

# SPLITTING FORMULAS FOR CERTAIN WALDHAUSEN NIL-GROUPS.

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ABSTRACT. We provide splitting formulas for certain Waldhausen Nil-groups. We focus on Waldhausen Nil-groups associated to *acylindrical* amalgamations  $\Gamma = G_1 *_H G_2$  of groups  $G_1, G_2$  over a common subgroup  $H$ . For these amalgamations, we explain how, provided  $G_1, G_2, \Gamma$  satisfy the Farrell-Jones isomorphism conjecture, the Waldhausen Nil-groups  $Nil_*^W(RH; R[G_1 - H], R[G_2 - H])$  can be expressed as a direct sum of Nil-groups associated to a specific collection of virtually cyclic subgroups of  $\Gamma$ . A special case covered by our theorem is the case of arbitrary amalgamations over a finite group  $H$ .

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## 1. INTRODUCTION

Waldhausen's Nil-groups were introduced in the two foundational papers [W78a], [W78b]. The motivation behind these Nil-groups originated in a desire to have a Mayer-Vietoris type sequence in algebraic  $K$ -theory. More precisely, if a group  $\Gamma = G_1 *_H G_2$  splits as an amalgamation of two groups  $G_1, G_2$  over a common subgroup  $H$ , one can ask how the algebraic  $K$ -theory of the group ring  $R\Gamma$  is related to the algebraic  $K$ -theory of the integral group rings  $RG_1, RG_2, RH$ . Motivated by the corresponding question in homology (or cohomology), one might expect a Mayer-Vietoris type exact sequence:

$$\dots \rightarrow K_{i+1}(R\Gamma) \rightarrow K_i(RH) \rightarrow K_i(RG_1) \oplus K_i(RG_2) \rightarrow K_i(R\Gamma) \rightarrow \dots$$

A major result in [W78a], [W78b] was the realization that the Mayer-Vietoris sequence above holds, provided one inserts suitable "error-terms", which are precisely the Waldhausen Nil-group associated to the amalgamation  $\Gamma = G_1 *_H G_2$ . In general, associated to any ring  $S$  (such as  $RH$ ), and any pair of flat  $S$ -bimodules  $M_1, M_2$  (such as the  $R[G_i - H]$ ), Waldhausen defines Nil-groups  $Nil_*^W(S; M_1, M_2)$ . The Waldhausen Nil-groups  $Nil_*^W(RH; R[G_1 - H], R[G_2 - H])$  are precisely the "error-terms" mentioned above.

Another context in which these Nil-groups make an appearance has to do with the *reduction to finites*. To explain this we recall the existence of a generalized equivariant homology theory, having the property that for any group  $\Gamma$ , one has an isomorphism:

$$H_n^\Gamma(*; \mathbb{K}\mathbb{R}^{-\infty}) \cong K_n(R\Gamma).$$

The term appearing to the right is the homology of the  $\Gamma$ -space consisting of a point  $*$  with the trivial  $\Gamma$ -action. Now for any  $\Gamma$ -space  $X$ , the obvious map  $X \rightarrow *$

is clearly  $\Gamma$ -equivariant, and hence induces an *assembly map* homomorphism:

$$H_n^\Gamma(X; \mathbb{K}R^{-\infty}) \rightarrow H_n^\Gamma(*; \mathbb{K}R^{-\infty}) \cong K_n(R\Gamma).$$

The Farrell-Jones isomorphism conjecture [FJ93] asserts that if  $X = E_{\mathcal{V}C}\Gamma$  is a model for the classifying space for  $\Gamma$ -actions with isotropy in the family of virtually cyclic subgroups, then the homomorphism described above is actually an isomorphism. Explicit models for  $E_{\mathcal{V}C}\Gamma$  are known for only a few classes of groups: virtually cyclic groups, crystallographic groups (Alvez and Ontaneda [AO06] and Connolly, Fehrman, and Hartglass [CFH]), hyperbolic groups (Juan-Pineda and Leary [JL06] and Lück [Lu05]), and relatively hyperbolic groups (Lafont and Ortiz [LO07]). In contrast, classifying spaces for *proper* actions, denoted  $E_{\mathcal{F}IN}\Gamma$  are known for many classes of groups. Now for any group  $\Gamma$ , one always has a unique (up to  $\Gamma$ -equivariant homotopy) map  $E_{\mathcal{F}IN}\Gamma \rightarrow E_{\mathcal{V}C}\Gamma$ , which induces a well-defined relative assembly map:

$$H_n^\Gamma(E_{\mathcal{F}IN}\Gamma; \mathbb{K}R^{-\infty}) \rightarrow H_n^\Gamma(E_{\mathcal{V}C}\Gamma; \mathbb{K}R^{-\infty}).$$

In view of the Farrell-Jones isomorphism conjecture, it is reasonable to ask whether this latter map is itself an isomorphism. Bartels [Ba03] has shown that the relative assembly map is split injective for arbitrary groups  $\Gamma$ , and arbitrary rings  $R$ . If the relative assembly map discussed above is actually an isomorphism, then we say that  $\Gamma$  satisfies the *reduction to finites*.

Let us now specialize to the case where  $R = \mathbb{Z}$ , i.e. we will be focusing on integral group rings. In this situation, the obstruction to the relative assembly map being an isomorphism lies in the Nil groups associated to the various infinite virtually cyclic subgroups of  $\Gamma$ . More precisely, if every infinite virtually cyclic subgroup  $V \leq \Gamma$  satisfies the the reduction to finites, then the entire group  $\Gamma$  satisfies the reduction to finites (see [FJ93, Thm. A.10]). Now for a virtually cyclic group  $V$ , the *failure* of the reduction to finites can be measured by the cokernel of the relative assembly map. Let us recall that infinite virtually cyclic groups  $V$  come in two flavors (see [SW79]):

- groups that surject onto the infinite dihedral group  $D_\infty = \mathbb{Z}_2 * \mathbb{Z}_2$ , and hence can be decomposed  $V = A *_C B$ , with  $A, B, C$  finite, and  $C$  of index two in the groups  $A, B$ .
- groups that do *not* surject onto  $D_\infty$ , which can always be written in the form  $V = F \rtimes_\alpha \mathbb{Z}$ , where  $F$  is a finite group and  $\alpha \in \text{Aut}(F)$ .

In the case where  $V$  surjects onto  $D_\infty$ , the cokernel of the relative assembly map coincides with the Waldhausen Nil-group associated to the splitting  $V = A *_C B$ . In the case where  $V$  does not surject onto  $D_\infty$ , the cokernel of the relative assembly map consists of two copies of the Farrell Nil-group associated to  $V = F \rtimes_\alpha \mathbb{Z}$ , denoted  $NK_*(\mathbb{Z}F, \alpha)$ .

We now have two contexts in which Waldhausen Nil-groups make an appearance: (1) they measure failure of the Mayer-Vietoris sequence in algebraic  $K$ -theory, and (2) they contain obstructions for groups to satisfy the reduction to finites. Having motivated our interest in these groups, we can now state our:

**Main Theorem.** *Let  $\Gamma = G_1 *_H G_2$  be an acylindrical amalgamation, and assume that the Farrell-Jones Isomorphism Conjecture holds for the groups  $\Gamma, G_1, G_2$ . Denote by  $\mathcal{V}$  a collection of subgroups of  $\Gamma$  consisting of one representative from each conjugacy class of subgroups  $V$  satisfying:*

- (1)  $V$  is virtually cyclic,
- (2)  $V$  is not conjugate to a subgroup of  $G_1$  or  $G_2$ , and
- (3)  $V$  is maximal with respect to subgroups satisfying (1) and (2).

Then for arbitrary rings  $R$  we have isomorphisms:

$$\text{Nil}_*^W(RH; R[G_1 - H], R[G_2 - H]) \cong \bigoplus_{V \in \mathcal{V}} H_*^V(E_{\mathcal{F}\mathcal{I}\mathcal{N}}V \rightarrow *; \mathbb{K}R^{-\infty})$$

where  $H_*^V(E_{\mathcal{F}\mathcal{I}\mathcal{N}}V \rightarrow *; \mathbb{K}R^{-\infty})$  are the cokernels of the relative assembly maps associated to the virtually cyclic subgroups  $V \in \mathcal{V}$ .

