THE WEAK SPECIFICATION PROPERTY FOR GEODESIC FLOWS ON CAT(-1) SPACES

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Abstract. We prove that the geodesic flow on a compact locally CAT(-1) space has the weak specification property, and give various applications of this property. We show that every Hölder continuous function on the space of geodesics has a unique equilibrium state, and as a result, that the Bowen-Margulis measure is the unique measure of maximal entropy. We establish the equidistribution of weighted periodic orbits and the large deviations principle for all such measures. For compact locally CAT(0) spaces, we give partial results, both positive and negative, on the specification property and the existence of a coding of the geodesic flow by a suspension flow over a compact shift of finite type.

1. Introduction

An important characteristic of hyperbolic dynamical systems is the *specification property*, introduced by Rufus Bowen in the early 1970's. The geodesic flow on the unit tangent bundle of a negatively curved Riemannian manifold is the primary example of a flow satisfying the specification property. Bowen used the specification property to establish a number of fundamental results about the ergodic properties of such geodesic flows (and more generally, for Anosov flows), showing for example the equidistribution of prime closed geodesics to an ergodic measure of maximal entropy [Bow72]. These results were obtained independently of Margulis' seminal work, and were proved before Bowen established the existence of Markov partitions and associated symbolic dynamics for these geodesic flows [Bow73]. Beyond uniform hyperbolicity, the paradigm remains that while proofs of the strongest properties of hyperbolic dynamics often require the system to be described by symbolic dynamics, an approach using the specification property affords much greater flexibility, and still yields many interesting results.

In the early 1980's, Gromov realized that many properties of negatively curved Riemannian manifolds held in much greater generality – for the class of *compact, locally* CAT(-1) *spaces*. The fundamental group of such a space is *non-elementary* if it is not isomorphic to $\mathbb{Z}$. Though they need not be Riemannian manifolds, compact locally CAT(-1) spaces admit a geodesic flow, as described in [Gro87]. More precisely, to any such space X, one can associate the space GX of all (bi-infinite) geodesics in X. The space GX is a compact metric space, and possesses a natural $\mathbb{R}$-flow by shifting the parametrization of geodesics – we call this the

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geodesic flow. A natural problem is to develop Bowen's approach in this broader class of flow spaces. Our first result is the following:

Theorem 1.1. *Let X be a compact, locally* $\mathrm{CAT}(-1)$*, geodesic metric space, with non-elementary fundamental group. Then the geodesic flow on GX satisfies the weak specification property.*

The weak specification property for a flow is a natural analogue of a well known discrete-time definition, and is a weakening of Bowen's original specification property. For the proof of Theorem 1.1, we use a coding of the geodesic flow due to Gromov [Gro87] (see also Coornaert and Papadopoulos [CP12]), which uses the topology of the setting to give a suspension on a subshift of finite type $\mathrm{Susp}(\Sigma, T)$, and an orbit semi-equivalence $h : \mathrm{Susp}(\Sigma, T) \to GX$. This gives us a "weak" symbolic description of GX: unlike the semi-conjugacy with a suspension flow which occurs in the negatively curved Riemannian setting, a priori, orbit semi-equivalence is too weak a relationship to preserve any refined dynamical information [GM10]. Our approach is to show that we can combine this weak symbolic description with an argument that uses the geometry of X to "push down" the weak specification property from $\mathrm{Susp}(\Sigma, T)$ to GX. Once we have the weak specification property for GX, we use this property directly to study thermodynamic formalism and large deviations for $\mathrm{CAT}(-1)$ spaces. We have the following:

Theorem 1.2. *Let X be a compact, locally* $\mathrm{CAT}(-1)$*, geodesic metric space, with non-elementary fundamental group, and φ a Hölder continuous function on GX. Then*

(1) *the potential function φ has a unique equilibrium measure μ_φ,*
(2) *the equilibrium measure μ_φ satisfies the Gibbs property,*
(3) *the φ-weighted periodic orbits for the geodesic flow equidistribute to μ_φ,*
(4) *the measure μ_φ satisfies the large deviations principle.*

In particular, for the special case $\varphi \equiv 0$, we see that the Bowen-Margulis measure μ_{BM} is the unique measure of maximal entropy, that μ_{BM} satisfies the Gibbs property, and that it satisfies the large deviations principle.

The dynamical notions that appear in the above theorem (equilibrium measures, large deviations principle, etc.) are defined in §6. For the geodesic flow on Riemannian manifolds of negative curvature (and more generally, for Axiom A flows), uniqueness of equilibrium states for Hölder potentials was proved by Bowen and Ruelle [BR75]. The uniqueness of the measure of maximal entropy (MME) was proved a little earlier in [Bow74]. In the context of the geodesic flow on locally $\mathrm{CAT}(-1)$ spaces, the Bowen-Margulis measure, which is defined using the Patterson-Sullivan construction, has been studied extensively, see e.g Roblin [Rob03]. This measure is known to be an MME, but the question of uniqueness does not seem to have been previously addressed.

Uniqueness of the MME beyond the negative curvature compact Riemannian case has received continued interest: notably, for non-positively curved Riemannian manifolds, this was proved in the deep work of Knieper [Kni98, Kni05]. Another notable result in this direction is Bufetov and Gurevic's proof of uniqueness of the MME for Teichmüller geodesic flow [BG11].

A beautiful theory of equilibrium states has been developed in the non-compact negative curvature Riemannian setting by Paulin, Pollicott and Schapira [PPS15], including results on uniqueness and equidistribution. They explicitly state that the reason they assume a smooth structure is due to the difficulties associated with controlling a Hölder potential function on GX for a CAT(-1) space (see remarks after [PPS15, Theorem1.10]. We sidestep these difficulties, providing techniques to handle Hölder potentials in the CAT(-1) setting.

The argument for obtaining the Large Deviations Principle from the specification property goes back to the 90's with notable results by Denker, Young, and Eizenberg, Kifer and Weiss [Den92, You90, EKW94]. We adapt this approach to the current setting. Large deviations results for flows with a weak specification property have also recently been obtained by [BV15].

Results on existence and non-existence of symbolic dynamics for various classes of dynamical systems have a long history [BR75, Adl98, BD04, Sar13]. When the dynamics can be described by a semi-conjugacy to a shift of finite type, this is a powerful technique to study the global statistical properties of the dynamical system [PP90]. It is reasonable to ask to what extent this approach can be carried out in the non-positively curved setting. The argument of Theorem 1.1 extends to the CAT(0) setting to give the following statement.

Theorem 1.3. *Let X be a compact, locally* $\mathrm{CAT}(0)$*, geodesic metric space with non-elementary fundamental group and topologically transitive geodesic flow. If there exists an orbit semi-equivalence $h : \mathrm{Susp}(\Sigma, T) \to GX$, where (Σ, T) is a compact subshift of finite type, then the geodesic flow on GX satisfies the weak specification property.*

This theorem can be used to give positive results on specification for some CAT(0) examples, including all those whose geodesic flow is orbit equivalent to geodesic flow on a CAT(−1) space. Another aspect of this result is that it can be used to *rule out* orbit semi-equivalence to a suspension of a shift of finite type in many cases. We show:

Corollary 1.4. *Let X be a compact, locally* $\mathrm{CAT}(0)$*, geodesic metric space with topologically transitive geodesic flow. Assume that $\tilde{X}$ contains a geodesic γ such that for some $w > 0$ the w-neighborhood $U = N_w(\gamma)$ of γ splits isometrically as $\mathbb{R} \times Y$. Then there does not exist any orbit semi-equivalence $h : \mathrm{Susp}(\Sigma, T) \to GX$, where (Σ, T) is a compact subshift of finite type.*

The hypotheses of Corollary 1.4 hold when M is a closed, irreducible, Riemannian manifold with non-positive sectional curvature which has an open neighborhood U of a closed geodesic where the sectional curvature is identically zero.

We point out that a basic obstruction to an orbit semi-equivalent coding is the existence of uncountably many closed geodesics. While this is the case for many examples covered by Corollary 1.4, it is not a consequence of our hypotheses, even for Riemannian 3-manifolds. For example, the flat strip could have holonomy an irrational rotation around a single central closed geodesic.

The paper is organized as follows. First, in §2, we summarize background material on the weak specification property, subshifts of finite type, suspension flows, and geodesic flows on locally CAT(−1) spaces. In §3, we establish Theorem 1. In §4, we establish Theorem 1.3 and Corollary 1.4. In §5, we prove that geodesic flows on CAT(−1) spaces are expansive, and that Hölder continuous functions on GX satisfy the Bowen property. In §6, we prove Theorem 1.2. Some additional technical results are proved in §7.

2. Background Material

The specification property is the ability to find an orbit segment which approximate the trajectories of finitely many given orbit segments. There are a number of quantifiers required to make the previous sentence rigorous, and there are many variations on the precise definition in the literature. We introduce here the specification properties which are relevant to our study. We also remind the reader of some basic dynamical notions, including subshifts of finite type, and suspension flows. Finally, we review properties of the geodesic flow on locally CAT(-1) spaces.

