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JEAN-FRANÇOIS LAFONT AND CHRISTOPHE PITTET

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We compare the l^1-seminorm $\|\cdot\|_1$ and the manifold seminorm $\|\cdot\|_{\mathrm{man}}$ on n-dimensional integral homology classes. Crowley and Löh showed that for any topological space X and any $\alpha \in H_n(X;\mathbb{Z})$, with $n \neq 3$, the equality $\|\alpha\|_{\mathrm{man}} = \|\alpha\|_1$ holds. We compute the simplicial volume of the 3-dimensional Tomei manifold and apply Gaifullin's desingularization to establish the existence of a constant $\delta_3 \approx 0.0115416$, with the property that for any X and any $\alpha \in H_3(X;\mathbb{Z})$, one has the inequality

$$\delta_3\|\alpha\|_{\mathrm{man}} \leq \|\alpha\|_1 \leq \|\alpha\|_{\mathrm{man}}.$$

1. Introduction

Let X be a topological space and let K be either the field of rational numbers or the field of real numbers. Let $\alpha \in H_n(X, K)$ be a class in the n-dimensional singular homology of X with coefficients in K. By definition there is a finite linear combination of continuous maps $\sigma_i : \Delta \to X$ defined on the standard n-dimensional simplex, with coefficients a_i in K, which represents α. The l^1-*(semi)norm* on singular homology is defined as

$$\|\alpha\|_1 = \inf\Big\{\sum |a_i| : \Big[\sum a_i\sigma_i\Big] = \alpha\Big\};$$

see [Gromov 1982, 0.2].

If $\alpha \in H_n(X, \mathbb{Z})$ is an *integral* class, we may apply to it the natural change-of-coefficients morphism

$$H_*(X, \mathbb{Z}) \to H_*(X, \mathbb{R})$$

and view it as a *real* class (which may vanish) and consider its l^1-norm, also denoted $\|\alpha\|_1$. This measures the optimal "size" (in the l^1-norm) of a real representative

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for the integral class. When M is a closed oriented manifold, the l^1-norm of its fundamental class $[M] \in H_n(M;\mathbb{Z})$ is called the *simplicial volume* of M, and will be denoted by $\|M\|$.

Rather than looking at *all* chains representing the class α, one could instead restrict oneself to chains which satisfy some additional geometric constraint. To this end, let us consider the set of all closed smooth oriented manifolds and continuous maps $(M, f : M \to X)$ such that f sends the fundamental class of M to α. Recall [Thom 1954, Théorème III.9] that if $n \geq 7$, this set may be empty, even if X is a finite polyhedron. On integral homology, we consider the subadditive function

$$\mu(\alpha) = \inf\{\|M\| : f_*[M] = \alpha\},$$

(with the usual convention that the infimum of the empty set is $+\infty$) and the corresponding *manifold* (*semi*)*norm*

$$\|\alpha\|_{\mathrm{man}} = \inf_{m\in\mathbb{N}} \left\{\frac{\mu(m\cdot\alpha)}{m}\right\}.$$

Thom [1954, Théorème III.4] has shown that the manifold norm is finite when X is a finite polyhedron. Since any homology class can be represented as the image of a finite polyhedron, it follows from Thom's result that the manifold norm is finite for any topological space.

It is immediate from the definitions that $\|\cdot\|_1 \leq \|\cdot\|_{\mathrm{man}}$ holds on $H_n(X,\mathbb{Z})$, for any n, and any topological space X.

Theorem 1.1. *For each degree n, there exists a constant $\delta_n > 0$, such that for any topological space X and any class $\alpha \in H_n(X,\mathbb{Z})$, we have*

$$\delta_n \|\alpha\|_{man} \leq \|\alpha\|_1 \leq \|\alpha\|_{man}.$$

One can take $\delta_n = 1$ if $n \neq 3$, and $\delta_3 \approx 0.0115416$.

