsilon}.$$

Hence we have found the desired k–term ε–almost arithmetic progression, completing the proof of the proposition. ☐

Remark. It is perhaps worth pointing out that there exist examples of Anosov flows that are distinct from the geodesic flow on the unit tangent bundle of a negatively curved manifold. For example, Eberlein [6] has constructed an example of aclosed non-positively curved Riemannian manifolds whose geodesic flow is Anosov, and which contain "large" open sets where the sectional curvature is identically zero. There are also examples of Anosov flows that do *not* come from geodesic flows, e.g. the suspension of an Anosov diffeomorphism on an odd dimensional manifold provides such an example.

3.2 Almost arithmetic progression - the density argument

An alternate route for showing that the primitive length spectrum $\mathscr{L}_p(M,g)$ of a negatively curved Riemannian manifold has arbitrarily long almost arithmetic progressions is to exploit Margulis' work on the growth rate of this sequence. More generally, consider a multiset $S \subset \mathbf{R}^+$ which is **discrete**, in that any bounded interval contains only finitely many elements of S. We can introduce the associated **counting function**

$$S(n) := |\{x \in S \,:\, x \le n\}|$$

We can then show:

Proposition 3.2. *Assume the function $S(x)$ has the property that there is some $t > 0$ such that*

$$\lim_{x\to\infty} \frac{S(x-t)}{S(x)}$$

exists and is not equal 1*. Then the multiset S has almost arithmetic progressions.*

Proof. Given an $\varepsilon > 0$, we want to find an ε–almost arithmetic progression of some given length N. Let us decompose

$$\mathbf{R}^+ = \bigcup_{k\in\mathbf{N}} \big((k-1)t, kt\big],$$

and form a subset $A \subset \mathbf{N}$ via

$$A := \{k \,:\, S \cap \big((k-1)t, kt\big] \neq \emptyset\}.$$

We now argue that the set $A \subset \mathbf{N}$ is the complement of a finite subset of $\mathbf{N}$.

If not, we could find an infinite sequence $k_i \subset \mathbf{N}$ with $k_i \notin A$. From the definition of A, we have that for each of these k_i, the set $S \cap \big((k_i-1)t, k_it\big]$ is empty. In terms of the counting function, this tells us that $S\big((k_i-1)t\big) = S(k_it)$. Now we divide by $S(k_it)$ and take the limit, giving

$$\lim_{i\to\infty} \frac{S(k_it-t)}{S(k_it)} = 1.$$

However, this contradicts the fact that the limit

$$\lim_{x\to\infty} \frac{S(x-t)}{S(x)}$$

exists and is not equal to 1. So $\mathbf{N} \setminus A$ is a finite set, as desired.

Next we choose an m sufficiently large so that all integers greater than or equal to m lie in the set A, and moreover

$$1 + \frac{2}{\varepsilon} < m.$$

Consider the sequence of natural numbers $\{m, 2m, \ldots Nm\}$. Since each of these natural numbers lies in the set A, we can choose numbers $x_j \in S \cap \big((jm-1)t, (jm)t\big]$, giving us a sequence of numbers $x_1 < x_2 < \cdots < x_N$ in the set S. We claim that this sequence forms an ε–almost arithmetic progression of length N. It suffices to estimate the ratio of the successive differences. Note that for any index j, we have the obvious estimate on the difference:

$$(m-1)t < \big|x_{j+1} - x_j\big| < (m+1)t.$$

Looking at the ratio between any two such successive differences, we obtain:

$$1 - \varepsilon < \frac{m-1}{m+1} < \frac{|x_{i+1} - x_i|}{\big|x_{j+1} - x_j\big|} < \frac{m+1}{m-1} < 1 + \varepsilon,$$

where the two outer inequalities follow from the fact that $1 + \frac{2}{\varepsilon} < m$. This completes the proof of the proposition. □

A celebrated result of Margulis [18] establishes that, for a closed negatively curved manifold, the counting function for the primitive length spectrum has asymptotic growth rate

$$S(x) \sim \frac{e^{hx}}{hx},$$

where $h > 0$ is the topological entropy of the geodesic flow on the unit tangent bundle. It is clear that, for any $t > 0$, we have

$$\lim_{x\to\infty} \frac{S(x-t)}{S(x)} = \lim_{x\to\infty} \frac{e^{h(x-t)}hx}{e^{hx}h(x-t)} = e^{-ht},$$

which is clearly not equal to 1 since both $h > 0, t > 0$. In particular, Margulis' work in tandem with Proposition 3.2 yields a second proof of Theorem 1.3.

Remark. Margulis' thesis actually establishes the asymptotics for the number of periodic orbits of Anosov flows. Hence, appealing to Margulis, one can recover Proposition 3.1 as a special case of Proposition 3.2. We chose to still include our proof of Proposition 3.1 for two reasons. First, it is relatively elementary, using only the specification property for Anosov flows, rather than the sophisticated result in Margulis' thesis. Secondly, it is constructive, allowing us to concretely "see" the sequence of periodic orbits whose lengths form the desired almost arithmetic progression.

4 Examples of manifolds with arithmetic progressions

In this section, we study the property of having genuine arithmetic progressions in the primitive length spectrum. We first show that this property is invariant under covering maps. Next, we prove that certain arithmetic manifolds have arithmetic progressions in their primitive length spectrum.

4.1 Commensurability invariance

The goal of this subsection is the following basic proposition.

Proposition 4.1. *Let $(\overline{M},\overline{g})$ be a finite orbifold cover of a closed orbifold (M,g), with covering map $p\colon \overline{M}\to M$. Then the following two statements are equivalent:*

(a) The primitive length spectrum $\mathscr{L}_p(M,g)$ has arithmetic progressions.

(b) The primitive length spectrum $\mathscr{L}_p(\overline{M},\overline{g})$ has arithmetic progressions.

Proof. We start by making a simple observation. For a closed curve $\gamma\colon S^1\to M$, we call a curve $\overline{\gamma}\colon S^1\to\overline{M}$ a lift of γ if there is a standard covering map $q\colon S^1\to S^1$ (given by $z\mapsto z^n$) with the property that $\gamma\circ q\equiv p\circ\overline{\gamma}$. If γ is a primitive geodesic in M, we observe that all of its lifts $\overline{\gamma}$ to $\overline{M}$ are also primitive geodesics. If d is the degree of the cover $p\colon \overline{M}\to M$, then the lift $\overline{\gamma}$ will always have length that is an integral multiple of γ. Moreover,

$$1\leq \ell(\overline{\gamma})/\ell(\gamma)\leq d,$$

for any geodesic γ on M and any lift $\overline{\gamma}$ of γ to $\overline{M}$.

Now for the direct implication that (a) implies (b), we assume that $\mathscr{L}_p(M,g)$ contains arithmetic progressions. Fixing some $k\geq 3$, our goal is to find a k–term arithmetic progression in the set $\mathscr{L}_p(\overline{M},\overline{g})$. From Van der Waerden's theorem (see for instance [36] or [10]), there is an integer $N:=N(d,k)$, so that if the set $\{1,\dots,N\}$ is d–colored, it contains a k–term monochromatic arithmetic progression. Since $\mathscr{L}_p(M,g)$ contains arithmetic progressions, we can find a collection of primitive closed geodesics $\gamma_1,\dots,\gamma_N$ such that the corresponding real numbers $\ell(\gamma_1),\dots,\ell(\gamma_N)$ form an N–term arithmetic progression. For each γ_i, choose a lift $\overline{\gamma_i}$ inside $\overline{M}$, and color the integer i by the color $\ell(\overline{\gamma_i})/\ell(\gamma_i)$. Looking at the monochromatic indices that form an arithmetic progression, we see that the corresponding $\ell(\gamma_i)$ form a k–term arithmetic progression. Moreover, by construction, the corresponding lifts $\overline{\gamma_i}$ are primitive geodesics whose lengths $\ell(\overline{\gamma_i})=m\cdot\ell(\gamma_i)$. Here m is a fixed integer which we view as the color of the monochromatic sequence. This gives the desired k–term arithmetic progression in the set $\mathscr{L}_p(\overline{M},\overline{g})$.