2.1. **Specification for flows.** Let $\mathcal{F} = \{f_s\}$ be a flow on a compact metric space (X, d). Given any $t > 0$, we can define a new metric by

$$d_t(x,y) = \max\{d(f_s x, f_s y) : s \in [0,T]\}.$$

Writing $\mathbb{R}^+ = [0,\infty)$, we view $X \times \mathbb{R}^+$ as the space of finite orbit segments for $(X, \mathcal{F})$ by associating to each pair (x,t) the orbit segment $\{f_s(x) \mid 0 \le s < t\}$. We say that $\mathcal{F}$ has *weak specification at scale* δ if there exists $\tau > 0$ such that for every collection of finite orbit segments $\{(x_i,t_i)\}_{i=1}^k$, there exists a point y and a sequence of *transition time* $\tau_1, \dots, \tau_{k-1} \in \mathbb{R}^+$ with $\tau_i \le \tau$ such that for $s_j = \sum_{i=1}^j t_i + \sum_{i=1}^{j-1} \tau_i$ and $s_0 = \tau_0 = 0$, we have

$$d_{t_j}(f_{s_{j-1}+\tau_{j-1}} y, x_j) < \delta \text{ for every } 1 \le j \le k. \tag{2.1}$$

We say $\mathcal{F}$ has *weak specification* if it has weak specification at every scale $\delta > 0$. We say $\mathcal{F}$ has *weak specification at scale δ with maximum transition time τ* if we want to declare a value of τ that plays the role described above. This definition of weak specification for flows appeared recently in the literature in [CT15], and under another name in [BV15].

Intuitively, (2.1) means that there is some point y whose orbit shadows the orbit of x_1 for time t_1, then after a transition period which takes time at most τ, shadows the orbit of x_2 for time t_2, and so on. Note that s_j is the time spent for the orbit y to approximate the orbit segments (x_1,t_1) up to (x_j,t_j). It is sometimes convenient to use the word 'shadowing' formally: For $y \in X$ and $s \in \mathbb{R}$, we say that $f_s y$ *δ-shadows* the orbit segment (x,t) if $d_t(f_s y, x) < \delta$.

The weak specification property clearly implies topological transitivity. Transitivity alone allows us to find an orbit which shadows a finite collection of orbit segments, but it does not give us any control on the size of the transition time. This is the crucial additional ingredient provided by weak specification: the size of the transition times is uniformly bounded above, depending only on the scale δ, and not on the orbit segments, or their lengths.

Remark. The specification property for flows which was originally introduced by Bowen is substantially stronger than weak specification. The main difference is that we ask that the approximating orbit y is periodic, and that, after modifiying by a small time change, we can take ANY $t \ge \tau$ to be a transition time. See [KH95, §18.3] or [Bow72] for the precise definition of specification for flows. Any topologically mixing Anosov flow has the specification property, see [KH95, §18.3], and these form the original motivating example for this property. Concrete examples are provided by the geodesic flow on any compact, negatively curved manifold.

Finally, we note that while the weak specification property only involves approximating *finitely* many orbit segments, it is easy to obtain an *infinitary version.* Since we will require this in the proof of Theorem 1.2, details are given in §7.1.

2.2. **Specification for discrete time systems.** Now let f be a continuous map on a compact metric space X. We view $X \times \mathbb{N}$ as the space of finite orbit segments for (X, f) by associating to each pair (x, n) the orbit segment $\{f^i x \mid i \in \{0, \dots n-1\}\}$. We say that f has *weak specification at scale* δ if there exists $\tau \in \mathbb{N}$ such that for every collection of finite orbit segments $\{(x_i, n_i)\}_{i=1}^k$, there exists a point y and a sequence of *transition times* $\tau_1, \dots, \tau_{k-1} \in \mathbb{N}$ with $\tau_i \leq \tau$ such that for $s_j = \sum_{i=1}^j n_i + \sum_{i=1}^{j-1} \tau_i$ and $s_0 = \tau_0 = 0$, we have

$$d_{t_j}(f^{s_{j-1}+\tau_{j-1}} y, x_j) < \delta \text{ for every } 1 \leq j \leq k. \tag{2.2}$$

We say f has *weak specification* if it has weak specification at every scale $\delta > 0$. We say f has *specification* if all transition times τ_i can be taken to be exactly τ.

2.3. **Shift spaces.** The full, two-sided shift on a finite alphabet $\mathcal{A}$ is a dynamical system on the set of bi-infinite sequences in the symbols of $\mathcal{A}$: $\Sigma_0 = \{\sigma : \mathbb{Z} \to \mathcal{A}\}$. The dynamics are given by the shift map $T : \Sigma_0 \to \Sigma_0$ defined by $T\sigma(n) = \sigma(n+1)$. Σ_0 is endowed with the usual product topology, and is compact and metrizable with the following metric:

$$d(\sigma, \tau) = \frac{1}{2^i} \quad \text{where} \quad i = \min\{|n| : \sigma(n) \neq \tau(n)\}.$$

A subshift of (Σ_0, T) is any closed, shift-invariant subset of Σ_0, with the induced topology, metric, and action of T.

Definition 2.1. Let $k > 0$ be an integer, and let $W \subset \mathcal{A}^{k+1}$ be any (non-empty) subset. Let

$$\Sigma(W) = \{\sigma \in \Sigma_0 : \text{ for all } n \in \mathbb{Z}, (\sigma(n), \dots, \sigma(n+k)) \in W\}.$$

Then $(\Sigma(W), T|_{\Sigma(W)})$ is a *subshift of finite type.*

Remark. To simplify notation, we will write Σ for $\Sigma(W)$ and T for $T|_{\Sigma(W)}$. Remark that, as a closed subset of the compact space Σ_0, Σ is compact.

Given a shift space (Σ, T), the *language of* Σ, denoted by $\mathcal{L} = \mathcal{L}(\Sigma)$, is the set of all finite words that appear in any sequence $x \in \Sigma$ – that is,

$$\mathcal{L}(\Sigma) = \{w \in \mathcal{A}^* \mid [w] \neq \varnothing\},$$

where $\mathcal{A}^* = \bigcup_{n \geq 0} \mathcal{A}^n$ and $[w]$ is the central cylinder for w. Given $w \in \mathcal{L}$, let $|w|$ denote the length of w. We now define the weak specification property for a shift space.

Definition 2.2. Given a shift space (Σ, T), and its language $\mathcal{L}$, we say that (Σ, T) has *weak specification* if there exists $\tau \in \mathbb{N}$ so for every $v, w \in \mathcal{L}$ there is $u \in \mathcal{L}$ such that $vuw \in \mathcal{L}$ and $|u| \leq \tau$.

It is a straightforward exercise to show that Definition 2.2 is equivalent to the more general weak specification property for maps defined in Section 2.2.

2.4. **Suspension flow.** We recall the definition of the suspension flow.

Definition 2.3. Let (X,T) be a (discrete) dynamical system. Then $\mathrm{Susp}(X,T)$ is the space $(X \times [0,1])/\sim$ where $(x,1) \sim (Tx,0)$, equipped with the flow $\phi_t(x,s) = (T^{\lfloor t+s \rfloor}x, [[t+s]])$ where $[[x]]$ denotes the fractional part of x.

If X is a metric space, we equip the space for the suspension flow with the Bowen-Walters metric (see [BW72]). For two point $(x,s),(y,s)$, we define the horizontal distance to be

$$d_H((x,s),(y,s)) = (1-s)d(x,y) + sd(Tx,Ty)$$

For two points $(x,s),(x,t)$, we define the vertical distance to be

$$d_V((x,s),(x,t)) = |s-t|$$

We define $d((x,s),(y,t))$ to be the smallest path length of a chain of horizontal and vertical paths connecting (x,s) and (y,t), where path length is calculated using d_H and d_V. The reason that we use this metric over a more naive choice is that the suspension flow is continuous in the Bowen-Walters metric.

We now establish the relationship between transitivity and weak specification for shifts of finite type and suspension flows.

Proposition 2.4. *Let Σ be a subshift of finite type. The following are equivalent.*

1. *Σ is transitive;*
2. *Σ satisfies the weak specification property;*
3. *$\mathrm{Susp}(\Sigma,T)$ is transitive;*
4. *$\mathrm{Susp}(\Sigma,T)$ satisfies the weak specification property.*

Proof. We prove $(1) \Longrightarrow (2) \Longrightarrow (4) \Longrightarrow (3) \Longrightarrow (1)$.

Proving $(1) \Longrightarrow (2)$ is a straightforward exercise: transitivity for a shift of finite type allows us to transition from any symbol i to another symbol j in bounded time. Thus, to glue two words $v,w \in \mathcal{L}$, it suffices to look at the final symbol of v and the first symbol of w and take a word which transitions between them.