After some preliminary material in Sections 2 and 3, we provide a proof of Theorem 1.1 in Sections 4 and 5. Section 4 shows the existence of the δ_n, whereas Section 5 is devoted to identifying the optimal values of the δ_n. It is straightforward to show that the norms are equal if $n \leq 2$ (that is, one can take $\delta_2 = 1$). Crowley and Löh [2012, Proposition 4.3] showed that for degree $n \geq 4$, one can take $\delta_n = 1$ (see Proposition 5.1 below). So in all cases except possibly in degree $= 3$, one actually has the equality $\|\alpha\|_1 = \|\alpha\|_{\mathrm{man}}$. We do not know if the optimal value of δ_3 is 1.

Shortly after this paper was written, Gaifullin posted a preprint [2012a] containing some closely related results. In fact, our Theorem 1.1 can be deduced from the results in [Gaifullin 2012a, Section 6], though without an explicit estimate for δ_3.

2. Gluing simplices along their faces

Our first goal is to realize an integral class β as the image of a Δ-complex [Hatcher 2002, Section 2.1] which is a disjoint union of n-dimensional pseudomanifolds [Spanier 1981, Chapter 3, Example C] whose number of n-simplices is controlled in terms of β. The precise statement we need is the following.

Proposition 2.1. *Let X be a topological space and $\beta \in H_n(X, \mathbb{Z})$ an integral class on X of degree n represented by a singular cycle $\sum_i m_i\sigma_i$, $m_i \in \mathbb{Z}$. Then there is a Δ-complex Q and a continuous map $g : Q \to X$ with the following properties.*

(1) *The number of n-dimensional simplices of Q is $\sum_i |m_i|$.*

(2) *The Δ-complex Q is topologically a finite disjoint union of oriented n-dimensional pseudomanifolds without boundary.*

(3) *$g_*[Q] = \beta$, that is, with appropriate orientations on each pseudomanifold, g sends the sum of the fundamental classes of the pseudomanifolds forming Q to the class β.*

Remark 2.2. If $n \leq 2$, we can choose Q so that the pseudomanifolds are manifolds.

All this is well-known and can be deduced from [Hatcher 2002, Chapter 2]. We sketch the proof for the convenience of the reader.

Proof. The statement is trivial if $n = 0$, hence we assume $n \geq 1$. In the cycle $\sum_i m_i\sigma_i$, we consider each singular n-simplex σ_i whose coefficient m_i is negative. We precompose σ_i with an affine automorphism of the standard n-simplex that reverses the orientation and changes the sign of m_i. This leads to a representative of the same class β with positive coefficients $m_i \in \mathbb{N}$. Let us define

$$T = \sum_i m_i,$$

and let U be the disjoint union of T standard n-simplices. Repeating m_i times each singular simplex σ_i, we write our cycle

$$\sum_{i=1}^{T} \sigma_i$$

and we obtain a continuous map

$$\sigma : U \to X$$

whose restriction to the i-th copy of the standard n-simplex is σ_i. Each term of the boundary

$$\partial\left(\sum_{i=1}^{T} \sigma_i\right)$$

is the restriction of some σ_i to an $(n-1)$-face of the i-th n-simplex of U (times a coefficient which is either 1 or -1 because we repeat the terms). If two such singular $(n-1)$-simplices are equal (as maps defined on the standard $(n-1)$-simplex) and if their coefficients are opposite, they form what we call a canceling pair. We choose a maximal collection of canceling pairs, and for each pair we identify the two $(n-1)$-faces of U on which the two terms of the pair coincide. The topological space defined as the quotient of U with respect to the equivalence relation defined by these identifications has a Δ-complex structure Q with T n-simplices. It has no boundary because we chose a maximal family of canceling pairs and because $\sum_{i=1}^{T} \sigma_i$ is a cycle. This also implies that each connected component of Q is an n-dimensional oriented pseudomanifold. The map $\sigma : U \to X$ factors through Q. The quotient map $g : Q \to X$ is continuous and $g_*[Q] = \beta$. This proves the proposition.

