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Roundness Properties of Groups

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Abstract. Roundness of metric spaces was introduced by Per Enflo as a tool to study uniform structures of linear topological spaces. The present paper investigates geometric and topological properties detected by the roundness of general metric spaces. In particular, we show that geodesic spaces of roundness 2 are contractible, and that a compact Riemannian manifold with roundness >1 must be simply connected. We then focus our investigation on Cayley graphs of finitely generated groups. One of our main results is that every Cayley graph of a free Abelian group on $\geqslant 2$ generators has roundness $=1$. We show that if a group has no Cayley graph of roundness $=1$, then it must be a torsion group with every element of order $2, 3, 5$, or 7.

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1. Introduction

In a series of papers, Per Enflo [6, 8, 9] used the idea of metric roundness to investigate the uniform structure of Banach spaces. Later the same idea was used in [19] to compare uniform structures between normed and quasi-normed linear topological spaces. An extension of this property (*generalized roundness*) was used by Enflo in the solution of Smirnov's problem [7]. Also, if a metric space has nontrivial generalized roundness, then some positive power of the distance function is a *negative kernel* on the space ([16]). Negative kernels on Cayley graphs of discrete groups were used for proving the coarse Baum–Connes Conjecture (and thus the Novikov Conjecture) for these groups [14, 15].

We investigate the roundness and generalized roundness properties of general metric spaces. The triangle inequality implies that any metric space has roundness at least 1. Using the results of Enflo, essentially, a metric space X has roundness p, $1 < p \leqslant 2$ if p is the supremum of all q so that quadrilaterals in X are *thinner* than the ones in an L_q-space. With this in mind, our first result is not surprising.

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THEOREM. *Every CAT(0)-space has roundness* 2.

On the other hand, it should be noticed that there are CAT(0)-spaces whose generalized roundness is equal to 0 [10]. All spaces with approximate midpoints have roundness between 1 and 2. It is reasonable to try to understand the extremal cases.

THEOREM. *Proper geodesic spaces that have roundness* 2 *are contractible.*

We also point out that, in Section 1.19_+ of [12], Gromov raises the question of determining what types of spaces one can obtain when one imposes a restriction on the distances achieved between all r-tuples of points. The previous theorem can be viewed as a partial answer to this question in the context where the restrictions on the distances between all 4-tuples of points are given by the roundness $=2$ condition.

Combining these two results, we recover the well-known result that any proper CAT(0)-space is contractible.

On the other hand, it is interesting to notice that the roundness properties of metric spaces with nontrivial closed geodesics are very poor. Mild assumptions on such a space imply that its roundness is 1. In particular, we have

THEOREM. *A non-simply connected, compact, Riemannian manifold has roundness* 1.

This is in fact a special case of a more general theorem applying to geodesic metric spaces with nontrivial fundamental group, and satisfying an additional hypothesis on existence of convex neighborhoods around every point. This more general result suggests that, as far as roundness is concerned, the most interesting spaces to look at are simply connected geodesic spaces or, at the other extreme, totally disconnected spaces.

Our main explicit calculations are on discrete metric spaces determined by graphs, in particular Cayley graphs of finitely generated groups. Roundness is not a quasi-isometric invariant and thus, in general, the roundness of a Cayley graph of a group depends on the choice of generating set. A more relevant algebraic invariant seems to be the *roundness spectrum* of a group, which is the collection of the roundness of all the Cayley graphs of the group. One of our main results is

THEOREM. *The roundness spectrum of a finitely generated free abelian group on more than one generator is* $\{1\}$.

In general, the roundness spectrum has the following property:

THEOREM. *If the roundness spectrum of G does not contain* 1 *then G is a purely torsion group in which every element has order* 2, 3, 5 *or* 7.

In [16] it was shown that in spaces with generalized roundness $p > 0$ the pth power of the distance function is a negative kernel. Using this result we show the following:

THEOREM. *Let G be a group having a presentation whose Cayley graph has positive generalized roundness. Then G satisfies the coarse Baum–Connes Conjecture and thus the strong Novikov Conjecture.*

In particular, if G be a group having a presentation whose Cayley graph isometrically embeds into an L_p-space with $1 \leqslant p \leqslant 2$, then G satisfies the coarse Baum–Connes Conjecture and thus the strong Novikov Conjecture (see also [18], Corollary 4.3).

On the other hand negative kernels are closely related to the Kazhdan property. Using this we can show that

THEOREM. *Every Cayley graph of a finitely generated infinite Kazhdan group has generalized roundness* 0.

This result follows from combining the fact that infinite Kazhdan groups do not admit negative kernels [3,4] and the equivalence between nontrivial generalized roundness and negative kernels [16]. It should be noted that generalized roundness is an easier condition to be checked than the existence of negative kernels because generalized roundness is a property of finite subspaces of the space.

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2. Preliminaries

DEFINITION 2.1. Let (X, d) be a metric space, $p \in [1, \infty]$.

(1) The *roundness* of (X, d) is p if p is the supremum of all q such that: for any four points x_{00}, x_{10}, x_{01}, x_{11} in X,

$$\begin{aligned} &d(x_{00}, x_{11})^q + d(x_{01}, x_{10})^q \\ &\quad \leqslant d(x_{00}, x_{01})^q + d(x_{00}, x_{10})^q + d(x_{11}, x_{01})^q + d(x_{11}, x_{10})^q. \end{aligned}$$

(2) The *generalized roundness* of (X, d) is the supremum of all q such that: for every $n \geqslant 2$ and any collection of $2n$-points $\{a_i\}_{i=1}^n$, $\{b_i\}_{i=1}^n$, we have that:

$$\sum_{1 \leqslant i < j \leqslant n} (d(a_i, a_j)^q + d(b_i, b_j)^q) \leqslant \sum_{1 \leqslant i, j \leqslant n} d(a_i, b_j)^q.$$

Remark 2.2. (1) Definition 2.1, Part (1), can be rephrased in terms of 2-cubes. Recall that the unit cube in $\mathbb{R}^n$ $(n \in \mathbb{N})$ is the set of n-vectors $\{0, 1\}^n$. An *n-cube* N in an arbitrary metric space (X, d) is a collection of 2^n (not necessarily distinct) points in X where each point in the collection is indexed by a distinct n-vector $\varepsilon \in \{0, 1\}^n$ from the unit cube. A *diagonal* in N is a pair of vertices $(x_\varepsilon, x_\delta)$ such that ε and δ differ in all coordinates. An *edge* in N is a pair of vertices $(x_\varepsilon, x_\delta)$ such that ε and δ differ in precisely one coordinate. The set of diagonals in N will be denoted $D(N)$ and the set of edges in N will be denoted $E(N)$. An n-cube N has 2^{n-1} diagonals and $n2^{n-1}$ edges. If $f = (x, y)$ is an edge or diagonal in N, we will let $l(f)$ denote the d-length of f in X. In other words, $l(f) = d(x, y)$. The analytic condition in Definition 2.1, Part (1), is a statement about 2-cubes N in X:

$$\sum_{d \in D(N)} l(d)^q \leqslant \sum_{e \in E(N)} l(e)^q.$$

(2) The triangle inequality implies that any metric space has roundness $\geqslant 1$. If the space has approximate midpoints, then its roundness is $\leqslant 2$.

(3) The collection of $2n$ points in the second part of the definition is usually called an n-double simplex.

3. Geometric Aspects of Roundness

Roundness and curvature bounded from above are two metric properties. In this section, we examine their connections. Our first observation is that in spaces with complicated topology, roundness cannot be large. We consider one of the simplest nonsimply connected space first.

LEMMA 3.1. *The roundness of the circle is* 1.

Proof. Let x_{00}, x_{01}, x_{10}, x_{11} be four points on S^1 so that:

$$d(x_{00}, x_{01}) + d(x_{01}, x_{11}) = d(x_{00}, x_{11}), \quad d(x_{01}, x_{11}) + d(x_{11}, x_{10}) = d(x_{01}, x_{10}).$$

Then, for $p > 1$,

$$\begin{aligned} d(x_{00}, x_{11})^p + d(x_{01}, x_{10})^p &= (d(x_{00}, x_{01}) + d(x_{01}, x_{11}))^p + (d(x_{01}, x_{11}) + d(x_{11}, x_{10}))^p \\ &> d(x_{00}, x_{01})^p + d(x_{01}, x_{11})^p + d(x_{01}, x_{11})^p + d(x_{11}, x_{10})^p \end{aligned}$$

Thus the roundness can not be larger than 1. □

Remark 3.2. The generalized roundness of the circle is 1: Lemma 3.1 implies that the generalized roundness is less than or equal to 1. But in [5], Theorem 6.4.5, it is shown that S^1 isometrically embeds into an ℓ_1-space, which has generalized roundness 1 [16].

