Solutions to homework on systems of linear differential equations.

1A) Here's a walkthrough of the first method for solving each of the five given systems of linear differential equations:

- Solving the first equation for v, get $v = u' - u$. Substituting into the second equation, get $(u' - u)' = u - 3(u' - u)$, which gives the second order differential equation $u'' + 2u' - 4u = 0$. From the characteristic polynomial, there are two distinct solutions $u_1 = e^{(-1+\sqrt{5})t}$ and $u_2 = e^{(-1-\sqrt{5})t}$. Substituting these into the equation $v = u' - u$, you get the corresponding $v_1 = (-2+\sqrt{5})e^{(-1+\sqrt{5})t}$ and $v_2 = (-2-\sqrt{5})e^{(-1-\sqrt{5})t}$.
- Solving the first equation for v, get $v = 2u - u'$. Substituting into the second equation, get $(2u - u')' = -2u + 3(2u - u')$, which gives the second order differential equation $u'' - 5u' + 4u = 0$. From the characteristic polynomial, there are two distinct solutions $u_1 = e^t$ and $u_2 = e^{4t}$. Substituting these into the equation $v = 2u - u'$, you get the corresponding $v_1 = e^t$ and $v_2 = -2e^{4t}$.
- Solving the first equation for v, get $v = u' + u$. Substituting into the second equation, get $(u'+u)' = u - (u'+u)$, which gives the second order differential equation $u'' + 2u' = 0$. From the characteristic polynomial, there are two distinct solutions $u_1 = e^{-2t}$ and $u_2 = 1$ (since 0 is a root). Substituting these into the equation $v = u' + u$, you get the corresponding $v_1 = -e^{-2t}$ and $v_2 = 1$.
- Solving the first equation for v, get $v = \frac{1}{2}u' + 2u$. Substituting into the second equation, get $(\frac{1}{2}u' + 2u)' = u - 3(\frac{1}{2}u' + 2u)$, which gives (after clearing denominators) the second order differential equation $u'' + 7u' + 10u = 0$. From the characteristic polynomial, there are two distinct solutions $u_1 = e^{-2t}$ and $u_2 = e^{-5t}$. Substituting these into the equation $v = \frac{1}{2}u' + 2u$, you get the corresponding $v_1 = e^{-2t}$ and $v_2 = -\frac{1}{2}e^{-5t}$.
- Solving the first equation for v, get $v = \frac{1}{3}u' - \frac{5}{3}u$. Substituting into the second equation, get $(\frac{1}{3}u' - \frac{5}{3}u)' = 2u + 4(\frac{1}{3}u' - \frac{5}{3}u)$, which gives (after clearing denominators) the second order differential equation $u'' - 9u' + 14u = 0$. From the characteristic polynomial, there are two distinct solutions $u_1 = e^{2t}$ and $u_2 = e^{7t}$. Substituting these into the equation $v = \frac{1}{3}u' - \frac{5}{3}u$, you get the corresponding $v_1 = -e^{2t}$ and $v_2 = \frac{2}{3}e^{7t}$.

1B) Here's a walkthrough of the second method for solving each of the five given systems of linear differential equations:

- The matrix in this problem is $\begin{bmatrix} 1 & 1 \\ 1 & -3 \end{bmatrix}$. The eigenvalues are given by the roots of the determinant of the matrix $\begin{bmatrix} x-1 & -1 \\ -1 & x+3 \end{bmatrix}$, which is the polynomial $(x-1)(x+3) - 1 = x^2 + 2x - 4$. This gives us the two eigenvalues: $\lambda_1 = -1 + \sqrt{5}$ and $\lambda_2 = -1 - \sqrt{5}$. The corresponding eigenvectors are $\begin{bmatrix} 1 \\ -2+\sqrt{5} \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -2-\sqrt{5} \end{bmatrix}$ respectively, giving the same solutions as the first method.
- The matrix in this problem is $\begin{bmatrix} 2 & -1 \\ -2 & 3 \end{bmatrix}$. The eigenvalues are given by the roots of the determinant of the matrix $\begin{bmatrix} x-2 & 1 \\ 2 & x-3 \end{bmatrix}$, which is the polynomial $(x-2)(x-3) - 2 = x^2 - 5x + 4$. This gives us the two

eigenvalues: $\lambda_1 = 1$ and $\lambda_2 = 4$. The corresponding eigenvectors are $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ respectively, giving the same solutions as the first method.