If $n \leq 2$, one checks that each link of each vertex of Q is a sphere. This proves the remark. □

3. Gaifullin's desingularization

We need a result of Gaifullin, which provides a *constructive* desingularization of an oriented pseudomanifold (see [[2008]; 2012b] for a more detailed explanation). Let us briefly describe this result. Gaifullin establishes the existence, in each dimension n, of a closed oriented n-manifold M having the following universal property. Given any oriented n-dimensional pseudomanifold P with K top-dimensional simplices, and with a regular coloring of the vertex set by $(n+1)$ colors (that is, any adjacent vertices are of different colors), there exists

- a finite cover $\pi : \widehat{M} \to M$, of degree $\frac{1}{2} K \, \Pi_\omega \, |P_\omega|$,
- a map $f : \widehat{M} \to P$ with the property that

$$f_*[\widehat{M}] = 2^{n-1} \, \Pi_\omega \, |P_\omega| \cdot [P] \in H_n(P; \mathbb{Z}).$$

The degrees of the maps involve the integer $\Pi_\omega |P_\omega|$ (which is the product of the cardinalities of the finite sets P_ω), whose precise definition [Gaifullin 2008, page 563] we will not need. We merely point out that the term $\Pi_\omega |P_\omega|$ depends *solely* on the combinatorics of P, and appears in the expressions for *both* the degree of the covering map π, *and* of the "desingularization" map f.

The universal manifolds M are explicitly described, and are the *Tomei manifolds*. For the convenience of the reader, we provide some discussion of the Tomei manifolds in the Appendix, which also establishes some specific properties of the 3-dimensional Tomei manifold which are used in the proof of Proposition 5.2.

Finally, we make a brief comment concerning simplicial complexes versus Δ-complexes. The difference between these two classes is that, for Δ-complexes, one does not restrict the gluing of simplices to be along a single face of distinct simplices. While Gaifullin's result is stated in the setting where P is a simplicial complex, the constraint on the gluings of simplices is not used in his proofs. As such, his desingularization process works equally well when applied to Δ-complexes (assuming of course that there exists a regular vertex $(n+1)$-coloring). We thank the anonymous referee for pointing this out to us.

4. Existence of the δ_n

In this section, we show that there exist constants δ_n satisfying the conclusion of Theorem 1.1.

Let $\alpha \in H_n(X, \mathbb{Z})$ and let $\epsilon > 0$. The change-of-coefficients morphism

$$H_n(X, \mathbb{Z}) \to H_n(X, \mathbb{R})$$

factors through $H_n(X, \mathbb{Q})$, and the map

$$H_n(X, \mathbb{Q}) \to H_n(X, \mathbb{R})$$

is an isometric injection. Hence we can find a representative

$$\sum_i r_i \sigma_i$$

of α with $r_i \in \mathbb{Q}$ such that

$$\sum_i |r_i| \leq \|\alpha\|_1 + \epsilon. \tag{1}$$

Let m be the least common multiple of all the denominators of the reduced fractions of the r_i. The chain

$$\sum_i m r_i \sigma_i$$

is an integral chain representing the class

$$\beta = m\alpha \in H_n(X, \mathbb{Z}).$$

Now we apply Proposition 2.1 to the integral class β. This gives us a Δ-complex Q and a continuous map $g : Q \to X$ with the following properties:

(i) The number of n-dimensional simplices of Q is

$$m \sum_i |r_i| \leq m(\|\alpha\|_1 + \epsilon).$$

(ii) Q consists of a finite disjoint union of oriented n-dimensional pseudomanifolds without boundary.

(iii) g maps the sum of the fundamental classes of the pseudomanifolds in Q to the class β, that is, $g_*[Q] = \beta$.

Notice that in the case where Q is a manifold (that is automatic if $n = 2$, as explained at the end of the proof of Proposition 2.1), the inequality

$$\|\alpha\|_{\text{man}} \leq \|\alpha\|_1$$

follows, since for any $\epsilon > 0$ we have

$$\|Q\|/m \leq \|\alpha\|_1 + \epsilon.$$

If Q is not a manifold — that is, if at least one of the connected components of Q is not a manifold but only a pseudomanifold — a desingularization process is needed to produce a manifold. We first consider the case when Q is connected. Let P denote the first barycentric subdivision of the Δ-complex Q. The number of n-dimensional simplices of the barycentric division of the standard n-simplex is $(n+1)!$, so the number K of top-dimensional simplices in P is