PROPOSITION 3.3. *Let* (X, d) *be a geodesic metric space that admits a globally minimizing closed geodesic. Then its roundness is* 1.

Proof. A globally minimizing closed geodesic γ is an isometric embedding of a circle of length $\ell(\gamma)$. The Proposition follows from Lemma 3.1. □

LEMMA 3.4. *Let* (X, d) *be a geodesic space. Suppose there is a closed curve* γ *such that*

$$\ell(\gamma) = \inf\{\ell(\xi) : \xi \text{ homotopically nontrivial rectifiable curve}\} > 0.$$

Then the roundness of X *is equal to* 1.

Proof. Assume that the roundness of X is greater than 1. Proposition 3.3 implies that such γ cannot be globally length minimizing. Hence, if $\gamma: [0, L] \to X$ is a unit length parametrization we have, after reparametrizing if necessary, that there is $s \in [0, L/2]$ such that $d(\gamma(0), \gamma(s)) < s$. As X is a geodesic space, there is a curve η from $\gamma(0)$ to $\gamma(s)$ whose length is equal to $d(\gamma(0), \gamma(s))$. Let γ_1 be γ restricted to $[0, s]$, γ_2 be γ restricted to $[s, L]$. Form two new loops $\eta_1 = \eta^{-1} * \gamma_1$, $\eta_2 = \gamma_2 * \eta$. Note that $\eta_2 * \eta_1 \simeq \gamma$. Since γ represents a nontrivial element in $\pi_1(X)$, one of the loops η_1, η_2 must likewise be nontrivial. We now compute the lengths of η_1, η_2:

$$\ell(\eta_1) = \ell(\gamma_1) + \ell(\eta) = s + \ell(\eta) < s + s = 2s \leqslant L,$$
$$\ell(\eta_2) = \ell(\gamma_2) + \ell(\eta) = (L - s) + \ell(\eta) < (L - s) + s = L.$$

So in both cases, we find a homotopically nontrivial loop with length shorter than the assumed minimum L, contradiction. □

The above lemma can be applied to a certain natural class of metric spaces.

DEFINITION 3.5. A metric space (X, d) is called *good* provided that, for each $p \in X$, there is a neighborhood N_p of p with:

(1) N_p is simply connected.
(2) N_p is geodesically convex i.e., for each $y, z \in N_p$ and for each geodesic γ joining y to z with $\ell(\gamma) = d(y, z)$, the trace of γ is contained in N_p.

Remark 3.6. If (X, d) is a Riemannian manifold then (X, d) is good; this follows from the existence of normal neighborhoods. More generally, any Finsler manifold is good (this is due to J. H. C. Whitehead [22]; the authors thank Z. Shen for informing us of this result).

PROPOSITION 3.7. *Let* (X, d) *be a good, compact, geodesic space with nontrivial fundamental group. Then there is a loop* γ *such that*

(1) γ *is not freely homotopic to a constant loop.*

(2) *For each loop* γ' *not freely homotopic to a constant loop,* $\ell(\gamma') \geqslant \ell(\gamma)$

Proof. Let $L = \inf\{\ell(\eta) \mid \eta$ not freely homotopic to a constant loop$\}$ and let $\{\gamma_i\}_{i \in \mathbb{N}}$ be a sequence of loops, each of which is not freely homotopic to a constant loop such that $\ell(\gamma_i) \to L$. We first observe that, without loss of generality, we can assume that γ_i is piecewise geodesic. Indeed, for a given γ_i, we can cover the trace of γ_i with a finite collection of simply connected, geodesically convex neighborhoods N_j, $j = 1, \ldots, k$, since (X, d) is a good geodesic space. Pick t_j, $j = 1, \ldots, k$, in S^1 such that $\gamma_j([t_j, t_{j+1}]) \subset N_j$ and replace $\gamma_j \| [t_j, t_{j+1}]$ by a geodesic lying in N_j and joining $\gamma_j(t_j)$ to $\gamma_j(t_{j+1})$. Since N_j is simply connected, the new loop is freely homotopic to the original γ_j, is piecewise geodesic, and it has length less than or equal to $\ell(\gamma_j)$. Hence this new sequence of loops also has lengths tending to L.

Now parametrize each of these loops with respect to arclength, scaled by $\ell(\gamma_i)$, and let $M = \sup\{\ell(\gamma_i) : i \in \mathbb{N}\}$. Note that $M < \infty$, and that for all i, all $x, y \in S^1$, we have

$$d(\gamma_i(x), \gamma_i(y)) \leqslant \ell(\gamma_i) d_{S^1}(x, y) \leqslant M d_{S^1}(x, y).$$

Hence the family of curves $\{\gamma_i\}$ is equicontinuous, and as X is compact, a subsequence (also denoted $\{\gamma_i\}$) converges to a closed loop γ_∞.

CLAIM 1. γ_∞ is freely homotopic to γ_i, for sufficiently large i.

Proof. The assumptions on X allow us to cover the trace of γ_∞ by a finite sequence of simply connected, geodesically convex neighborhoods N_j, $j = 1, \ldots, k$. As before, choose t_j, $j = 1, \ldots, k$, in S^1 such that $\gamma_j([t_j, t_{j+1}]) \subset N_j$. Note that, since $\gamma_i \to \gamma_\infty$ uniformly, we can also have that, for i sufficiently large, that $\gamma_i([t_j, t_{j+1}]) \subset N_j$. For each $1 \leqslant j \leqslant k$, pick a geodesic η_j joining $\gamma(t_j)$ to $\gamma_\infty(t_j)$. Let $\gamma_i^j = \gamma_i \| [t_j, t_{j+1}]$, $\gamma_\infty^j = \gamma_\infty \| [t_j, t_{j+1}]$. Consider the closed loops $(\gamma_j^i)^{-1} * (\eta_{j+1})^{-1} * \gamma_\infty^j * \eta_j$, and observe that this closed loop lies entirely in N_j. Since N_j is simply connected, this loop is contractible. Concatenating the homotopies on the various pieces, we see that γ_∞ is freely homotopic to γ_i, for i sufficiently large, proving the claim.

Claim 1 implies:

(1) γ_∞ is not freely homotopic to a constant loop.

(2) From the definition of L, we derive that $\ell(\gamma_\infty) \geqslant L$.

The rest of this proof is fairly standard. Replace γ_∞ by a curve γ which is piecewise geodesic, with geodesics joining $\gamma_\infty(t_j)$ and $\gamma_\infty(t_{j+1})$. As before, γ is freely homotopic to γ_∞, hence $\ell(\gamma) \geqslant L$.

CLAIM 2. $\ell(\gamma)=L$.

Proof. Assume not. Then $\ell(\gamma)>L$. Let

$$\varepsilon=\frac{\ell(\gamma)-L}{2k+1}>0.$$

Then there exists a positive integer i such that

(1) $\ell(\gamma_j)-L<\varepsilon$.

(2) $d(\gamma_j(t),\gamma_\infty(t))<\varepsilon$.

As before, set $\gamma_i^j=\gamma_i|[t_j,t_{j+1}]$, $\gamma^j=\gamma|[t_j,t_{j+1}]$. We have that

$$\sum_{j=1}^{k}\left(\ell(\gamma_i^j)+2\varepsilon\right)=\ell(\gamma_i)+2k\varepsilon<L+(2k+1)\varepsilon=\ell(\gamma)=\sum_{j=1}^{k}\ell(\gamma^j).$$

Hence there is j such that $\ell(\gamma_i^j)+2\varepsilon<\ell(\gamma^j)$. But this contradicts the fact that each γ^j is a geodesic. Hence $\ell\leqslant L$. That completes the proof of Claim 2 and the proposition. $\square$

Combining Lemma 3.4, Remark 3.6 and Proposition 3.7 we have:

COROLLARY 3.8. *Let* (X,d) *be a good, compact, geodesic space with non-trivial fundamental group. Then the roundness of* X *is equal to* 1. *In particular, a compact non-simply connected Riemannian manifold has roundness* 1.

Remark 3.9. Corollary 3.8 implies that, from the roundness point of view, the most interesting Riemannian manifolds are the simply connected ones.

PROPOSITION 3.10. *Let* (X,d) *be a CAT(0)-space. Then* (X,d) *has roundness* 2.