For the converse implication, we assume (b), that $\mathscr{L}_p(\overline{M},\overline{g})$ has arithmetic progressions. Given a primitive closed geodesic $\overline{\gamma}$ in $\overline{M}$, one can look at the image geodesic $p\circ\overline{\gamma}$ in M, and ask whether or not this geodesic is primitive. Since $\overline{\gamma}$ is primitive, the only way $p\circ\overline{\gamma}$ could fail to be primitive is if the map p induced a non-trivial covering from $\overline{\gamma}$ to the image curve $p\circ\overline{\gamma}$. Of course, the degree $d_{\overline{\gamma}}$ of this covering is smaller than or equal to d, and the quotient curve will be a primitive geodesic γ_i of length $\ell(\overline{\gamma})/d_{\overline{\gamma}}$. Now as before, to produce a k–term arithmetic progression in $\mathscr{L}_p(M,g)$, we let N be the Van der Waerden number $N(d,k)$, and choose a sequence of primitive closed geodesics $\overline{\gamma}_1,\dots,\overline{\gamma}_N$ in $\overline{M}$ whose lengths form an arithmetic progression. For each of these, we consider the corresponding primitive closed geodesic γ_i in M of length $\ell(\overline{\gamma})/d_{\overline{\gamma}}$. We color the index i according to the color $d_{\overline{\gamma}_i}$. Then from Van der Waerden's theorem, there is a monochromatic arithmetic subprogression $S \subset \{1,\dots,N\}$. The corresponding family of primitive geodesics $\{\gamma_i\}_{i\in S}$ have lengths which form a k–term arithmetic progression inside $\mathcal{L}_p(M,g)$, as required. □

Remark. The argument in the proof of Proposition 4.1 applies almost verbatim in the setting of almost arithmetic progressions, and shows that the following two statements are also equivalent:

(a) The primitive length spectrum $\mathcal{L}_p(M,g)$ has almost arithmetic progressions.

(b) The primitive length spectrum $\mathcal{L}_p(\overline{M},\overline{g})$ has almost arithmetic progressions.

As we will not need this result, we leave the details to the interested reader.

4.2 Example: $\mathrm{PSL}(2,\mathbf{Z})$

We will start with the modular curve $X = \mathbf{H}^2/\,\mathrm{PSL}(2,\mathbf{Z})$, which is an arithmetic, hyperbolic 2–orbifold.

4.2.1 Preliminaries

The closed geodesics c_γ on X are in bijective correspondence with the conjugacy classes $[\gamma]$ of hyperbolic elements $\gamma \in \mathrm{PSL}(2,\mathbf{Z})$. The trace $\mathrm{Tr}(\gamma)$ is well defined up to sign and the length $\ell(c_\gamma)$ is related to the trace via the formula (see [17, p. 384])

$$2\cosh\left(\frac{\ell(c_\gamma)}{2}\right) = \pm\,\mathrm{Tr}(\gamma).$$

The geodesic c_γ will be primitive when γ is primitive. Namely, γ is not a proper power of some $\eta \in \mathrm{PSL}(2,\mathbf{Z})$. Every hyperbolic element $\gamma \in \mathrm{PSL}(2,\mathbf{R})$ can be diagonalized with the form

$$\gamma \sim \begin{pmatrix} \pm\lambda_\gamma & 0 \\ 0 & \pm\lambda_\gamma^{-1} \end{pmatrix}.$$

Up to the sign of the trace, the characteristic polynomial of γ will be of the form

$$P_\gamma(t) = t^2 - \mathrm{Tr}(\gamma)t + 1.$$

As $|\mathrm{Tr}(\gamma)| > 2$ (see [17, p. 51]), we see that λ_γ is a real number. In the case that $\gamma \in \mathrm{PSL}(2,\mathbf{Z})$, since $\mathrm{Tr}(\gamma) \in \mathbf{Z}$, we see that $\mathbf{Q}(\lambda_\gamma)$ is always a real quadratic extension K_γ, since m^2-4 is never a square for any integer m with $|m| > 2$. Moreover, $\lambda_\gamma \in \mathcal{O}_{K_\gamma}$ and λ_γ^{-1} is the Galois conjugate of λ_γ. In particular, $\lambda_\gamma \in \mathcal{O}_{K_\gamma}^1$ is a unit in $\mathcal{O}_{K_\gamma}$. By Dirichlet's Unit Theorem (see [16, Theorem 38, p. 142]), the group of units $\mathcal{O}_{K_\gamma}^1$ of $\mathcal{O}_{K_\gamma}$ is isomorphic to $\{\pm 1\} \times \mathbf{Z}$, where $\mathbf{Z}$ is generated by a fundamental unit. We will say that γ is **absolutely primitive** if λ_γ is a fundamental unit in $\mathcal{O}_{K_\gamma}^1$. Namely, we want λ_γ to be primitive in the group $\mathcal{O}_{K_\gamma}^1$. We have two basic lemmas. The first is the following.

Lemma 4.2. *Given a real quadratic extension $K/\mathbf{Q}$, there exists an absolutely primitive element $\gamma \in \mathrm{PSL}(2,\mathbf{Z})$ with $K_\gamma = K$.*

Proof. Let $K/\mathbf{Q}$ be a real quadratic extension with $\mathbf{Z}[a_1,a_2] = \mathcal{O}_K$. Left multiplication of K on itself is a $\mathbf{Q}$–linear map and in the $\mathbf{Q}$–basis $\{a_1,a_2\}$, we have a map

$$K^\times \longrightarrow \mathrm{GL}(2,\mathbf{Q}), \quad \mathcal{O}_K^\times \longrightarrow \mathrm{GL}(2,\mathbf{Z}).$$

The group of norm 1 element $\mathcal{O}_K^1$ maps into $\mathrm{SL}(2,\mathbf{Z})$. The image of a fundamental unit will be an absolutely primitive hyperbolic element. □

Next, we have our second lemma.

Lemma 4.3. *If $\gamma, \eta \in \mathrm{PSL}(2,\mathbf{Z})$ are hyperbolic elements with $K_\gamma = K_\eta$, then there are powers $j_\gamma, j_\eta \in \mathbf{N}$ such that* $\mathrm{Tr}(\gamma^{j_\gamma}) = \mathrm{Tr}(\eta^{j_\eta})$.

Proof. After taking inverses of γ and/or η, if necessary, we can assume that each has a diagonal form

$$\begin{pmatrix} \lambda_\gamma & 0 \\ 0 & \lambda_\gamma^{-1} \end{pmatrix}, \quad \begin{pmatrix} \lambda_\eta & 0 \\ 0 & \lambda_\eta^{-1} \end{pmatrix}$$

with $\lambda_\gamma, \lambda_\eta > 1$. Each is a power then of the matrix

$$\begin{pmatrix} \mu_K & 0 \\ 0 & \mu_K^{-1} \end{pmatrix},$$

where μ_K is a fundamental unit for $\mathscr{O}_K^1$ with $K = K_\gamma = K_\eta$. In particular, if we set L to be the least common multiple of these powers t_γ, t_η, we can take

$$j_\gamma = \frac{L}{t_\gamma}, \quad j_\eta = \frac{L}{t_\eta}.$$

□

As a consequence of Lemma 4.2 and Lemma 4.3, we have the following result.