**Remarks:** (1) An amalgamation  $\Gamma = G_1 *_H G_2$  is said to be *acylindrical* if there exists an integer  $k$  such that, for every path  $\eta$  of length  $k$  in the Bass-Serre tree  $T$  associated to the splitting of  $\Gamma$ , the stabilizer of  $\eta$  is finite. The notion of an acylindrical amalgamation was first formulated by Sela [Sel97] in relation to his work on the accessibility problem for finitely generated groups. We use a generalization of Sela's original definition which is due to Delzant [Del99]. More generally, given a family  $\mathcal{C}$  of subgroups, Delzant calls an amalgamation  $(k, \mathcal{C})$ -acylindrical provided that, for the associated action of  $\Gamma$  on its Bass-Serre tree, the stabilizer of any path of length  $k$  lies in  $\mathcal{C}$ . The version we are using corresponds to the case where  $\mathcal{C} = \mathcal{FIN}$ , the family of finite subgroups of  $\Gamma$ .

Let us now give some examples of acylindrical amalgamations. Observe that if the amalgamating subgroup  $H$  is finite, the amalgamation is automatically acylindrical (with  $k = 1$ ), as every edge will have a finite stabilizer. If the amalgamating subgroup  $H$  has the property that  $|gHg^{-1} \cap H| < \infty$  for every  $g \in G_i - H$  (for either  $i$ ), then the amalgamation is acylindrical (with  $k = 3$ ). In particular, we note that this is satisfied for any acylindrical amalgamation where  $H$  is malnormal in either  $G_i$ . We refer the reader to Remark (8) for a more thorough discussion of this notion, and to Corollary 4 for a concrete application.

(2) The cokernels  $H_*^{V_i}(E_{\mathcal{F}\mathcal{I}\mathcal{N}}V_i \rightarrow *; \mathbb{K}R^{-\infty})$  are the familiar Waldhausen or (two copies of the) Farrell Nil groups, according to whether the virtually cyclic group  $V_i$  surjects onto  $D_\infty$  or not. Note that every virtually cyclic subgroup  $V$  that maps onto  $D_\infty$  contains a canonical index two subgroup  $V'$  which does *not* map onto  $D_\infty$  (the pre-image of the obvious index two subgroup  $\mathbb{Z} \triangleleft D_\infty$ ). Recent independent work by various authors (Davis [D], Davis-Khan-Ranicki [DKR], Quinn, and Reich), has established that the Waldhausen Nil-group of  $V$  is isomorphic to the Farrell Nil-group of  $V'$ .

(3) From the computational viewpoint, the Main Theorem combined with the previous remark completely reduces (modulo the Isomorphism Conjecture) the computation of Waldhausen Nil-groups associated to acylindrical amalgamations to that of Farrell Nil-groups.

(4) Consider the simple case of a *free product*  $\Gamma = G_1 * G_2$ . In this situation, the group  $\Gamma$  is known to be (strongly) relatively hyperbolic, relative to the subgroups  $G_1, G_2$ . Assuming the Farrell-Jones isomorphism conjecture for  $\Gamma$ , previous work of the authors [LO, Cor. 3.3] yields the following expression for  $K_n(R\Gamma)$ :

$$H_n^\Gamma(E_{\mathcal{F}\mathcal{I}\mathcal{N}}\Gamma) \oplus \left( \bigoplus_{i=1,2} H_n^{G_i}(E_{\mathcal{F}\mathcal{I}\mathcal{N}}G_i \rightarrow E_{\mathcal{V}\mathcal{C}}G_i) \right) \oplus \left( \bigoplus_{V \in \mathcal{V}} H_n^V(E_{\mathcal{F}\mathcal{I}\mathcal{N}}V \rightarrow *) \right)$$

where  $\mathcal{V}$  is the collection of virtually cyclic subgroups mentioned in our Main Theorem (we omitted the coefficients  $\mathbb{K}\mathbb{R}^{-\infty}$  to simplify notation). Now, morally speaking, the Waldhausen Nil-group is the portion of the  $K$ -theory of  $\Gamma$  that does *not come from the  $K$ -theory of the factors  $G_i$* . In the expression above, it is clear that the second term is determined by the  $K$ -theory of the factors. Furthermore, recalling the well known fact that every finite subgroup of  $\Gamma$  has to be conjugate into one of the  $G_i$ , one sees that the first term also comes from the  $K$ -theory of the factors. Hence the Waldhausen Nil-group should correspond to the last term in the expression above. Our Main Theorem came about from trying to make this heuristic precise.

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## 2. PROOF OF MAIN THEOREM

Given the group  $\Gamma = G_1 *_H G_2$  satisfying the hypotheses of our theorem, let us form the family  $\mathcal{F}$  of subgroups of  $\Gamma$  consisting of all virtually cyclic subgroups that can be conjugated into either  $G_1 \leq \Gamma$  or  $G_2 \leq \Gamma$ . Observe that we have a containment of families  $\mathcal{F} \subset \mathcal{VC}$  of subgroups of  $\Gamma$ , which in turn induces an assembly map:

$$\rho : H_n^\Gamma(E_{\mathcal{F}}\Gamma; \mathbb{K}\mathbb{R}^{-\infty}) \rightarrow H_n^\Gamma(E_{\mathcal{VC}}\Gamma; \mathbb{K}\mathbb{R}^{-\infty}).$$

Our proof will focus on analyzing the map  $\rho$ , and in particular on gaining an understanding of the cokernel of that map. Let us start by describing the Waldhausen Nil-group as the cokernel of a suitable assembly map. Corresponding to the splitting  $\Gamma = G_1 *_H G_2$ , we have a simplicial action of  $\Gamma$  on the corresponding Bass-Serre tree  $T$  (see [Ser80]). From the natural  $\Gamma$ -equivariant map  $T \rightarrow *$ , we get an assembly map:

$$\rho' : H_n^\Gamma(T; \mathbb{K}\mathbb{R}^{-\infty}) \rightarrow H_n^\Gamma(*; \mathbb{K}\mathbb{R}^{-\infty}) \cong K_*(R\Gamma).$$

We start out with the important:

**Fact:** The map  $\rho'$  is split injective, and

$$\text{coker}(\rho') \cong \text{Nil}_*^W(RH; R[G_1 - H], R[G_2 - H]).$$

A proof of this **Fact** can be found in Davis [D, Lemma 7] (see also Remark (5) at the end of this section). In view of this result, we are merely trying to identify the cokernel of the map  $\rho'$ . The first step is to relate the cokernel of  $\rho'$  with the cokernel of the map  $\rho$ .

**Claim 1:** The map  $\rho$  is split injective, and there is a canonical isomorphism

$$\text{coker}(\rho) \cong \text{coker}(\rho')$$

We observe that we have four families of subgroups that we are dealing with: the three we have looked at so far are  $\mathcal{VC}$ ,  $\mathcal{ALL}$ , and the family  $\mathcal{F}$  we introduced at the

beginning of our proof (consisting of virtually cyclic subgroups conjugate into one of the  $G_i$ ). In addition, there is the family  $\mathcal{G}$  consisting of *all* subgroups of  $\Gamma$  which can be conjugated into either  $G_1$  or  $G_2$ . Now observe that we have containments of families  $\mathcal{F} \subset \mathcal{G} \subset \mathcal{ALL}$ , and  $\mathcal{F} \subset \mathcal{VC} \subset \mathcal{ALL}$ . Furthermore, we have that  $T$  is a model for  $E_{\mathcal{G}}\Gamma$ . This yields the following commutative diagram:

$$\begin{array}{ccc} H_*^\Gamma(E_{\mathcal{F}}\Gamma; \mathbb{KR}^{-\infty}) & \xrightarrow{\rho} & H_*^\Gamma(E_{\mathcal{VC}}\Gamma; \mathbb{KR}^{-\infty}) \\ \downarrow & & \downarrow \cong \\ H_*^\Gamma(T; \mathbb{KR}^{-\infty}) & \xrightarrow{\rho'} & H_*^\Gamma(*; \mathbb{KR}^{-\infty}) \end{array}$$

where all the maps are relative assembly maps corresponding to the inclusions of the various families of subgroups. Note that the horizontal maps are precisely the ones we are trying to relate. Now recall that we are assuming that  $\Gamma$  satisfies the Farrell-Jones isomorphism conjecture. This immediately implies that the second vertical map is an isomorphism, as indicated in the commutative diagram. So in order to identify the cokernels of the two horizontal maps, we are left with showing that the first vertical map is also an isomorphism.