To prove $(2) \Longrightarrow (4)$, we show that if (X,T) is a dynamical system with the weak specification property, then $\mathrm{Susp}(X,T)$ satisfies weak specification. Suppose (X,T) has weak specification at scale δ with maximum transition time τ. Suppose that we wish to find an orbit for the suspension flow which approximates the orbit segments $((x_1,s_1),t_1),\ \dots,\ ((x_k,s_k),t_k)$ at scale δ. We can apply the weak specification property to approximate the orbit segments $(x_1,\lfloor t_1 \rfloor + 2),\ \dots,\ (x_k,\lfloor t_k \rfloor + 2)$ in the base with an orbit segment (y,n). It is straightforward to check that if $y \in B_n(x,\delta)$ in the base, then $(y,s) \in B_{n-1}((x,s),\delta)$ in the Bowen-Walters metric. Using this fact, we can verify that the orbit segment for the flow starting at (y,s_1) approximates the orbit segments $((x_1,s_1),t_1),\dots,((x_k,s_k),t_k)$ as required (in the sense of (2.2)), with maximum transition time $\tau + 2$.

$(4) \Longrightarrow (3)$ is trivial. All that remains is to show that $(3) \Longrightarrow (1)$, and we prove the contrapositive . If Σ is not transitive, then there exists cylinder sets $[w_1],[w_2]$ so that $\sigma^k[w_1] \cap [w_2] = \varnothing$ for all k. Clearly, the open sets $A = [w_1] \times (0,\frac{1}{2})$, $B = [w_2] \times (0,\frac{1}{2})$ satisfy $\phi_t A \cap B = \varnothing$ for all t, so $\mathrm{Susp}(\Sigma,\sigma)$ is not transitive. □

2.5. **Orbit equivalences.** Our arguments will rely on the existence of an orbit semi-equivalence from a flow space which is well understood (suspension flow on a subshift of finite type) to a flow space we are interested in (geodesic flow for a CAT(-1) space). We recall:

Definition 2.5. Flows (X, ϕ_t) and (Y, ψ_t) are *orbit equivalent* if there is a homeomorphism $h : X \to Y$ sending orbits of ϕ_t to orbits of ψ_t homeomorphically and preserving the orientation along those orbits. A *orbit semi-equivalence* of flows is a continuous surjection $h : X \to Y$, whose restriction to any ϕ-orbit in X is an orientation preserving local homeomorphism onto some ψ-orbit in Y.

We note that orbit semi-equivalence is too weak a relationship to preserve any refined dynamical information [GM10]. In particular, weak specification is not preserved by orbit equivalence in general. To see this, a convenient source of examples of orbit equivalences comes from considering suspension flows with varying roof function $r : X \to \mathbb{R}^+$ over a discrete dynamical system X. In this well-known modification of the suspension construction, the underlying space $\mathrm{Susp}_r(X, T)$ is the quotient of $\{(x,t) : 0 \le t \le r(x)\} \subset X \times [0, \infty)$ by the equivalence relation $(x, r(x)) \sim (Tx, 0)$ for all $x \in X$, and the flow is given by unit speed shift in the t-direction. We refer the reader to Parry and Pollicott [PP90] for more details. It is clear that, for any choice of continuous roof functions $r_1, r_2 : X \to \mathbb{R}^+$, the suspension flows $\mathrm{Susp}_{r_1}(X, T), \mathrm{Susp}_{r_2}(X, T)$ will always be orbit equivalent.

It is possible to construct a suspension flow over the full shift with more than one measure of maximal entropy, which rules out the possibility that this flow has weak specification. We do not give full details of this construction here as it is beyond the scope of this paper, but we note that the main tool is the description of the measures of maximal entropy for the flow in terms of equilibrium states for the base map given by Proposition 6.1 of [PP90]. This reduces the problem to finding a roof function r so that $P(-r) = 0$ and $-r$ has more than one equilibrium state.

2.6. CAT(-1) **spaces and their geodesic flows.** We now remind the reader of some basic results on the geometry and dynamics of locally CAT(-1) space. Given any geodesic triangle $\Delta(x, y, z)$ inside a geodesic space X, one can construct a comparison triangle $\Delta(\bar{x}, \bar{y}, \bar{z})$ inside the hyperbolic plane $\mathbb{H}^2$ having exactly the same side lengths. Corresponding to any pair of points p, q on the triangle $\Delta(x, y, z)$, there is a corresponding pair of comparison points $\bar{p}, \bar{q}$ on $\Delta(\bar{x}, \bar{y}, \bar{z})$. The triangle is said to satisfy the CAT(-1) inequality if, for every such pair of points, one has the inequality $d_X(p, q) \le d_{\mathbb{H}^2}(\bar{p}, \bar{q})$. A geodesic space is CAT(-1) if every geodesic triangle in the space is CAT(-1), and it is locally CAT(-1) if every point has a neighborhood which is $CAT(-1)$. In this paper, we are interested in compact locally CAT(-1) spaces. Any such space X has a universal cover $\tilde{X}$ which is CAT(-1), with $\Gamma := \pi_1(X)$ acting isometrically on $\tilde{X}$.

We assume from now on that $\Gamma := \pi_1(X)$ is *non-elementary*, i.e. not isomorphic to $\mathbb{Z}$. This is the generic case. When $\Gamma \cong \mathbb{Z}$ (e.g. $X = S^1$), the geodesic flow on X behaves differently from other examples, and is simple to investigate. GX consists of two disjoint circles, with the flow acting by rotations on the circles. Note that specification clearly *fails* in this case, as two orbit segments on the distinct circles can never be approximated by a single orbit segment.

To a CAT(-1) space $\tilde{X}$, one can associate a *boundary at infinity* $\partial^\infty \tilde{X}$, consisting of equivalence classes of geodesic rays $\eta : [0, \infty) \to \tilde{X}$, where rays are considered equivalent if they remain at bounded distance apart. Note that any geodesic $\gamma : \mathbb{R} \to \tilde{X}$ naturally gives rise to a pair of points $\gamma^\pm \in \partial^\infty \tilde{X}$. If we form $G\tilde{X}$ the space of all geodesics in $\tilde{X}$, there is thus a natural identification $G\tilde{X} \cong \partial^\infty X \times \partial^\infty X \times \mathbb{R}$.

There is a natural flow on $G\tilde{X}$, given by translating in the $\mathbb{R}$-factor,which we call the *geodesic flow* on $\tilde{X}$. This geodesic flow on $G\tilde{X}$ can be written as $g_t(\gamma(s)) = \gamma(s+t)$.

Now if X is locally CAT(-1), then one can similarly form the space GX of geodesics in X, where a geodesic is a locally isometric map $\gamma : \mathbb{R} \to X$. This comes equipped with a natural flow, given by pre-composing by translations on $\mathbb{R}$, which we call the *geodesic flow on* X. The fundamental group Γ acts isometrically on the universal cover $\tilde{X}$, hence on the boundary at infinity $\tilde{X}$, and on the space of geodesics $G\tilde{X}$. The flow on $G\tilde{X}$ commutes with the Γ-action, hence descends to a flow on $(G\tilde{X})/\Gamma$, and there is a flow equivariant homeomorphism $GX \cong (G\tilde{X})/\Gamma$.

Finally, if we also have that the locally CAT(-1) space X is compact, then the fundamental group Γ is a Gromov hyperbolic group, see [Gro87]. Such a group has a well-defined boundary at infinity $\partial^\infty\Gamma$, and there is a Γ-equivariant homeomorphism $\partial^\infty\Gamma \cong \partial^\infty\tilde{X}$. This allows us to take results on $\partial^\infty\Gamma$ obtained from the theory of Gromov hyperbolic groups, and to apply them to the boundary $\partial^\infty\tilde{X}$.

The space GX of all geodesics in X can be endowed with the metric

$$d_{GX}(\gamma_1,\gamma_2) = \int_{-\infty}^{\infty} d_X(\gamma_1(t),\gamma_2(t))e^{-2|t|}dt.$$

For a geodesic $\gamma \in GX$, we use the notation $\gamma([0,T]) := \{\gamma(s) : s \in [0,T]\}$ for a geodesic segment of γ, considered as a path in X. We want to compare geodesic segments after a possible time change, and it is convenient to make the following definition.

Definition 2.6. We say that $\rho : [0,T_1] \to [0,T_2]$ is a *time-change* function if it is a continuous, increasing and surjective function.

A detailed discussion of the geodesic flow on locally CAT(-1) spaces can be found in Ballmann's book [Bal95] or in Roblin's monograph [Rob03].

2.7. **Background results on** CAT(-1) **spaces.** We collect some background results on CAT(-1) spaces that we use in this paper.