Corollary 4.4. *If $\gamma \in \mathrm{PSL}(2,\mathbf{Z})$ is absolutely primitive, then γ is primitive. Moreover, if γ is primitive, then there exists an absolutely primitive $\eta \in \mathrm{PSL}(2,\mathbf{Z})$ such that* $\mathrm{Tr}(\gamma) = \mathrm{Tr}(\eta^j)$.

4.2.2 Producing long progressions

The idea for producing arbitrarily long arithmetic progression in the primitive length spectrum of X is as follows. To diminish the notational burden on the reader, set $\Gamma = \mathrm{PSL}(2,\mathbf{Z})$. Given $\eta \in \mathrm{PGL}(2,\mathbf{Q})$, we know (see [24, Chapter 10]) that

$$\Gamma_\eta = (\eta \Gamma \eta^{-1}) \cap \Gamma$$

is a finite index subgroup of Γ and $\eta\Gamma\eta^{-1}$. We define

$$P\colon \Gamma \times \mathrm{PGL}(2,\mathbf{Q}) \longrightarrow \mathbf{N}$$

by

$$P(\gamma,\eta) = \min\left\{ j \in \mathbf{N} \ : \ (\eta\gamma\eta^{-1})^j \in \Gamma \right\}.$$

For a fixed element $\gamma \in \Gamma$, we can restrict the map P to the fiber $\{\gamma\} \times \mathrm{PGL}(2,\mathbf{Q})$ to obtain the subset

$$\mathscr{P}(\gamma) = \{P(\gamma,\eta) \ : \ \eta \in \mathrm{PGL}(2,\mathbf{Q})\} \subseteq \mathbf{N}.$$

We set

$$\theta_{\gamma,\eta} = \eta\gamma^{P(\gamma,\eta)}\eta^{-1} \in \Gamma$$

and notice that

$$\ell(c_{\theta_{\gamma,\eta}}) = P(\gamma,\eta)\ell(c_\gamma).$$

In particular, in the geodesic length spectrum $\mathscr{L}(X)$, we have

$$\left\{ P(\gamma,\eta)\ell(c_\gamma) \ : \ \eta \in \mathrm{PGL}(2,\mathbf{Q}) \right\} = \mathscr{P}(\gamma)\ell(c_\gamma) \subset \mathscr{L}(X).$$

In order to produce arbitrarily long arithmetic progressions in $\mathscr{L}_p(X)$, we proceed in two steps. First, we will use a particularly nice family of elements $\left\{\eta_j\right\} \subset \mathrm{PGL}(2,\mathbf{Q})$ to show that, for any hyperbolic element γ, the set of natural numbers $\mathscr{P}(\gamma)$ contains arbitrarily long arithmetic progressions. Second, we will show that when γ is absolutely primitive, the resulting elements θ_{γ,η_j} are always primitive. Combining these two steps will establish the following:

Theorem 4.5. *Let γ be a primitive hyperbolic element in* $\mathrm{PSL}(2,\mathbf{Z})$ *with associated geodesic length $\ell = \ell(c_\gamma)$. Then for each $k \in \mathbf{N}$, there exists an arithmetic progression $\left\{C_{\gamma,k}\ell n\right\}_{n=1}^{k} \subset \mathscr{L}_p(X)$ where $C_{\gamma,k} \in \mathbf{Q}$. Moreover, there exists $D_\gamma \in \mathbf{N}$ such that $C_{\gamma,k}D_\gamma \in \mathbf{N}$ for all k (i.e. for each fixed γ, the set of rational numbers $C_{\gamma,k}$ have uniformly bounded denominators).*

Remark. If one simply seeks arithmetic progressions in the set of traces of primitive elements in $\mathrm{PSL}(2,\mathbf{Z})$, one can take the elements

$$\begin{pmatrix} a & 1 \\ -1 & 0 \end{pmatrix}.$$

We will see that the failure of $C_{\gamma,k}$ to be an integer is controlled by the failure of γ to be absolutely primitive. Specifically, Theorem 4.5 is a consequence of the following theorem in combination with Corollary 4.4.

Theorem 4.6. *Let γ be an absolutely primitive element of* $\mathrm{PSL}(2,\mathbf{Z})$ *with associated geodesic length $\ell = \ell(c_\gamma)$. Then for each $k \in \mathbf{N}$, there exists an arithmetic progression $\left\{C_{\gamma,k}\ell n\right\}_{n=1}^{k} \subset \mathscr{L}_p(X)$ where $C_{\gamma,k} \in \mathbf{N}$.*

One can get an explicit estimate on the constant $C_{\gamma,k}$ as a function of k (Remark 4.2.2 gives a rough estimate for the constant $C_{\gamma,k}$).

Proof of Theorem 4.6. For $\alpha \in \mathbf{R}$, we define

$$\eta_\alpha = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}$$

and note that $\eta_{\alpha^{-1}} = \eta_\alpha^{-1}$. Our interest will be in $\alpha = m$ or m^{-1} for an integer $m \in \mathbf{N}$. Given

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

we see that

$$\eta_m \gamma \eta_m^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & m \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & m^{-1} \end{pmatrix} = \begin{pmatrix} a & m^{-1}b \\ mc & d \end{pmatrix}.$$

It is a simple matter to see that

$$P(\gamma, \eta_m) = \min\left\{ j \in \mathbf{N} \ : \ m \mid b_j \right\}$$

where

$$\gamma^j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix}.$$

Set

$$\mathbf{B}_L(\mathbf{Z}/m\mathbf{Z}) = \left\{ \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \ : \ a,c,d \in \mathbf{Z}/m\mathbf{Z} \right\} < \mathrm{PSL}(2,\mathbf{Z}/m\mathbf{Z}).$$

We have the homomorphism

$$r_m \colon \Gamma \longrightarrow \mathrm{PSL}(2,\mathbf{Z}/m\mathbf{Z})$$

given by reducing the matrix coefficients modulo m and $P(\gamma, \eta_m)$ is the smallest integer j such that $r_m(\gamma^j) \in \mathbf{B}_L(\mathbf{Z}/m\mathbf{Z})$. Note that since γ is hyperbolic, we have both $b, c \neq 0$ and for all $j \geq 1$, $b_j, c_j \neq 0$. Indeed, if this were not the case, then some power γ^j of γ would have either the form

$$\begin{pmatrix} a_j & 0 \\ c_j & d_j \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} a_j & b_j \\ 0 & d_j \end{pmatrix}.$$

Being an element of $\mathrm{PSL}(2,\mathbf{Z})$, this forces $a_j, d_j = \pm 1$ and thus γ would be virtually unipotent, which is impossible for an infinite order hyperbolic element.