$$K = (n+1)!m \sum_i |r_i|.$$

Moreover, the vertex set of P clearly has a regular coloring by $(n+1)$ colors: each vertex v lies in the interior of a unique cell σ_v from the original Δ-complex Q, and we can color the vertex v with the color $1 + \dim(\sigma_v) \in \{1, \ldots, n+1\}$. So we can now apply Gaifullin's desingularization process to the pseudomanifold P, obtaining the following diagram of spaces and maps:

$$M \xleftarrow{\pi} \widehat{M} \xrightarrow{f} P \xrightarrow{g} X .$$

We also know that

(a) $g_*[P] = \beta = m \cdot \alpha \in H_n(X; \mathbb{Z})$,

(b) $f_*[\widehat{M}] = 2^{n-1} \Pi_\omega |P_\omega| \cdot [P] \in H_n(P; \mathbb{Z})$.

The map π is a covering map of degree $\frac{1}{2} K \, \Pi_\omega |P_\omega|$, so we can also compute the simplicial volume of $\widehat{M}$:

$$\|\widehat{M}\| = \tfrac{1}{2} K \, \Pi_\omega |P_\omega| \, \|M\|.$$

Combining (a) and (b), we see that the composite map $g \circ f : \widehat{M} \to X$ allows us to represent the homology class $[m \cdot 2^{n-1} \Pi_\omega |P_\omega|] \cdot \alpha \in H_n(X; \mathbb{Z})$ as the image of the fundamental class of the oriented manifold $\widehat{M}$. From the definition of the manifold seminorm, we obtain

$$\|\alpha\|_{\mathrm{man}} \leq \frac{1}{m \cdot 2^{n-1} \prod_\omega |P_\omega|} \|\widehat{M}\| = \frac{\frac{1}{2} K \prod_\omega |P_\omega|}{m \cdot 2^{n-1} \prod_\omega |P_\omega|} \|M\|$$
$$= \frac{(n+1)!\, m \sum_i |r_i|}{m \cdot 2^n} \|M\| \leq \|M\| \frac{(n+1)!}{2^n} (\|\alpha\| + \epsilon).$$

Letting ϵ go to zero completes the proof, with the explicit value

$$\delta_n = \frac{2^n}{(n+1)!\, \|M\|}$$

where M is the n-dimensional Tomei manifold appearing in Gaifullin's desingularization procedure. In the case where $P = \bigsqcup_i P_i$ has several connected components P_i, let d be the least common multiple of the $\prod_\omega |(P_i)_\omega|$, and for each i, let $m_i = d / \prod_\omega |(P_i)_\omega|$. Exactly the same proof applies with $\widehat{M} = \bigsqcup_i \bigsqcup_{m_i} \widehat{M}_i$, $f = \bigsqcup_i \bigsqcup_{m_i} f_i$, and $\pi = \bigsqcup_i \bigsqcup_{m_i} \pi_i$.

5. Estimating the δ_n

In this section, we complete the proof of Theorem 1.1 by estimating the δ_n. As explained in the previous section, one can take $\delta_2 = 1$. Crowley and Löh [2012] have shown that for $n \geq 4$, one can take $\delta_n = 1$. Their result is stated in the a priori more restrictive setting of finite CW-complexes, but it is straightforward to deduce the general case from that special case. For completeness, we include a proof of this result.

Proposition 5.1. *In degrees $n \geq 4$, we can take $\delta_n = 1$, that is, for any topological space X and any class $\alpha \in H_n(X, \mathbb{Z})$ of degree $n \geq 4$, one has the equality*

$$\|\alpha\|_1 = \|\alpha\|_{man}.$$

Proof. The inequality $\|\alpha\|_1 \leq \|\alpha\|_{\mathrm{man}}$ is immediate from the definitions, so let us focus on the converse. Proceeding as in the proof of Theorem 1.1, given any $\epsilon > 0$, we can find a corresponding *integral* chain