Lemma 2.7. *Let X be a compact, locally* CAT(-1)*, geodesic metric space. Then the geodesic flow on $GX = G(\tilde{X}/\Gamma) = (G\tilde{X})/\Gamma$ is topologically transitive.*

Proof. Since Γ is non-elementary, the Γ-action on $\partial^\infty\Gamma$ has dense orbits (see [Gro87, Section 8.2]), and hence so does the Γ-action on $\partial^\infty\tilde{X}$. The lemma is now an immediate consequence of [Bal95, Theorem III.2.3]. □

Lemma 2.8. *Let X be a compact, locally* CAT(-1)*, geodesic metric space. Then there exists a topologically transitive subshift of finite type (Σ, T), and an orbit semi-equivalence $h : \mathrm{Susp}(\Sigma,T)$ to GX. Moreover, h is finite-to-one.*

Proof. To a Gromov hyperbolic group Γ, one can associate a metric space $\hat{G}(\Gamma)$, equipped with both a Γ-action, and a Γ-equivariant $\mathbb{R}$-flow. The space $\hat{G}(\Gamma)$ is constructed to satisfy certain universal properties. The construction was outlined by Gromov in [Gro87, Theorem 8.3.C], with detailed arguments worked out by Champetier [Cha94, Section 4] (see also Mathéus [Mat91]).

The quotient metric space $\bar{G}(\Gamma) := \hat{G}(\Gamma)/\Gamma$, equipped with the induced $\mathbb{R}$-flow, has a orbit semi-equivalence $h_1 : \mathrm{Susp}(\Sigma,T) \to \bar{G}(\Gamma)$ which is uniformly finite-to-one, where Σ is a shift of finite type. This was explained by Gromov in [Gro87, Section 8.5.Q], and a careful proof can be found in the paper by Coornaert and

Papadopoulos [CP12]. Finally, as noted on [CP12, pg. 484, Facts 4 and 5], in the case where X is locally CAT(-1) and $\Gamma = \pi_1(X)$, one has a Γ-equivariant orbit equivalence $G\tilde{X} \to \hat{G}(\Gamma)$ (this is deduced from the universal properties of the flow space $\hat{G}(\Gamma)$). This descends to an orbit equivalence $h_2 : GX \to \bar{G}(\Gamma)$. Defining $h := h_2^{-1} \circ h_1 : \mathrm{Susp}(\Sigma, T) \to GX$ provides the claimed orbit equivalence.

To see that Σ can be taken to be transitive, we sketch a general argument in §7.3 that shows that since h is an orbit semi-equivalence onto a transitive flow, we still get an orbit equivalence if we restrict to a suitable transitive component of Σ. □

Remark. If the symbolic description for GX above could be improved by finding a roof function $r : \Sigma \to \mathbb{R}^+$ and a finite-to-one semi-conjugacy $\pi : \mathrm{Susp}_r(\Sigma, T) \to GX$, then the full power of symbolic dynamics could be applied to GX. In particular, the theory developed by Parry and Pollicott in [PP90] could be bought to bear, yielding refined results on the periodic orbit structure via the study of dynamical zeta functions. This stronger symbolic description is not currently available for geodesic flow on CAT(-1) spaces, and its existence is an interesting open question.

The following result may well be standard.

Lemma 2.9. *Let X be a compact, locally* CAT(-1) *space. Then there is some $\epsilon_0 > 0$ such that for all $x \in X$, $B(x, \epsilon_0)$ is (globally)* CAT(-1).

Proof. For each $x \in X$, let $\hat{\epsilon}(x)$ be $\sup\{\epsilon : B(x, \epsilon)$ is (globally) CAT$(-1)\}$. Suppose that $\hat{\epsilon}(x)$ is not bounded below, and take a sequence $x_n \to x^*$ with $\epsilon(x_n) \to 0$. $\hat{\epsilon}(x^*) > 0$ so for sufficiently large n, $x_n \in B(x^*, \hat{\epsilon}(x^*)/2$. But then for such x_n, $B(x_n, \hat{\epsilon}(x^*)/2) \subset B(x^*, \hat{\epsilon}(x^*))$ and so $B(x_n, \hat{\epsilon}(x^*)/2)$ is (globally) CAT(-1). This contradicts $\hat{\epsilon}(x_n) \to 0$. □

Corollary 2.10. *For all $x \in X$ and all $\epsilon < \epsilon_0$, $B(x, \epsilon)$ is simply connected.*

Proof. If not, there is a non-trivial geodesic loop contained in the globally CAT(-1) metric space $B(x, \epsilon)$. But, such a loop, divided in thirds, contradicts the CAT(-1) condition. □

The following lemma shows that geodesics which are close in GX are close when evaluated at time 0 on X.

Lemma 2.11. *For all $\epsilon > 0$, there exists a constant $K = K(\epsilon) > 0$ so that for $\gamma_1, \gamma_2 \in GX$,*

$$d_{GX}(\gamma_1, \gamma_2) < \epsilon \quad \textit{implies} \quad d_X(\gamma_1(0), \gamma_2(0)) < K\epsilon.$$

Furthermore, for $s, t \in \mathbb{R}$, $d_{GX}(g_s\gamma_1, g_t\gamma_2) < \epsilon$ implies $d_X(\gamma_1(s), \gamma_2(t)) < K\epsilon$.

Proof. Recall that ϵ_0 is such that $B(x, \epsilon_0)$ is (globally) CAT(-1) for all $x \in X$. Fix $\epsilon > 0$ and assume that $d_{GX}(\gamma_1, \gamma_2) < \epsilon$. We prove the Lemma in two cases:

Case 1: $d_X(\gamma_1(0), \gamma_2(0)) < \frac{\epsilon_0}{2}$. In this case, for $|s| < \frac{\epsilon_0}{2}$, $\gamma_i(s) \in B(\gamma_1(0), \epsilon_0)$. Therefore, for such s, $d_X(\gamma_1(s), \gamma_2(s))$ is a convex function of s. From this we have that for either $s \in [0, \frac{\epsilon_0}{2}]$ or $[-\frac{\epsilon_0}{2}, 0]$, $d_X(\gamma_1(s), \gamma_2(s)) \geq d(\gamma_1(0), \gamma_2(0))$.

Let $I_0 = \int_0^{\frac{\epsilon_0}{2}} e^{-2s} ds$. Suppose that $K \geq 1/I_0$; we claim $d_X(\gamma_1(0), \gamma_2(0)) < K\epsilon$. Indeed, if $d_X(\gamma_1(0), \gamma_2(0)) \geq K\epsilon$, then

$$d_{GX}(\gamma_1, \gamma_2) \geq \int_0^{\frac{\epsilon_0}{2}} K\epsilon e^{-2s} = KI_0\epsilon \geq \epsilon.$$

This contradicts our choice of K, so we conclude that $d_X(\gamma_1(0), \gamma_2(0)) < K\epsilon$.

Case 2: $d(\gamma_1(0), \gamma_2(0)) \geq \frac{\epsilon_0}{2}$. Let $M = d_X(\gamma_1(0), \gamma_2(0))$. Since the geodesic flow is unit-speed, for all s,

$$d_X(\gamma_1(s), \gamma_2(s)) \geq \max\{M - 2|s|, 0\}.$$

Thus,

$$\begin{aligned} d_{GX}(\gamma_1, \gamma_2) &\geq \int_{-\infty}^{\infty} \max\{M - 2|s|, 0\} e^{-2|s|} ds \\ &= 2\int_0^{\frac{M}{2}} (M - 2s) e^{-2s} ds \\ &= e^{-M} + M - 1. \end{aligned}$$

As a function of M, this expression is increasing, concave up, and runs through the origin. Therefore, taking $\frac{1}{K} = \frac{e^{-\epsilon_0/2} + \epsilon_0/2 - 1}{\epsilon_0/2}$, under this case

$$d_{GX}(\gamma_1, \gamma_2) \geq \frac{1}{K} d_X(\gamma_1(0), \gamma_2(0)).$$

Thus, if $d_X(\gamma_1(0), \gamma_2(0)) \geq K\epsilon$, we again contradict our assumption on $d_{GX}(\gamma_1, \gamma_2)$.

Combining these cases, and taking $K = \min\{1/I_0, \frac{\epsilon_0/2}{e^{-\epsilon_0/2} + \epsilon_0/2 - 1}\}$ finishes the proof for γ_1, γ_2 with $d_{GX}(\gamma_1, \gamma_2) < \epsilon$. Now assume that $d_{GX}(g_s\gamma_1, g_t\gamma_2) < \epsilon$. We have already shown that $d_X(g_s\gamma_1(0), g_t\gamma_2(0)) < K\epsilon$. Observing that $g_s\gamma_1(0) = \gamma_1(s)$ and $g_t\gamma_2(0) = \gamma_2(t)$ completes the proof. $\square$

Conversely, the following Lemma shows that geodesic segments which stay close in X are close in GX.

