HYPERBOLIC GROUPS WITH BOUNDARY AN $n$-DIMENSIONAL SIERPINSKI SPACE

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Abstract. For $n \geq 7$, we show that if $G$ is a torsion-free hyperbolic group whose visual boundary $\partial_\infty G \simeq \mathcal{S}^{n-2}$ is an $(n-2)$-dimensional Sierpinski space, then $G = \pi_1(W)$ for some aspherical $n$-manifold $W$ with nonempty boundary. Concerning the converse, we construct, for each $n \geq 4$, examples of aspherical manifolds with boundary, whose fundamental group $G$ is hyperbolic, but with visual boundary $\partial_\infty G$ not homeomorphic to $\mathcal{S}^{n-2}$.

1. Introduction

One of the basic invariants for a hyperbolic group is its boundary at infinity, and a fundamental question is to determine what properties of the group are captured by the topology of the boundary at infinity. For example, the famous Cannon conjecture postulates that a hyperbolic group whose boundary at infinity is the 2-sphere $S^2$ must admit a properly discontinuous, isometric, cocompact action on hyperbolic 3-space $\mathbb{H}^3$.

In [19], Kapovich and Kleiner study groups whose boundary at infinity is a Sierpinski carpet – a boundary version of the Cannon conjecture. In [4], Bartels, Lück, and Weinberger study groups whose boundary at infinity is a sphere $S^n$ of dimension $n \geq 5$ – a high-dimensional version of the Cannon conjecture. In this paper, we consider groups whose boundary at infinity are high-dimensional Sierpinski spaces – thus lying somewhere between the work of Kapovich-Kleiner and that of Bartels-Lück-Weinberger.

The two main theorems are as follows. Let $\mathcal{S}^{n-2}$ denote an $(n-2)$-dimensional Sierpinski space. See Section 2 for the definition.

**Theorem 1.** Fix $n \geq 7$ and let $G$ be a torsion-free hyperbolic group. If the visual boundary $\partial_\infty G$ is homeomorphic to $\mathcal{S}^{n-2}$, then there exists an $n$-dimensional compact aspherical topological manifold $W$ with nonempty boundary such that $\pi_1(W) \cong G$. Furthermore, $W$ is unique up to homeomorphism.

Note that the fundamental group $\pi$ of a closed aspherical manifold $M$ is an example of a Poincaré duality group. Whether or not all finitely presented Poincaré duality groups arise in this fashion is an open problem that goes back to Wall [16]. So the existence portion of Theorem 1 addresses a relative version of Wall realization problem for a special class of groups. On the other hand, the uniqueness portion of Theorem 1 verifies the Borel conjecture for this same class of groups.

Our second result shows that the converse of Theorem 1 is false – even if one imposes additional strong constraints on the geometry of the aspherical manifold.

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Theorem 2. For each \( n \geq 4 \), there exists a compact aspherical manifold \( M^n \) with nonempty connected boundary \( \partial M^n = N^{n-1} \) such that:

1. \( G = \pi_1(M) \) is hyperbolic, and \( H = \pi_1(N) \) is a proper quasi-convex subgroup in \( G \).
2. \( \partial_\infty(\pi_1(N)) \) is homeomorphic to \( S^{n-2} \), but
3. \( \partial_\infty G \cong \partial_\infty \tilde{M} \) is not homeomorphic to \( S^{n-2} \).

Moreover, when \( n \geq 5 \), the manifold \( M^n \) supports a locally \( \text{CAT}(-1) \) metric with totally geodesic boundary.

Remark 3. If one just wants a simple counterexample to the converse of Theorem 1, one can proceed as follows: start with a \( k \)-dimensional closed hyperbolic manifold \( K \) with fundamental group \( G \), where \( k < n \). Now embed the hyperbolic \( k \)-plane \( \mathbb{H}^k \) isometrically inside \( \mathbb{H}^n \). Then the \( G \)-action on the embedded \( \mathbb{H}^k \) extends to an action on the \( r \)-neighborhood \( X \) of the \( \mathbb{H}^k \).

Let \( M = X/G \), and note that \( M \) is aspherical, diffeomorphic to \( K \times \mathbb{D}^{n-k} \), with fundamental group \( G \). Clearly \( \partial_\infty G \) is homeomorphic to the \( (k-1) \)-sphere \( S^{k-1} \), and not to Sierpinski \( (n-2) \)-space \( \mathcal{S}^{n-2} \). Of course, in this example, \( N = K \times S^{n-k-1} \), so the example fails to have property (1) from Theorem 2.

Remark 4. In Theorem 2 one can construct, in dimensions \( n \geq 5 \), manifolds satisfying property (1), but failing to have (2). Start with a Davis-Januszkiewicz example of a locally \( \text{CAT}(-1) \) closed \( (n-1) \)-manifold \( N \) with \( \partial_\infty \tilde{N} \) not homeomorphic to \( S^{n-2} \), chosen so that \( N = \partial W^{n+1} \) for some compact manifold \( W^{n+1} \). Then take \( M \) to be the relative hyperbolization of \( W \), relative to \( N \) (see [15]). Properties of relative hyperbolization readily yield statement (1), while the choice of \( N \) ensures that (2) fails. It seems likely that such manifolds \( M \) would also have property (3).

Indeed, one could visualize the boundary at infinity of \( \tilde{M} \) to be similar to a Sierpinski curve, but instead of having peripheral spheres (see Section 2), it would have peripheral subspaces which are \( \check{\text{C}} \)ech homology spheres instead of genuine spheres (since (2) fails). Such a space is probably not homeomorphic to \( \mathcal{S}^{n-2} \). We point out, however, that this approach could not possibly work in dimension \( n = 4 \), as in this case the boundary would be a closed 3-manifold, which forces (2) to hold.