We first consider the case when $m = p_i$ is a prime. We then have $P(\gamma, \eta_{p_i})$ is the smallest power j such that $r_{p_i}(\gamma^j) \in \mathbf{B}_L(\mathbf{F}_{p_i})$. We have

$$\left|\mathrm{PSL}(2,\mathbf{F}_{p_i})\right| = \frac{(p_i^2-1)(p_i^2-p_i)}{2(p_i-1)} = \frac{p_i(p_i-1)(p_i+1)}{2}, \quad \left|\mathbf{B}_L(\mathbf{F}_{p_i})\right| = \frac{p_i(p_i-1)}{2}$$

and so

$$[\mathrm{PSL}(2,\mathbf{F}_{p_i}) : \mathbf{B}_L(\mathbf{F}_{p_i})] = p_i + 1.$$

From this, we see that $P(\gamma, \eta_{p_i})$ divides $p_i + 1$. For $\eta_{p_i^2}$, we have again that $P(\gamma, \eta_{p_i^2})$ is the smallest j such that $r_{p_i^2}(\gamma^j) \in \mathbf{B}_L(\mathbf{Z}/p_i^2\mathbf{Z})$. We have the short exact sequence (see [1, Corollary 9.3], [5, Chapter 9], or [14, Lemma 16.4.5])

$$1 \longrightarrow V_{p_i} \longrightarrow \mathrm{PSL}(2,\mathbf{Z}/p_i^k\mathbf{Z}) \longrightarrow \mathrm{PSL}(2,\mathbf{Z}/p_i^{k-1}\mathbf{Z}) \longrightarrow 1, \tag{8}$$

where $V_{p_i} \cong \mathbf{F}_{p_i}^3$, as an abelian group; in fact, V_{p_i} is the $\mathbf{F}_{p_i}$–Lie algebra of $\mathrm{SL}(2,\mathbf{F}_{p_i})$. We also have an exact sequence

$$1 \longrightarrow W_{p_i} \longrightarrow \mathbf{B}_L(\mathbf{Z}/p_i^k\mathbf{Z}) \longrightarrow \mathbf{B}_L(\mathbf{Z}/p_i^{k-1}\mathbf{Z}) \longrightarrow 1,$$

where $W_{p_i} \cong \mathbf{F}_{p_i}^2$, as an abelian group. Since $P(\gamma, \eta_{p_i^k})$ is the smallest power j such that $r_{p_i^k}(\gamma^j) \in \mathbf{B}_L(\mathbf{Z}/p_i^k\mathbf{Z})$, from the above sequences, we see that

$$P(\gamma, \eta_{p_i^k}) = p_i^{s_k} P(\gamma, \eta_{p_i^{k-1}}),$$

where $s_k = 0, 1$. Thus, we see that for

$$t_k = \sum_{m=2}^{k} s_m$$

that

$$P(\gamma, \eta_{p_i^k}) = p_i^{t_k} P(\gamma, \eta_{p_i}),$$

where $P(\gamma, \eta_{p_i}) \mid p_i + 1$. We require the following lemma.

Lemma 4.7. *If $\tau \in \mathrm{PSL}(2,\mathbf{Z})$ satisfies $r_{p_i^k}(\tau) \in \mathbf{B}_L(\mathbf{Z}/p_i^k\mathbf{Z})$ for all sufficiently large $k \in \mathbf{N}$, then $\tau \in \mathbf{B}_L(\mathbf{Z})$.*

Proof of Lemma 4.7. Assume that $\tau \in \mathrm{PSL}(2,\mathbf{Z})$ is such that $r_{p_i^k}(\tau) \in \mathbf{B}_L(\mathbf{Z}/p_i^k\mathbf{Z})$ for all sufficiently large k. Write

$$\tau = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and note that the condition $p_i^k \mid b$ for all sufficiently large k forces $b = 0$. □

As a consequence, we see that $P(\gamma, \eta_{p_i^k}) = j_{i,k}$ is an unbounded sequence, for otherwise, it would eventually be a constant, say j_0, and Lemma 4.7 would force $\gamma^{j_0} \in \mathbf{B}_L(\mathbf{Z})$. However, we already noted that this is impossible, since γ is hyperbolic. Now, since $j_{i,k}$ is unbounded, there exists a subsequence $n_{i,t}$ such that

$$P(\gamma, \eta_{p_i^{n_{i,t}}}) = p_i^t P(\gamma, \eta_{p_i}),$$

where t ranges over $\mathbf{N}$. In particular, we have

$$\left\{P(\gamma, \eta_{p_i}), p_i P(\gamma, \eta_{p_i}), p_i^2 P(\gamma, \eta_{p_i}), p_i^3 P(\gamma, \eta_{p_i}), \dots, \right\} \subset \mathscr{P}(\gamma).$$

This subset of powers is very far from being an arithmetic progression. In order to produce long arithmetic progression, we will need to use additional primes.

An important feature of the phenomena we are studying is that distinct primes behave independently from each other. Specifically, via the Chinese Remainder Theorem, we have for any collection of distinct primes $p_1, \ldots, p_v$ and any collection of powers $r_1, \ldots, r_v$, an isomorphism

$$\mathrm{PSL}\left(2, \mathbf{Z}/\left(\prod_{u=1}^{v} p_u^{r_u}\right)\mathbf{Z}\right) \cong \prod_{u=1}^{v} \mathrm{PSL}(2, \mathbf{Z}/p_u^{r_u}\mathbf{Z})$$

which restricts to an isomorphism between the subgroups

$$\mathbf{B}_L\left(\mathbf{Z}/\left(\prod_{u=1}^{v} p_u^{r_u}\right)\mathbf{Z}\right) \cong \prod_{u=1}^{n} \mathbf{B}_L(\mathbf{Z}/p_u^{r_u}\mathbf{Z}).$$

Thus,

$$P(\gamma, \eta_{p_1^{r_1}\ldots p_v^{r_v}}) = \mathrm{LCM}\left\{P(\gamma, \eta_{p_1^{r_1}}), \ldots, P(\gamma, \eta_{p_v^{r_v}})\right\}.$$

However, since for each prime p_i, the sequence $P(\gamma, \eta_{p_i^k})$ is of the form $p_i^{t_k} P(\gamma, \eta_{p_i})$, we see that

$$P(\gamma, \eta_{p_1^{r_1}\ldots p_v^{r_v}}) = \left(\prod_{u=1}^{v} p_u^{t_{r_u}}\right) \mathrm{LCM}\left\{P(\gamma, \eta_{p_1}), \ldots, P(\gamma, \eta_{p_v})\right\}.$$

Set

$$C_{\gamma, p_1, \ldots, p_v} = \mathrm{LCM}\left\{P(\gamma, \eta_{p_1}), \ldots, P(\gamma, \eta_{p_v})\right\}. \tag{9}$$

This gives us that

$$\left\{C_{\gamma, p_1, \ldots, p_v} p_1^{w_1} \ldots p_u^{w_u}\right\} \subset \mathscr{P}(\gamma),$$

where $w_1, \ldots, w_u$ range independently over all possible non-negative integers. From this fact, it is now a trivial matter to produce arithmetic progressions in $\mathscr{P}(\gamma)$.

Let k be a given integer, and set $p_1, \ldots, p_{u_k}$ to be all the prime divisors of the numbers $\{1, \ldots, k\}$. Using these primes, and setting $C_k := C_{\gamma, p_1, \ldots, p_{u_k}}$, the discussion in the previous paragraph gives us

$$\{C_k, 2C_k, \ldots, kC_k\} \subset \left\{C_k \cdot p_1^{w_1} \ldots p_{u_k}^{w_{u_k}}\right\} \subset \mathscr{P}(\gamma).$$

Now, for each $1 \leq r \leq k$, we have associated to the number $C_k r \in \mathscr{P}(\gamma)$ an element

$$\theta_{\gamma, \eta_r} = \eta_r \gamma^{C_k r} \eta_r^{-1} \in \mathrm{PSL}(2, \mathbf{Z}).$$

The associated geodesic for θ_{γ, η_r} has length

$$\ell(c_{\theta_{\gamma, \eta_r}}) = C_k r \ell(c_\gamma).$$

In particular, as r ranges over $1 \leq r \leq k$, we have a k–term arithmetic progression involving an integral multiple of the length of γ, where each of these lengths arises as the length of some closed geodesic. This completes the first step of our proof. Of course, if we remove the "primitivity" condition and consider the full length spectrum, this statement is trivial: just take powers of γ.