Lemma 2.12. *Let $\epsilon > 0$ be given. Then there exists $T = T(\epsilon) > 0$ such that if $d_X(\gamma_1(t), \gamma_2(t)) < \epsilon/2$ for all $t \in [a - T, b + T]$, then $d_{GX}(g_t\gamma_1, g_t\gamma_2) < \epsilon$ for all $t \in [a, b]$. For small ϵ, we can take $T(\epsilon) = -\log(\epsilon)$.*

Proof. Choose $T = T(\epsilon)$ so that $\int_T^{\infty} (\epsilon/2 + 2(\sigma - T)) e^{-2\sigma} d\sigma < \epsilon/4$. Analysis of this integral shows that for small ϵ, we could take $T(\epsilon) = \log(\epsilon^{-1})$. We have

$$\begin{aligned} d_{GX}(g_t\gamma_1, g_t\gamma_2) &= \int_{-\infty}^{\infty} d_X(\gamma_1(s+t), \gamma_2(s+t)) e^{-2|s|} ds \\ &= \int_{-\infty}^{a-T} d_X(\gamma_1(\tau), \gamma_2(\tau)) e^{-2|\tau - t|} d\tau \\ &\quad + \int_{b+T}^{\infty} d_X(\gamma_1(\tau), \gamma_2(\tau)) e^{-2|\tau - t|} d\tau \\ &\quad + \int_{a-T}^{b+T} d_X(\gamma_1(\tau), \gamma_2(\tau)) e^{-2|\tau - t|} d\tau, \end{aligned}$$

where we have made the change of variables $\tau = s + t$. In the third integral, we can bound $d_X(\gamma_1(\tau), \gamma_2(\tau))$, and thus the whole integral regardless of T, by $\epsilon/2$. Since $a \leq t \leq b$, over the domain of the first integral $|\tau - t| = -(\tau - t)$, and over the domain of the second interval $|\tau - t| = (\tau - t)$. In the first, we may bound $d_X(\gamma_1(\tau), \gamma_2(\tau)) < \epsilon/2 + 2(a - T - \tau)$ and in the second, $d_X(\gamma_1(\tau), \gamma_2(\tau)) < \epsilon/2 + 2(\tau - b - T)$ using triangle inequality. Then,

$$
\begin{aligned}
d_{GX}(g_t\gamma_1, g_t\gamma_2) < &\int_{-\infty}^{a-T} (\epsilon/2 + 2(a - T - \tau))e^{2(\tau-t)} d\tau \\
&+ \int_{b+T}^{\infty} (\epsilon/2 + 2(\tau - b - T))e^{-2(\tau-t)} d\tau \\
&+ \epsilon/2.
\end{aligned}
$$

The first integral is largest when $t = a$, the second when $t = b$. Making these substitutions and changing variables by $\sigma = \tau - a$, $\sigma = \tau - b$, respectively,

$$
\begin{aligned}
d_{GX}(g_t\gamma_1, g_t\gamma_2) < &\int_{-\infty}^{-T} (\epsilon/2 + 2(T - \sigma))e^{2\sigma} d\sigma \\
&+ \int_{T}^{\infty} (\epsilon/2 + 2(\sigma - T))e^{-2\sigma} d\sigma \\
&+ \epsilon/2.
\end{aligned}
$$

Our choice of T finishes the proof. □

3. Weak specification for the geodesic flow

In this section, we prove Theorem 1.1. We are given a compact, locally CAT(-1), geodesic space X, and we wish to establish the weak specification property for GX. By Lemma 2.8, there exists a topologically transitive subshift of finite type (Σ, T), and an orbit semi-equivalence $h : \mathrm{Susp}(\Sigma, T) \to GX$.

On $\mathrm{Susp}(\Sigma, T)$, Proposition 2.4 shows that transitivity immediately bootstraps to weak specification. We now show that this property can be transported to GX using the orbit semi-equivalence h. While the weak specification property is not preserved under a general orbit semi-equivalence, the geometry of our setting provides more structure to carry out our argument.

First, we show that geodesic segments that are close (after time change) on X are close after lifting to the universal cover.

Lemma 3.1. *Let $\epsilon < \epsilon_0$ and let $\gamma_1([0, T_1])$, $\gamma_2([0, T_2])$ be geodesic segments and $\rho : [0, T_2] \to [0, T_1]$ a time change such that $d_X(\gamma_1(\rho(t)), \gamma_2(t)) < \epsilon$ for all $t \in [0, T_2]$. Then for any lift $\tilde{\gamma}_1$ of γ_1, there exists a lift $\tilde{\gamma}_2$ of γ_2 with $\tilde{\gamma}_i(0)$ lying above $\gamma_i(0)$ such that $d_{\tilde{X}}(\tilde{\gamma}_1(\rho(t)), \tilde{\gamma}_2(t)) < \epsilon$ for all $t \in [0, T_2]$.*

Proof. Using compactness of $\gamma_2([0, T_2])$ there exists a finite sequence $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = T_2$ such that $B(\rho(t_i), \epsilon) \supset \gamma_2([t_i, t_{i+1}])$ for all $i = 0, \ldots, t_{n-1}$. Since, by Corollary 2.10, $B(\rho(t_i), \epsilon)$ is simply connected and (globally) CAT(-1), there is a (geodesic) homotopy $h_i(s, x)$ of paths from $\gamma_2([t_i, t_{i+1}])$ to $\gamma_1([\rho(t_i), \rho(t_{i+1})])$ with $h(s, \gamma_2(t)) \in B(\rho(t_i), \epsilon)$ for all $t \in [t_i, t_{i+1}]$ and $s \in [0, 1]$, such that $h_i(1, \gamma_2(t)) = \gamma_1(\rho(t))$ for all $t \in [t_i, t_{i+1}]$. In particular, we may assume this homotopy takes $\gamma_2(t_j)$ to $\gamma_1(\rho(t_j))$ along the (unique) shortest geodesic segment connecting them at constant speed $1/d(\gamma_2(t_j), \gamma_1(t_j))$ for $j = i, i + 1$.

By their definitions at the endpoints, the homotopies h_i and h_{i+1} agree in how they move $\gamma_2(t_{i+1})$ to $\gamma_1(\rho(t_{i+1}))$, so these local homotopies may be patched together into a global homotopy $h(s, x)$ such that $h(1, \gamma_2(t)) = \gamma_1(\rho(t))$ for all $t \in [0, T_2]$.

Fix a lift $\tilde{\gamma}_1$ of γ_1 parametrized so that $\tilde{\gamma}_1(0)$ projects to $\gamma_1(0)$ and lift the homotopy h to a homotopy $\tilde{h}$ with $\tilde{h}(1,\cdot) = \tilde{\gamma}_1([0,T_1])$. The lift $\tilde{\gamma}_2$ desired is given by the (properly parametrized) geodesic $\tilde{h}(0,\cdot)$. □

The following lemma allows us to show that geodesic segments which are close after a time change are in fact close without the time change. This is where the assumption that the geodesic flow is on a space of negative curvature is used. The proof requires only that geodesics in the universal cover are globally length minimising, so a non-positive curvature assumption would be sufficient.

Proposition 3.2. *Let X be a* CAT(-1) *space, and $\gamma_1, \gamma_2 \in GX$ be geodesics. Suppose there exists a time change $\rho : [0,T_2] \to [0,T_1]$ so that $d_X(\gamma_1(\rho(t)), \gamma_2(t)) < \epsilon$ for all $t \in [0,T_2]$. Then $d(\gamma_1(t), \gamma_2(t)) < 3\epsilon$ for all $t \in [0, T_1 - 2\epsilon]$.*

Proof. First, using Lemma 3.1, we lift γ_i to geodesic segments on the universal cover so that $d_{\tilde{X}}(\tilde{\gamma}_1(\rho(t)), \tilde{\gamma}_2(t)) < \epsilon$ for all $t \in [0,T_2]$. If we prove the statement in the universal cover, we have proven it in the original space. In the universal cover, the geodesics are globally length minimizing, and $d_{\tilde{X}}(\tilde{\gamma}_i(t_1), \tilde{\gamma}_i(t_2)) = |t_1 - t_2|$.

We fix $t \in [0,T_2]$, and we know that $\tilde{\gamma}_2(t)$ is within distance ϵ of $\tilde{\gamma}_1(\rho(t))$. Then one can reach $\tilde{\gamma}_2(t)$ from $\tilde{\gamma}_2(0)$ by the geodesic $\tilde{\gamma}_2$, or by following the path $\tilde{\gamma}_2(0) \to \tilde{\gamma}_1(0) \to \tilde{\gamma}_1(\rho(t)) \to \tilde{\gamma}_2(t)$ (see Figure 1). By the length-minimizing property of $\tilde{\gamma}_2$,

$$t = d_{\tilde{X}}(\tilde{\gamma}_2(0), \tilde{\gamma}_2(t)) < 2\epsilon + d_{\tilde{X}}(\tilde{\gamma}_1(0), \tilde{\gamma}_1(\rho(t))) = 2\epsilon + \rho(t).$$

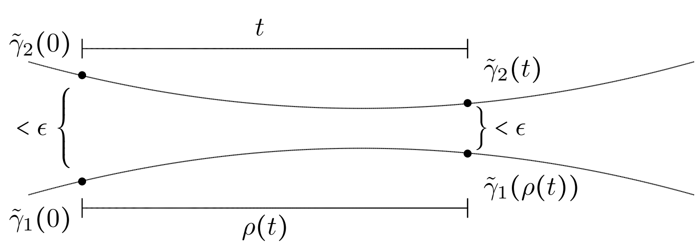

FIGURE 1. Nearby geodesics in the CAT(-1) space $\tilde{X}$ must shadow each other.