Structure of paper. In Section 2 we recall the definition of an \( n \)-dimensional Sierpinski space. In Sections 3 and 4 we prove Theorems 1 and 2, respectively. In Section 5 we remark on a generalization of Theorem 1 to \( \text{CAT}(0) \) groups.

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(i) \( \overline{D_i} \cap \overline{D_j} = \emptyset \) for \( i \neq j \),
(ii) \( \text{diam}(D_i) \to 0 \) with respect to the round metric on \( S^{n+1} \), and
(iii) \( \bigcup D_i \subset S^{n+1} \) is dense.

Then \( S^n := S^{n+1} \setminus \bigcup D_i \) is an \( n \)-dimensional Sierpinski space. The spheres \( S^n \cong \partial(D_i) \subset S^{n+1} \) are called peripheral spheres.

Example. A 0-dimensional Sierpinski space \( S^0 \) is a Cantor set, while the space \( S^1 \) is the classical Sierpinski carpet. The Sierpinski space \( S^{n-2} \) arises as the visual boundary of hyperbolic groups (in the sense of Gromov [17]). For example, if \( W^n \) is a hyperbolic \( n \)-manifold with nonempty totally geodesic boundary, then \( \pi_1(W) \) is a hyperbolic group whose visual boundary is a Sierpinski \((n-2)\)-space. To see this, observe that the universal cover \( \tilde{W} \) can be embedded as a submanifold of hyperbolic space \( \tilde{W} \hookrightarrow \mathbb{H}^n \). Using the disk model, the visual boundary \( \partial_{\infty} \tilde{W} \) is a subspace of \( \partial_{\infty} \mathbb{H}^n \cong S^{n-1} \). The boundary components of \( \tilde{W} \) lift to countably many disjoint geodesic hyperplanes \( \mathbb{H}^{n-1} \subset \mathbb{H}^n \). Each hyperplane has boundary a sphere \( \partial_{\infty} \mathbb{H}^{n-1} \cong S^{n-2} \), which bounds an open ball \( \mathbb{D}^{n-1} \subset S^{n-1} \). The visual boundary of \( \tilde{W} \) is obtained by removing this countable collection of open balls, yielding a Sierpinski space \( S^{n-2} \).

![Figure 1. A torus with one boundary component, and its universal cover inside the hyperbolic plane.](image)

The simplest example of this is when \( W \) is a torus with one boundary component (see Figure 1). More examples are furnished by the following general theorem of Lafont [20].

**Theorem 5** (Lafont). Let \( M^n \) be a compact, negatively curved Riemannian manifold with nonempty totally geodesic boundary. Then \( \partial_{\infty} \tilde{M} \) is homeomorphic to \( S^{n-2} \).

We remark that the dimension restriction in the statement of [20, Theorem 1.1] is unnecessary thanks to work of Freedman and Quinn (c.f. the MathSciNet review of [25]). As a consequence of this result, the “locally \( \text{CAT}(-1) \) metric” statement in Theorem 2 cannot be replaced by “negatively curved Riemannian metric”.

3. Proof of Theorem 1

**Proof.** We first prove the existence part of the statement, proceeding in three steps.

**Step 1 (Peripheral subgroups and Poincaré duality pairs).** Recall that \( G \) is a torsion-free hyperbolic group such that \( \partial_{\infty} G \cong S^{n-2} \). The stabilizer \( H \leq G \) of a peripheral sphere \( S^{n-2} \subset \mathscr{S}^{n-2} \) is called a peripheral subgroup. By Kapovich-Kleiner [19] Theorem 8(1)], there are finitely many peripheral subgroups, up to conjugacy in \( G \). Choose representatives \( H_1, \ldots, H_p \) for the conjugacy classes.

In order to show that \( G \) is the fundamental group of a manifold with boundary, we first need to establish that \( G \) has the same Poincaré duality as a manifold with boundary. To be
precise, Kapovich-Kleiner [19] Corollary 12] show that \((G, \{H_i\})\) is a group \(PD(n)\) pair in the sense of Bieri-Eckmann [7]. This has the following topological consequence (see [18] Theorem 1] and [6] Section 6): let \((X, Y)\) be the CW-complex pair obtained by taking \(Y = \bigcup_{i=1}^p B_{H_i}\) and defining \(X\) to be the mapping cylinder of the map \(\prod B_{H_i} \to BG\). Then \((X, Y)\) is a CW-complex \(PD(n)\) pair in the sense of Wall [28]. In particular this means that there are isomorphisms \(H^i(X; \mathbb{Z}) \cong H_{n-i}(X, Y; \mathbb{Z})\) and \(H^{i-1}(Y; \mathbb{Z}) \cong H_{n-1}(Y)\) induced by cap product with \([X] \in H_n(X)\) and \(\partial[X] \in H_{n-1}(Y)\), respectively, and that \(X\) is a finitely dominated CW complex (i.e. there exists a finite CW complex \(L\) and maps \(X \xrightarrow{r} L \xleftarrow{i} X\) such that \(r \circ i = id_X\)).

**Step 2 (Preparing for surgery).** Let \((X, Y)\) be the pair from Step 0. We now explain why \((X, Y)\) is homotopy equivalent to a pair \((K, N)\) such that

- (A) \(K\) is a finite CW complex, and
- (B) \(N\) is a manifold.

This will allow us to employ the total surgery obstruction in Step 3.