Proof. Since CAT(0)-spaces have approximate midpoints ([1], Proposition 1.11), the roundness of (X,d) is $\leqslant 2$. Now we will show that the roundness is at least 2. So let x_{00}, x_{01}, x_{10}, x_{11} be four points in X. Proposition 1.11 in [1] implies that there is a subembedding of the four points in $\mathbb{R}^2$. More precisely, there are points $\overline{x}_{00}$, $\overline{x}_{01}$, $\overline{x}_{10}$, $\overline{x}_{11}$ in $\mathbb{R}^2$ such that $d(x_{ij},x_{k\ell})=d(\overline{x}_{ij},\overline{x}_{k\ell})$, whenever (i,j) and (k,ℓ) are different in one coordinate, and $d(x_{ij},x_{k\ell})\leqslant d(\overline{x}_{ij},\overline{x}_{k\ell})$ whenever they differ in both coordinates. Thus

$$\begin{aligned}
&d(x_{00},x_{11})^2+d(x_{01},x_{10})^2\\
&\quad\leqslant d(\overline{x}_{00},\overline{x}_{11})^2+d(\overline{x}_{01},\overline{x}_{10})^2\\
&\quad\leqslant d(\overline{x}_{00},\overline{x}_{01})^2+d(\overline{x}_{00},\overline{x}_{10})^2+d(\overline{x}_{11},\overline{x}_{01})^2+d(\overline{x}_{11},\overline{x}_{10})^2\\
&\quad= d(x_{00},x_{01})^2+d(x_{00},x_{10})^2+d(x_{11},x_{01})^2+d(x_{11},x_{10})^2
\end{aligned}$$

The second inequality holds because $\mathbb{R}^2$, with the standard metric, has roundness 2. $\square$

Roundness 2 imposes geometric and metric restrictions on the space.

PROPOSITION 3.11. *Let (X, d) be a geodesic metric space of roundness 2. For any two points A and B in X, there is a unique geodesic connecting them.*

Proof. Assume that there are two geodesics between A and B. Let M_i, $i = 1, 2$, be the midpoints on the corresponding geodesics. Apply the roundness 2 inequality:

$$|M_1M_2|^2 + |AB|^2 \leqslant |AM_1|^2 + |M_1B|^2 + |AM_2|^2 + |M_2B|^2 = |AB|^2,$$

where $|xy|$ denotes the distance between the points x and y. The inequality above immediately forces $M_1 = M_2$. Iterating this procedure we see that the two geodesics coincide on a dense set of points, so that by continuity, they must coincide. □

PROPOSITION 3.12. *Let (X, d) be a proper geodesic space such that any pair of points in X can be joined by a unique geodesic segment. Then X is contractible.*

Proof. Let $p \in X$ be the base point, and let I denote the interval $[0, 1]$. Define F: $X \times I \to X$ by letting $F(q, t)$ to be the time-one reparametrization of the geodesic segment joining q to p. To show that F is continuous, let (q, t) be a point in $X \times I$ and $\{(q_n, t_n)\}_{n \geqslant 1}$ a sequence of points that converges to (q, t). If F fails to be continuous at (q, t), then there exists a subsequence, also denoted $\{F(q_n, t_n)\}_{n \geqslant 1}$ with $d(F(q_n, t_n), F(q, t)) \geqslant \varepsilon$, for all n, for some $\varepsilon > 0$. We also obtain that, since $F(q_n, t_n)$ lies on a geodesic joining q_n to p:

$$d(p, F(p_n, t_n)) \leqslant \sup_n \{d(p, q_n)\}.$$

Since $\{q_n\}$ converges to q, the supremum on the right is bounded, hence the points $F(q_n, t_n)$ lie in some closed ball of radius R at p. The properness of the metric of X ensures that there is a convergent subsequence of $\{F(q_n, t_n)\}_{n \geqslant 1}$. After reparametrizing we assume that

$$\lim_{n \to \infty} F(q_n, t_n) = z \neq F(q, t).$$

Set $S = \{q_n \mid n \in \mathbb{N}\}$ with the metric induced from X.

CLAIM. Under the above hypotheses, $d(p, z) = d(p, F(q, t))$ and $d(q, z) = d(q, F(q, t))$.

Proof. The continuity of the distance function implies that the function

$$d(p, -)\colon\ S \to \mathbb{R}$$

is continuous. Notice that $d(p, F(q, t)) = td(p, q)$. Thus the continuity of multiplication implies that

$$\phi = d(p, F(-, -))\colon\ S \times I \to \mathbb{R}$$

is also continuous. Continuity of ϕ along with the fact that $\{(q_n, t_n)\}_{n\geqslant 1}$ converges to (q, t) implies that

$$\lim_{n\to\infty} \phi(q_n, t_n) = \phi(q, t) \Rightarrow \lim_{n\to\infty} d(p, F(q_n, t_n)) = d(p, F(q, t))$$
$$\Rightarrow d(p, \lim_{n\to\infty} F(q_n, t_n)) = d(p, F(q, t)) \Rightarrow d(p, z) = d(p, F(q, t)).$$

As before, the function

$$\psi = d(q, F(-, -)) : S \times I \to \mathbb{R}$$

is continuous. Then

$$\lim_{n\to\infty} \psi(q_n, t_n) = \psi(q, t) \Rightarrow \lim_{n\to\infty} d(q, F(q_n, t_n)) = d(q, F(q, t))$$
$$\Rightarrow d(f(q), \lim_{n\to\infty} F(q_n, t_n)) = d(q, F(q, t)) \Rightarrow d(q, z) = d(q, F(q, t)).$$

This proves the claim.

Using the Claim, one can find a path η joining p to q by concatenating the unique geodesic from p to z and the unique geodesic from z to q. The Claim shows that the length of η is

$$\ell(\eta) = d(p, F(q, t)) + d(F(q, t), q) = d(p, q).$$

The last equality follows because $F(q, t)$ is a point on the geodesic joining p to q. Since $\ell(\eta)$ is equal to the distance between its two end-points, η is a geodesic. Since z belongs to the unique geodesic from p to q, z must lie on η, and the Claim forces $z = F(q, t)$. This contradicts the fact that $d(F(q_n, t_n), F(q, t)) \geqslant \varepsilon > 0$, for all n. □

4. Roundness Properties of Groups

In this section we look at the geometric properties of Cayley graphs of finitely generated groups. The graphs will be considered as discrete metric spaces equipped with the combinatorial distance. In the remainder of this paper, we will consider finite, symmetric (i.e., $g \in \Sigma \Rightarrow g^{-1} \in \Sigma$) generating sets Σ which do not contain the identity. Note that if the group G does not contain any elements of order 2, then the generating sets of G have even cardinality. For a 4-tuple of points w, x, y, z in a metric space X, we use the notation $[w, x, y, z]$ to denote the 1-double simplex whose diagonals are $\{w, y\}$ and $\{x, z\}$. By the roundness of a 1-double simplex we will mean the supremum of exponents for which the roundness inequality holds *for that specific 1-double simplex*. This of course provides an upper bound for the roundness of the space X. We will similarly use the term generalized roundness of a n-double simplex to refer to the supremum of exponents for which the generalized roundness inequality holds for that specific n-double simplex.

The following is well known ([17], Proposition 2). We outline the proof for completeness.

Figure 1.

PROPOSITION 4.1. *Let X be an $\mathbb{R}$-tree. Then the roundness of X is* 2.

Proof. Geodesics in $\mathbb{R}$-trees have midpoints. So the roundness of X is $\leqslant 2$. Now, any four points in an $\mathbb{R}$-tree have a convex hull as in Figure 1.

There are two cases to be considered. One is the quadrilateral $[A, B, C, D]$ and the other is the quadrilateral $[A, C, B, D]$. Direct calculation shows that in both cases the inequality holds for $p=2$. It is also easy to see that the only time that equality holds is if the points A, B, C and D are colinear in that order and $d(A, B)=d(C, D)$. Then the quadrilateral $[A, B, C, D]$ has roundness 2. □

COROLLARY 4.2. *The Cayley graph of a nontrivial free group with the standard set of generators has roundness* 2. *Also the Cayley graph of the free product of finitely many copies of the cyclic group of order* 2 *has roundness* 2.

Remark 4.3. The generalized roundness of a tree is $\geqslant 1$: in [5], Example 19.1.4, it is shown that finite trees can be isometrically embedded into the cube of a finite ℓ_1-space. Notice that any n-double simplex in the tree will be embedded isometrically into an ℓ_1-space. Thus it will have generalized roundness $\geqslant$ 1. Since the roundness of the tree is 2, the generalized roundness is between 1 and 2.

Remark 4.4. Roundness is not an invariant of quasi-isometries of metric spaces: let $\mathrm{Cay}(F_2, \{x, y\})$ be the Cayley graph of the standard presentation of the free group F_2 on two generators x and y. Then $\mathcal{G}$ is a tree and thus it has roundness 2. We will give a different presentation of F_2:

$$F_2=\langle x, y, z_1, z_2, z_3, z_4 : z_1=x^{-1}y,\ z_2=xy,\ z_3=xy^{-1},\ z_4=x^{-1}y^{-1}\rangle.$$

Then in the new Cayley graph $\mathrm{Cay}(F_2, \Sigma)$ there is a quadrilateral as in Figure 2.

The lengths of the sides is 1 and the diagonals have length 2. That implies that the roundness of $\mathrm{Cay}(F_2, \Sigma)$ is equal to 1. But $\mathrm{Cay}(F_2, \{x, y\})$ and $\mathrm{Cay}(F_2, \Sigma)$ are quasi-isometric as they are Cayley graphs of the same group.