- The matrix in this problem is $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$. The eigenvalues are given by the roots of the determinant of the matrix $\begin{bmatrix} x+1 & -1 \\ -1 & x+1 \end{bmatrix}$, which is the polynomial $(x+1)(x+1) - 1 = x^2 + 2x$. This gives us the two eigenvalues: $\lambda_1 = 0$ and $\lambda_2 = -2$. The corresponding eigenvectors are $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ respectively, giving the same solutions as the first method.
- The matrix in this problem is $\begin{bmatrix} -4 & 2 \\ 1 & -3 \end{bmatrix}$. The eigenvalues are given by the roots of the determinant of the matrix $\begin{bmatrix} x+4 & -2 \\ -1 & x+3 \end{bmatrix}$, which is the polynomial $(x+4)(x+3) - 2 = x^2 + 7x + 10$. This gives us the two eigenvalues: $\lambda_1 = -2$ and $\lambda_2 = -5$. The corresponding eigenvectors are $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1/2 \end{bmatrix}$ respectively, giving the same solutions as the first method.
- The matrix in this problem is $\begin{bmatrix} 5 & 3 \\ 2 & 4 \end{bmatrix}$. The eigenvalues are given by the roots of the determinant of the matrix $\begin{bmatrix} x-5 & -3 \\ -2 & x-4 \end{bmatrix}$, which is the polynomial $(x-5)(x-4) - 6 = x^2 - 9x + 14$. This gives us the two eigenvalues: $\lambda_1 = 2$ and $\lambda_2 = 7$. Since this matrix shows up again in problem 4, let me give some details on how to find the corresponding eigenvectors.

  For the eigenvalue $\lambda_1 = 2$, we want to find solutions A, B to the matrix equation:

  $$2\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 5A+3B \\ 2A+4B \end{bmatrix}$$

  Looking at the first entry, we see that we must have $2A = 5A + 3B$, giving us that $A = -B$. Note that the second entry would give you the exact same relationship between A and B. So for an eigenvector, we can take $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. This gives the pair of solutions $u_1 = e^{2t}$, $v_1 = -e^{2t}$.

  For the eigenvalue $\lambda_1 = 7$, we want to find solutions A, B to the matrix equation:

  $$7\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 5A+3B \\ 2A+4B \end{bmatrix}$$

  Looking at the first entry, we see that we must have $7A = 5A + 3B$, giving us that $2A = 3B$. Note that, again, the second entry would give you the exact same relationship between A and B. So for an eigenvector, we can take $\begin{bmatrix} 1 \\ 2/3 \end{bmatrix}$. This gives the pair of solutions $u_1 = e^{7t}$, $v_1 = \frac{2}{3}e^{7t}$.

  In particular, we recover exactly the same solutions as from the first method.

2) Recall that u_1, v_1 and u_2, v_2 are two distinct pairs of solutions to a system of differential equations of the form: $u' = au + bv$ and $v' = cu + dv$, with a, b, c, d constants. We now verify the two statements:

- We need to check that if C is a constant, then Cu_1 and Cv_1 also form a pair of solutions. To do this, we substitute these functions into the system of differential equations, and we need to check that the resulting two equations hold:

$$(Cu_1)' = a(Cu_1) + b(Cv_1)$$
$$(Cv_1)' = c(Cu_1) + d(Cv_1)$$

But after canceling out a C from both sides of both equations, we see that we are left with checking that the two equations: $u_1' = au_1 + bv_1$ and $v_1' = cu_1 + dv_1$ both hold. Since we started with u_1, v_1 a solution to the system of differential equations, these two equations automatically hold, and we're done.

- We need to check that the sums $u_1 + u_2$ and $v_1 + v_2$ also form a solution to the same system of differential equations. Again, we substitute this into the system of differential equations, and see that we need to check that the resulting two equations hold:

$$(u_1 + u_2)' = a(u_1 + u_2) + b(v_1 + v_2)$$
$$(v_1 + v_2)' = c(u_1 + u_2) + d(v_1 + v_2)$$

But we *know* that the following equations hold: (1) $u_1' = au_1 + bv_1$, $v_1' = cu_1 + dv_1$ (since u_1, v_1 is a solution to the system) and (2) $u_2' = au_2 + bv_2$, $v_2' = cu_2 + dv_2$ (since u_2, v_2 is a solution to the system). Adding the equations for u_1' and u_2' together, we obtain precisely the first equation we needed to check, so we see that the first equation holds. Likewise, adding the equations for v_1' and v_2' together yields the second equation, and we're done.