(A) Wall’s finiteness obstruction \(\partial(X) \in \tilde{K}_0(X)\) vanishes if and only if \(X\) is homotopy equivalent to a finite CW complex [24]. Thus to show (A), it suffices to show \(\tilde{K}_0(X) = 0\). This is a corollary of the following powerful result (see [4] Proof of Theorem 1.2] for more information):

**Theorem 6** (Bartels-Lück [2], Bartels-Lück-Reich [3]). Let \(G\) be a torsion-free hyperbolic group \(G\). Then

- (†) the (non-connective) \(K\)-theory assembly map \(H_1(BG; \mathbb{K}_\mathbb{Z}) \to K_i(ZG)\) is an isomorphism for \(i \leq 0\) and surjective for \(i = 1\);
- (‡) the (non-connective) \(L\)-theory assembly map \(H_1(BG; \mathbb{K}_\mathbb{Z}^{(-\infty)}) \to L_i^{(-\infty)}(ZG, w)\) is bijective for every \(i \in \mathbb{Z}\) and every orientation homomorphism \(w : G \to \{±1\}\).

The conditions (†) and (‡) are called the Farrell-Jones conjectures in \(K\)- and \(L\)-theory, respectively. Note that, since \(G\) is a torsion-free hyperbolic group, a constructive alternative is to take \(X\) a large enough Rips complex (which is automatically a finite simplicial complex). We included the non-constructive proof above, as this “obstruction” point of view will reappear in later arguments.

(B) It remains to see that \(Y\) is homotopy equivalent to a closed manifold \(N^{n-1}\). By definition \(Y\) is homotopy equivalent to \(\prod_{i=1}^p B_{H_i}\). The peripheral subgroups \(H_i\) are all hyperbolic groups, and \(\partial_{\infty}H_i\) is identified with the sphere \(S^{n-2} \subset \mathcal{S}^{n-2}\) stabilized by \(H_i\) (see [19] Theorem 8)). The following result from [3] Theorem A] implies that \(Y \simeq \prod_{i=1}^p B_{H_i}\) is homotopy equivalent to a manifold:

**Theorem 7** (Bartels-Lück-Weinberger [4]). Fix \(n \geq 7\), and let \(H\) be a torsion-free hyperbolic group. If \(\partial_{\infty}H \cong S^{n-2}\), then there is a closed aspherical manifold \(N^{n-1}\) such that \(\pi_1(N) \cong H\).

**Step 3 (The total surgery obstruction).** Let \((K, N)\) be the pair from Step 2. The structure set \(S^{TOP}_0(K)\) is defined as the set of equivalence classes of homotopy equivalences \(f : (M, \partial M) \to (K, N)\) where \((M, \partial M)\) is a manifold with boundary and \(f|_{\partial M} : \partial M \to N\) is a homeomorphism (the equivalence relation is \(h\)-cobordism rel \(\partial\); see [24] Chapter 18]). Surgery theory provides computable obstructions to determine whether or not \((K, N)\) is homotopy equivalent to a manifold with boundary, i.e. whether or not \(S^{TOP}_0(K) \neq \emptyset\). We will follow the algebraic approach detailed in Ranicki [24]. The total surgery obstruction \(s_0(K)\) lives in the structure group \(S_n(K)\) and has the property that \(s_0(K) = 0\) if and only if
The quadratic structure groups \( S_1 \). The group \( H \) and as noted above, \( \partial \pi \) is the \( 1 \)-connective surgery spectrum whose 0th space is \( G/TOP \) and whose homotopy groups are \( \pi_i(\mathcal{L}_\bullet) = L_i(\mathbb{Z}) \) for \( i \geq 1 \).

To show that \( S_n^{TOP}(K) \neq 0 \), we will show that \( S_n(K) = 0 \). For this, we need to consider two other versions of the structure groups.

- The quadratic structure groups \( S_i(\mathbb{Z}, K) \) are defined in [24, Definition 14.6].
- The group \( \overline{S}_n(K) \) (see [24, Chapter 25]) belongs to the 4-periodic algebraic surgery exact sequence

\[
\cdots \to H_n(K; \mathcal{L}_\bullet) \xrightarrow{A} L_n(\pi_1(K)) \to \overline{S}_n(K) \to H_{n-1}(K; \mathcal{L}_\bullet) \to \cdots
\]

where \( \mathcal{L}_\bullet \) is the 0-connective surgery spectrum whose 0th space is \( L_0(\mathbb{Z}) \times G/TOP \cong \mathbb{Z} \times G/TOP \) and whose homotopy groups are \( \pi_i(\mathcal{L}_\bullet) = L_i(\mathbb{Z}) \) for \( i \geq 0 \).

In order to show that \( S_n(K) = 0 \), we use the following three facts.

(a) The groups \( \overline{S}_n(K) \) and \( S_n(\mathbb{Z}, K) \) are equal. This follows directly from Ranicki [24, Proposition 15.11(iii)-(iv)]. Here we have used that \( \dim K \geq 6 \). Note that \( L_q(\mathbb{Z}) = 0 \) for \( q = -1 \), and in Ranicki’s notation \( S_n(0)(\mathbb{Z}, K) = \overline{S}_n(K) \) (compare with [24, Page 289]).

(b) The quadratic structure groups \( S_i(\mathbb{Z}, K) \cong S_i(\mathbb{Z}, BG) \) are 0 for all \( i \in \mathbb{Z} \). For the proof, see [3, Proof of Theorem 1.2]. Note that this also uses Theorem 6.

(c) There is an exact sequence

\[
H_n(K; L_0(\mathbb{Z})) \to S_n(K) \to \overline{S}_n(K).
\]

See Ranicki [24, Theorem 25.3(i)].