We suggest another invariant that comes closer into being a quasi-isometry invariant, at least for infinite groups.

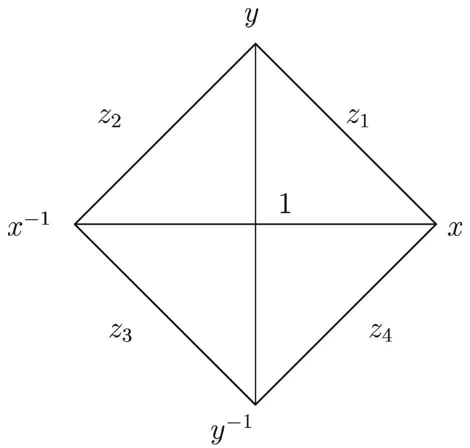

Figure 2.

DEFINITION 4.5. Let G be a finitely presented discrete group. The *roundness spectrum* of G is defined as

$$\rho(G) = \{\rho(\mathrm{Cay}(G, \Sigma)) : \Sigma \text{ a generating set for } G\}.$$

Remark 4.6. (1) In general, $\rho(G) \subseteq [1, \infty]$.

(2) In Remark 4.4, we have shown that $\rho(F_2) \supseteq \{1, 2\}$. If we use the presentation

$$F_2 = \langle x, y, z : z = y^{-1}x \rangle,$$

then the roundness of the $\mathrm{Cay}(F_2, \Sigma)$ is $\leqslant \ln 3/\ln 2$; the authors suspect that the previous inequality is actually an equality.

PROPOSITION 4.7. *Let G be an infinite, finitely generated group. Then* $\rho(G) \subset [1, 2]$.

Proof. Let Σ be a finite presentation of G. Assume that $\mathrm{Cay}(G, \Sigma)$ contains three points, x, y, and z, such that $d(x, y) = d(y, z) = 1$, $d(z, x) = 2$. Then $\rho(\mathrm{Cay}(G, \Sigma)) \leqslant 2$ because y is the midpoint of x and z. If there is no such triple, then the triangle inequality implies that, for all triples x, y, z,

$$d(x, y) = d(y, z) = 1 \Longrightarrow d(z, x) = 1.$$

Therefore, if g and h are generators so is gh. That implies $\Sigma = G$, a contradiction, since Σ is finite and G is infinite. $\square$

Remark 4.8. The previous result is not true for finite groups. For a finite group G, let G be the set of generators. The Cayley graph of this presentation is a finite complete graph. But the roundness of a complete finite graph is ∞. So the roundness spectrum of a finite group always contains ∞.

Actually, the example in the Remark 4.4 suggests a way of constructing Cayley graphs for almost any group whose roundness is 1.

PROPOSITION 4.9. *Let G be a finitely generated group, containing two elements x and y with the property that:*

(1) *x and y do not have order* 2.

(2) $x \neq y^{\pm 1}$.

(3) $x^3 \neq y^{\pm 1}$ *and* $y^3 \neq x^{\pm 1}$.

Then $1 \in \rho(G)$.

Proof. Let Σ be a finite symmetric set of generators of G. Include x and y in the set of generators. If x^2 or y^2 belong to Σ, then remove those generators. Also, include the generators and relations:

$$z_1 = x^{-1}y, \quad z_2 = xy, \quad z_3 = xy^{-1}, \quad z_4 = x^{-1}y^{-1}.$$

Let $\mathcal{G}$ be the Cayley graph in the new presentation. The quadrilateral $[x, y, x^{-1}, y^{-1}]$ has all vertices distinct (by (1) and (2)), and the edges all have length 1, since we added z_i, $i = 1, \ldots, 4$, as generators. The diagonals $d_1 = [x, x^{-1}]$ and $d_2 = [y, y^{-1}]$ have length two. That is because x^2 and y^2 do not belong to the generating set and Conditions (2) and (3) ensure that $x^{\pm 2}$, $y^{\pm 2}$ are not equal to z_i, $i = 1, \ldots, 4$. This 4-point configuration implies that the roundness is 1. □

COROLLARY 4.10. *Assume G is a finitely generated group with $1 \notin \rho(G)$. Then G is a torsion group with every element of order* 2, 3, 5 *or* 7.

Proof. Let $g \in G$ have order n bigger than or equal to 7. A simple counting argument shows that in $\langle g \rangle$, there exists an element g' such that $g' \notin \{g, g^{n-1}, g^3, g^{n-3}\}$ and $(g')^3 \neq g^{\pm 1}$. Then the pair $\{g, g'\}$ satisfies the conditions of Proposition 4.9.

Let G contain an element g of order 4. Then include g in the generating set of G. If g^2 or g^4 are in the generating set then delete them from the generating set. Then the quadrilateral $[1, g, g^2, g^3]$ has roundness 1. That is because from the construction the edges have length 1 and the diagonals have length 2.

If G contains an element g of order 6, include g in the generating set. Delete any generator from the original set which is a power of g. Then $\{g, g^2\}$ satisfies the conditions of Proposition 4.9. □

Remark 4.11. An argument identical to that of Proposition 4.9 and Corollary 4.10 implies that if a graph $\mathcal{G}$ has minimal cycles of length different from 3, 5 and 7, then the roundness of $\mathcal{G}$ is 1.

Let $\mathbb{Z}^2$ denote the free Abelian group on two generators. We will consider $\mathbb{Z}^2$ as the integral lattice in $\mathbb{R}^2$ and we will use coordinates to denote elements of $\mathbb{Z}^2$. Let $\vec{i}=(1,0)$, $\vec{j}=(0,1)$ denote the standard basis of $\mathbb{Z}^2$.

THEOREM 4.12. *If Σ is a generating set for $\mathbb{Z}^2$, then the Cayley graph $\mathcal{G}_\Sigma$ of $(\mathbb{Z}^2,\Sigma)$ has roundness* 1. *In other words, $\rho(\mathbb{Z}^2)=\{1\}$.*

Proof. Before starting the proof we make two simple observations:

(1) If Σ is a finite, symmetric generating set for the group G, and $\phi\in\mathrm{Aut}(G)$, then there is a canonical isometry between $\mathrm{Cay}(G,\Sigma)$ and $\mathrm{Cay}(G,\phi(\Sigma))$; in fact, ϕ induces the isometry.

(2) If Σ is a finite symmetric generating set for $\mathbb{Z}^2$, and there exist g and h in Σ $(g\neq\pm h)$ with $g\pm h\notin\Sigma$, then $[0,g,g+h,h]$ is a 4-tuple with roundness equal to 1.

We will consider cases depending on $|\Sigma|$.

Case 1. $|\Sigma|=4$. Then $\Sigma=\{\pm u,\pm v\}$ where $u,v\in\mathbb{Z}^2$ are linearly independent. Observation (2) immediately applies, hence the roundness of $\mathcal{G}_\Sigma$ equals 1.

Case 2. $|\Sigma|=6$. If Σ is not of the form $\{\pm u,\pm v,\pm(u+v)\}$, then Observation (2) applies and we are done. Hence assume that Σ is of the form above, and observe that $\{\pm u,\pm v\}\subseteq\Sigma$ is already a generating set for $\mathbb{Z}^2$. But we know that $\mathrm{Aut}(\mathbb{Z}^2)$ acts transitively on pairs of generating elements. Hence from Observation (1), it is sufficient to compute the roundness of $\mathcal{G}_\Sigma$ where

$$\Sigma=\{\pm\vec{i},\pm\vec{j},\pm(\vec{i}+\vec{j})\}.$$

We will show that roundness is 1 by contradiction. Assume that the roundness is equal to $p>1$. Consider the quadrilateral with vertices $(0,0)$, $(0,1)$, $(n,0)$, $(n,1)$. Then:

$$(n+1)^p+n^p\leqslant n^p+n^p+1^p+1^p \implies (n+1)^p-n^p\leqslant 2.$$

But, taking limits, and noting that $p>1$,

$$\lim_{n\to\infty}[(n+1)^p-n^p]=\infty.$$

So there is $n\in\mathbb{N}$, such that $(n+1)^p-n^p>2$. Contradiction.

Case 3. $|\Sigma|=2k\geqslant 8$. Theorem 7.1 (proved in the Appendix) implies that Σ contains two elements u and v $(u\neq\pm v)$ with $u\pm v\notin\Sigma$. Hence Observation (2) applies and we are done.

This concludes the argument for Theorem 4.12. $\square$

The proof of Theorem 4.12 can be modified to work for any finitely generated free Abelian group.