3) Recall that we are considering a second order differential equation (with constant coefficients a, b, c):

$$ay'' + by' + cy = 0$$

- Using the substitution $u = y$, $v = y'$ we can rewrite the second order differential equation as a linear system of two first order differential equations. We need to write u' and v' as a linear combination of copies of u and v. One of these is easy to do: indeed, since $u = y$, we see that $u' = y' = v$. This gives the *first* equation of the linear system: $u' = v$. For the *second* equation of the linear system, we note that since $v = y'$, we have that $v' = y''$. Now the second order differential equation allows us to write:

$$y'' = \frac{-c}{a}y + \frac{-b}{a}y'$$

(note that we are assuming $a \neq 0$, else the original differential equation wasn't really second order to begin with). Rewriting this in terms of u and v, we see that the original second order differential equation is equivalent to solving the following system of differential equations:

$$u' = v$$
$$v' = \frac{-c}{a}u + \frac{-b}{a}v$$

- The matrix for this system of differential equations is:

$$\begin{bmatrix} 0 & 1 \\ \frac{-c}{a} & \frac{-b}{a} \end{bmatrix}$$

Hence the characteristic polynomial of the matrix is given by the determinant of the matrix:

$$\begin{bmatrix} x & -1 \\ \frac{c}{a} & x + \frac{b}{a} \end{bmatrix}$$

which is the polynomial:

$$x(x + \frac{b}{a}) + \frac{c}{a} = x^2 + \frac{b}{a}x + \frac{c}{a}$$

Note that the characteristic polynomial of the original second order differential equation is just $ax^2 + bx + c$, i.e. the two polynomials differ by multiplication by the constant a. In particular, they have *exactly* the same roots, and hence will give rise to exactly the same two exponential solutions to the differential equation (as they should).

4) We are considering the 2nd order system: $u'' = 5u + 3v$, $v'' = 2u + 4v$.

- Trying for a solution of the form $u = Ae^{rt}$, $v = Be^{rt}$, and substituting into the pair of differential equations, we see (after canceling the common e^{rt} terms) that the following matrix equation must be satisfied:

$$r^2 \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix}$$

  Note that the r^2 term in this equation comes from differentiating the e^{rt} terms *twice*.

- From the discussion in class, this implies that r^2 is an eigenvalue of the matrix $\begin{bmatrix} 5 & 3 \\ 2 & 4 \end{bmatrix}$, with corresponding eigenvector $\begin{bmatrix} A \\ B \end{bmatrix}$ Observe that the matrix that appears in this problem is precisely the one that was considered in the last set of equations in problem 1. In particular, we know that this matrix has eigenvalues $\lambda = 2$ and $\lambda = 7$, with corresponding eigenvectors $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2/3 \end{bmatrix}$ respectively. Since r^2 must be an eigenvalue, we get the following four sets of solutions:

  1. $r = \sqrt{2}$ and corresponding eigenvector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ gives $u_1 = e^{\sqrt{2}t}$, $v_1 = -e^{\sqrt{2}t}$.
  2. $r = -\sqrt{2}$ and corresponding eigenvector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ gives $u_2 = e^{-\sqrt{2}t}$, $v_2 = -e^{-\sqrt{2}t}$.
  3. $r = \sqrt{7}$ and corresponding eigenvector $\begin{bmatrix} 1 \\ 2/3 \end{bmatrix}$ gives $u_3 = e^{\sqrt{7}t}$, $v_3 = \frac{2}{3}e^{\sqrt{7}t}$.
  4. $r = -\sqrt{7}$ and corresponding eigenvector $\begin{bmatrix} 1 \\ 2/3 \end{bmatrix}$ gives $u_4 = e^{-\sqrt{7}t}$, $v_4 = \frac{2}{3}e^{-\sqrt{7}t}$.