From (a) and (b), it follows that \( \overline{S}_n(K) = 0 \). Then, by (c), to show \( S_n(K) = 0 \) is suffices to show \( H_n(K; L_0(\mathbb{Z})) = H_n(K; \mathbb{Z}) = 0 \). This can be seen from the long exact sequence in homology of a pair \((K, N)\):

\[
H_n(N; \mathbb{Z}) \to H_n(K; \mathbb{Z}) \to H_n(K, N; \mathbb{Z}) \xrightarrow{\partial} H_{n-1}(N; \mathbb{Z}).
\]

The group \( H_n(N; \mathbb{Z}) = 0 \) because \( N \) is a PD\((n-1)\) complex. Also \( H_n(K, N; \mathbb{Z}) \cong \mathbb{Z} \) is generated by the fundamental class \([K]\), and \( \partial[K] \) is a sum of fundamental classes of the components of \( N \). In particular, \( \partial[K] \neq 0 \), so \( H_n(K; \mathbb{Z}) = 0 \), as desired.

This concludes the proof of existence.

**Uniqueness.** So far we have proven the existence of a compact aspherical manifold \( W \) with \( \pi_1(W) = G \). To show \( W \) is unique, we want to show that \( S_n^{TOP}(W) \) is a singleton. By [23, Corollary 1 (rel \( \partial \))] it suffices to show that \( S_{n+1}(W) = 0 \). By [24, Theorem 25.3(i)], there is an exact sequence

\[
0 \to S_{n+1}(W) \to \overline{S}_{n+1}(W) \to H_n(W; \mathbb{Z}),
\]

and as noted above, \( H_n(W; \mathbb{Z}) = 0 \). Thus, it suffices to show that \( \overline{S}_{n+1}(W) = 0 \). This follows because \( \overline{S}_{n+1}(W) = S_{n+1}(Z, W) \) (by the same reason as in Step 3, Fact (a) above), and \( S_{n+1}(Z, W) = 0 \) (see Step 3, Fact (b)).
4. Proof of Theorem 2

The proof of Theorem 2 is an adaptation of [14] Section (5a), (5c). We briefly explain the relative version of [14] and the problem with extending it directly to our case.

The paper [14] uses hyperbolization to construct a closed, locally CAT(−1) manifold $M^n$ with the unusual property that $\partial_\infty \tilde{M}$ is not homeomorphic to $S^{n-1}$. To show this, they establish that $\partial_\infty \tilde{M} - \{\gamma_+, \gamma_\}$ is not simply connected, where $\gamma_+, \gamma_-$ are the endpoints of a geodesic $\gamma : (-\infty, \infty) \to \tilde{M}$ whose link is a homology sphere $H$ with $\pi_1(H) \neq 1$. In order to find nontrivial elements of $\pi_1(\partial_\infty \tilde{M} - \{\gamma_+, \gamma_\})$, [14] studies metric spheres $S_p(r)$ centered at $p = \gamma(0)$. When $s > r$, there are natural geodesic contraction maps $\rho^* : S_p(s) \to S_p(r)$, which allow one to relate the topology of small spheres to the topology of $\partial_\infty \tilde{M}$. The central property of the maps $\rho^*$ that makes the comparison work is that they are cell-like.

Following [14], we will construct a triangulated, locally CAT(−1) manifold $M$ with totally geodesic boundary $\partial M$ whose universal cover $\tilde{M}$ contains a geodesic $\gamma : (-\infty, \infty) \to \tilde{M}$ whose link is a homology sphere $H$ with $\pi_1(H) \neq 1$. As above, we wish to show $\pi_1(\partial_\infty \tilde{M} - \{\gamma_+, \gamma_\}) \neq 1$ (Lemma 8 below) then implies that $\partial_\infty \tilde{M}$ is not homeomorphic to $\mathbb{S}^{n-2}$. In this case $\tilde{M}$ is a manifold with boundary, and the maps $\rho^* : S_p(s) \to S_p(r)$ are not surjective for $s >> r$. This prevents us from proceeding directly as in [14]. To bypass this issue, we "cap off" the boundary components of $\tilde{M}$ to obtain a CAT(−1) manifold $\tilde{M} \supset M$ to which the arguments of [14] apply; in particular, $\pi_1(\partial_\infty \tilde{M} - \{\gamma_+, \gamma_\}) \neq 1$. At this point it will be clear from the capping procedure (see specifically Lemma 9 below) that $\pi_1(\partial_\infty \tilde{M} - \{\gamma_+, \gamma_-\}) \neq 1$.

For the proof of Theorem 2 we need the following elementary fact.

Lemma 8. For $n \geq 2$, the $n$-dimensional Sierpinski space $\mathcal{S}^n$ is simply-connected. Moreover, if $F \subset \mathcal{S}^n$ is any finite collection of points in $\mathcal{S}^n$, then $\mathcal{S}^n \setminus F$ is still simply-connected.

Proof. Model $\mathcal{S}^n$ as the complement, in the standard sphere $S^{n+1}$, of the interiors of a dense collection of pairwise disjoint round disks $D_i$ whose radii $r_i$ tend to zero. If $\gamma$ is a curve in $\mathcal{S}^n \subset S^{n+1}$, we can find a bounding disk $\phi : D^2 \to S^{n+1}$. Perturbing the map a little bit, we can assume that $\phi$ is transverse to all the $D_i$. Inductively define $\phi_k : D^2 \to S^{n+1}$ to have image disjoint from $D_1, \ldots, D_k$, as follows. $\phi^{-1}(\partial D_k)$ is a finite collection of curves in $D^2$, and each of these curves maps to a curve $\eta_j$ on $\partial D_k \cong \mathbb{S}^n$. Since $n \geq 2$, we can redefine $\phi_{k-1}$ on the interior of these finitely many curves in $D^2$, by sending each of these to a bounding disk in $\partial D_k$ for the corresponding $\eta_j$. Since the diameter of the $D_i$ shrinks to zero, the maps $\phi_k$ converge to a map $\phi_\infty : D^2 \to S^{n+1}$ whose boundary coincides with $\gamma$, and whose image is disjoint from the interiors of all the $D_i$, i.e. the image of $\phi_\infty$ lies in $\mathcal{S}^n$. A similar argument works even after removing finitely many points in $\mathcal{S}^n$.