We now move to the second step of the proof – showing that θ_{γ, η_r} is a primitive element, and thus has a corresponding primitive geodesic. This ensures that one can find primitive geodesics whose lengths realize the k–term arithmetic progression produced in the first step of our proof.

To verify that the above collection of elements are primitive, we will use the absolute primitivity assumption on our given element γ. To this end, let $\eta \in \mathrm{PGL}(2, \mathbf{Q})$ and let $j = P(\gamma, \eta)$ with

$$\theta_{\gamma, \eta} = \eta \gamma^j \eta^{-1}.$$

By way of contradiction, assume there exists $\mu \in \mathrm{PSL}(2,\mathbf{Z})$ with $\mu^{j'} = \theta_{\gamma,\eta}$. Diagonalizing via some $D \in \mathrm{PGL}(2,\mathbf{C})$, we see that

$$D\mu^{j'}D^{-1} = D\theta_{\gamma,\eta}D^{-1} = D\eta\gamma^j\eta^{-1}D^{-1}$$

and

$$\begin{pmatrix} \lambda_{\theta_{\gamma,\eta}} & 0 \\ 0 & \lambda_{\theta_{\gamma,\eta}}^{-1} \end{pmatrix} = \begin{pmatrix} \lambda_\mu^{j'} & 0 \\ 0 & \lambda_\mu^{-j'} \end{pmatrix} = \begin{pmatrix} \lambda_\gamma^{j} & 0 \\ 0 & \lambda_\gamma^{-j} \end{pmatrix}.$$

Since γ is absolutely primitive, we know that λ_μ is a power of λ_γ, say L. Thus, we see that

$$D\mu D^{-1} = \begin{pmatrix} \lambda_\mu & 0 \\ 0 & \lambda_\mu^{-1} \end{pmatrix} = \begin{pmatrix} \lambda_\gamma^{L} & 0 \\ 0 & \lambda_\gamma^{-L} \end{pmatrix} = D\eta\gamma^L\eta^{-1}D^{-1}.$$

Consequently, we have

$$\eta\gamma^L\eta^{-1} = \mu \in \mathrm{PSL}(2,\mathbf{Z}).$$

Since j is the smallest power of γ whose η–conjugate lands in $\mathrm{PSL}(2,\mathbf{Z})$, this tells us that $L \geq j$. On the other hand, the fact that $\mu^{j'} = \theta_{\gamma,\eta}$ immediately tells us that $j'L = j$, which gives us $L \leq j$ (as these are non-negative integers). Combining these inequalities we get $L = j$, and hence $j' = 1$. Thus, $\theta_{\gamma,\eta}$ is primitive, as desired. The proof of Theorem 4.6 is now complete. □

Since every noncompact arithmetic hyperbolic 2–orbifold is commensurable with the modular curve (see [17]), our work above in tandem with Proposition 4.1 yields:

Corollary 4.8. *If M is a noncompact, arithmetic hyperbolic 2–orbifold, then $\mathscr{L}_p(M)$ contains arithmetic progressions.*

Remark. The following gives an estimate for the constant $C_{\gamma,k}$ from our Theorem 4.6. The constant $C_{\gamma,k}$ is given by (9), where the primes p_i are all the possible prime divisors of $\{1,\dots,k\}$. Since $P(\gamma,\eta_{p_i})$ divides p_i+1, we see that

$$\begin{aligned} C_{\gamma,k} &= \mathrm{LCM}\{P(\gamma,p) \ : \ p \text{ is prime}, \ p \leq k\} \\ &\leq \mathrm{LCM}\{p+1 \ : \ p \text{ is prime}, \ p \leq k\} \end{aligned}$$

As the map

$$\mathrm{PSL}(2,\mathbf{Z}) \longrightarrow \prod_{\substack{2<p\leq k,\\ p \text{ prime}}} \mathrm{PSL}(2,\mathbf{Z}/p\mathbf{Z})$$

is onto, there exists an element $\gamma \in \mathrm{PSL}(2,\mathbf{Z})$ with

$$C_{\gamma,k} = \mathrm{LCM}\{p+1 \ : \ p \text{ is prime}, \ p \leq k\}.$$

The elements τ in $\mathrm{PSL}(2,\mathbf{Z}/p\mathbf{Z})$ that have τ^{p+1} as the smallest power that resides in $\mathbf{B}(\mathbf{Z}/p\mathbf{Z})$ are constructed from generators of $\mathbf{F}_{p^2}^\times$, which is a cyclic group of order p^2-1. These elements, after diagonalizing in $\mathrm{PSL}(2,\mathbf{F}_{p^2})$, are of the form

$$\tau = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix},$$

where $a \in \mathbf{F}_{p^2}$ is a generator of $\mathbf{F}_{p^2}^\times$. Though a typical element will have a much smaller constant $C_{\gamma,k}$, the above observation shows that our estimate for the constant is sharp. We can give a further (crude) upper bound on the LCM via

$$C_{\gamma,k} \leq \mathrm{LCM}\{p+1 \ : \ p \text{ is prime}, \ p \leq k\} < 6\left(\prod_{\substack{2<p\leq k,\\ p \text{ prime}}} \frac{p+1}{2}\right).$$

4.3 Generalizations and improvements

With a slight modification of the above construction, we can prove a slightly better result than Theorem 4.5. Namely, we have the following corollary.

Corollary 4.9. *Let ℓ be a primitive geodesic length on X, where X is the modular curve. Then for any integer $k \in \mathbf{N}$, there exists a constant $C_{\ell,k} \in \mathbf{N}$ such that the set*

$$\left\{C_{\ell,k} n\ell\right\}_{n=1}^{k} \subset \mathscr{L}_p(X).$$

Proof. Let $\ell' = \ell/D_\ell$ be the length of the associated absolutely primitive geodesic for the primitive length ℓ. Set

$$S = \{D_\ell, 2D_\ell, \dots, kD_\ell\}$$

and let $\mathscr{P}_S$ be the set of distinct prime factors for the elements of S. Using our construction above, we can find a constant $C_{\ell',S} \in \mathbf{N}$ such that

$$\left\{C_{\ell',S} D_\ell n\ell'\right\}_{n=1}^{k} \subset \mathscr{L}_p(X).$$

For that, note that we can simply replace S with the larger set

$$\{1, \dots, kD_\ell\}$$

and then run our construction to produce the desired progression using the length ℓ' as in the proof of Theorem 4.6. Returning to the proof of the corollary, we see that

$$C_{\ell',S} D_\ell n\ell' = C_{\ell',S} n\ell$$

and so

$$\left\{C_{\ell',S} n\ell\right\} \subset \mathscr{L}_p(X).$$

□

We say that a primitive length $\ell \in \mathscr{L}_p(M,g)$ **occurs in arithmetic progressions**, if for any k, there exists an integer k–term arithmetic progression $\{a+bs\}_{s=1}^{k} \subset \mathbf{N}$ such that

$$\{\ell(a+bs)\}_{s=1}^{k} \subset \mathscr{L}_p(M,g).$$

Corollary 4.9 shows that every primitive length for the modular curve occurs in arithmetic progressions. Our final result of this subsection shows that all non-compact arithmetic hyperbolic 2–orbifolds also have this property.