$$\sum_i m r_i \sigma_i$$

representing a class

$$\beta = m\alpha \in H_n(X, \mathbb{Z})$$

and where the rational numbers r_i satisfy

$$\sum_i |r_i| \leq \|\alpha\|_1 + \epsilon/2 \tag{2}$$

Now apply Proposition 2.1 to the integral class β, obtaining a Δ-complex Q and a continuous map $g : Q \to X$ such that $g_*[Q] = \beta$. As Q itself is a finite CW-complex of dimension $n \geq 4$, [Crowley and Löh 2012, Prop. 4.3] implies that $\|[Q]\|_1 = \|[Q]\|_{\text{man}}$. Since we have a realization of Q as a Δ-complex with exactly $m \sum_i |r_i|$ top-dimensional simplices, we obtain

$$\|[Q]\|_{\text{man}} = \|[Q]\|_1 \leq m \sum_i |r_i|.$$

Consider the positive real number $m\epsilon/2 > 0$. From the definition of the manifold norm, we can find a closed oriented manifold N, and a continuous map $h : N \to Q$ of degree d, with the property that $h_*[N] = d \cdot [Q]$, and satisfying

$$\frac{\|N\|}{d} \leq \|Q\|_{\text{man}} + m\epsilon/2 \leq m \sum_i |r_i| + m\epsilon/2. \tag{3}$$

The composite map $g \circ h : N \to X$ sends the fundamental class $[N]$ to $d \cdot \beta = d \cdot m\alpha$. Using this map to estimate the manifold norm of α, we obtain

$$\begin{aligned}\|\alpha\|_{\text{man}} &\leq \frac{\|N\|}{d\, m} \\ &\leq \frac{1}{m}\Big(m \sum_i |r_i| + m\epsilon/2\Big) \\ &\leq \sum_i |r_i| + \epsilon/2 \\ &\leq \|\alpha\|_1 + \epsilon,\end{aligned}$$

where the second inequality was deduced from (3), and the last inequality from (2). Finally, letting $\epsilon > 0$ go to zero, we obtain $\|\alpha\|_{\text{man}} \leq \|\alpha\|_1$, completing the proof. □

It is tempting to guess that the optimal value of δ_3 is also 1. Our method of proof gives a substantially lower value of δ_3, which is explicitly given by the following.

Proposition 5.2. *The optimal value of* δ_3 *is* $\geq V_3/(24V_8) \approx 0.0115416$, *where* V_3 *and* V_8 *are the volumes of the* 3*-dimensional regular ideal hyperbolic tetrahedron and octahedron, respectively.*

Proof. The proof of Theorem 1.1 yields the general value

$$\delta_n = \frac{2^n}{(n+1)!\, \|M\|}$$

where M is the n-dimensional Tomei manifold. Specializing to dimension $n = 3$, and using the fact that $\|M^3\| = 8V_8/V_3$ (see Lemma A.2 below), we obtain the claim. □

Appendix: Tomei manifolds

The universal manifolds M used in Gaifullin's desingularization are the *Tomei manifolds*. For the convenience of the reader, we provide a brief description of these manifolds. We also establish some results concerning the 3-dimensional Tomei manifold that are used in estimating the constant δ_3 arising in our proof of Theorem 1.1 (see Proposition 5.2).

A matrix $A = [a_{ij}]$ is *tridiagonal* if $a_{ij} = 0$ for all indices satisfying $|i - j| > 1$. The n-dimensional Tomei manifold consists of all $(n+1) \times (n+1)$ real symmetric tridiagonal matrices, with fixed simple spectrum $\lambda_0 < \lambda_1 < \cdots < \lambda_n$ (the manifold is independent of the choice of simple spectrum). These manifolds were introduced by Tomei [1984] and further studied by Davis [1987]. An important result of Tomei is that these manifolds support a very natural cellular decomposition, which we now describe.