Proof of Theorem 2 We proceed in several steps.

Step 1 (Construction). We construct $M$ using the strict hyperbolization construction of Charney-Davis [13]. For simplicity we will focus primarily on the case $n \geq 5$. The case $n = 4$ will be explained at the end of Step 2.

The case $n \geq 5$ is modeled on [14] Section (5c). Fix a smooth $n$-manifold $X$ with non-empty connected boundary $Y$, equipped with a PL-triangulation. Choose a smooth homology sphere $H^{n-2}$ with non-trivial fundamental group, take a PL-triangulation of $H$, and consider the double suspension $\Sigma^2 H \cong S^n$, with the obvious induced (no longer PL) triangulation. Take the triangulated connect sum $X \# \Sigma^2 H$, obtained by using the interior of a pair of $n$-simplices
in the triangulated $X$, $\Sigma^2H$ to take the connect sum (and chosen so that simplex in $X$ does not intersect the boundary of $X$). Note that, topologically $X\Sigma^2H$ is homeomorphic to $X$, but now has a triangulation that fails to be PL – there is precisely one 4-cycle in the 1-skeleton of the triangulation whose link is $H$ (instead of $S^n-2$). Finally, we let $M^n = h(X\Sigma^2H)$, an $n$-manifold with boundary $N^{n-1} = h(Y)$, and set $G = \pi_1(M)$.

Properties of hyperbolization implies statement (1) in our Theorem, while statement (2) follows from the fact that the triangulation of $Y$ is PL (applying Davis-Januszkiewicz [14, Theorem (3b.2)]). The rest of our proof thus focuses on establishing statement (3) in the theorem – that $\partial_\infty G$ is not homeomorphic to $\mathcal{S}^{n-2}$.

**Step 2 (Capping procedure).** To show that $\partial_\infty G \neq \mathcal{S}^{n-2}$, first identify $\partial_\infty G \cong \partial_\infty \tilde{M}$. We use Lemma 8 and show that $\pi_1(\partial_\infty \tilde{M} \setminus F) \neq 1$, where $F = \{\gamma_+, \gamma_-\}$ consists of two points.

$\tilde{M}$ is a non-compact CAT($-1$) manifold with non-empty boundary, each component of which is isometric to $h(Y)$. To understand $\partial_\infty \tilde{M}$, we first define an isometric embedding $\tilde{M} \hookrightarrow \hat{M}$ into a CAT($-1$) space without boundary. It will be easier to analyze $\hat{M}$, which is obtained from $\tilde{M}$ by gluing a certain space $Z$ to each component of $\partial_\infty \tilde{M}$. Next we define $Z$ and describe its key features.

Let $DX$ be the double of $X$ across $Y$, with the induced triangulation. We apply a strict hyperbolization of Charney-Davis [13] to obtain a closed $n$-manifold $h(DX)$ equipped with a locally CAT($-1$) metric. The universal cover $h(DX)$ has boundary at infinity homeomorphic to $S^{n-1}$ (see [14, Theorem (3b.2)]). Take any lift $\hat{h}(Y)$ of the separating codimension one submanifold $h(Y) \subset h(DX)$. Then $\hat{h}(Y)$ separates $h(DX)$ into two (isometric) convex subsets. Denote by $Z$ the closure of one of these convex subsets. Then $Z$ is a non-compact locally CAT($-1$) $n$-manifold with totally geodesic boundary $\hat{h}(Y)$.

**Lemma 9.** The boundary at infinity $\partial_\infty Z$ of $Z$ is homeomorphic to $D^{n-1}$. The inclusion $\hat{h}(Y) = \partial Z$ induces, at the boundary at infinity, an identification $\partial_\infty \hat{h}(Y) = S^{n-2} = \partial(D^{n-1})$.

Let us momentarily assume Lemma 9 and finish the proof. Form the CAT($-1$) space $\tilde{M}$ by gluing a copy of $Z$ to each boundary component of $\partial \tilde{M}$, by isometrically identifying the copy of $\hat{h}(Y)$ inside $Z$ with the boundary component. We have an isometric embedding $\tilde{M} \hookrightarrow \hat{M}$, inducing an embedding $\partial_\infty \tilde{M} \hookrightarrow \partial_\infty \hat{M}$. Let $\gamma$ be a lift in $\hat{M}$ of the singular geodesic in $M$, i.e. the geodesic whose link is the homology sphere $H$. The argument in [14, Proof of Theorem 5c.1(iv), pg. 385] applies verbatim to show that $\partial_\infty \hat{M} - \{\gamma_+, \gamma_-\}$ is not simply-connected. If $\eta$ denotes a homotopically non-trivial loop in $\partial_\infty \hat{M} - \{\gamma_+, \gamma_-\}$, then Lemma 8 allows us to use the same argument as in Lemma 8 to homotope $\eta$ into the subset $\partial_\infty \hat{M} = \partial_\infty G$. We conclude that $\partial_\infty G - \{\gamma_+, \gamma_-\}$ fails to be simply connected. From Lemma 8 we conclude that $\partial_\infty G$ is not homeomorphic to $\mathcal{S}^{n-2}$.