Corollary 4.10. *Let M be a noncompact, arithmetic hyperbolic 2–orbifold. Then every primitive length occurs in arithmetic progressions.*

Proof. For a non-compact, arithmetic hyperbolic 2–orbifold M, we know that there is a finite cover Y of M that is also a finite cover of the modular curve X, since M is commensurable with X. For each primitive length $\ell \in \mathscr{L}_p(M)$ and for each $k \in \mathbf{N}$, we must provide

$$\{\ell(a+bs)\}_{s=1}^{k} \subset \mathscr{L}_p(M)$$

with $a, b \in \mathbf{N}$. To that end, we will make two coloring arguments in the spirit of Proposition 4.1. Set d_M, d_X to be the degree of the covers $Y \to M, X$, respectively and for any natural number s, let $\tau(s)$ be the number of positive divisors of s (e.g. $\tau(p) = 2$ if p is a prime). Set

$$D = \left(\prod_{1 \leq d \leq d_M} d\right)\left(\prod_{1 \leq d \leq d_X} d\right).$$

By Van der Waerden's theorem, there is an integer N_1 with the property that any $\tau(d_M)$ coloring of the set $\{1,\dots,N_1\}$ contains a monochromatic k–term arithmetic progression, and there is an integer N_2 such that any $\tau(d_X)$ coloring of the set $\{1,\dots,N_2\}$ contains a monochromatic N_1–term arithmetic progression.

Fix a closed lift to Y of the geodesic associated to ℓ, which gives us a primitive geodesic in Y of length $j\ell$ for some divisor j of d_M. This will descend to a (cover of a) primitive geodesic on X of length $(j/i)\ell$ where i is a divisor of d_X; the reader should look back at the initial discussion on lifts and projections of geodesics given in the proof of Proposition 4.1. Since $\ell' = (j/i)\ell$ is the length of a primitive geodesic in X, Corollary 4.9 tells us there is a constant $C := C_{\ell',DN_2} \in \mathbf{N}$ such that

$$\left\{CDn\ell'\right\}_{n=1}^{N_2} \subset \left\{Cn\ell'\right\}_{n=1}^{DN_2} \subset \mathscr{L}_p(X).$$

For each integer $1 \le n \le N_2$, we take a primitive geodesic in X of length $CDn\ell'$, and look at a lift in Y. The length of this lift will be of length $i_n \cdot CDn\ell'$, for some divisor i_n of d_X, and we can color each integer n in the set $\{1,\dots,N_2\}$ by the corresponding i_n. This gives a coloring of the set $\{1,\dots,N_2\}$ by $\tau(d_X)$ colors, so from Van der Waerden's theorem, we can now extract a monochromatic N_1–term subsequence $\{a'+b'r\}_{r=1}^{N_1} \subset \{1,\dots,N_2\}$, corresponding to some fixed color i_0. Notice that this gives us a sequence of N_1 primitive geodesics in Y, whose lengths are $\{(CDi_0)(a'+b'r)\ell'\}_{r=1}^{N_1}$. Now for each r, the corresponding primitive geodesic in Y projects back down to a (cover of a) primitive geodesic in M of length $\big((CDi_0)(a'+b'r)\ell'\big)/j_r$ for some divisor j_r of d_M. So we can color the set of indices $\{1,\dots,N_1\}$ by the corresponding divisor j_r, giving us a coloring with $\tau(d_M)$ colors. Again, from Van der Waerden's theorem, we can conclude that there exists a k–term monochromatic subsequence $\{a''+b''s\}_{s=1}^{k}$ of indices, corresponding to some fixed color j_0.

Looking at the corresponding primitive geodesics in M, we see that they have lengths given in terms of s by the equation:

$$\left(\frac{CDi_0}{j_0}\right)\big(a'+b'(a''+b''s)\big)\ell'$$

Since $\ell' = (j/i)\ell$, we can substitute in and simplify the expression to obtain:

$$\left\{\left(\frac{CDi_0 j}{j_0 i}\right)\big((a'+b'a'')+b'b''s)\big)\ell\right\}_{s=1}^{k} \subset \mathscr{L}_p(M).$$

Notice that all the constants appearing in the above expression are integers, and that moreover, the product $j_0 i$ is a divisor of D. So defining the integers

$$a = \left(\frac{CDi_0 j}{j_0 i}\right)(a'+b'a''), \quad b = \left(\frac{CDi_0 j}{j_0 i}\right)(b'b''),$$

we obtain the desired k–term arithmetic progression $\{\ell(a+bs)\}_{s=1}^{k} \subset \mathscr{L}_p(M)$, completing the proof. □

4.4 More examples

The method employed for $\mathrm{PSL}(2,\mathbf{Z})$ extends to $\mathrm{PSL}(n,\mathbf{Z})$. One instead takes diagonal matrices

$$\eta_{j,p_i^k} = \mathrm{diag}(1,\dots,1,p_i^k,1,\dots,1),$$

where we place p_i^k at the (j,j)–diagonal coefficient. The construction is essentially identical except now the role of the Borel subgroup $\mathbf{B}_L$ is played by various maximal, proper, parabolic subgroups. For instance, in

$\mathrm{PSL}(3,\mathbf{Z})$, we see that

$$\begin{pmatrix} p_i & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} \begin{pmatrix} p_i^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a_{1,1} & p_i a_{1,2} & p_i a_{1,3} \\ p_i^{-1} a_{2,1} & a_{2,2} & a_{2,3} \\ p_i^{-1} a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & p_i & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & p_i^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a_{1,1} & p_i^{-1} a_{1,2} & a_{1,3} \\ p_i a_{2,1} & a_{2,2} & p_i a_{2,3} \\ a_{3,1} & p_i^{-1} a_{3,2} & a_{3,3} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & p_i \end{pmatrix} \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & p_i^{-1} \end{pmatrix} = \begin{pmatrix} a_{1,1} & a_{1,2} & p_i^{-1} a_{1,3} \\ a_{2,1} & a_{2,2} & p_i^{-1} a_{2,3} \\ p_i a_{3,1} & p_i a_{3,2} & a_{3,3} \end{pmatrix}.$$

The associated subgroups modulo p_i that play the role of $\mathbf{B}_L(\mathbf{F}_{p_i})$ are the parabolic subgroups

$$\mathbf{P}_1 = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \right\}$$

$$\mathbf{P}_2 = \left\{ \begin{pmatrix} * & 0 & * \\ * & * & * \\ * & 0 & * \end{pmatrix} \right\}$$

$$\mathbf{P}_3 = \left\{ \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ * & * & * \end{pmatrix} \right\}.$$

If we instead conjugate by the inverses, we get:

$$\begin{pmatrix} p_i^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} \begin{pmatrix} p_i & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a_{1,1} & p_i^{-1} a_{1,2} & p_i^{-1} a_{1,3} \\ p_i a_{2,1} & a_{2,2} & a_{2,3} \\ p_i a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & p_i^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & p_i & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a_{1,1} & p_i a_{1,2} & a_{1,3} \\ p_i^{-1} a_{2,1} & a_{2,2} & p_i^{-1} a_{2,3} \\ a_{3,1} & p_i a_{3,2} & a_{3,3} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & p_i^{-1} \end{pmatrix} \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & p_i \end{pmatrix} = \begin{pmatrix} a_{1,1} & a_{1,2} & p_i a_{1,3} \\ a_{2,1} & a_{2,2} & p_i a_{2,3} \\ p_i^{-1} a_{3,1} & p_i^{-1} a_{3,2} & a_{3,3} \end{pmatrix}.$$

The associated parabolic subgroups are given by:

$$\mathbf{P}_1' = \left\{ \begin{pmatrix} * & 0 & 0 \\ * & * & * \\ * & * & * \end{pmatrix} \right\}$$

$$\mathbf{P}_2' = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & 0 \\ * & * & * \end{pmatrix} \right\}$$

$$\mathbf{P}_3' = \left\{ \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \right\}.$$

Specifically, for η_{1,p_i}, for instance, we see that $P(\gamma, \eta_{1,p_i})$ is the smallest integer j such that $r_{p_i}(\gamma^j) \in \mathbf{P}_1(\mathbf{F}_{p_i})$. For any infinite order element, one of the six options will work since

$$\bigcap_{j=1}^{3} \mathbf{P}_j(\mathbf{Z}) \cap \bigcap_{j=1}^{3} \mathbf{P}_j'(\mathbf{Z}) = \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix}.$$

For elements not virtually in one of the Borel subgroups

$$\mathbf{B}_U(\mathbf{Z}) = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \right\}$$

or

$$\mathbf{B}_L(\mathbf{Z}) = \left\{ \begin{pmatrix} * & 0 & 0 \\ * & * & 0 \\ * & * & * \end{pmatrix} \right\},$$

we can get away with two of the six choices; edges meet at vertices. For example, for $\mathbf{B}_U$, we can use $\mathbf{P}_3'$ and $\mathbf{P}_1$. Note that elements in $\mathrm{PSL}(n,\mathbf{Z})$ that are conjugate into one of the Borel subgroups do not correspond to geodesics in the associated arithmetic Riemannian orbifold.