First, recall the definition of the n-dimensional permutahedron Π^n. The permutahedron is an n-dimensional, simple, convex polytope, obtained as the convex hull of a specific configuration of points in $\mathbb{R}^{n+1}$. If the symmetric group S_{n+1} acts on $\mathbb{R}^{n+1}$ by permuting the coordinates, the permutahedron Π^n is defined to be the convex hull of the S_{n+1}-orbit of the point $(1, 2, \dots, n+1) \in \mathbb{R}^{n+1}$. Denote by $\mathcal{S}$ this specific S_{n+1}-orbit, so that $\Pi^n = \mathrm{Conv}(\mathcal{S})$ (see Figure 1 for an illustration of Π^3).

The facets (codimension one faces) of the permutahedron Π^n are indexed by the $2^{n+1} - 2$ nonempty proper subsets $\omega \subsetneq \{1, \dots, n+1\}$, as follows. Given a subset ω, define the subset $\mathcal{S}_\omega \subset \mathcal{S}$ by

$$\mathcal{S}_\omega := \{\vec{x} \in \mathcal{S} \mid \forall i \in \omega, \forall j \notin \omega,\ x_i < x_j\}.$$

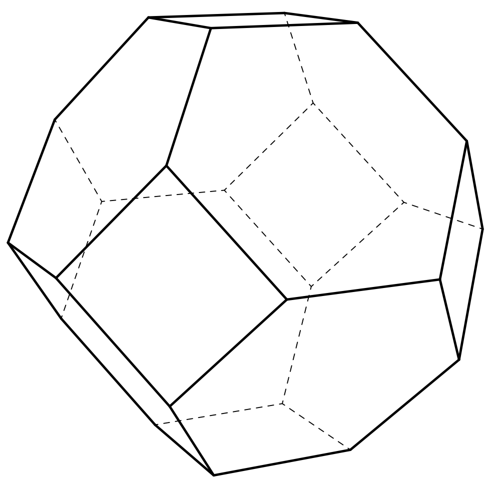

Figure 1. The 3-dimensional permutahedron Π^3.

In other words, a vertex $\vec{x} \in \mathcal{S}$ lies in $\mathcal{S}_\omega$ if the integers $\{1, \ldots, |\omega|\}$ occur precisely in the coordinates whose index lies in ω. The facet F_ω is then defined to be the convex hull $\mathrm{Conv}(\mathcal{S}_\omega)$. From this, it easily follows that two distinct facets $F_{\omega_1}, F_{\omega_2}$ intersect if and only if $\omega_1 \subsetneq \omega_2$ or $\omega_2 \subsetneq \omega_1$. One also has that any codimension k face of Π^n, being of the form $F_{\omega_1} \cap \cdots \cap F_{\omega_k}$ for some choice of distinct facets, corresponds (after possibly reindexing) to a unique length k chain $\omega_1 \subsetneq \omega_2 \subsetneq \cdots \subsetneq \omega_k$ of nonempty proper subsets of $\{1, \ldots, n+1\}$.

Tomei [1984] showed that the n-dimensional Tomei manifold M has a particularly simple tiling by 2^n copies of the n-dimensional permutahedron Π^n. Let $e_1, \ldots, e_n$ be the standard generators for $\mathbb{Z}_2^n$. Then the n-dimensional Tomei manifold can be identified with $(\mathbb{Z}_2^n \times \Pi^n)/\sim$, where the equivalence relation is given by $(g, x) \sim (e_{|\omega|}g, x)$ whenever $x \in F_\omega$.

Example. For a concrete example, when $n = 3$, the permutahedron Π^3 is the truncated octahedron (see Figure 1). It has 6 square facets (parametrized by subsets $\omega \subsetneq \{1, 2, 3, 4\}$ with $|\omega| = 2$) and 8 hexagonal facets (parametrized by the ω with $|\omega| = 1, 3$). Figure 2 includes some vertex coordinates and labels some of the facets with the corresponding subset of $\{1, 2, 3, 4\}$.