The first vertical map is a relative assembly map, corresponding to the inclusion of the families  $\mathcal{F} \subset \mathcal{G}$  of subgroups of  $\Gamma$ . In order to show that the relative assembly map is an isomorphism, one merely needs to establish that for every maximal subgroup  $H \in \mathcal{G} - \mathcal{F}$ , the corresponding relative assembly map induced by the inclusions of families  $\mathcal{F}(H) \subset \mathcal{G}(H)$  of subgroups of  $H$  is an isomorphism (see [FJ93, Thm. A.10], [LS00, Thm. 2.3]). But observe that the maximal subgroups in  $\mathcal{G} - \mathcal{F}$  are precisely the (conjugates of) the subgroups  $G_i \leq \Gamma$ . Furthermore, for these subgroups, we have that  $\mathcal{G}(G_i) = \mathcal{ALL}(G_i)$ , and that  $\mathcal{F}(G_i) = \mathcal{VC}(G_i)$ . Hence the relative assembly maps which we require to be isomorphisms are exactly those induced by  $E_{\mathcal{VC}}G_i \rightarrow E_{\mathcal{ALL}}G_i \cong *$ , i.e. those that arise in the Farrell-Jones isomorphism conjecture. Since we are assuming the isomorphism conjecture holds for the groups  $G_1, G_2$ , we conclude that the first vertical map is indeed an isomorphism, completing the proof of the claim.

At this point, combining **Claim 1** with the **Fact**, we have an identification:

$$\text{coker}(\rho) \cong \text{coker}(\rho') \cong \text{Nil}_*^W(RH; R[G_1 - H], R[G_2 - H]).$$

In order to complete the proof, we now focus entirely on studying the map  $\rho$ , with a goal of showing that one can express its cokernel as a direct sum of the desired Nil-groups associated to the virtually cyclic subgroups  $V \in \mathcal{V}$ . We remind the reader that  $\rho$  is the relative assembly map induced by the map  $E_{\mathcal{F}}\Gamma \rightarrow E_{\mathcal{VC}}\Gamma$ , where  $\mathcal{F}$  is the family of subgroups consisting of all virtually cyclic subgroups of  $\Gamma$  that can be conjugated into either  $G_i$ .

In order to analyze this relative assembly map, we will need to make use of some properties of the  $\Gamma$ -action on the Bass-Serre tree. Particularly, we would like to understand the behavior of virtually cyclic subgroups  $V \in \mathcal{VC} - \mathcal{F}$ . The specific result we will require is contained in our:

**Claim 2:** In the case of an acylindrical amalgamation, the stabilizer of any geodesic  $\gamma$  in the Bass-Serre tree  $T$  is a virtually cyclic subgroup of  $\Gamma$ . Furthermore, every

virtually cyclic subgroup  $V \leq \Gamma$  satisfying  $V \in \mathcal{VC} - \mathcal{F}$  stabilizes a unique geodesic  $\gamma \subset T$ .

Let us start by recalling some basic facts concerning the action of  $\Gamma$  on the Bass-Serre tree  $T$  corresponding to the amalgamation  $\Gamma = G_1 *_H G_2$ :

- the action is without inversions, i.e. if an element stabilizes an edge  $e$ , then it automatically preserves the chosen orientation of  $e$ ,
- the stabilizer of any vertex  $v \in T$  is isomorphic to a conjugate of  $G_1$  or  $G_2$ ,
- the stabilizer of any edge  $e \subset T$  is isomorphic to a conjugate of  $H$ ,
- any finite subgroup of  $\Gamma$  fixes a vertex in  $T$ ,

The first three statements above are built into the definition of the Bass-Serre tree (see [Ser80]), while the last statement is a well known general facts about group actions on trees. We remind the reader that a *geodesic* in a tree  $T$  will be a subcomplex simplicially isomorphic to  $\mathbb{R}$ , with the standard simplicial structure (i.e. vertices at the integers, and edges between).

To show the first statement in our claim, we note that  $\text{Stab}_\Gamma(\gamma)$  clearly fits into a short exact sequence:

$$0 \rightarrow \text{Fix}_\Gamma(\gamma) \rightarrow \text{Stab}_\Gamma(\gamma) \rightarrow \text{Simp}_{\Gamma, \gamma}(\mathbb{R}) \rightarrow 0$$

where  $\text{Fix}_\Gamma(\gamma)$  is the subgroup fixing  $\gamma$  pointwise, and  $\text{Simp}_{\Gamma, \gamma}(\mathbb{R})$  is the induced simplicial action on  $\mathbb{R}$  (obtained by simplicially identifying  $\gamma$  with  $\mathbb{R}$ ). Note that the group of simplicial automorphisms of  $\mathbb{R}$  is  $D_\infty$ , the infinite dihedral group. In particular, we see that  $\text{Simp}_{\Gamma, \gamma}(\mathbb{R})$  is virtually cyclic (in fact is isomorphic to either the trivial group,  $\mathbb{Z}_2$ ,  $\mathbb{Z}$ , or  $D_\infty$ ).

Next we observe that  $\text{Fix}_\Gamma(\gamma)$  is finite. To see this, we recall that the amalgamation  $\Gamma = G_1 *_H G_2$  was assumed to be *acylindrical*, which means that there exists an integer  $k \geq 1$  with the property that the stabilizer  $\text{Stab}_\Gamma(\eta) \leq \Gamma$  of any combinatorial path  $\eta \subset T$  of length  $\geq k$  is finite. Since  $\gamma \subset T$  is a geodesic, it contains combinatorial subpaths  $\eta$  of arbitrarily long length (in particular, length  $\geq k$ ). The obvious containment  $\text{Fix}_\Gamma(\gamma) \leq \text{Stab}_\Gamma(\eta)$  now completes the argument for the first statement in our claim.

For the second statement, we note that  $\Gamma$  acts simplicially on the Bass-Serre tree, and hence the given virtually cyclic subgroup  $V \in \mathcal{VC} - \mathcal{F}$  likewise inherits an action on  $T$ . Since  $V \notin \mathcal{F}$ , we have that the  $V$  action on  $T$  has no globally fixed point, hence cannot be finite. In particular,  $V$  must be an *infinite* virtually cyclic subgroup, and hence contains a finite index infinite cyclic normal subgroup  $V' \triangleleft V$ , with  $V'$  generated by  $g \in V$ . We now claim that  $g$  cannot fix any vertex in  $T$ .

Assume, by way of contradiction, that there exists a vertex  $v$  fixed by  $g$  (and hence by  $V'$ ). Let  $T' \subset T$  be the subset consisting of points that are fixed by  $V'$ . Note that  $T'$  is non-empty (since  $v \in T'$ ), and is a subtree of  $T$  (since  $\Gamma$  acts simplicially on  $T$ ). Furthermore, observe that the group  $F := V/V'$  inherits a simplicial action on  $T'$ . But note that  $F$  is finite, and hence the  $F$ -action on  $T'$  has a fixed vertex  $w \in T' \subset T$ . But this immediately implies that the original group  $V$  fixes  $w$ , a contradiction as  $V \notin \mathcal{F}$ .

Now establishing that  $V$  stabilizes a geodesic is a straightforward application of standard techniques in the geometry of group actions on trees (applied to  $T$ ). For the convenience of the reader, we give a quick outline of the argument. For an arbitrary element  $g$  of infinite order in  $V$ , one can look at the associated displacement

function on  $T$ , i.e. the distance from  $v$  to  $g \cdot v$ . The previous paragraph establishes that this function is strictly positive. One then considers the set  $\text{Min}(g)$  of points in  $T$  which minimize the displacement function, call this minimal value  $\mu_g$ . It is easy to see that:

- $\text{Min}(g)$  contains a geodesic  $\gamma$ : take a vertex  $v \in \text{Min}(g)$ , and consider  $\gamma := \bigcup_{i \in \mathbb{Z}} g^i \cdot \eta$ , where  $\eta$  is the geodesic segment from  $v$  to  $g \cdot v$  (note that such a non-trivial segment exists by the previous paragraph),
- in fact,  $\gamma = \text{Min}(g)$ : any point at distance  $r > 0$  from  $\gamma$  will be displaced  $2r + \mu_g > \mu_g$ , and so cannot lie in  $\text{Min}(g)$ ,
- for any non-zero integer  $i$ ,  $\text{Min}(g) = \text{Min}(g^i)$ : any point at distance  $r > 0$  from  $\gamma$  will be displaced  $2r + |i| \cdot \mu_g > |i| \cdot \mu_g$ , while points on  $\gamma$  will clearly only be displaced  $|i| \cdot \mu_g$  by the element  $g^i$ ,
- for any two elements  $g, h$  of infinite order in  $V$ , we have  $\text{Min}(g) = \text{Min}(h)$ : two such elements have a common power, and apply the previous statement.