THEOREM 4.13. *If* Σ *is a generating set for* $\mathbb{Z}^n$ $(n \geqslant 2)$, *then the Cayley graph* $\mathcal{G}_\Sigma$ *of* $(\mathbb{Z}^n, \Sigma)$ *has roundness* 1. *In other words,* $\rho(\mathbb{Z}^n) = \{1\}$ *whenever* $n \geqslant 2$.

Proof. In Theorem 4.12, the case $n=2$ has been dealt with, so we assume that $n \geqslant 3$. As it was already observed in the proof of Theorem 4.12, if there is a pair u, v in Σ with $u \neq \pm v$ and $u \pm v \notin \Sigma$, the quadrilateral $[0, u, u+v, v]$ has roundness 1 forcing the roundness of the Cayley graph to be 1. Hence if we have a generating set Σ such that the roundness of $\mathcal{G}_\Sigma$ is not 1, then Σ has the property

$$\text{for each } u, v \in \Sigma, u \neq \pm v, \text{ either } u+v \in \Sigma \text{ or } u-v \in \Sigma \tag{$*$}$$

If u and v are two linearly independent elements in Σ, they span a subgroup of $\mathbb{Z}^n$ that is isomorphic to $\mathbb{Z}^2$. Furthermore, the set $\Sigma = \Sigma \cap \langle u, v\rangle$ also satisfies property $(*)$. But the argument in Theorem 7.1 shows that any generating set of $\mathbb{Z}^2$ having property $(*)$ has cardinality 6 and it has (up to relabeling) the form $\Sigma = \{\pm u, \pm v, \pm(u \pm v)\}$. Hence the generating set Σ has the stronger property

$$\text{for all } u, v \in \Sigma, \{u, v\} \text{ linearly independent, either } u+v \in \Sigma \text{ or } u-v \in \Sigma$$
$$\text{but not both.} \tag{$**$}$$

Since $n \geqslant 3$, Σ contains at least three linearly independent elements u, v and w. Using Property $(**)$ we see that, up to relabeling, there are two possible cases:

Case 1. $u+v \in \Sigma$, $u+w \in \Sigma$, $v-w \in \Sigma$.

Case 2. $u+v \in \Sigma$, $u+w \in \Sigma$, $v+w \in \Sigma$.

We now discuss each case separately.

Case 1. Since $u+v \in \Sigma$, $w \in \Sigma$, Property $(**)$ implies that either $u+v+w \in \Sigma$ or $u+v-w \in \Sigma$ but not both.

Let us assume that $u+v+w \in \Sigma$. Since $u+w \in \Sigma$, $v \in \Sigma$, Property $(**)$ implies that $u-v+w \notin \Sigma$. Since $u \in \Sigma$, $v-w \in \Sigma$ property $(*)$ again forces $u+v-w \in \Sigma$. But now we have that $u+v \in \Sigma$, $w \in \Sigma$ and both $(u+v) \pm w \in \Sigma$, contradicting $(**)$.

On the other hand, if $u+v-w \in \Sigma$, since $u+v \in \Sigma$, $w \in \Sigma$, $(**)$ implies that $u+v+w \notin \Sigma$. As $u+w \in \Sigma$, $v \in \Sigma$, $(**)$ forces $u-v+w \in \Sigma$. But now we have $u \in \Sigma$, $v-w \in \Sigma$, and both $u \pm (v-w) \in \Sigma$, contradicting $(**)$. Thus Case 1 cannot occur.

Case 2. In this case, we claim that the assumption implies that Σ must contain $nu+nv+(n-1)w$, $nu+(n-1)v+nw$, $(n-1)u+nv+nw$ for infinitely many $n \in \mathbb{N}$. If this were the case, linear independence of u, v and w implies that all these elements are distinct, contradicting the finiteness of Σ.

We show the Claim by recursion on n. The fact that this triple of vectors with $n=1$ lie in the generating set follows from the hypotheses for Case 2. Notice that if Σ contains $nu+nv+(n-1)w$, $nu+(n-1)v+nw$, and $(n-1)u+nv+nw$, then it must contain the elements

$$2nu+(2n-1)v+(2n-1)w, \quad (2n-1)u+2nv+(2n-1)w,$$
$$(2n-1)u+(2n-1)v+2nw.$$

To see this observe that $(**)$ along with the hypotheses for Case 2 implies that $u-v$, $u-w$ and $v-w$ are not in Σ. The hypotheses along with $(**)$ and

$$\begin{aligned}&[nu+nv+(n-1)w]-[nu+(n-1)v+nw]=v-w\notin\Sigma\\ &\Rightarrow[nu+nv+(n-1)w]+[nu+(n-1)v+nw]=2nu+(2n-1)v+(2n-1)w\in\Sigma\end{aligned}$$

One applies the same reasoning to obtain the other two elements. So we obtain that indeed:

$$\left.\begin{array}{l} nu+nv+(n-1)w\in\Sigma\\ nu+(n-1)v+nw\in\Sigma\\ (n-1)u+nv+nw\in\Sigma\end{array}\right\}\Longrightarrow\left\{\begin{array}{l}2nu+(2n-1)v+(2n-1)w\in\Sigma\\ (2n-1)u+2nv+(2n-1)w\in\Sigma\\ (2n-1)u+(2n-1)v+2nw\in\Sigma\end{array}\right.$$

But now for this second set of elements of Σ, we see that the three differences are $u-v$, $u-w$, and $v-w$ are not in Σ, hence their sums must be in Σ; so we have:

$$\left.\begin{array}{l} nu+nv+(n-1)w\in\Sigma\\ nu+(n-1)v+nw\in\Sigma\\ (n-1)u+nv+nw\in\Sigma\end{array}\right\}\Longrightarrow\left\{\begin{array}{l}(4n-1)u+(4n-1)v+(4n-2)w\in\Sigma\\ (4n-1)u+(4n-2)v+(4n-1)w\in\Sigma\\ (4n-2)u+(4n-1)v+(4n-1)w\in\Sigma\end{array}\right.$$

Finally, observe that for $n\in\mathbb{N}$, $4n-1>n$. We conclude that

$$\left.\begin{array}{l} nu+nv+(n-1)w\in\Sigma\\ nu+(n-1)v+nw\in\Sigma\\ (n-1)u+nv+nw\in\Sigma\end{array}\right\}\quad\text{for } n=1,3,11,43,171,\ldots$$

giving the desired contradiction in Case 2.

As we obtain a contradiction in all case, we conclude that there is no finite symmetric generating set Σ having property $(*)$, and hence $\mathcal{G}_\Sigma$ has roundness 1. □

COROLLARY 4.14. *Let Σ be a finite generating set of $\mathbb{Z}^n$. Then the Cayley graph $\mathcal{G}_\Sigma$ has generalized roundness* $\leqslant 1$.

Remark 4.15. The attentive reader might wonder whether there is a simpler proof for Theorem 4.13, and indeed might be tempted to argue as follows. Take a pair of linearly independent vectors from the generating set for $\mathbb{Z}^n$, and consider the $\mathbb{Z}^2$ subgroup they generate. From Theorem 4.12, this subgroup has generalized roundness $=1$, hence there are configurations in the subgroup whose roundness is $=1$, which would force the roundness of $\mathbb{Z}^n$ to likewise be $=1$. The problem with this approach is that the distance on the $\mathbb{Z}^2$ subgroup induced by the ambient $\mathbb{Z}^n$ might *not*, *à priori*, be isometric to a Cayley graph of $\mathbb{Z}^2$. In fact, this approach can be tweaked to give an easy proof in most cases. As long as there is a pair of linearly independent vectors $u,v\in\Sigma$ with the property that $|\langle u,v\rangle\cap\Sigma|\neq 6$, the argument outlined above can be modified to work.

5. Generalized Roundness and Baum–Connes Conjecture

Generalized roundness is connected with the existence of negative kernels which are used in proving certain forms of the Baum–Connes Conjecture.

DEFINITION 5.1. Let X be a set. A real valued function h on $X \times X$ is called a negative kernel provided that

(1) $h(x, x) = 0$, for all $x \in X$.
(2) $h(x, y) = h(y, x)$, for all $x, y \in X$.
(3) For all n-tuples $x_1, x_2, \ldots, x_n$ in X and $a_1, a_2, \ldots, a_n$ in $\mathbb{R}$ satisfying $\sum_{i=1}^n a_i = 0$, we have that

$$\sum_{i,j=1}^{n} a_i a_j h(x_i, x_j) \leqslant 0.$$

In [16], it was shown that

PROPOSITION 5.2. *In a metric space X, the pth power of the distance function is a negative kernel if and only if it has generalized roundness $\geqslant p$.*

An immediate application of the above result is to the generalized roundness of Kazhdan groups [3,4].

PROPOSITION 5.3. *Let Σ be a finite generating set for an infinite Kazhdan group G and $\mathcal{G}_\Sigma$ the corresponding Cayley graph. Then the generalized roundness of $\mathcal{G}_\Sigma$ is* 0.