In the corresponding Tomei manifold M^3, tessellated by eight copies of Π^3, one can easily see that each edge of the tessellation lies on exactly four copies of Π^3. Now consider the 24 squares appearing in the tessellation of M. The union of all these squares forms a collection of six tori embedded in M, each tessellated by four squares. Note that, from the definition of the gluings, each square bounds two copies of Π^3, whose indices in $\mathbb{Z}^3$ differ in the middle coordinate (corresponding to the generator e_2). This implies that the collection of six tori separate M^3 into two copies of a manifold N. Each of the two copies of N is tessellated by four copies of Π^3, and there is a $\mathbb{Z}_2$-involution on M^3 which fixes the collection of tori and interchanges the two copies of N. The involution can be easily described in terms of the description $M = (\mathbb{Z}_2^3 \times \Pi^3)/\sim$: it sends each element (g, x) to $(e_2 \cdot g, x)$.

A nice consequence of Gaifullin's work is the following elementary result.

Lemma A.1. *If M is a Tomei manifold, $\|M\| > 0$.*

Proof. Let N be a closed hyperbolic manifold of the same dimension as M. It follows from work of Gromov and Thurston that $\|N\| > 0$ (see [Thurston 1980, Chapter 6]). Take an arbitrary triangulation of N, pass to the barycentric subdivision, and apply Gaifullin's desingularization. This gives us a finite cover $\widehat{M} \to M$ with a map $f : \widehat{M} \to N$, of degree $d \neq 0$. Since $\|N\| > 0$, the obvious inequality $\|\widehat{M}\|/d \geq \|N\|$ immediately forces $\|\widehat{M}\| > 0$. But the simplicial volume scales under covering maps, so we conclude that $\|M\| > 0$, as desired. □

In general, the computation of the exact value of the simplicial volume is an extremely difficult problem. For the 3-dimensional Tomei manifold, we can, however, give an exact computation. Let V_8 denote the volume of a regular ideal hyperbolic octahedron and V_3 the volume of a regular ideal hyperbolic tetrahedron. These volumes can be expressed in terms of the Lobachevsky function

$$\Lambda(\theta) := -\int_0^\theta \log|2\sin t|\,dt$$

and are exactly equal to $V_8 = 8\Lambda(\pi/4)$ and $V_3 = 2\Lambda(\pi/6)$ (see [Thurston 1980, Section 7.2]). Up to five decimal places, $V_8 \approx 3.66386$ and $V_3 \approx 1.01494$.

Lemma A.2. *The* 3*-dimensional Tomei manifold* M^3 *has simplicial volume* $\|M\| = 8V_8/V_3$ (*which is* ≈ 28.8794).

Proof. Closed 3-manifolds are one of the few classes of manifolds for which the simplicial volume is known. Recall that for hyperbolic 3-manifolds, the simplicial volume is proportional to the hyperbolic volume, with constant of proportionality $1/V_3$. For Seifert fibered 3-manifolds, the existence of an S^1-action immediately implies that the simplicial volume is zero. For a general closed, orientable 3-manifold, the validity of Thurston's geometrization conjecture (recently established

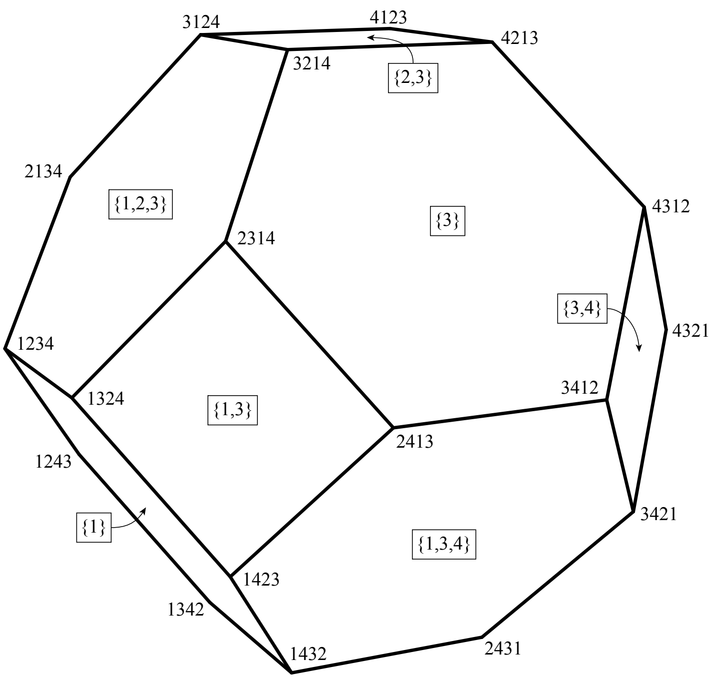

Figure 2. A portion of Π^3. Vertices are labeled by their coordinates in $\mathbb{R}^4$ (parentheses and commas omitted to avoid cluttering the picture). Facets are labeled with the corresponding subset $\omega \subset \{1, 2, 3, 4\}$.