The $n = 4$ case proceeds similarly, but is modeled instead on [14, Section (5a)]. Briefly, one lets $X$ be a 4-dimensional simplicial complex whose geometric realization is a homology manifold with non-empty boundary $Y$, and which contains a singular point in the interior of $X$ (whose link is, for example, the Poincare homology 3-sphere $H$). One then looks at the universal cover of the hyperbolization $W = h(X)$. We can “cap off” the boundary components of $\tilde{W}$ as in the last paragraph to obtain $\hat{W}$. Then the arguments in [14, Section 3d] shows that $\pi_1(\partial_\infty \hat{W})$ is non-trivial. Again, using Lemma 9 we can push a homotopically non-trivial loop in $\partial_\infty \hat{W}$ into
For points $p$ well-defined close to $\rho$ where the bonding maps are given by the maps $S_\rho(x,y) = \rho(x) + v(y)$, we rely on the following:

![Image](image.png)

This is again a minor adaptation of the arguments in [14, Sections 3b, 3c]. Choose a basepoint $x \in \partial Z$, and consider the closed metric $r$-balls $B_Z(r)$, $B_{\partial Z}(r)$ in the spaces $Z$, $\partial Z$, centered at $x$, as well as the metric $r$-spheres $S_Z(r)$ and $S_{\partial Z}(r)$. The proof of Lemma 9 will rely on the following:

**Claim 1:** For all $r$, the metric spheres $S_Z(r)$ are manifolds with boundary $S_{\partial Z}(r)$.

**Claim 2:** For points $p \in S_{\partial Z}(r)$, the complement $Lk(p) \setminus B_{Lk(p)}(v; \pi)$ of the metric ball of radius $\pi$, centered at $v \in \partial (Lk(p))$ in the link of $p$, is a cell-like set.

From these two claims, it is easy to conclude. If one takes a small enough $r$, then clearly $S_Z(r)$ is homeomorphic to a disk $D^{n-1}$. In view of Claim 2 and the discussion in [14] pg. 372, there is an $\epsilon > 0$ such that each of the geodesic contraction maps $\rho^\epsilon_r : S_Z(s) \to S_Z(r)$ is a cell-like map when $r < s < r + \epsilon$. So by Claim 1, the maps $\rho^\epsilon_r$ are cell-like maps between manifolds with boundaries. From the work of Siebenmann [26], Quinn [22], and Armentrout [1] it follows that each $\rho^\epsilon_r$ is a *near-homeomorphism* (i.e. can be approximated arbitrarily closely by homeomorphisms), and hence, that all the $S_Z(r)$ are homeomorphic to a disk $D^{n-1}$, with boundary $\partial S_Z(r) = S_{\partial Z}(r)$.

Finally, we identify the pair $(\partial_\infty Z, \partial_\infty(\partial Z))$ with the inverse limit $\lim_{r>0} \{ (S_Z(r), S_{\partial Z}(r)) \}$, where the bonding maps are given by the maps $\rho^\epsilon_r$ (where $0 < r < s$), which we saw are all near-homeomorphisms. Lemma 9 now follows by applying the main result of Brown [10].

This reduces the proof of Lemma 9 (and hence also of the theorem) to checking Claim 1 and Claim 2 – which are the last two steps of the proof.

**Step 4 (Proof of Claim 1).** We first argue that the ball $B_Z(r)$ of radius $r$ is a manifold with boundary. It is clear that points $p \in \operatorname{Int}(\tilde{M})$ at distance $< r$ from the basepoint have manifold neighborhoods. It is also immediate that points $p \in \partial \tilde{M}$ at distance $< r$ from the basepoint have neighborhoods homeomorphic to $\mathbb{R}^{n-1} \times \mathbb{R}_+$. Points at distance $= r$ from the basepoint are either in $\operatorname{Int}(\tilde{M})$ or on $\partial \tilde{M}$.

For points $p$ in $\operatorname{Int}(\tilde{M})$, the argument in [14] pg. 372 shows that $p$ has a neighborhood homeomorphic to $\mathbb{R}^{n-1} \times \mathbb{R}_+$. So the only possible points to worry about are points at distance $= r$, and lying on the subset $\partial \tilde{M}$. But for such a point $p$, a similar argument works with no trouble. Let $v$ be the point in $Lk(p)$ pointing from $p$ to the basepoint $x$, and consider the closed ball $\overline{B}_{Lk(p)}(v; \pi/2)$ in the link of $p$, centered at $v$, of radius $\pi/2$. For any vector $w \in \overline{B}_{Lk(p)}(v; \pi/2)$, one can look at the geodesic $\gamma_w$ emanating from $p$, in the direction $w$ ($\gamma_w$ is well-defined close to $p$). If the direction $w$ is at distance $< \pi/2$ from $v$, then for a small interval of time $[0, s(w)]$, the geodesic $\gamma_w$ lies entirely in $B_Z(r)$, with $\gamma_w(s(w)) \in S_Z(r) \cup B_{\partial Z}(r)$. Note that $s$ varies continuously and $s(w) \to 0$ as $w \to S_{Lk(p)}(v; \pi/2)$. It follows that $p$ has a neighborhood homeomorphic to the set $\hat{X}$ constructed as follows: take the product $I \times \overline{B}_{Lk(p)}(v; \pi/2)$, collapse the fibers over the subset $S_{Lk(p)}(v; \pi/2)$ to 0, and then collapse the subset $\{0\} \times \overline{B}_{Lk(p)}(v; \pi/2)$ to a single point (which is identified with $p$) – see Figure 2. By an inductive argument (note...
that $\dim(\text{Lk}(p)) = \dim(\tilde{M}) - 1$ one can assume that $\overline{B}_{\text{Lk}(p)}(v; \pi/2)$ is homeomorphic to a disk $\mathbb{D}^{n-1}$, with the subset $S_{\text{Lk}(p)}(v; \pi/2)$ corresponding to an embedded $\mathbb{D}^{n-2}$ inside $\partial \mathbb{D}^{n-1} \cong S^{n-2}$.