Proof. Assume that the generalized roundness of $\mathcal{G}_\Sigma$ is $p > 0$. Then by Proposition 5.2 we have that $d_\Sigma^p \colon G \times G \to \mathbb{R}$ is a negative kernel. Define $\Phi_p \colon G \to \mathbb{R}$ by

$$\Phi_p(g) = d_\Sigma^p(g, e).$$

Then by the left invariance of the metric on $\mathcal{G}_\Sigma$, we get that $d_\Sigma^p(x, y) = \Phi_p(x^{-1}y)$. Furthermore, observe that if $z_j \in \mathbb{C}$, $j = 1, \ldots, n$ satisfy $\sum_{j=1}^n z_j = 0$, then for any collection of n elements g_j of G, an easy computation yields that:

$$\sum_{j,k=1}^{n} z_j \overline{z_k} d_\Sigma^p(g_j, g_k) \leqslant 0.$$

Since G is Kazhdan, this implies that Φ_p is bounded (see Delorme [4]). But $p > 0$ and G is infinite, hence we obtain a contradiction. □

To apply the above to the coarse Baum–Connes conjecture we need the following definition:

DEFINITION 5.4. Let X, Y be a pair of metric spaces. A (not necessarily continuous) map $f: X \to Y$ is a coarse embedding if there are nondecreasing proper function $\rho_{\pm}$: $[0, \infty) \to [0, \infty)$ such that:

$$\rho_{-}(d_X(x, y)) \leqslant d_Y(x, y) \leqslant \rho_{+}(d_X(x, y)), \quad \text{for all } x, y \in X,$$

and with $\lim_{t \to \infty} \rho_{-}(t) = \infty$. Of particular interest is the case where Y is a Hilbert space, with distance induced by the norm. A discrete metric space X is said to have bounded geometry, provided that for every $r > 0$, there exists a uniform upper bound $N(r)$ on the cardinality of the metric balls of radius r.

Note that a composition of coarse embeddings is still a coarse embedding. Furthermore, if Γ is a finitely generated group, then the identity map provides a coarse embedding from any Cayley graph of Γ to any other Cayley graph of Γ. Hence if one Cayley graph coarsely embedds into Hilbert space, they all coarsely embedd into Hilbert space. In this situation we will say that the group Γ coarsely embedds into Hilbert space, and ignore any reference to a Cayley graph.

Now Yu ([23]) has shown that discrete metric spaces with bounded geometry that are coarsely embeddable into a Hilbert space satisfy the coarse Baum–Connes conjecture. In particular, since Cayley graphs of finitely generated groups have bounded geometry, if a finitely generated group coarsely embedds into Hilbert space, then the coarse Baum–Connes Conjecture holds for the space, and hence the strong Novikov conjecture holds for the group in question (see [23]). Recall that the strong Novikov conjecture asserts the injectivity of the classical assembly map for topological K-theory, and implies (amongst other things) the original Novikov conjecture: that the higher signatures are homotopy invariants.

THEOREM 5.5. *Let Γ be a finitely generated group, and assume that Γ coarsely embedds into a metric space X with generalized roundness $p > 0$. Then Γ coarsely embedds in Hilbert space. In particular, Γ must satisfy the coarse Baum–Connes conjecture, and hence the strong Novikov conjecture.*

Proof. We start by observing that, since X has generalized roundness $p > 0$, the pth power of the distance function is a negative kernel. Next we recall that a classic result of Schoenberg [20] states that given a negative kernel h on a set X, there exists a map f: $X \to \mathcal{H}$ into a Hilbert space $\mathcal{H}$ with the property that $h(x, y) = ||f(x) - f(y)||^2$. So in our setting, there exists a map f: $X \to \mathcal{H}$ with the property that $d_X^p(x, y) = ||f(x) - f(y)||^2$ for all $x, y \in X$. In particular, the map f is a coarse embedding, with $\rho_{-}(t) = \rho_{+}(t) = t^{p/2}$. Since Γ coarsely embedds into X by hypothesis, the composition yields the desired coarse embedding into $\mathcal{H}$. $\square$

Two special cases are worth pointing out. Note that an isometric embedding is a coarse embedding, and a quasi-isometric embedding is also a coarse embedding. Furthermore, if a group acts properly discontinuously, cocompactly, freely, and isometrically, on a space X, then Γ and X are quasi-isometric. This immediately yields:

COROLLARY 5.6. *Let* Γ *be a finitely generated group,* X *a metric space with generalized roundness* >0*, and assume that either of the following holds:*

(1) *a Cayley graph of* Γ *isometrically embedds into* X*, or*
(2) Γ *acts properly discontinuously, cocompactly, with finite stabilizers, by isometries on* X.

Then Γ *is coarsely embeddable into Hilbert space. In particular,* Γ *must satisfy the coarse Baum–Connes conjecture, and hence the strong Novikov conjecture.*

Note that a special case of the above corollary is the situation where some Cayley graph of Γ has generalized roundness >0. To obtain some further examples, we note that in [16], it was proved that the Banach spaces $L_p(\mu)$ (with $1 \leqslant p \leqslant 2$) have generalized roundness $\geqslant p$. Hence we have

COROLLARY 5.7. *Assume that the Cayley graph of a group* Γ *admits an isometric embedding into an* $L_p(\mu)$ *space with* $1 \leqslant p \leqslant 2$*. Then* Γ *satisfies the coarse Baum–Connes conjecture and thus the strong Novikov conjecture.*

We point out that a somewhat more general version of Corollary 5.7 can be found in the work of Nowak [18]. We also mention that in the book by Deza–Laurent ([5], Chapter 19), conditions are given for graphs to be embeddable into an ℓ_1-space. A natural question to ask is whether a converse to Corollary 5.6 can hold. Our next result is a partial counterexample to the converse:

PROPOSITION 5.8. *There exists a group* Γ *which is coarsely embeddable into Hilbert space, but fails to satisfy the hypotheses in Corollary 5.6.*

Proof. In Proposition 5.3, we showed that all Cayley graphs of finitely generated Kazhdan groups have generalized roundness $= 0$. In particular, if Γ is a uniform lattice in $Sp(n,1)$ or $F_{4(-20)}$, then Γ is Kazhdan (see [3]) and, hence, every Cayley graph of Γ has generalized roundness $=0$. This implies that Γ cannot be isometrically embedded into any space X with generalized roundness >0 and, hence, fails to satisfy hypothesis (1) in Theorem 5.7.

Next note that if Γ satisfies hypothesis (2) in Theorem 5.7, then picking a point $x \in X$, one can define a new distance d_Γ on Γ by setting $d_\Gamma(g,h) := d_X(g \cdot x, h \cdot x)$. Note that this distance is left-invariant under the natural Γ action on itself. Furthermore, with this distance, the map $\phi\colon \Gamma \to X$ given by $\phi(g) = g \cdot x$ is an isometric embedding, and hence d_Γ must have generalized roundness >0. But now the argument given in Proposition 5.3 applies verbatim and yields a contradiction.

Finally, we note that Γ acts isometrically on a quaternionic hyperbolic space or on the Cayley hyperbolic plane, hence Γ is δ-hyperbolic. But Sela [21] has proved that δ-hyperbolic groups uniformly embedd into Hilbert space, giving the desired result. □

Let us point out that a consequence of work of Faraut–Harzallah [10] implies that the generalized roundness of quaternionic hyperbolic spaces and of the Cayley hyperbolic plane is $=0$. Note however that this does not, *à priori*, imply our Proposition 5.3 for uniform lattices in $\mathrm{Sp}(n,1)$ or $F_{4(-20)}$. Indeed, the difficulty again lies in that generalized roundness is not well behaved with respect to coarse embeddings.

We conclude this section by pointing out that Gromov [13] has established the existence of finitely generated groups whose Cayley graph *cannot* be uniformly embedded into Hilbert space. An immediate consequence of Corollary 5.7 is the following:

COROLLARY 5.9. *The groups constructed by Gromov in* [13] *cannot:*

(1) *have a Cayley graph that isometrically embedds into a space of generalized roundness* >0*,*
(2) *act properly discontinuously, cocompactly, with finite stabilizers, by isometries on a space with generalized roundness* >0.

6. Open Problems

The calculations presented in this paper suggest a few of questions related to roundness and generalized roundness.

QUESTION. Is every CAT(0) space coarsely equivalent to a space with positive generalized roundness?

Using Theorem 5.5, a positive answer to this question would imply the coarse Baum–Connes Conjecture for groups acting properly discontinuously, freely and cocompactly by isometries on $CAT(0)$-spaces. Note that while the Novikov Conjecture is known for these groups [2, 11], the coarse Baum–Connes is still open.

Concerning compact Riemannian manifolds, one can ask:

QUESTION. Does every compact Riemannian manifold contain a globally minimizing closed geodesic? Do they always have roundness $=1$?