Ensuring the constructed elements $\theta_{\gamma,\eta}$ are primitive uses an identical argument as well. In this case, the eigenvalues of a hyperbolic element will be units in the splitting field of the characteristic polynomial of γ. We require that the eigenvalues be primitive again in $\mathscr{O}^1_{K_\gamma}$, which is now a Galois extension of degree at most $n!$. Overall, this yields the same pair of results (here X_n is the associated arithmetic Riemannian orbifold for $\mathrm{PSL}(n,\mathbf{Z})$):

Theorem 4.11. *Let γ be a primitive hyperbolic element in $\mathrm{PSL}(n,\mathbf{Z})$ with associated geodesic length $\ell = \ell(c_\gamma)$. Then for each $k \in \mathbf{N}$, there exists an arithmetic progression $\{C_{\gamma,k}\ell m\}_{m=1}^k \subset \mathscr{L}_p(X_n)$ where $C_{\gamma,k} \in \mathbf{Q}$. Moreover, there exists $D_\gamma \in \mathbf{N}$ such that $C_{\gamma,k}D_\gamma \in \mathbf{N}$ for all k.*

We call an element absolutely primitive if one of the eigenvalues is primitive in the group of units in the splitting field of the associated characteristic polynomial.

Theorem 4.12. *Let γ be an absolutely primitive element of $\mathrm{PSL}(n,\mathbf{Z})$ with associated geodesic length $\ell = \ell(c_\gamma)$. Then for each $k \in \mathbf{N}$, there exists an arithmetic progression $\{C_{\gamma,k}\ell m\}_{m=1}^k \subset \mathscr{L}_p(X_n)$ where $C_{\gamma,k} \in \mathbf{N}$.*

Corollary 4.9 and Corollary 4.10 can also be extended to this setting though we have opted to not explicitly state them here.

4.5 Larger field examples

For a number field $K/\mathbf{Q}$ we can consider the groups $\mathrm{PSL}(2,\mathscr{O}_K)$. These are the orbifold fundamental groups of the associated **Hilbert–Blumenthal modular varieties**. If K has r_1 real places and r_2 complex places, up to conjugation, then

$$X_K = ((\mathbf{H}^2)^{r_1} \times (\mathbf{H}^3)^{r_2})/\mathrm{PSL}(2,\mathscr{O}_K).$$

When K is a real quadratic field, these orbifolds are called **Hilbert modular surfaces**. When K is an imaginary quadratic field, the groups $\mathrm{PSL}(2,\mathscr{O}_K)$ are called **Bianchi groups** and the associated orbifolds are noncompact arithmetic hyperbolic 3–orbifolds.

Let us now focus on the Bianchi groups, and observe that when K is an imaginary quadratic field, $\mathbf{Z}$ always lies within the ring of integers $\mathscr{O}_K$. This induces an embedding of the modular surface $\mathrm{PSL}(2,\mathbf{Z})$ into the corresponding Bianchi group $\mathrm{PSL}(2,\mathscr{O}_K)$, which extends to a Lie group embedding $\mathrm{PSL}(2,\mathbf{R})$ into $\mathrm{PSL}(2,\mathbf{C})$. As a consequence, we see that the non-compact arithmetic hyperbolic 3–orbifolds associated to the Bianchi groups always contain an embedded copy of the modular curve X. We now state the obvious:

Lemma 4.13. *Assume that M,N are a pair of negatively curved orbifolds, and that $N \hookrightarrow M$ is a locally isometric orbifold embedding. Then we have an inclusion $\mathscr{L}_p(N) \hookrightarrow \mathscr{L}_p(M)$.*

Since the modular curve X has arithmetic progressions, we immediately obtain:

Corollary 4.14. *Let $X_K = \mathbf{H}^3/\mathrm{PSL}(2,\mathscr{O}_K)$ be the non-compact arithmetic hyperbolic 3–orbifold associated to a Bianchi group $\mathrm{PSL}(2,\mathscr{O}_K)$. Then $\mathscr{L}_p(X_K)$ has arithmetic progressions.*

Since every noncompact arithmetic hyperbolic 3–orbifold is commensurable with one of the manifolds associated to a Bianchi group (see [17]), via Proposition 4.1, we obtain:

Corollary 4.15. *If M is any noncompact, arithmetic hyperbolic 3–orbifold, then $\mathscr{L}_p(M)$ has arithmetic progressions.*

Remark. We expect the stronger analogue of Corollary 4.10 to hold for non-compact arithmetic hyperbolic 3–orbifolds, namely every primitive length occurs in arithmetic progressions. Indeed, in the Bianchi group case, we again have a function

$$P\colon\ \mathrm{PSL}(2,\mathscr{O}_K)\times\mathrm{PGL}(2,K)\longrightarrow\mathbf{N}$$

given by

$$P(\gamma,\eta)=\min\left\{j\in\mathbf{N}\ :\ \eta\gamma^j\eta^{-1}\in\mathrm{PSL}(2,\mathscr{O}_K)\right\}.$$

The general methods used for $\mathrm{PSL}(n,\mathbf{Z})$ can then be used in this setting to prove the strong form that every primitive length arises in arbitrarily long arithmetic progressions. Important here is that we still have the exact sequence (8) and that the kernel is a p–group. To be explicit, taking a prime ideal $\mathfrak{p}$ in $\mathscr{O}_K$, we still have a sequence

$$1\longrightarrow V_{\mathfrak{p}}\longrightarrow\mathrm{PSL}(2,\mathscr{O}_K/\mathfrak{p}^{j+1})\longrightarrow\mathrm{PSL}(2,\mathscr{O}_K/\mathfrak{p}^{j})\longrightarrow 1$$

where $V_{\mathfrak{p}}$ is a 3–dimensional $(\mathscr{O}_K/\mathfrak{p})$–vector space. We can also conjugate by elements of the form

$$\eta_\alpha=\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}$$

for $\alpha\in K^\times$. In fact, this works for any number field K and also in $\mathrm{PSL}(n,\mathscr{O}_K)$. We leave this task of generalization to the interested reader.

The general approach that we take for producing arithmetic progressions in arithmetic manifolds can be implemented in great generality. The large commensurator for such manifolds is the driving force. For the general case, it is better to work in the framework of Galois cohomology. Our desire to avoid the use of Galois cohomology is the sole reason we have focused on the limited examples explored here, for which a substantially simpler and somewhat elementary approach is sufficient. The general case will be taken up in a future paper.

5 Final remarks

We conclude this article with some final remarks, questions, and conjectures.