Following the construction of $\tilde{X}$ given above, we see that $\tilde{X}$ is homeomorphic to $\mathbb{D}^n$, with the point corresponding to $p$ lying on $\partial \mathbb{D}^n$. This shows that $B_Z(r)$ is indeed a manifold with boundary, and that the boundary of $B_Z(r)$ naturally decomposes as the union of $S_Z(r) \cup B_{\partial Z}(r)$, where the union is over the common subset $S_{\partial Z}(r)$.

Finally, we check that $S_Z(r)$ is an $(n-1)$-manifold with boundary. For points $p \in S_Z(r)$ lying in $\text{Int}(\tilde{M})$, it follows easily from [14, pg. 372] that these points have neighborhoods homeomorphic to $\mathbb{D}^{n-1}$ with $p$ lying as an interior point. In the case where $p \in S_Z(r)$ lies on $\partial \tilde{M}$, we look at the neighborhood $\hat{X}$ of $p$ after quotienting by the gray region.

**Step 5 (Proof of Claim 2).** We want to show that the complement $\text{Lk}(p) \setminus B_{\text{Lk}(p)}(v; \pi)$ is cell-like. The set $\text{Lk}(p)$ is homeomorphic to a disk $\mathbb{D}^{n-1}$, so we can think of the set we are interested in as lying within the double $D(\text{Lk}(p)) \cong S^{n-1}$. Given an $r \in (0, \pi)$, consider the subset $U_r \subset D(\text{Lk}(p)) \cong S^{n-1}$ defined to be the union of $D(\text{Lk}(p)) \setminus \text{Lk}(p)$ and the set $B_{\text{Lk}(p)}(v; r)$. See Figure 3. We will show each such $U_r$ is homeomorphic to $\mathbb{R}^{n-1}$. Then by a result of Brown [11] it follows that the union $U_\infty := \bigcup_{r \in (0, \pi)} U_r \subset D(\text{Lk}(p)) \cong S^{n-1}$ is also

![Figure 2](image1.png)  
**Figure 2.** Left: The link $L = \text{Lk}(p)$. Right: The space $I \times \overline{B}_{\text{Lk}(p)}(v; \pi/2)$, which is identified with a neighborhood $\tilde{X}$ of $p$ after quotienting by the gray region.

![Figure 3](image2.png)  
**Figure 3.** The link $L = \text{Lk}(p)$ and its double $DL$. 


homeomorphic to $\mathbb{R}^{n-1}$. But if a subset of $S^{n-1}$ is homeomorphic to $\mathbb{R}^{n-1}$, its complement is automatically cell-like. Since the complement of $U_\infty$ coincides with $Lk(p) \setminus B_{Lk(p)}(v;\pi)$, this would establish Claim 2.

To see that each $U_\tau$ is homeomorphic to $\mathbb{R}^{n-1}$, we consider their closures $\overline{U_\tau}$. We have that $U_\tau = \text{Int}(\overline{U_\tau})$, and that $\overline{U_\tau}$ can be written as the union of a copy of $Lk(p)$ along with $B_{Lk(p)}(v;r)$, where the union is taken over the common subset of $B_{\partial Lk(p)}(v;r)$. Let us analyze the two pieces in this decomposition.

On one of the sides, the subset $Lk(p)$ is homeomorphic to $\mathbb{D}^{n-1}$, and the common subset $B_{\partial Lk(p)}(v;r)$ is homeomorphic to an embedded $(n-2)$-disk $\mathbb{D}^{n-2}$ inside the boundary sphere $\partial Lk(p) \cong S^{n-2}$. Note that, by varying the parameter $r$, we see that

$$S^{n-3} \cong \partial B_{\partial Lk(p)}(v;r) \subset \partial Lk(p) \cong S^{n-2}$$

is bicolliared. On the other side, the subset $B_{Lk(p)}(v;r)$ is also homeomorphic to $\mathbb{D}^{n-1}$, and the gluing disk $\mathbb{D}^{n-2} \cong B_{\partial Lk(p)}(v;r)$ inside the boundary sphere $S^{n-2} \cong \partial B_{Lk(p)}(v;r)$ also has complement a disk (by the argument in Claim 1). Thus, we see that $\overline{U_\tau}$ is obtained by gluing together two closed $(n-1)$-disks, by identifying together two copies of an $(n-2)$-disk, where each copy is nicely embedded in the respective boundary spheres $S^{n-2} \cong \mathbb{D}^{n-1}$. It follows that $\overline{U_\tau}$ is also homeomorphic to $\mathbb{D}^{n-1}$. This completes the proof of Claim 2 and the proof of the theorem. □

Remark 10. Let us make a small comment on approximating cell-like maps by homeomorphisms, in the case of manifolds with boundary. The attentive reader will probably notice that, in Siebenmann’s work [20], there are two cases that require special care. In the 5-dimensional case, he requires the restriction of the map to the boundary to be a homeomorphism (rather than just a cell-like map). This is due to the fact that, at the time [26] was written, it was unclear whether or not cell-like maps of (closed) 4-manifolds could be approximated by homeomorphisms—hence the need of a stronger hypothesis on the boundary map. In view of Quinn’s subsequent proof of the 4-dimensional case [22], this stronger hypothesis is no longer needed in the 5-dimensional boundary case. Note that, in our context, the bonding maps, when restricted to the boundary, are always cell-like (but are not homeomorphisms).