By interchanging the roles of the geodesics, $\rho(t) < 2\epsilon + t$, and so $|t - \rho(t)| < 2\epsilon$. Thus,

$$\begin{aligned} d_{\tilde{X}}(\tilde{\gamma}_1(t), \tilde{\gamma}_2(t)) &\le d_{\tilde{X}}(\tilde{\gamma}_1(t), \tilde{\gamma}_1(\rho(t))) + d_{\tilde{X}}(\tilde{\gamma}_1(\rho(t)), \tilde{\gamma}_2(t)) \\ &\le |t - \rho(t)| + \epsilon < 3\epsilon. \end{aligned}$$

Since $d_{\tilde{X}}(\tilde{\gamma}_1(T_1), \tilde{\gamma}_2(T_2)) < \epsilon$, a similar argument shows that $|T_1 - T_2| < 2\epsilon$. Thus, the above estimate holds for $t \in [0, T_1 - 2\epsilon]$. □

The proof of Theorem 1.1 is an immediate corollary, via Proposition 2.7 and Lemma 2.8, of the following result.

Theorem 3.3. *Suppose that (Y, ϕ_t) is a flow on a compact space satisfying the weak specification property. Suppose that $h : (Y, \phi_t) \to (GX, g_t)$ is a continuous, surjective orbit semi-equivalence to the geodesic flow on a compact, locally* CAT(-1) *space X. Then (GX, g_t) satisfies the weak specification property.*

Proof. Let $\epsilon > 0$ be given. We fix a collection of orbit segments $\{(\gamma_i, t_i)\}_{i=1}^k$ for (GX, g_t), and show how to glue them together. Let K and $T = T(\epsilon)$ be the constants from Lemma 2.11 and Lemma 2.12 respectively. As h is uniformly continuous, let $\delta > 0$ be so small that $d_Y(y_1, y_2) < \delta$ implies $d_{GX}(h(y_1), h(y_2)) < \epsilon/3K$. Thus, writing $\gamma_1 = h(y_1), \gamma_2 = h(y_2)$, it follows from Lemma 2.11 that $d_X(\gamma_1(0), \gamma_2(0)) < \epsilon/3$.

Fix lifts $\{(y_i, \hat{t}_i)\}_{i=1}^k$ under h of orbit segments $\{(g_{-T}\gamma_i, t_i + 2\epsilon + 2T)\}_{i=1}^k$. That is, each $(y_i, \hat{t}_i)$ is an orbit segment for (Y, ϕ_t) such that

$$\{h(\phi_s y_i) : s \in [0, \hat{t}_i]\} = \{g_s\gamma_i : s \in [-T, t_i + T + 2\epsilon]\}.$$

The first step is to apply the specification property to these lifted orbit segments. Let $\hat{\tau}$ be provided by the weak specification property for (Y, ϕ_t) at scale δ. There is a point $z \in Y$ and a sequence of transition times $\hat{\tau}_1, \dots \hat{\tau}_{k-1} \leq \hat{\tau}$ such that

$$d_{\hat{t}_j}(\phi_{\hat{s}_{j-1}+\hat{\tau}_{j-1}} z, y_j) < \delta \text{ for every } 1 \leq j \leq k,$$

where $\hat{s}_j = \sum_{i=1}^j \hat{t}_i + \sum_{i=1}^{j-1} \hat{\tau}_i$. Fix an index j, and write $z' = \phi_{\hat{s}_{j-1}+\hat{\tau}_{j-1}} z$. Consider the image under h of the orbit segment $(z', \hat{t}_j)$. Then for all $s \in [0, \hat{t}_j]$,

$$d_{GX}(h(\phi_s z'), h(\phi_s y_j)) < \epsilon/3K.$$

Thus, writing $h(z') = \gamma'$ and reparameterizing, we see there is a time change ρ so that for all $s \in [0, t_j + 2\epsilon + 2T]$,

$$d_{GX}(g_{\rho(s)}\gamma', g_s(g_{-T}\gamma_j)) < \epsilon/3K.$$

Using Lemma 2.11, we see that for all $s \in [0, t_j + 2\epsilon + 2T]$,

$$d_X(\gamma'(\rho(s)), g_{-T}\gamma_j(s)) < \epsilon/3.$$

Now we apply Proposition 3.2 to obtain that for all $s \in [0, t_j + 2T]$

$$d_X(\gamma'(s), g_{-T}\gamma_j(s)) < \epsilon.$$

Now we apply Lemma 2.12 to obtain that for all $s \in [T, t_j + T]$,

$$d_{GX}(g_s\gamma', g_s(g_{-T}\gamma_j)) < 2\epsilon,$$

and thus for all $s \in [0, t_j]$,

$$d_{GX}(g_s(g_T\gamma'), g_s(\gamma_j)) < 2\epsilon.$$

Now consider $\gamma = g_T(h(z))$. Noting that $g_T\gamma'$ is an appropriate iterate of γ under (GX, g_t), the argument above shows that for each j, an appropriate iterate of γ is 2ϵ-shadowing for (γ_j, t_j).

It only remains to show that the transition times for γ remain controlled. An argument based on continuity of the orbit equivalence and compactness of the phase space shows that there exists κ so that for all $y \in Y$, the image of an orbit segment $(y, \hat{\tau})$ under the orbit equivalence h is contained in the orbit segment $(h(y), \kappa)$. That is,

$$h(\{\phi_s(y) : s \in [0, \hat{\tau}]\}) \subseteq \{g_s(h(y)) : s \in [0, \kappa]\}.$$

The details of this argument are given in §7.2. The segments of γ that correspond to transitions between the shadowed orbit segments comprise of images of orbit segments of the form $(y, \hat{\tau}_i)$ with $\hat{\tau}_i \leq \hat{\tau}$, and an additional run of length at most $2T$ coming from the application of Lemma 2.12. Thus the transition times are bounded above by $\kappa + 2T$. It follows that (GX, g_t) satisfies weak specification. □

4. CAT(0) spaces and codings for the geodesic flow

In this section, we consider the case of non-positive curvature, and we prove Theorem 1.3 and Corollary 1.4.

4.1. Specification for a class of CAT(0) spaces.

Theorem 1.3 states that if X is a compact, locally CAT(0), geodesic metric space with non-elementary fundamental group and topologically transitive geodesic flow, and there exists an orbit semi-equivalence $h : \mathrm{Susp}(\Sigma, T) \to GX$, where (Σ, T) is a compact subshift of finite type, then the geodesic flow on GX satisfies the weak specification property. We observe that this follows from our proof of Theorem 1.1, where we used the assumption of CAT(−1) in only two places; the first was to provide the orbit-equivalent symbolic description of GX (Lemma 2.8), which we now *assume* to hold; the second was in the proof of Proposition 3.2 and we already observed that a CAT(0) assumption was sufficient for that argument. We conclude that our proof of Theorem 1.1 also gives the statement of Theorem 1.3.

A class of examples that is covered by Theorem 1.3 is given by CAT(0) spaces whose geodesics can be mapped homeomorphically to the geodesics for a CAT(−1) metric. For example, on a Riemannian surface with genus at least 2, non-positive curvature metrics can be found so that a single closed geodesic has curvature zero, and geodesics can be mapped homeomorphically to those for a hyperbolic metric.

We note that even for these CAT(0) geodesic flows where the specification property holds, extensions of the dynamical properties of Theorem 1.2 would still require a great deal of new theory: our proofs will require Bowen regularity of the potentials under consideration, and the argument for Hölder regularity to imply Bowen regularity (Propostion 5.4) requires negative curvature globally.

4.2. Non-existence of symbolic coding.

We establish Corollary 1.4: if X is a complete, locally CAT(0), geodesic metric space with topologically transitive geodesic flow containing a geodesic γ, and there exists $w > 0$ such that some w-neighborhood $N_w(\tilde{\gamma})$ of a lift of γ to $\tilde{X}$ splits isometrically as $\mathbb{R} \times Y$, then there does not exist any orbit semi-equivalence $h : \mathrm{Susp}(\Sigma, T) \to GX$, where (Σ, T) is a compact shift of finite type. The idea is to show that the weak specification property does not hold for these geodesic flows, and we can thus conclude that it has no orbit semi-equivalent symbolic coding.