We have answered both questions (see Proposition 3.7) for compact Riemannian manifolds with nontrivial fundamental group. If the answer to the first question were affirmative in general, our Proposition 3.1 would imply that all compact Riemannian manifolds have roundness $=1$.

In view of the fact that one of our main results is the computation of the roundness of Cayley graphs of finitely generated free Abelian groups, it is natural to ask:

QUESTION. What is the generalized roundness of a Cayley graph of $\mathbb{Z}^n$?

And more generally, for the application to the Novikov conjecture, we can ask:

QUESTION. Which finitely generated groups have a Cayley graph with positive generalized roundness?

It is clear from this paper that many of the difficulties in working with roundness and generalized roundness arise from the fact that these metric invariants are not coarse invariants. The authors believe that the development of coarse analogues of roundness and generalized roundness would be useful. The main hope would be that such a generalization would allow the results in Theorem 5.5 to apply to a broader class of groups.

7. Appendix

We will show the combinatorial result used in the proof of Theorem 4.12. As before, let $\mathbb{Z}^2$ denote the free Abelian group on two generators. Also, we embed $\mathbb{Z}^2$ as the integral lattice in $\mathbb{R}^2$ and we will use coordinates to denote elements of $\mathbb{Z}^2$. Let $\vec{i}=(1,0)$, $\vec{j}=(0,1)$ denote the standard basis of $\mathbb{Z}^2$. Let $\|-\|$ denote the usual norm on $\mathbb{R}^2$.

THEOREM 7.1. *Given a finite symmetric generating set* Σ *with* $|\Sigma|\geqslant 8$, *then there is a pair* g *and* h *in* Σ, *such that* $g\pm h\notin\Sigma$.

Proof. Let Σ be a minimal generating set of cardinality bigger than or equal to 8, that satisfies property $(*)$:

$$\text{for each } g,h\in\Sigma, g\neq\pm h, \text{ either } g+h\in\Sigma \text{ or } g-h\in\Sigma \qquad (*)$$

Then for any pair α, β of linearly independent elements of Σ we have that either

(1) $\langle\alpha,\beta\rangle=\mathbb{Z}^2$ or

(2) $|\Sigma\cap\langle\alpha,\beta\rangle|=6$.

Indeed, if $\langle\alpha,\beta\rangle$ does not generate all of $\mathbb{Z}^2$, then it generates a proper subgroup (isomorphic to $\mathbb{Z}^2$) and, hence, $|\Sigma\cap\langle\alpha,\beta\rangle|<|\Sigma|$. But the subset $|\Sigma\cap\langle\alpha,\beta\rangle|$ is a generating set for the subgroup $\langle\alpha,\beta\rangle$ (which is abstractly a $\mathbb{Z}^2$), and inherits the property (*). By minimality of the cardinality of Σ, this implies that $|\Sigma\cap\langle\alpha,\beta\rangle|=6$.

Notice that in case (2) above, we have that either $\alpha+\beta\in\Sigma$, or $\alpha-\beta\in\Sigma$, but *not both*. We now break up the argument into cases.

(i) Assume that Σ contains two elements that generate $\mathbb{Z}^2$. Then, after applying an element of $\mathrm{SL}(2,\mathbb{Z})$, we may assume that $\vec{i}$ and $\vec{j}$ and $\vec{i}+\vec{j}$ are in Σ.

(i.a) Assume that Σ contains a vector $\vec{v}=(v_1,v_2)$ such that $\min\{|v_1|,|v_2|\}>2$. We will show the proof when $\vec{v}$ is in the first quadrant. The other cases follow similarly. Notice that the pair $\{\vec{i},\vec{v}\}$ is a linearly independent subset but it does not generate $\mathbb{Z}^2$, since $|v-2|>2$. The same true is for the pair $\{\vec{j},\vec{v}\}$, since $|v_1|>2$. Thus the two pairs satisfy condition (2). Therefore either $\vec{i}+\vec{v}\in\Sigma$ or $\vec{i}-\vec{v}\in\Sigma$ but not both. Choose $\vec{v}$ to have maximal norm among all elements of Σ with both coordinates bigger than 2. Since $\|\vec{v}+\vec{i}\|>\|\vec{v}\|$, and $\|\vec{v}+\vec{j}\|>\|\vec{v}\|$ the maximality of $\|\vec{v}\|$ implies that $\vec{v}-\vec{i}\in\Sigma$, and $\vec{v}-\vec{j}\in\Sigma$. Now consider the pair $\{\vec{v}-\vec{i},\vec{v}-\vec{j}\}$. It is a linearly independent subset and it does not generate $\mathbb{Z}^2$. Hence, either $(\vec{v}-\vec{i})+(\vec{v}-\vec{j})\in\Sigma$, or $(\vec{v}-\vec{i})-(\vec{v}-\vec{j})\in\Sigma$, but not both. Since $\|(\vec{v}-\vec{i})+(\vec{v}-\vec{j})\|>\|\vec{v}\|$, the maximality of $\|\vec{v}\|$ implies that $(\vec{v}-\vec{i})-(\vec{v}-\vec{j})=\vec{j}-\vec{i}\in\Sigma$. Again $\{\vec{j}-\vec{i},\vec{v}\}$ are linearly independent and they do not generate $\mathbb{Z}^2$, so the sum or the difference, but not both are in Σ. Assume, without loss of generality, that $\vec{v}+(\vec{j}-\vec{i})\in\Sigma$. Then

$$\vec{v}-(\vec{j}-\vec{i})\notin\Sigma. \tag{\#}$$

Since $\{\vec{v}-\vec{i},\vec{j}\}$ are linearly independent, do not generate $\mathbb{Z}^2$, and their sum is in Σ, their difference is not:

$$\vec{v}-\vec{i}-\vec{j}\notin\Sigma. \tag{\#\#}$$

So $\{\vec{v}-\vec{j},\vec{i}\}$ are linearly independent, do not generate $\mathbb{Z}^2$, and both $(\vec{v}-\vec{j})+\vec{i}\notin\Sigma$ (by #) and $(\vec{v}-\vec{j})-\vec{i}\notin\Sigma$ (by ##), a contradiction.

(i.b) Assume that there is $\vec{v}=(v_1,v_2)$ in Σ with $|v_1|$ maximal and bigger than 2. We assume $v_1>0$, the case $v_1<0$, follows from an identical argument. Since $\vec{v}+\vec{j}$ or $\vec{v}-\vec{j}$ belongs to Σ, we may assume $v_2\neq 0$.

Case 1. Let $v_2>0$. Choose $\vec{v}$ so that $|v_1|$ is maximal and bigger than 2, and v_2 is positive and maximal. Then we have

$\vec{v}-\vec{i}\in\Sigma$ (maximality of v_1, and $v_1>0$),
$\vec{v}-\vec{j}\in\Sigma$ (maximality of v_2 amongst $\vec{v}$ with v_1 maximal),
$\vec{v}-\vec{i}-\vec{j}\in\Sigma$ (maximality of v_1 and $\{\vec{v},\vec{i}+\vec{j}\}\subset\Sigma$ with v_1 maximal).

Since $\vec{v}-\vec{j}$ and $\vec{j}$ are linearly independent, do not generate $\mathbb{Z}^2$ and $\vec{v}\in\Sigma$, $\vec{v}-2\vec{j}\notin\Sigma$. Also, since $\vec{v}-\vec{i}-\vec{j}$ and $\vec{j}$ are linearly independent, do not generate $\mathbb{Z}^2$ and $\vec{v}-\vec{i}\in\Sigma$, $\vec{v}-\vec{i}-2\vec{j}\notin\Sigma$. But since $\vec{v}-\vec{j}\in\Sigma$, $\vec{i}+\vec{j}\in\Sigma$ are linearly independent and do not generate $\mathbb{Z}^2$ we have

$$\begin{aligned}\text{either: } &(\vec{v}-\vec{j})-(\vec{i}+\vec{j})=\vec{v}-\vec{i}-2\vec{j}\in\Sigma\\ \text{or: } &(\vec{v}-\vec{j})+(\vec{i}+\vec{j})=\vec{v}+\vec{i}\in\Sigma\end{aligned}$$

But, as explained before, the first case could not occur. Thus $\vec{v}+\vec{i}\in\Sigma$, contradicting the maximality of v_1.

Case 2. We assume, as before, that $\vec{v} \in \Sigma$ with $v_1 > 0$ maximal and bigger than 2 and $v_2 < 0$. As in Case 1, $\vec{v} + \vec{j} \in \Sigma$ from the minimality of v_2. As in Case 1 again, This forces $\vec{v} - \vec{j} \notin \Sigma$ (minimality of v_2) and $\vec{v} - 2\vec{i} - \vec{j} \in \Sigma$ because $\{\vec{v} - \vec{i} - \vec{j}, \vec{i}\}$ are linearly independent, do not generate $\mathbb{Z}^2$ and $\vec{v} - \vec{j} \notin \Sigma$. Hence both $(\vec{v} - \vec{i}) \pm (\vec{i} + \vec{j}) \in \Sigma$. However, these are linearly independent and do not generate $\mathbb{Z}^2$, contradiction.