The other special case has to do with 3-dimensions. Here there is an added hypothesis that every point pre-image has a neighborhood $N$ which isn’t just contractible, but in addition is prime (i.e. if $N = M_1 \# M_2$, then one of the $M_i$ is a standard 3-sphere). The only way this could fail is if one of the $M_i$ were instead a homotopy 3-sphere – but by Perelman’s resolution of the Poincaré Conjecture, such a manifold is automatically $S^3$. So again, in the 3-dimensional case, this additional hypothesis is now unnecessary.

5. Remarks on CAT(0) groups

In this section we remark on generalizing the main result from hyperbolic groups to CAT(0) groups. A proper geodesic space $X$ is called CAT(0) if geodesic triangles in $X$ are at least as thin as triangles in Euclidean space [8]. A group $G$ is called CAT(0) if there exists a CAT(0) space $X$ on which $G$ acts geometrically (that is, isometrically, properly, and compactly).

A CAT(0) space $X$ has a visual boundary $\partial_\infty X$, and if $G$ acts geometrically on $X$, then $G$ acts on $\partial_\infty X$ by homeomorphisms. In this case $\partial_\infty X$ is called a boundary of $G$. With this terminology we have the following theorem.
**Theorem 11.** Let $G$ be a CAT(0) group for which $S^{n-1}$ is a boundary. If $n \geq 6$, then there exists a closed $n$-dimensional aspherical manifold $W$ such that $\pi_1(W) \simeq G$.

The proof is almost identical to the proof of Theorem [7] in [4]. We give a short explanation for how to extend that argument to the CAT(0) case.

*Proof of Theorem [7]* By assumption $G$ acts geometrically on an $X$ with $\partial_\infty X = S^{n-1}$. Denote $\overline{X} = X \cup \partial_\infty X$. We proceed in three steps.

*Step 1.* $BG$ is homotopy equivalent to a closed aspherical homology $n$-manifold $W$ such that $W$ has the disjoint disk property. To show this, it suffices to show that $G$ is a PD($n$) group and to note that CAT(0) groups satisfy the Farrell-Jones conjectures in $K$- and $L$-theory. For then we may use [4, Theorem 1.2], which says that for such a group, $BG$ is homotopy equivalent to a closed aspherical homology $n$-manifold $M$ with the disjoint disk property.

We explain why $G$ is PD($n$) group. First, we know $G$ is of type FP once we know that there exists a finite CW complex $K \simeq BG$. For then the cellular chain complex of the universal cover $\tilde{K}$ is a finite length resolution of $Z$ by finitely generated free $G$ modules. A finite CW complex $K \simeq BG$ for a group $G$ that acts geometrically on a proper CAT(0) space is shown to exist by Lück [21]. Now $G$ is a PD($n$) group because

$$H^i(G;ZG) \cong H^i_c(X) \cong \bar{H}^{i-1}(\partial_\infty X) = \bar{H}^{i-1}(S^{n-1}) = \begin{cases} Z & \text{if } i = n \\ 0 & \text{else} \end{cases}$$

The first two isomorphisms are described by Bestvina [5]. That this implies $G$ is a PD($n$) group is explained in [4, VIII.10.1].

*Step 2.* The universal cover $\tilde{W}$ can be compactified $N = \tilde{W} \cup \partial_\infty X$ such that $N$ is a homology manifold with boundary. To show that $N$ is a homology manifold with boundary it suffices to show that $N$ is a finite-dimensional locally compact ANR and $\partial_\infty X$ is a $Z$-set in $N$ (see [4, Proposition 2.5]). The pair $(\overline{X},\partial_\infty X)$ is a $Z$-structure on $G$ by Bestvina [5, Example 1.2(ii)]. Furthermore, by [5, Lemma 1.4] for any other finite model $K$ for $BG$, there is a natural $Z$-structure on $(\overline{K},\partial_\infty X)$, where $\overline{K} = K \cup \partial_\infty X$. Thus $(N,\partial_\infty X)$ admits a $Z$-set structure; in particular, $N$ is a Euclidean retract, finite dimensional, and $S^{n-1}$ is a $Z$-set inside $N$.

*Step 3.* $\tilde{W}$ (and hence also $W$) is a manifold. This part of the argument is identical to that given in [4, Theorem A]. Quinn’s invariant allows one to recognize manifolds among homology manifolds with the disjoint disk property. By the local nature of Quinn’s invariant, if $(B,\partial B)$ is a homology manifold with boundary and $\partial B$ is a manifold, then $\text{int}(B)$ is a manifold. □

In light of this result and Theorem [1] above, it is natural to ask the following question.

**Question.** Let $G$ be a CAT(0) group which admits $\mathcal{F}^{n-2}$ as a boundary. Is $G$ the fundamental group of an $n$-dimensional aspherical manifold with boundary?

Examples of $G$ satisfying the hypothesis of this Question are given by Ruane [25]: every nonuniform lattice $\Gamma \leq \text{SO}(n,1)$ is an example. For these examples, an aspherical manifold with boundary can be obtained by “truncating the cusps” of $\mathbb{H}^n/\Gamma$.

There are some basic problems with answering this Question with the techniques of this paper. For example, it is not obvious that peripheral subgroups of a CAT(0) group with Sierpinski space boundary are CAT(0), or that the double of a CAT(0) group along peripheral subgroups is CAT(0).
References


HYPERBOLIC GROUPS WITH BOUNDARY AN n-DIMENSIONAL SIERPINSKI SPACE

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