We note that in many cases, there is a more elementary way to rule out the existence of a symbolic coding: if GX has uncountably many closed geodesics, since $Susp(\Sigma, T)$ has only countably many periodic orbits, a symbolic coding is impossible. So, for example, if X is the surface given by gluing together two tori using a flat cylinder, this simpler argument suffices. However, we note that having uncountably many closed geodesics is not a consequence of our hypotheses, even for Riemannian 3-manifolds. For example, the flat strip could have holonomy an irrational rotation around a single central closed geodesic. A construction like this requires dimension at least three. In dimension two, the main theorem of [CX08] implies that whenever a zero curvature neighborhood U of a geodesic in the universal cover exists, there is always a closed flat cylinder in the surface, allowing the simpler argument.

Proof of Corollary 1.4. Suppose that (GX, g_t) satisfies the weak specification property. Let K be as provided by Lemma 2.11, let $\delta = \frac{w}{10K}$, and let $\tau(\delta)$ be the corresponding maximum gap size.

Let $\gamma_1 = \gamma$ and γ_2 be a geodesic with $\gamma_2(0) \notin N_w(\gamma)$. Let $t_1 = \tau$ and $t_2 = 1$. For the weak specification property to hold in GX, there must be some geodesic γ^* which δ-shadows γ for time t_1, then after transition time at most τ, δ-shadows γ_2.

By Lemma 2.11, $d(\gamma(t), \gamma^*(t)) < K\delta = w/10$ for all $t \in [0, t_1]$. By the geometry of the flat neighborhood $N_w(\gamma)$ (or, lifting to the universal cover, the flat strip $N_w(\tilde{\gamma})$), $\gamma^*(t)$ travels at most distance $w/5$ perpendicular to the image of γ over $t \in [0, t_1]$, remaining all the while in the $w/10$-neighborhood of γ. Therefore, over the subsequent $\tau = t_1$ units of time, it can again travel at most distance $w/5$ perpendicularly away from the image of γ. Therefore at any time $t \in [\tau, 2\tau]$, $\gamma^*(t)$ is at least distance $w/5$ from $\gamma_2(0)$. To fulfill the desired shadowing, for some such t, $g_t\gamma^*$ should be within δ of γ_2. At such a time, $d_{GX}(g_t\gamma^*, \gamma_2) < \delta = \frac{w}{10K}$. Using Lemma 2.11, we must at this point have $d(\gamma^*(t), \gamma_2(0)) < K\delta = \frac{w}{10}$. Since this is not the case, we have a contradiction and γ^* cannot achieve the shadowing required.

We have shown that (GX, g_t) cannot have the weak specification property. Now suppose that there were an orbit semi-equivalence $h : \mathrm{Susp}(\Sigma, T) \to GX$, where (Σ, T) is a compact shift of finite type. Then by §7.3, (Σ, T) could be taken to be topologically transitive and the arguments of §3 would show that (GX, g_t) has weak specification. Therefore, no such $h : \mathrm{Susp}(\Sigma, T) \to GX$ exists. □

One of the hypotheses of Corollary 1.4 is that the geodesic flow on M is topologically transitive. By [Bal95, Theorem III.2.4], if the geodesic flow is not topologically transitive, every geodesic of $\tilde{X}$ is contained in a flat plane. In the Riemannian case, by [Ebe96, Prop 4.7.3 and 4.7.4], if M is rank one, the geodesic flow is transitive. If M is not rank one, then as noted above, every geodesic belongs to a flat plane. By the rank rigidity theorem [Bal85, BS87], M is a locally symmetric space of non-compact type, since we have assumed it is irreducible. But $\tilde{\gamma}$ has a flat neighborhood, implying M is flat, contradicting the irreducibility assumption. We remark that if M is flat, it will have uncountably many closed geodesics, which immediately rules out the existence of a symbolic coding.

Remark. Corollary 1.4 rigorously confirms the expected phenomenon that a compact shift of finite type can not capture the dynamics of this setting. The idea of using the failure of the specification property to rule out the existence of a coding by a shift of finite type was first used by Lind [Lin79], who used this argument to show that quasi-hyperbolic toral automorphisms (i.e. ergodic automorphisms of the torus with some eigenvalues of modulus 1) do not admit Markov partitions.

Beyond uniform hyperbolicity, the best hope to capture the dynamics symbolically is often to code using a shift of finite type on a countable alphabet. The existence of countable state symbolic dynamics for smooth flows on three dimensional Riemannian manifolds was recently established by Lima and Sarig [LS14]. This kind of phenomenon is not ruled out by Corollary 1.4.

Remark. The work of Coornaert and Papadopoulos on existence of an orbit semi-equivalent symbolic coding holds if $\pi_1(X)$ is a hyperbolic group. However, it provides a symbolic description of the geodesic flow on the group $\bar{G}(\Gamma)$ rather than GX (see proof of Lemma 2.8). This does not extend to GX under only a CAT(0) assumption because a flat strip of parallel geodesics in X will correspond to a single geodesic in $\bar{G}(\Gamma)$.

5. Expansivity and the Bowen property

Before turning to applications of the weak specification property, we require two further properties of the geodesic flow on a compact CAT(-1) space. The first property we want to check is expansivity (see [BW72]).

Definition 5.1. A continuous flow $\{f_t\}$ on X is *expansive* if there is for all $\epsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in X$ and all continuous $\tau : \mathbb{R} \to \mathbb{R}$ with $\tau(0) = 0$, if $d(f_t(x), f_t(y)) < \epsilon$ for all $t \in \mathbb{R}$, then $y = f_s(x)$ for some s, where $|s| < \epsilon$.

That this property is satisfied by geodesic flows on a CAT(-1) space is not hard to see:

Proposition 5.2. *(GX, g_t) is expansive.*

Proof. First, for a fixed geodesic $\tilde{\gamma}$ in $\tilde{X}$, let

$$Opp(\tilde{\gamma}) = \{\tilde{\gamma}' : \tilde{\gamma}'(t) = \tilde{\gamma}(-t+s), \text{ for some } s \text{ and all } t\}.$$

That is, $Opp(\tilde{\gamma})$ is the set of all linear reparametrizations of $\tilde{\gamma}$ with the opposite orientation. Let $\delta = \min_{\tilde{\gamma}' \in Opp(\tilde{\gamma})} d_{G\tilde{X}}(\tilde{\gamma}, \tilde{\gamma}')$. Using the definition of $d_{G\tilde{X}}$ it is easy to check that δ does not depend on $\tilde{\gamma}$ and is positive.

Take ϵ smaller than δ and smaller than ϵ_0/K (K from Lemma 2.11 and ϵ_0 from Corollary 2.10). Consider any $\tau : \mathbb{R} \to \mathbb{R}$ with $\tau(0) = 0$. Suppose that γ_1 and γ_2 are geodesics in GX with $d_{GX}(g_t\gamma_1, g_{\tau(t)}\gamma_2) < \epsilon$ for all t. Then by Lemma 2.11, $d(\gamma_1(t), \gamma_2(\tau(t))) < K\epsilon$ which is less than ϵ_0.

Using Lemma 3.1, we can lift the geodesics γ_1 and γ_2 to the universal cover in such a way that $d(\tilde{\gamma}_1(t), \tilde{\gamma}_2(\tau(t))) < K\epsilon < \epsilon_0$ for all t. Using the length-minimizing properties of geodesics in $\tilde{X}$ and the triangle inequality:

$$d(\tilde{\gamma}_1(0), \tilde{\gamma}_1(t)) - 2\epsilon_0 < d(\tilde{\gamma}_2(0), \tilde{\gamma}_2(\tau(t))) < d(\tilde{\gamma}_1(0), \tilde{\gamma}_1(t)) + 2\epsilon_0.$$

Examining this inequality for $t > 0$ (resp. $t < 0$), we conclude that $|\tau(t)| \to \infty$ as $t \to \infty$ (resp. as $t \to -\infty$).

Together with the fact that $d(\tilde{\gamma}_1(t), \tilde{\gamma}_2(\tau(t))) < \epsilon_0$ for all t, this implies that $\tilde{\gamma}_2(+\infty), \tilde{\gamma}_2(-\infty) \in \{\tilde{\gamma}_1(\infty), \tilde{\gamma}_1(-\infty)\}$. Since the endpoints at infinity for $\tilde{\gamma}_2$ must be distinct, $\tilde{\gamma}_2$ has the same pair of endpoints at infinity as $\tilde{\gamma}_1$. Since both are unit-speed geodesics, we have $\tilde{\gamma}_2(t) = \tilde{\gamma}_1(\pm t + s)$ for some s. Now the assumption that $\tau(0) = 0$ implies that $d_{G\tilde{X}}(\tilde{\gamma}_1, \tilde{\gamma}_2) < \delta$ and, by the choice of δ, that $\tilde{\gamma}_2$ does not belong to $Opp(\tilde{\gamma}_1)$. Hence $\tilde{\gamma}_2(t) = \tilde{\gamma}_1(t+s)$ for some s. A straightforward calculation with the definition of d_{GX} implies that $|s| < \epsilon$. □

The second property we want is a dynamical regularity property for functions on the space GX.