(i.c) The same argument shows that we can also exclude the case $|v_2| > 2$. Also, Σ is invariant under taking negatives. That is we need to exclude the following points:

$$(2, \pm 1), (2, \pm 2), (2, 0), (1, 2), (1, -1), (2, 1), (0, 2), (-1, 2).$$

We just consider each case separately:

(a) Assume that $\vec{v} = (2, 2) \in \Sigma$. Then, $(2, 1)$ and $(1, 2)$ are in Σ. Since the sum of the last two vectors is not in Σ, $(2, 1) - (1, 2) = (1, -1) \in \Sigma$. But then $\vec{v} + \langle 1, -1 \rangle$ or $\vec{v} + (1, -1)$ must be in Σ. Contradiction because one of the coordinates is greater than 2.

(b) Assume that $\vec{v} = (1, 2) \in \Sigma$. Then, $(0, 2)$ is in Σ. Since $(1, 1)$ is in Σ, $(-1, 1)$ is in Σ, which implies that $(2, 1)$ is in Σ. By (a), $(2, 1) + \vec{j} \notin \Sigma$, we get $(2, 0)$ is in Σ. Since $(1, 1)$ and $(-1, 1)$ are in Σ and do not generate $\mathbb{Z}^2$ then only one of $(1, 1) \pm (-1, 1)$ can be in Σ. That is a contradiction, because both $(2, 0)$ and $(0, 2)$ are in Σ.

(c) In all the other cases, it is easy to see that $(-1, 1) \in \Sigma$. By applying the matrix

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

we can exclude $(-1, 2)$ $(-2, 2)$ (corresponding to cases (a) and (b) above). That implies that $(0, 2) \notin \Sigma$, because $(0, 2) \pm \vec{i} \notin \Sigma$, from the previous cases. Also, $(2, 1)$ can be excluded because $(2, 1) \pm \vec{j} \notin \Sigma$. All the other cases can be excluded similarly, except when $(-1, 1) \in \Sigma$. If $(-1, 1) \in \Sigma$ then $(-1, 1) \pm (1, 1) \notin \Sigma$, from the previous cases.

(ii) Assume that no two elements of Σ generate $\mathbb{Z}^2$. Let $\overrightarrow{x}$ and $\overrightarrow{y}$ be two linearly independent elements of Σ. Then, after applying an element of $\mathrm{SL}(2, \mathbb{Z})$ we can assume that $m\vec{i}$ and $n\vec{j}$ belong to Σ. If neither of the vectors $m\vec{i} \pm n\vec{j}$ are in Σ, then we are done. So let us assume that $m\vec{i} + n\vec{j} \in \Sigma$.

We now define three sets of points in $\mathbb{Z}^2$:

- $\mathcal{L}_x$ consists of the integral points lying on the lines $y = 0$, $y = \pm n$, $y = \pm 2n$,
- $\mathcal{L}_y$ consists of the integral points lying on the lines $x = 0$, $x = \pm m$, $x = \pm 2m$,
- $\mathcal{L}_{xy}$ consists of the integral points lying on the lines $y = (n/m)x$, $y = (n/m)x \pm n$, $y = (n/m)x \pm 2n$.

Note that these subsets have the property that all of their pairwise intersections lie in the subgroup of $\mathbb{Z}^2$ generated by the pair $(m\vec{i}, n\vec{j})$. This implies that the generating set Σ must contain some vector $\vec{v}$ with the property that $\vec{v} \notin (\mathcal{L}_x \cap \mathcal{L}_y) \cup (\mathcal{L}_x \cap \mathcal{L}_{xy}) \cup (\mathcal{L}_y \cap \mathcal{L}_{xy})$. But now from basic set theory, we can conclude that $\vec{v} \notin (\mathcal{L}_x \cup \mathcal{L}_y) \cap (\mathcal{L}_x \cup \mathcal{L}_{xy}) \cap (\mathcal{L}_y \cup \mathcal{L}_{xy})$. Hence the vector $\vec{v}$ fails to lie in one of the pairwise intersection. At the cost of applying an automorphism of $\mathbb{Z}^2$, we may assume that we have a $\vec{v} \in \Sigma$ satisfying $\vec{v} \notin \mathcal{L}_x \cup \mathcal{L}_y$.

But now observe that the argument in Case (i.a) works equally well in this setting. Indeed, the fact that $\vec{v} \notin \mathcal{L}_x \cup \mathcal{L}_y$ implies that all of the vectors $\vec{v} + (\epsilon_1 m\vec{i}) + (\epsilon_2 n\vec{j})$ are linearly independent from both $m\vec{i}$ and $n\vec{j}$, where each $\epsilon_i \in \{0, \pm 1, \pm 2\}$. In particular, carrying out the argument in Case (i.a) but replacing each $\vec{i}, \vec{j}$ in that argument by $m\vec{i}, n\vec{j}$, we still have linear independence at all the required steps. Hence we again obtain a contradiction. This concludes the proof of Theorem 7.1. $\square$

References

1. Bridson, M. R. and Haefliger, A.: *Metric Spaces of Non-positive Curvature*, Grundlehren Math. Wiss. 319, Springer-Verlag, Berlin, 1999.
2. Carlsson, G. and Pedersen, E. K.: Controlled algebra and the Novikov conjectures for K- and L-theory, *Topology* **34** (1995), 731–758.
3. de la Harpe, P. and Valette, A.: *La propriété (T) de Kazhdan pour les groupes localement compacts* (avec un appendice de Marc Burger), *Astrisque* No. 175 (1989), 158 pp.
4. Delorme, P.: 1-cohomologie des représentations unitaires des groupes de Lie semisimples et résolubles, *Bull. Soc. Math. France* **105** (1977), 281–336.
5. Deza, M. M. and Laurent, M.: *Geometry of Cuts and Metrics*, Algorithms Combin. 15, Springer-Verlag, Berlin, 1997.
6. Enflo, P.: On the non-existence of uniform homeomorphisms between $\mathrm{L_p}$-spaces, *Ark. Mat.* **8** (1969), 103–105.
7. Enflo, P.: On a problem of Smirnov, *Ark. Mat.* **8** (1969), 107–109.
8. Enflo, P.: Uniform structures and square roots in topological groups, I, *Israel J. Math.* **8** (1970), 230–252.
9. Enflo, P.: Uniform structures and square roots in topological groups, II, *Israel J. Math.* **8** (1970), 253–272.
10. Faraut, J. and Harzallah, K.: Distances hilbertiennes invariantes sur un espace homogène, *Ann. Inst. Fourier (Grenoble)* **24** (1974), 171–217.
11. Farrell, F. T. and Lafont, J.-F.: EZ-structures and topological applications, *Comment. Math. Helv.* **80** (2005) 103–121.
12. Gromov, M.: *Metric Structures for Riemannian and Non-Riemannian Spaces*, (English summary) Progr. in Math. 152, Birkhäuser, Boston, 1999.
13. Gromov, M.: Random walk in random groups, *Geom. Funct. Anal.* **13** (2003), 73–146.
14. Guntner, E., Higson, N. and Weinberger, A.: The Novikov Conjecture for Linear Groups, preprint.
15. Guentner, E. and Kaminker, J.: Exactness and the Novikov conjecture, *Topology* **41** (2002), 411–418.
16. Lennard, C. J., Tonge, A. M. and Weston, A.: Generalized roundness and negative type, *Michigan Math. J.* **44** (1997), 37–45.
17. Naor, A. and Schechtman, G.: Remarks on non linear type and Pisier's inequality, *J. Reine Angew. Math.* **552** (2002), 213–236.

18. Nowak, P. W.: Coarse embeddings of metric spaces into Banach spaces, *Proc. Amer. Math. Soc.* **133** (2005), 2589–2596.
19. Prassidis, E. and Weston, A.: Uniform Banach groups and structures, *C.R. Math. Acad. Sci. Soc. R. Can*. **26** (2004), 25–32.
20. Schoenberg, I. J.: Metric spaces and positive definite functions, *Trans. Amer. Math. Soc*. **44** (1938), 522–536.
21. Sela, Z.: Uniform embeddings of hyperbolic groups in Hilbert spaces, *Israel J. Math*. **80** (1992), 171–181.
22. Whitehead, J. H. C.: Convex regions in the geometry of paths, *Quart. J. Math. Oxford, Ser*. **3** (1932), 33–42.
23. Yu, G.: The coarse Baum–Connes conjecture for spaces which admit a uniform embedding into Hilbert space, *Invent. Math*. **139** (2000), 201–240.