5.1 Conjectural characterization of arithmeticity

In this article, we have shown that for negatively curved metrics, despite the fact that almost arithmetic progressions are abundant, genuine arithmetic progressions are rare. We have provided several examples of arithmetic negatively curved (and non-positively curved) manifolds which have arithmetic progressions. It is tempting to conjecture that *all* arithmetic manifolds have arithmetic progressions. In fact, we have little doubt that this holds. It is tempting to conjecture that the presence of arithmetic progressions in the primitive length spectrum can be used to characterize arithmetic manifolds. One should be a bit careful, as we have the following easy consequence of Lemma 4.13 and Theorem 4.11:

Corollary 5.1. *Let M be a non-positively curved manifold, and assume that M contains a totally geodesic submanifold commensurable to some locally symmetric space $\mathrm{PSL}(n,\mathbf{Z})$. Then $\mathscr{L}_p(M)$ has arithmetic progressions.*

Using Corollary 5.1, one can easily produce examples of non-arithmetic, negatively curved manifolds whose length spectrum has arithmetic progressions. Start with a high-dimensional hyperbolic manifold M which contains a non-compact arithmetic hyperbolic surface as a totally geodesic submanifold N; every non-compact, arithmetic hyperbolic n–manifold has such a surface (see, for instance, Theorem 5.1 in [19] for a description of the non-compact arithmetic lattices in $\mathrm{Isom}(\mathbf{H}^n)$). Pick an arbitrary point $p \in M \setminus N$, and slightly perturb the metric in a small enough neighborhood of p. If the perturbation is small enough, the resulting Riemannian manifold (M,g) will still be negatively curved, though no longer hyperbolic. Since the perturbation is performed away from the submanifold N, the latter will still be totally geodesic inside (M,g). So Corollary 5.1 ensures that the resulting $\mathscr{L}_p(M,g)$ has arithmetic progressions, even though (M,g) is not arithmetic (in fact, not even locally symmetric). One simple result of this discussion is the following:

Corollary 5.2. *The set of metrics whose primitive length spectrum have arithmetic progressions need not be discrete.*

Note that the non-arithmetic examples of Gromov–Piatetski-Shapiro [11] are built by gluing together two arithmetic manifolds along a common totally geodesic hypersurface. Being arithmetic, we would expect this hypersurface to contain arithmetic progressions, and from our Lemma 4.13, the hybrid non-arithmetic manifold would then also have arithmetic progressions. Reid [25, Theorem 3] constructed infinitely many commensurability classes of non-arithmetic hyperbolic 3–manifolds with a totally geodesic surface. The surface is a non-compact, arithmetic surface and so contains arithmetic progressions. By Corollary 5.1, these non-arithmetic hyperbolic 3–manifolds have arithmetic progressions. The commensurability classes are commensurability classes of hyperbolic knot complements in S^3

However, recall that our constructions actually show that the arithmetic manifolds we consider satisfy a much stronger condition than just having arithmetic progressions. Namely, *every primitive geodesic length occurs in arithmetic progressions*. The hybrid manifolds of Gromov–Piatetski-Shapiro are unlikely to satisfy this much stronger condition, as a generic primitive geodesic is unlikely to reside on an arithmetic submanifold. In particular, it is unclear where one might find infinitely many primitive geodesics that have the same length (up to rational multiples) as our given primitive geodesic. More generally, we do not expect non-arithmetic hyperbolic manifolds to contain many (or even any) totally geodesic submanifolds. In addition, the deformation argument that we mentioned above (that proves that the set of metrics containing arithmetic progressions need not be discrete) cannot simply be employed with the stronger condition since we know any neighborhood about any point will always intersect some closed geodesics.

Conjecture A. *Let (M,g) be a closed or finite volume, complete Riemannian manifold. If $\mathscr{L}_p(M,g)$ has every primitive length occurring in arithmetic progressions (in the sense of Section 4.3), then (M,g) is arithmetic.*

A much weaker version of Conjecture A, where we restrict the topological type of the manifold M, would already be of considerable interest:

Conjecture B. *Let M be a closed manifold that admits a locally symmetric metric, and assume that the universal cover of M has no compact factors and M is irreducible. Given a metric (M,g) on M, assume that $\mathscr{L}_p(M,g)$ has every primitive length occurring in arithmetic progressions (in the sense of Section 4.3). Then g is a locally symmetric metric, and is arithmetic.*

At present, it is still an open problem as to whether higher rank, locally symmetric manifolds (M, g_{sym}) are determined in the space of Riemannian metrics by their primitive length spectrum. The local version of this type of rigidity is often referred to as **spectral isolation**. The spectral isolation of symmetric or locally symmetric metrics seems to be a folklore conjecture that has been around for some time; see [9] for some recent work and history on this problem. Conjecture B implies the stronger global spectral rigidity conjecture immediately for locally symmetric metrics; one might say the locally symmetric metric is **spectrally isolated globally** in that case.

Our last conjecture is weaker than Conjecture A and B.

Conjecture C. *Let M be a closed manifold that admits a negatively curved metric and let $\mathscr{M}(M)$ denote the space of negatively curved metrics with the Lipschitz topology. Consider the metrics with the property that*

$\mathcal{L}_p(M,g)$ has every primitive length occurring in arithmetic progressions (in the sense of Section 4.3). Then the set of such metrics forms a discrete (or even better, finite) subset of $\mathcal{M}(M)$.

We do not know whether Conjecture C holds when M is a closed surface of genus at least two. Higher genus closed surfaces are an obvious test case for this conjecture.

5.2 Other proposed characterizations of arithmeticity

Sarnak [29] proposed a characterization for arithmetic surfaces that is also of a geometric nature. For a Fuchsian group $\Gamma < \mathrm{PSL}(2,\mathbf{R})$, set

$$\mathrm{Tr}(\Gamma) = \{|\mathrm{Tr}(\gamma)| \ : \ \gamma \in \Gamma\}.$$

A Fuchsian group satisfies the **bounded clustering property** if there exists a constant C_Γ such that, for all integers n, we have

$$|\mathrm{Tr}(\Gamma) \cap [n,n+1]| < C_\Gamma.$$

It was verified by Luo–Sarnak [15] that arithmetic surfaces satisfy the bounded clustering property. Schmutz [30] proposed a characterization of arithmeticity based on the function

$$F(x) = |\mathrm{Tr}(\Gamma) \cap [0,x]|.$$

Specifically, Γ is arithmetic if and only if $F(x)$ grows at most linearly in x. Geninska–Leuzinger [8] verified Sarnak's conjecture in the case where Γ contains a non-trivial parabolic isometry. In [8], they also point out a gap in [30] that verified the linear growth characterization for lattices with a non-trivial parabolic isometry. At present, this verification seems to still be open.

These characterizations of arithmeticity are based on the fact that arithmetic manifolds have unusually high multiplicities in the primitive geodesic length spectrum, a phenomenon first observed by Selberg. One explanation for the high multiplicities can be seen from our proof that arithmetic, noncompact surfaces have arithmetic progressions. Specifically, from one primitive length ℓ, via the commensurator, we can produce infinitely many primitive lengths of the form $\left(\frac{m}{d}\right)\ell$, where m ranges over an infinite set of integers and d ranges over a finite set of integers. When ℓ is the associated length of an absolutely primitive element, we obtain lengths of the form $m\ell$ as m ranges over an infinite set of integers. Given the freedom on the production of these lengths, it is impossible to imagine that huge multiplicities will not arise.

Other characterizations of arithmeticity given by Cooper–Long–Reid [3] (see also Reid [27]) and Farb–Weinberger [7] exploit the abundant presence of symmetries, and thus are still in the realm of Margulis' characterization via commensurators. Reid [26], Chinburg–Hamilton–Long–Reid [2], and Prasad–Rapinchuk [23] also recover arithmeticity using spectral invariants, and so we feel our proposed characterization sits somewhere between the commensurator and spectral sides.

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