by Perelman) implies that there is a decomposition into geometric pieces. Since simplicial volume is additive under connected sums (in dimensions ≥ 3) and under gluings along tori (see [Gromov 1982, Section 3.5]), this implies that the simplicial volume of any closed, orientable 3-manifold is proportional (with constant $1/V_3$) to the sum of the (hyperbolic) volumes of the hyperbolic pieces in its geometric decomposition.

Let us apply this procedure to the Tomei manifold M. Recall that M is the double of a 3-manifold N with ∂N consisting of four tori. From the gluing formula we deduce that $\|M\| = 2\|N\|$. To compute $\|N\|$, recall that N is tessellated by four copies of the 3-dimensional permutahedron Π^3, with the collection of square faces of all the Π^3 forming the boundary tori of N. This implies that the interior of N is tessellated by copies of Π^3 *with the square boundary faces removed*. Next we claim that $\mathrm{Int}(N)$ supports a finite volume hyperbolic metric.

Under this tessellation, each interior edge of N lies on exactly *four* of the Π^3. Let $\mathbb{O} \subset \mathbb{H}^3$ denote the regular ideal hyperbolic octahedron. This octahedron has all six vertices on the boundary at infinity of $\mathbb{H}^3$, and has all incident pairs of faces forming angles of $\pi/2$. A copy of the permutahedron Π^3 can be obtained by removing small horoball neighborhoods of each of the ideal vertices. Each hexagonal face of Π^3 corresponds to a triangular face of $\mathbb{O}$. So one can form a manifold N^0 by gluing together four copies of $\mathbb{O}$, using the same gluing pattern as in the formation of N. Using isometries to glue together the sides of $\mathbb{O}$, one obtains a metric on N^0 which is hyperbolic, except possibly along the 1-skeleton of N^0. To check whether or not one has a singularity along the edges of N^0, one just needs to calculate the total angle transverse to the edge. But recall that along each edge in N^0, one has four copies of $\mathbb{O}$ coming together. Since each edge in $\mathbb{O}$ has an internal angle of $\pi/2$, the total angle transverse to each edge of N^0 is equal to 2π. We conclude that N^0 supports a complete hyperbolic metric, with hyperbolic volume $= 4V_8$.

N is obtained from N^0 by removing a neighborhood of the ideal vertices in each $\mathbb{O}$ in the tessellation of N^0. This means that N is obtained from the noncompact, finite volume, hyperbolic manifold N^0 by truncating the cusps. It follows that $\mathrm{Int}(N)$ is diffeomorphic to N^0. Since cutting M open along the collection of tori results in two copies of $\mathrm{Int}(N) = N^0$, a manifold supporting a hyperbolic metric, we have that this is exactly the geometric decomposition of M predicted by Thurston's geometrization conjecture (cf. [Davis 1987, page 105, footnote 2]). Our discussion above implies that $\|M\| = 2\,\mathrm{Vol}(N^0)/V_3 = 8V_8/V_3$, completing the proof. □

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JEAN-FRANÇOIS LAFONT
DEPARTMENT OF MATHEMATICS
THE OHIO STATE UNIVERSITY
100 MATH TOWER
231 WEST 18TH AVENUE
COLUMBUS, OHIO 43210-1174
UNITED STATES
jlafont@math.ohio-state.edu

CHRISTOPHE PITTET
CENTRE DE MATHÉMATIQUES ET INFORMATIQUE
AIX-MARSEILLE UNIVERSITÉ
39 RUE FRÉDÉRIC JOLIOT-CURIE
13453 MARSEILLE CEDEX 13
FRANCE
pittet@cmi.univ-mrs.fr

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