From the observations above, we see that every single element in  $V$  of infinite order stabilizes the exact same geodesic  $\gamma \subset T$ .

Hence the only elements we might have to worry about are elements  $h \in V$  of *finite order*. For these, we just note that  $V$  contains  $V' \triangleleft V$ , a finite index cyclic normal subgroup generated by an element  $g$  of infinite order. We have a natural morphism from  $H = \langle h \rangle$  to  $\text{Aut}(V') \cong \mathbb{Z}/2$ . In particular, we have that  $hgh^{-1} = g^{\pm 1}$  and hence for any vertex  $v \in \text{Min}(g)$  we have the obvious equalities:

$$d(g \cdot hv, hv) = d(h^{-1}gh \cdot v, v) = d(g^{\pm 1} \cdot v, v) = \mu_{g^{\pm 1}} = \mu_g$$

Since  $hv$  is minimally displaced by  $g$ , it must also lie on  $\gamma = \text{Min}(g)$ . This deals with elements of finite order, hence completes the verification that  $\gamma$  is  $V$ -invariant. Finally, from the fact that  $V$  has a (finite index) subgroup that acts on  $\gamma$  via a translation, it is easy to see that there are no other  $V$ -invariant geodesics in  $T$ , yielding uniqueness. This finishes the argument for our **Claim 2**.

Having established some basic properties of the  $\Gamma$ -action on  $T$ , we now return to the main argument. Recall that by combining our **Claim 1** with the **Fact**, we have reduced the proof of the Main Theorem to understanding the cokernel of the relative assembly map:

$$\rho : H_n^\Gamma(E_{\mathcal{F}} \Gamma; \mathbb{K}\mathbb{R}^{-\infty}) \rightarrow H_n^\Gamma(E_{\mathcal{V}\mathcal{C}} \Gamma; \mathbb{K}\mathbb{R}^{-\infty})$$

where  $\mathcal{F}$  is the family of subgroups consisting of all virtually cyclic subgroups of  $\Gamma$  that can be conjugated into either  $G_i$ .

Now recall that in [LO07], the authors introduced the notion of a collection of subgroups to be *adapted* to a nested family of subgroups, and in the presence of an adapted family, showed how a model for the classifying space with isotropy in the smaller family could be “promoted” to a model for the classifying space with isotropy in the larger family. In a subsequent paper [LO], the authors used some recent work of Lueck-Weiermann [LW] to give an alternate model for this classifying space, which had the additional advantage of providing *explicit splittings* for the cokernel of the relative assembly maps. Let us briefly recall the relevant definitions.

Given a nested pair of families  $\mathcal{F} \subset \tilde{\mathcal{F}}$  of subgroups of  $\Gamma$ , we say that a collection  $\{H_\alpha\}_{\alpha \in I}$  of subgroups of  $\Gamma$  is *adapted* to the pair  $(\mathcal{F}, \tilde{\mathcal{F}})$  provided that:

- (1) For all  $H_1, H_2 \in \{H_\alpha\}_{\alpha \in I}$ , either  $H_1 = H_2$ , or  $H_1 \cap H_2 \in \mathcal{F}$ .

- (2) The collection  $\{H_\alpha\}_{\alpha \in I}$  is *conjugacy closed* i.e. if  $H \in \{H_\alpha\}_{\alpha \in I}$  then  $gHg^{-1} \in \{H_\alpha\}_{\alpha \in I}$  for all  $g \in \Gamma$ .
- (3) Every  $H \in \{H_\alpha\}_{\alpha \in I}$  is *self-normalizing*, i.e.  $N_\Gamma(H) = H$ .
- (4) For all  $G \in \tilde{\mathcal{F}} \setminus \mathcal{F}$ , there exists  $H \in \{H_\alpha\}_{\alpha \in I}$  such that  $G \leq H$ .

In [LO], we applied this result to the nested family  $\mathcal{FIN} \subset \mathcal{VC}$  for relatively hyperbolic groups (for which an adapted collection of subgroups is easy to find).

In our present context, we would like to find a collection of subgroups adapted to the nested pair of families  $\mathcal{F} \subset \mathcal{VC}$ . Recall that  $\mathcal{F}$  consists of all virtually cyclic subgroups that can be conjugated into either  $G_i$ , and  $\mathcal{VC}$  consists of all virtually cyclic subgroups of  $\Gamma$ .

**Claim 3:** The collection  $\{H_\alpha\}$  of subgroups of  $\Gamma$  consisting of all *maximal* virtually cyclic subgroups in  $\mathcal{VC} - \mathcal{F}$  is adapted to the pair  $(\mathcal{F}, \mathcal{VC})$ .

To verify this, we first note that properties (2) and (4) in the definition of an adapted family are immediate. Property (3) follows easily from **Claim 2**: let  $V \in \{H_\alpha\}$  be given, and consider  $N_\Gamma(V)$ . We know that  $V$  leaves invariant a unique geodesic  $\gamma \subset T$ . Furthermore, for every  $g \in \Gamma$ , we see that  $gVg^{-1}$  leaves  $g \cdot \gamma$  invariant. Uniqueness of the  $V$ -invariant geodesic  $\gamma$  now implies that  $\gamma$  is actually  $N_\Gamma(V)$ -invariant. In particular, we have containments  $V \leq N_\Gamma(V) \leq \text{Stab}_\Gamma(\gamma)$ . But from **Claim 2**, we know that  $\text{Stab}_\Gamma(\gamma)$  is virtually cyclic, and maximality of  $V$  now forces all the containments to be equalities, and in particular,  $V = N_\Gamma(V)$  as required by property (3).

For property (1), let  $V_1, V_2 \in \{H_\alpha\}$ . We want to establish that either  $V_1 = V_2$ , or that  $V_1 \cap V_2 \in \mathcal{F}$ . So let us assume that  $V_1 \neq V_2$ . We know from **Claim 2** that each  $V_i$  stabilizes a unique geodesic  $\gamma_i$ , and from the maximality of the groups  $V_i$ , we actually have  $V_i = \text{Stab}_\Gamma(\gamma_i)$ . Since  $V_1 \neq V_2$ , we have that  $\gamma_1 \neq \gamma_2$ . There are now two possibilities: (i) either  $\gamma_1 \cap \gamma_2 = \emptyset$ , or (ii)  $\gamma_1 \cap \gamma_2$  is a path in  $T$ . We claim that in both cases, the intersection  $H = V_1 \cap V_2 \leq V_i$  has the property that the  $H$ -action on the corresponding  $\gamma_i$  *fixes a point*.

To see this, let us first consider possibility (i): since  $\gamma_1 \cap \gamma_2 = \emptyset$ , one can consider the (unique) minimal length geodesic segment  $\eta$  joining  $\gamma_1$  to  $\gamma_2$ . We observe that, since  $H$  stabilizes both  $\gamma_i$ , it must leave the segment  $\eta$  invariant. In particular,  $H$  must fix the vertex  $v_i = \eta \cap \gamma_i \in \gamma_i$ , as desired. Next consider possibility (ii): if  $\gamma_1 \cap \gamma_2 \neq \emptyset$ , then the intersection will be a subpath  $\eta \subset \gamma_i$ . Note that  $\eta$  is either a geodesic *segment*, or is a geodesic *ray*, and in both cases, will be invariant under the group  $H$ . If  $\eta$  is a geodesic ray (i.e. homeomorphic to  $[0, \infty)$ ), then there is a (topologically) distinguished point inside  $\eta$ , which will have to be fixed by  $H$ . If  $\eta$  is a geodesic segment, then each element in  $H$  either fixes  $\eta$ , or reverses  $\eta$  (note that the latter can only occur if  $\eta$  has *even* length, as  $\Gamma$  acts on  $T$  without inversions). In particular, we see that if  $\eta$  has odd length, then every point in  $\eta$  is fixed by  $H$ , while if  $\eta$  has even length, then the (combinatorial) midpoint is fixed.

Finally, we observe that  $H \leq V_i$  acts on  $\gamma_i$ , and fixes a point. This immediately implies that  $H$  contains a subgroup of index at most two which *acts trivially* on  $\gamma_i$ , i.e.  $H' \leq \text{Fix}_\Gamma(\gamma_i)$ . But recall that the latter group is finite (see the proof of **Claim 2**), completing the proof of property (1). We conclude that the collection  $\{H_\alpha\}$  is an adapted collection of subgroups for the nested families  $(\mathcal{F}, \mathcal{VC})$ , as desired.



Finally, we exploit the adapted family we just constructed to establish:

**Claim 4:** We have an identification:

$$\operatorname{coker}(\rho) \cong \bigoplus_{V \in \mathcal{V}} H_*^V(E_{\mathcal{F}\mathcal{I}\mathcal{N}}V \rightarrow *; \mathbb{K}\mathbb{R}^{-\infty})$$

where the groups  $H_*^V(E_{\mathcal{F}\mathcal{I}\mathcal{N}}V \rightarrow *; \mathbb{K}\mathbb{R}^{-\infty})$  are the cokernels of the relative assembly maps associated with the virtually cyclic groups  $V \in \mathcal{V}$

The argument for this is virtually identical to the one given in [LO, Cor. 3.2]; we reproduce the argument here for the convenience of the reader. From **Claim 3**, we have an adapted family for the pair  $(\mathcal{F}, \mathcal{V}\mathcal{C})$ , consisting of all maximal subgroups in  $\mathcal{V}\mathcal{C} - \mathcal{F}$ . From this adapted collection of subgroups, [LO, Prop. 3.1] establishes (using [LW, Thm. 2.3]) a method for constructing an  $E_{\mathcal{V}\mathcal{C}}\Gamma$ . Namely, if  $\mathcal{V}$  is a complete set of representatives of the conjugacy classes within the adapted collection of subgroups  $\{H_\alpha\}$ , form the cellular  $\Gamma$ -pushout:

$$\begin{array}{ccc} \coprod_{V \in \mathcal{V}} \Gamma \times_V E_{\mathcal{F}}V & \xrightarrow{\beta} & E_{\mathcal{F}}\Gamma \\ \alpha \downarrow & & \downarrow \\ \coprod_{V \in \mathcal{V}} \Gamma \times_V E_{\mathcal{V}\mathcal{C}}V & \longrightarrow & X \end{array}$$

Then the resulting space  $X$  is a model for  $E_{\mathcal{V}\mathcal{C}}\Gamma$  (we refer the reader to [LO, Prop. 3.1] for a more precise discussion of this result, including a description of the maps  $\alpha, \beta$  in the above cellular  $\Gamma$ -pushout). Note that the map  $\rho$  whose cokernel we are trying to understand is precisely the map on (equivariant) homology induced by the second vertical arrow in the above cellular  $\Gamma$ -pushout.

Since  $X$  is the double mapping cylinder of the maps  $\alpha, \beta$  in the above diagram, one has a natural  $\Gamma$ -equivariant decomposition of  $X$  by taking  $A$  (respectively  $B$ ) to be the  $[0, 2/3]$  (respectively  $(1/3, 1]$ ) portions of the double mapping cylinder. Applying the homology functor  $H_*^\Gamma(-; \mathbb{K}\mathbb{R}^{-\infty})$  (and omitting the coefficients to shorten notation), we have the Mayer-Vietoris sequence:

$$\dots \rightarrow H_*^\Gamma(A \cap B) \rightarrow H_*^\Gamma(A) \oplus H_*^\Gamma(B) \rightarrow H_*^\Gamma(X) \rightarrow H_{*-1}^\Gamma(A \cap B) \rightarrow \dots$$

But now observe that we have obvious  $\Gamma$ -equivariant homotopy equivalences between:

- (1)  $A \simeq_\Gamma \coprod_{V \in \mathcal{V}} \Gamma \times_V E_{\mathcal{V}\mathcal{C}}V$
- (2)  $B \simeq_\Gamma E_{\mathcal{F}}\Gamma$
- (3)  $A \cap B \simeq_\Gamma \coprod_{V \in \mathcal{V}} \Gamma \times_V E_{\mathcal{F}}V$

Furthermore, the homology theory we have takes disjoint unions into direct sums. Combining this with the induction structure, we obtain isomorphisms:

$$\begin{aligned} H_*^\Gamma(A) &\cong \bigoplus_{V \in \mathcal{V}} H_*^\Gamma(\Gamma \times_V E_{\mathcal{V}\mathcal{C}}V) \cong \bigoplus_{V \in \mathcal{V}} H_*^V(E_{\mathcal{V}\mathcal{C}}V) \\ H_*^\Gamma(A \cap B) &\cong \bigoplus_{V \in \mathcal{V}} H_*^\Gamma(\Gamma \times_V E_{\mathcal{F}}V) \cong \bigoplus_{V \in \mathcal{V}} H_*^V(E_{\mathcal{F}}V) \end{aligned}$$

Now observing that the groups  $V \in \mathcal{V}$  are all virtually cyclic, we have that each  $E_{\mathcal{V}\mathcal{C}}V$  can be taken to be a point. Furthermore, for the groups  $V \in \mathcal{V}$ , we have that the restriction of the family  $\mathcal{F}$  to  $V$  coincides with the family of finite subgroups

of  $V$ , i.e.  $E_{\mathcal{F}}V = E_{\mathcal{F}IN}V$ . Substituting all of this in the above Mayer-Vietoris sequence, we get the long exact sequence:

$$\dots \rightarrow \bigoplus_{V \in \mathcal{V}} H_*^V(E_{\mathcal{F}IN}V) \rightarrow H_*^\Gamma(E_{\mathcal{F}}\Gamma) \oplus \bigoplus_{V \in \mathcal{V}} H_*^V(*) \rightarrow H_*^\Gamma(E_{\mathcal{V}C}\Gamma) \rightarrow \dots$$

Now observe that each of the maps  $H_*^V(E_{\mathcal{F}IN}V) \rightarrow H_*^V(*)$  are split injective from Bartels result [Ba03]. Since the map  $\rho : H_*^\Gamma(E_{\mathcal{F}}\Gamma) \rightarrow H_*^\Gamma(E_{\mathcal{V}C}\Gamma)$  is also split injective (from **Claim 1**), we now have an identification:

$$\text{coker}(\rho) \cong \bigoplus_{V \in \mathcal{V}} \text{coker}(H_*^V(E_{\mathcal{F}IN}V) \rightarrow H_*^V(*))$$

Finally, combining the **Fact** with **Claim 1** and **Claim 4**, we see that we have the desired splitting:

$$\text{Nil}_*^W(RH; R[G_1 - H], R[G_2 - H]) \cong \bigoplus_{V \in \mathcal{V}} H_*^V(E_{\mathcal{F}IN}V \rightarrow *; \mathbb{K}R^{-\infty})$$

where the groups  $H_*^V(E_{\mathcal{F}IN}V \rightarrow *; \mathbb{K}R^{-\infty})$  denote the cokernels appearing in **Claim 4**. This completes the proof of the Main Theorem.

**Remark: (5)** One of the key ingredients in our proof was the **Fact** established by Davis in [D, Lemma 7]. Prior to learning of Davis' preprint, the authors had an alternate argument for the **Fact**. For the sake of the interested readers, we briefly outline our alternate approach.

Anderson-Munkholm [AM00, Section 7] defined a functor, continuously controlled  $K$ -theory, from the category of diagrams of holink type to the category of spectra. Munkholm-Prassidis [MP01, Theorem 2.1] showed that the Waldhausen Nil-group we are interested in can be identified with the cokernel of a natural split injective map  $\tilde{K}_{*+1}^{cc}(\xi^+) \rightarrow K_*(\mathbb{Z}\Gamma)$ , where  $\xi^+$  is a suitably defined diagram of holink type associated to the splitting  $\Gamma = G_1 *_H G_2$  (see [AM00, Section 9]). Furthermore, Anderson-Munkholm have shown [AM00, Theorem 9.1] that there is a natural isomorphism  $\tilde{K}_{*+1}^{cc}(\xi^+) \cong \tilde{K}_{*+1}^{bc}(\xi^+)$ , where the latter is the boundedly controlled  $K$ -theory defined by Anderson-Munkholm in [AM90]. Finally, there are Atiyah-Hirzebruch spectral sequences computing both the groups  $\tilde{K}_{*+1}^{bc}(\xi^+)$  (see [AM90, Theorem 4.1]) and  $H_*^\Gamma(T; \mathbb{K}Z^{-\infty})$  (see [Qu82, Section 8]). It is easy to verify that the two spectral sequences are *canonically identical*: they have the same  $E^2$ -terms and the same differentials. Combining these results, and keeping track of the various maps appearing in the sequence of isomorphisms, one can give an alternate proof of the **Fact**.

**(6)** We also point out that, from the  $\Gamma$ -action on the Bass-Serre tree  $T$ , it is easy to obtain constraints on the isomorphism type of groups inside the collection  $\mathcal{V}$ . Indeed, any such group must be the stabilizer of a bi-infinite geodesic  $\gamma \subset T$  (see **Claim 2**), and must act cocompactly on  $\gamma$ . Recall that infinite virtually cyclic subgroups are of two types: those that surject onto the infinite dihedral group  $D_\infty$ , and those that do not.

The groups that surject onto  $D_\infty$  always split as an amalgamation  $A *_C B$ , with all three groups  $A, B, C$  finite, and  $C$  of index two in both  $A, B$ . Observe that if  $V \in \mathcal{V}$  is of this type, then under the action of  $V$  on  $\gamma \subset T$ ,  $A$  and  $B$  can be identified with the stabilizers of a pair of vertices  $v, w$ , and  $C$  can be identified with

the stabilizer of the segment joining  $v$  to  $w$ . In particular,  $C$  must be a subgroup of an edge stabilizer, and hence is conjugate (within  $\Gamma$ ) to a finite subgroup of  $H$ . Furthermore, since  $A, B$  both stabilize a pair of vertices, they must be conjugate (within  $\Gamma$ ) to a finite subgroup of either  $G_1$  or  $G_2$ .

The groups that do not surject onto  $D_\infty$  are of the form  $F \rtimes_\alpha \mathbb{Z}$ , where  $F$  is a finite group, and  $\alpha \in \text{Aut}(F)$  is an automorphism. If  $V \in \mathcal{V}$  is of this form, then for the action of  $V$  on  $\gamma \subset T$ , one has that  $F$  can be identified with the subset of  $V$  that pointwise fixes  $\gamma$ , while the  $\mathbb{Z}$  component acts on  $\gamma$  via a translation. In particular,  $F$  is again conjugate (within  $\Gamma$ ) to a finite subgroup of  $H$ .

In particular, if we are given an explicit amalgamation  $\Gamma = G_1 *_H G_2$ , and we have knowledge of the finite subgroups inside the groups  $H, G_1, G_2$ , then we can readily identify up to isomorphism the possible groups arising in the collection  $\mathcal{V}$ . If one has knowledge of the Nil-groups associated with these various groups, our Main Theorem can be used to get corresponding information about the Waldhausen Nil-group associated to  $\Gamma$ .

(7) The attentive reader will notice that the hypothesis of “acylindricity” was used only in the proof of **Claim 2**, where it was used to show that for every geodesic  $\gamma \subset T$ , the subgroup  $\text{Fix}_\Gamma(\gamma) \leq \Gamma$  consisting of elements that fix  $\gamma$  pointwise is in fact finite. In particular, the conclusion of the Main Theorem holds in a slightly more general setting. Namely, rather than requiring amalgamations  $\Gamma = G_1 *_H G_2$  to be acylindrical, it is sufficient to require them to have the property that, for the action of  $\Gamma$  on the associated Bass-Serre tree, every geodesic  $\gamma \subset T$  satisfies  $|\text{Fix}_\Gamma(\gamma)| < \infty$ . The authors do not know of any way to ensure this *a priori* more general condition, other than by showing the amalgamation is acylindrical.

(8) We remark that there is also a version of the Waldhausen Nil-groups associated to HNN extensions. The proof of our Main Theorem should extend to provide a similar splitting for HNN extensions, under an identical hypothesis on the action of  $\Gamma$  on the Bass-Serre tree associated to the HNN extension (i.e. acylindricity, or more generally, finiteness of the subgroups fixing geodesics in the Bass-Serre tree). The key missing step in the HNN case is the analogue of the **Fact**, which the authors have not verified, but believe to be true. Modulo this result, the rest of our argument extends verbatim to the HNN setting.

It is also perhaps worth mentioning that, unlike the amalgamation case, in the HNN case there are easy examples of extensions satisfying the more general condition discussed in Remark (7). For example, if we consider the classical Baumslag-Solitar groups  $BS(1, n) := \{a, b \mid aba^{-1} = b^n\}$ , these come with a natural realization as an HNN extension, with vertex group generated by the subgroup  $\langle b \rangle$ , edge group isomorphic to  $\mathbb{Z} = \langle c \rangle$ , and the two edge to vertex inclusions given by  $c \mapsto b$  and  $c \mapsto b^n$  respectively. In the associated Bass-Serre tree, we note that there is a single distinguished end which is fixed by the entire  $BS(1, n)$ , while all other ends of the Bass-Serre tree are only left invariant by the trivial element. In particular, since every geodesic  $\eta \subset T$  has two distinct ends, one of the two has trivial stabilizer, and hence the fixed group of  $\eta$  must likewise be trivial. This implies that the HNN extension satisfies the property discussed in our Remark (7). On the other hand, every finite subtree of  $T$  has infinite stabilizer (isomorphic to  $\mathbb{Z}$ ), and hence the amalgamation is *not* acylindrical. We thank M. Forester for bringing this simple example to our attention.

## 3. APPLICATIONS

Having completed the proof of our Main Theorem, we now proceed to isolate a few interesting corollaries. As mentioned earlier, from the viewpoint of topological applications, the most interesting situation is the case where  $R = \mathbb{Z}$ , i.e. integral group rings.

**Corollary 1.** *Let  $\Gamma = G_1 *_H G_2$  be an amalgamation, and assume that the Farrell-Jones Isomorphism Conjecture holds for the groups  $\Gamma, G_1, G_2$ , and that  $H$  is finite. Then the associated Waldhausen Nil-group  $Nil_*^W(\mathbb{Z}H; \mathbb{Z}[G_1 - H], \mathbb{Z}[G_2 - H])$  is either trivial, or an infinitely generated torsion group.*

*Proof.* Note that since  $H$  is finite, the amalgamation is acylindrical, and our Main Theorem applies. So the Waldhausen group we are interested in splits as a direct sum of Nil-groups associated to a particular collection  $\mathcal{V}$  of virtually cyclic subgroups. It is well known that the Nil-groups associated to virtually cyclic groups are either trivial or infinitely generated (see [F77], [G07], [R]). Furthermore, these groups are known to be purely torsion (see [We81], [CP02], [KT03], [G08]), giving us the second statement.

A special case of the above corollary is worth mentioning:

**Corollary 2.** *Let  $\Gamma = G_1 *_H G_2$  be an amalgamation, and assume that  $G_1, G_2$ , and  $H$  are all finite. Then the associated Waldhausen Nil-group  $Nil_*^W(\mathbb{Z}H; \mathbb{Z}[G_1 - H], \mathbb{Z}[G_2 - H])$  is either trivial, or an infinitely generated torsion group.*

*Proof.*  $G_1, G_2$  trivially satisfy the Farrell-Jones isomorphism conjecture, as they are finite. Furthermore, the group  $\Gamma$  is  $\delta$ -hyperbolic, and hence by a recent result of Bartels-Lück-Reich [BLR08a], also satisfies the isomorphism conjecture. Hence the hypotheses of our previous corollary are satisfied.

As was kindly pointed out to the authors by J. Grunewald, the “torsion group” conclusion in our Corollary 2 has also been independently obtained, via different methods, by Bartels-Lueck-Reich (see [BLR08b, Thm. 0.15]).

Finally, we observe that the Bartels-Lück-Reich result [BLR08a] establishes the Isomorphism conjecture for  $\Gamma$  a  $\delta$ -hyperbolic group, in all dimensions, and for arbitrary coefficient rings  $R$ . In particular, the hypotheses of our Main Theorem hold for arbitrary amalgamations of finite groups, giving:

**Corollary 3.** *Let  $\Gamma = G_1 *_H G_2$  be an amalgamation, where  $G_1, G_2$ , and  $H$  are all finite. Then, for arbitrary rings with unity  $R$ , we have isomorphisms:*

$$Nil_*^W(RH; R[G_1 - H], R[G_2 - H]) \cong \bigoplus_{V \in \mathcal{V}} H_*^V(E_{\mathcal{F}IN}V \rightarrow *; \mathbb{K}R^{-\infty})$$

where  $H_*^V(E_{\mathcal{F}IN}V \rightarrow *; \mathbb{K}R^{-\infty})$  are the cokernels of the relative assembly maps associated to the virtually cyclic subgroups  $V \in \mathcal{V}$ , and the collection  $\mathcal{V}$  is as in the statement of our Main Theorem.

We point out that the special case of the modular group  $\Gamma = PSL_2(\mathbb{Z}) = \mathbb{Z}_2 * \mathbb{Z}_3$  has also been independently studied by Davis-Khan-Ranicki (see [DKR, Section 3.3]).

**Remark: (9)** As was mentioned in Remark (1), another convenient class of acylindrical amalgamations are those of the form  $\Gamma = G_1 *_H G_2$ , where  $H$  has the property that  $|gHg^{-1} \cap H| < \infty$  for every  $g \in G_1 - H$  (we call this property *conjugacy*

*$\mathcal{FLN}$ -separability*). In the case where both  $G_i$  are  $\delta$ -hyperbolic, and  $H$  is also assumed to be quasi-convex in both  $G_i$ , then the resulting amalgamation  $\Gamma$  is known to be  $\delta$ -hyperbolic (see Bestvina-Feighn [BF92], [BF96], and also Kharlampovich-Myasnikov [KM98, Theorem 2]). Combining these two comments, we see that in the situation where  $H \leq G$  is a quasi-convex, conjugacy  $\mathcal{FLN}$ -separable subgroup of a  $\delta$ -hyperbolic group  $G$ , the resulting amalgamation  $\Gamma = G *_H G$  is an acylindrical amalgamation, with both vertex groups  $G$  and the amalgamated group  $\Gamma$  satisfying the Farrell-Jones isomorphism conjecture (by the recent work of Bartels-Lück-Reich [BLR08a]). In particular, this would give rise to examples of amalgamations satisfying all the hypotheses of our Main Theorem.

In the case where  $G$  is torsion-free, the property of conjugacy  $\mathcal{FLN}$ -separability reduces to the more familiar notion of *malnormality*. Examples of quasi-convex malnormal subgroups in torsion-free  $\delta$ -hyperbolic groups are plentiful. For example, in the situation where  $G$  is non-elementary, any maximal virtually cyclic subgroup  $H < G$  is automatically quasi-convex and malnormal. As another example of how common such subgroups are, we mention a result of I. Kapovich [Ka99]: every non-elementary subgroup  $K$  of a torsion-free  $\delta$ -hyperbolic group  $G$  contains a further subgroup  $F_2 \cong H < K$ , isomorphic to a free group on two generators, which is malnormal and quasi-convex in the ambient  $G$ . Of course, these results do not yield interesting examples, since the lack of torsion elements implies that all the groups involved will have vanishing  $K$ -theory (by [BLR08a]), and hence both sides of the isomorphism in our Main Theorem would trivially vanish.

It would perhaps be of some interest to see whether, in a non-elementary  $\delta$ -hyperbolic group  $G$  with non-trivial torsion, one can find examples of quasi-convex subgroups  $H < G$  which are conjugacy  $\mathcal{FLN}$ -separable. By the discussion above, such a pair would give rise, by doubling along the subgroup  $H$ , to a group  $\Gamma = G *_H G$  satisfying the hypotheses of our Main Theorem, and for which the algebraic  $K$ -theory could potentially be non-trivial. The authors suspect that such subgroups should exist, and in view of the results mentioned in the previous paragraph for the torsion-free case, we expect such examples to be plentiful.

To give a concrete example which fits into the general context discussed in our Remark (9) above, we establish our final:

**Corollary 4.** *Let  $G_i \leq O^+(3, 1) = \text{Isom}(\mathbb{H}^3)$  be a pair of uniform lattices in the isometry group of hyperbolic 3-space. Assume that  $H_i \leq G_i$  are a pair of maximal infinite virtually cyclic subgroups, which are abstractly isomorphic to a group  $H$ . Consider the amalgamated group  $\Gamma = G_1 *_H G_2$ , where  $H$  is identified with the subgroups  $H_i$ . Then for  $* \leq 1$ , the associated Waldhausen Nil-group satisfies*

$$\text{Nil}_*^W(\mathbb{Z}H; \mathbb{Z}[G_1 - H_1], \mathbb{Z}[G_2 - H_2]) = 0.$$

*Proof.* Since the groups  $G_i$  are uniform lattices in the semi-simple Lie group  $O^+(3, 1)$ , they both satisfy the Farrell-Jones isomorphism conjecture (see [FJ93]). Next we observe that, since the  $H_i$  are maximal in the respective  $G_i$ , they are automatically malnormal, and hence the amalgamation  $\Gamma$  is acylindrical. Finally, since we are amalgamating  $\delta$ -hyperbolic groups along malnormal infinite virtually cyclic subgroups, the resulting group  $\Gamma$  is also  $\delta$ -hyperbolic (see [BF92], [BF96], [KM98]), and so also satisfies the Farrell-Jones isomorphism conjecture ([BLR08a]). This completes the verification of the hypotheses of our Main Theorem, which allows us

to conclude that there is a splitting:

$$\mathrm{Nil}_*^W(\mathbb{Z}H; \mathbb{Z}[G_1 - H_1], \mathbb{Z}[G_2 - H_2]) \cong \bigoplus_{V \in \mathcal{V}} H_*^V(E_{\mathcal{FIN}}V \rightarrow *; \mathbb{K}\mathbb{Z}^{-\infty})$$

We are now left with two goals: to identify the infinite virtually cyclic groups in the collection  $\mathcal{V}$ , and to show that the cokernels appearing on the left hand side of the above expression all vanish.

Next we recall that in the proof of our Main Theorem, we were able to identify groups in  $\mathcal{V}$  with infinite stabilizers of geodesics  $\gamma$  in the Bass-Serre tree associated to the splitting  $\Gamma = G_1 *_H G_2$  (see our Claim 2). Returning to the notation of our Claim 2, we now proceed to exploit the short exact sequence:

$$0 \rightarrow \mathrm{Fix}_\Gamma(\gamma) \rightarrow \mathrm{Stab}_\Gamma(\gamma) \rightarrow \mathrm{Simp}_{\Gamma, \gamma}(\mathbb{R}) \rightarrow 0.$$

Recall that  $\mathrm{Fix}_\Gamma(\gamma)$  is the subgroup fixing  $\gamma$  pointwise, and  $\mathrm{Simp}_{\Gamma, \gamma}(\mathbb{R})$  is the induced simplicial action on  $\mathbb{R}$  (and in the present situation, is isomorphic to either  $\mathbb{Z}$ , or  $D_\infty$ ). We now claim that, under our hypotheses, we also have control of the group  $\mathrm{Fix}_\Gamma(\gamma)$ .

Indeed, taking any two consecutive edges  $e_1, e_2$  along the geodesic  $g$ , and letting  $v$  be the common vertex, we note that  $\mathrm{Fix}_\Gamma(\gamma)$  can be identified with a subgroup  $F$  of  $\mathrm{Stab}_\Gamma(v)$ . The latter is conjugate to one of the two groups  $G_i$ , a subgroup of the isometry group of  $\mathbb{H}^3$ . Now recall that the two subgroups  $\mathrm{Stab}_\Gamma(e_i) \leq \mathrm{Stab}_\Gamma(v) \cong G_i$  can be identified geometrically as follows: if we think of  $\mathrm{Stab}_\Gamma(v)$  acting isometrically on  $\mathbb{H}^3$  (via the identification with the appropriate  $G_i$ ), then each of the two groups  $\mathrm{Stab}_\Gamma(e_1) \cong H \cong \mathrm{Stab}_\Gamma(e_2)$  correspond to the stabilizers of two distinct geodesics  $\gamma_1, \gamma_2 \subset \mathbb{H}^3$ . Now since  $\mathrm{Fix}_\Gamma(\gamma)$  also fixes the two incident edges  $e_1, e_2$ , the corresponding subgroup  $F \leq \mathrm{Stab}_\Gamma(v)$  acting on  $\mathbb{H}^3$  is also a subgroup in both  $\mathrm{Stab}_\Gamma(e_i)$ , i.e. *must leave both the geodesics  $\gamma_1, \gamma_2$  invariant*. But given two distinct geodesics in  $\mathbb{H}^3$ , the subgroup of  $\mathrm{Isom}(\mathbb{H}^3)$  leaving both geodesics invariant is either trivial or isomorphic to  $\mathbb{Z}_2$ . This implies that the group  $\mathrm{Fix}_\Gamma(\gamma)$  is either trivial or  $\mathbb{Z}_2$ . It is not too hard from the geometry (a slight modification of the argument in [LO07, Prop. 3.6]) to see that the short exact sequence must split as a direct product, forcing the groups we are interested in to each be abstractly isomorphic to one of the four groups  $\mathbb{Z}, D_\infty, \mathbb{Z}_2 \times \mathbb{Z}$ , or  $\mathbb{Z}_2 \times D_\infty$ . Finally, when  $V$  is any of the four virtually cyclic groups listed above, the cokernels of the relative assembly map  $H_*^V(E_{\mathcal{FIN}}V \rightarrow *; \mathbb{K}\mathbb{Z}^{-\infty})$  are known to vanish for  $* \leq 1$  (see [FJ95] for  $\mathbb{Z}, D_\infty$ , and [P98] for  $\mathbb{Z}_2 \times \mathbb{Z}$  and  $\mathbb{Z}_2 \times D_\infty$ ). We conclude that every term in the direct sum splitting for the Waldhausen Nil group vanishes, completing the proof of our last corollary.

## REFERENCES

- [AO06] A. Alves, P. Ontaneda. *A formula for the Whitehead group of a three-dimensional crystallographic group*, *Topology* **45** (2006), 1-25.
- [AM90] D. R. Anderson and H. J. Munkholm. *Geometric modules and algebraic K-homology theory*, *K-theory* **3** (1990), 561-602.
- [AM00] D. R. Anderson and H. J. Munkholm. *Continuously controlled K-theory with variable coefficients*, *J. Pure Appl. Algebra* **145** (2000), 215-266.
- [Ba03] A. Bartels. *On the domain of the assembly map in algebraic K-theory*, *Algebr. Geom. Topol.* **3** (2003), 1037-1050.
- [BLR08a] A. Bartels, W. Lück, and H. Reich. *The K-theoretic Farrell-Jones Conjecture for hyperbolic groups*, *Invent. Math.* **172** (2008), 29-70.

- [BLR08b] A. Bartels, W. Lück, and H. Reich. *On the Farrell-Jones Conjecture and its applications*, J. Topology **1** (2008), 57-86.
- [BF92] M. Bestvina and M. Feighn. *A combination theorem for negatively curved groups*, J. Differential Geom. **35** (1992), 85-101.
- [BF96] M. Bestvina and M. Feighn. *Addendum and correction to: "A combination theorem for negatively curved groups"*, J. Differential Geom. **43** (1996), 783-788.
- [BH] M. Bridson and A. Haefliger. *Metric spaces of non-positive curvature*, Springer-Verlag, Berlin (1999).
- [CFH] F. X. Connolly, B. Fehrman, and M. Hartglass. *On the dimension of the virtually cyclic classifying space of a crystallographic group*, preprint available on the arXiv: math.AT/0610387
- [CP02] F. X. Connolly and S. Prassidis. *On the exponent of the cokernel of the forget-control map on  $K_0$ -groups*, Fund. Math. **172** (2003), no. 3, 201-216.
- [D] J. Davis. *Some remarks on Nil groups in algebraic K-theory*, preprint available on the arXiv: 0803.1641
- [DKR] J. Davis, Q. Khan, and A. Ranicki. *Algebraic K-theory over the infinite dihedral group*, preprint available on the arXiv: 0803.1639
- [Del99] T. Delzant. *Sur l'accessibilité acylindrique des groupes de présentation finie*, Ann. Inst. Fourier (Grenoble) **49** (1999), 1215-1224.
- [F77] F. T. Farrell. *The nonfiniteness of Nil*, Proc. Amer. Math. Soc. **65** (1977), 215-216.
- [FJ93] F. T. Farrell and L. Jones. *Isomorphism conjectures in algebraic K-theory*, J. Amer. Math. Soc. **6** (1993), 249-297.
- [FJ95] F. T. Farrell and L. Jones. *The lower algebraic K-theory of virtually infinite cyclic groups*, K-theory **9** (1995), 13-30.
- [G07] J. Grunewald. *Non-finiteness results for Nil-groups*, Algebr. Geom. Topol. **7** (2007), 1979-1986.
- [G08] J. Grunewald. *The Behavior of Nil-Groups under localization and the relative assembly map*, Topology **47** (2008), 160-202.
- [JL06] D. Juan-Pineda and I. J. Leary. *On classifying spaces for the family of virtually cyclic subgroups*, in "Recent Developments in Algebraic Topology" (Contemp. Math. **407**, pgs. 135-145), Amer. Math. Soc., 2006.
- [Ka99] I. Kapovich. *A non-quasiconvexity embedding theorem for hyperbolic groups*, Math. Proc. Cambridge Philos. Soc. **127** (1999), 461-486.
- [KM98] O. Kharlampovich and A. Myasnikov. *Hyperbolic groups and free constructions*, Trans. Amer. Math. Soc. **350** (1998), 571-613.
- [KT03] A. O. Kuku and G. Tang. *Higher K-theory of group-rings of virtually cyclic infinite groups*, Math. Ann. **325** (2003), 711-726.
- [LO07] J. F. Lafont and I. J. Ortiz. *Relative hyperbolicity, classifying spaces, and lower algebraic K-theory*, Topology **46** (2007), 527-553.
- [LO] J. F. Lafont and I. J. Ortiz. *Lower algebraic K-theory of hyperbolic 3-simplex reflection groups*, to appear in Comment. Math. Helv.
- [Lu05] W. Lück. *Survey on classifying spaces for families of subgroups*, in "Infinite groups: geometric, combinatorial and dynamical aspects" (Progr. Math. **248**, pgs. 269-322), Birkhauser, 2005.
- [LS00] W. Lück and R. Stamm. *Computations of K- and L-theory of cocompact planar groups*, K-theory **21** (2000), 249-292.
- [LW] W. Lück and M. Weiermann. *On the classifying space of the family of virtually cyclic subgroups*, preprint available on the arXiv math.AT/070264.
- [MP99] H. J. Munkholm and S. Prassidis. *Waldhausen's Nil groups and continuously controlled K-theory*, Fund. Math. **161** (1999), 217-224.
- [MP01] H. J. Munkholm and S. Prassidis. *On the vanishing of certain K-theory Nil-groups*, in "Cohomological methods in homotopy theory (Bellaterra, 1998)" (Progr. Math. **196**, pgs. 307-321), Birkhauser, 2001.
- [P98] K. Pearson. *Algebraic K-theory of two dimensional crystallographic groups*, K-Theory **14** (1998), 265-280.
- [Qu82] F. Quinn. *Ends of maps II*, Invent. Math. **68** (1982), 353-424.
- [R] R. Ramos. *Non finiteness of twisted Nils*, preprint (2006).

- [SW79] P. Scott and T. Wall. *Topological methods in group theory*, in “Homological group theory, Proceedings held in Durham in September 1977” (London Math. Soc. Lect. Note Ser. **36**, 137-203), Cambridge Univ. Press, 1979.
- [Ser80] J. - P. Serre. *Trees*, Springer-Verlag, Berlin (1980).
- [Sel97] Z. Sela. *Acylindrical accessibility for groups*, Invent. math. **129** (1997), 527-565.
- [W78a] F. Waldhausen. *Algebraic K-theory of generalized free products. I, II.*, Ann. of Math. (2), **108** (1978), 135-204.
- [W78b] F. Waldhausen. *Algebraic K-theory of generalized free products. III, IV.*, Ann. of Math. (2), **108** (1978), 205-256.
- [We81] C. A. Weibel. *Mayer-Vietoris sequences and module structures on  $NK_*$* , in “Algebraic K-theory, Evanston 1980 (Proc. Conf., Northwestern Univ., Evanston, IL, 1980)” (Springer Lecture Notes in Math. **854**, pgs. 466-493), Springer-Verlag, 1981.

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