Definition 5.3 (see [Fra77])**.** Let $\{f_t\}$ be a continuous flow on a compact metric space (X, d). A continuous function φ on X is said to have the *Bowen property* (for ϕ_t) if there exists $V > 0$ so that for any sufficiently small $\epsilon > 0$,

$$d(f_t(x), f_t(y)) < \epsilon \text{ for all } t \in [0, S] \text{ implies } \left|\int_0^S \varphi(f_t x)dt - \int_0^S \varphi(f_t y)dt\right| < V$$

for any $x, y \in X$ and any $S > 0$.

We claim that Hölder functions on GX satisfy this property.

Proposition 5.4. *If φ is a Hölder continuous function on GX, then φ satisfies the Bowen property for the geodesic flow g_t.*

Proof. We actually prove the Walters property for φ: for any $V > 0$, there exists an $\epsilon > 0$ such that

$$d_{GX}(g_t(\gamma_1), g_t(\gamma_2)) < \epsilon \text{ for all } t \in [0,S] \text{ implies } |\int_0^S \varphi(g_t\gamma_1)dt - \int_0^S \varphi(g_t\gamma_2)dt| < V$$

for any $\gamma_1, \gamma_2 \in GX$ and any $S > 0$. Clearly, if φ has the Walters property, then φ has the Bowen property. The basic idea of the proof is that, using the CAT(-1) property for a comparison with $\mathbb{H}^2$, geodesics in X which stay close over $[0,S]$ are in fact exponentially close over that range, from which the result follows. The need to move between the metrics on GX and X adds some technicalities to the proof.

Let $V > 0$ be given, and let $C, \alpha > 0$ be the Hölder constants for φ so that $|\varphi(\gamma_1, \gamma_2)| < C d_{GX}(\gamma_1, \gamma_2)^\alpha$. We fix $\epsilon > 0$ to be specified later. Suppose that $d_{GX}(g_t\gamma_1, g_t\gamma_2) < \epsilon$ for $t \in [0,S]$. By Lemma 2.11, $d_X(\gamma_1(t), \gamma_2(t)) < K\epsilon$ for $t \in [0,S]$. By Lemma 3.1, assuming that $K\epsilon < \epsilon_0$, lifting to the universal cover, we have $d_{\tilde{X}}(\tilde{\gamma}_1(t), \tilde{\gamma}_2(t)) < \epsilon$ for $t \in [0,S]$.

We construct a comparison pair of geodesic segments $c_1(t), c_2(t)$ in $\mathbb{H}^2$ with lengths S and with distance at most $K\epsilon$ between their endpoints using the pair of triangles shown in Figure 2. By convexity of the distance function, $d_{\mathbb{H}^2}(c_1(t), c_2(t)) < K\epsilon$. We translate the time parameter for c_2 by a constant r so that at the point of their nearest approach in $\mathbb{H}^2$, both have the same time parameter. By interchanging the roles of c_1 and c_2 if necessary, we can assume that $r \geq 0$. Since the flow is unit speed, $r \leq K\epsilon$, and we write $S' := S - r$. Then, by a standard argument for the behavior of geodesics in $\mathbb{H}^2$, we have that

$$d_{\mathbb{H}^2}(c_1(t), c_2(t+r)) < K\epsilon e^{-\min\{t, S'-t\}} \text{ for all } t \in [0, S'].$$

Applying the CAT(-1) property, we have that

$$d_{\tilde{X}}(\tilde{\gamma}_1(t), \tilde{\gamma}_2(t+r)) < K\epsilon e^{-\min\{t, S'-t\}} \text{ for all } t \in [0, S'],$$

and we can push this estimate back down to X.

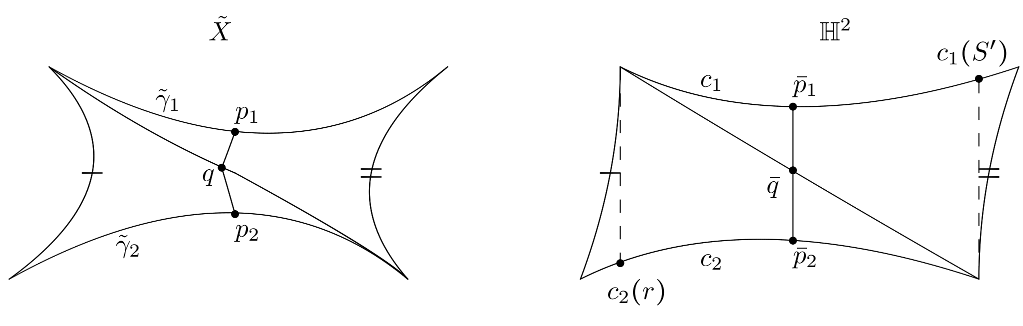

FIGURE 2. Comparison quadrilateral for Proposition 5.4. By the CAT(-1) condition, $d_X(p_1, p_2) \leq d_{\mathbb{H}^2}(\bar{p}_1, \bar{p}_2)$.

Next, using Lemma 2.12 we see that that there is a constant $T = T(2K\epsilon)$ such that

$$d_{GX}(g_t\gamma_1, g_{t+r}\gamma_2) < 2d_X(\gamma_1(t), \gamma_2(t+r)) < 2K\epsilon e^{-\min\{t,S'-t\}} \text{ for all } t \in [T, S'-T].$$

We recall from Lemma 2.12 that for small ϵ, we can take $T(2K\epsilon) = -\log(2K\epsilon)$, and thus $\lim_{\epsilon\to 0}\epsilon^\alpha T(2K\epsilon) = 0$. We assume ϵ is so small that $2C(2K\epsilon)^\alpha T < V/3$.

To control $|\int_0^S \varphi(g_t\gamma_1)dt - \int_0^S \varphi(g_t\gamma_2)dt|$, we first note that

$$|\int_0^S \varphi(g_t\gamma_1)dt - \int_0^S \varphi(g_t\gamma_2)dt| \le |\int_0^{S'} \varphi(g_t\gamma_1)dt - \int_r^S \varphi(g_t\gamma_2)dt| + 2r\|\varphi\|.$$

Therefore, picking ϵ so small that $2K\epsilon\|\varphi\| < V/3$, and writing $\gamma_2' = g_r\gamma_2$, it suffices to control $|\int_0^{S'} \varphi(g_t\gamma_1)dt - \int_0^{S'} \varphi(g_t\gamma_2')dt|$.

We cover $[0,S']$ by the intervals $I_1 = [0,T], I_2 = (T, S'-T)$, and $I_3 = [S'-T, S']$. Note that I_2 may be empty and I_1 and I_3 may overlap, depending on the values of S' and ϵ. Then,

$$\begin{aligned}|\int_0^{S'} \varphi(g_t\gamma_1)dt - \int_0^{S'} \varphi(g_t\gamma_2')dt| &\le \int_0^{S'} |\varphi(g_t\gamma_1) - \varphi(g_t\gamma_2')|dt \\ &\le \int_{I_1} |\varphi(g_t\gamma_1) - \varphi(g_t\gamma_2')|dt + \int_{I_3} |\varphi(g_t\gamma_1) - \varphi(g_t\gamma_2')|dt \\ &+ \int_{I_2} |\varphi(g_t\gamma_1) - \varphi(g_t\gamma_2')|dt.\end{aligned}$$

Over I_1 and I_3, $d_{GX}(g_t\gamma_1, g_t\gamma_2') < d_{GX}(g_t\gamma_1, g_t\gamma_2) + d_{GX}(g_t\gamma_2, g_t\gamma_2') < \epsilon + K\epsilon$, so by the Hölder condition, $|\varphi(g_t\gamma_1) - \varphi(g_t\gamma_2')| \le C(2K\epsilon)^\alpha$. Thus

$$\int_{I_1} |\varphi(g_t\gamma_1) - \varphi(g_t\gamma_2')|dt + \int_{I_3} |\varphi(g_t\gamma_1) - \varphi(g_t\gamma_2')|dt < 2C(2K\epsilon)^\alpha T < V/3.$$

To bound the integral over I_2, we use the Hölder property again to obtain

$$\begin{aligned}\int_{I_2} |\varphi(g_t\gamma_1) - \varphi(g_t\gamma_2')|dt &< \int_{I_2} C d_{GX}(g_t\tilde\gamma_1, g_t\tilde\gamma_2')^\alpha dt \\ &< \int_{I_2} C2^\alpha K^\alpha \epsilon^\alpha e^{-\alpha\min\{t,S-t\}}dt \\ &< \epsilon^\alpha \int_0^\infty C2^\alpha K^\alpha e^{-\alpha\min\{t,S-t\}}dt < V/3,\end{aligned}$$

where the last inequality comes from making a sufficiently small choice of ϵ. Thus, $|\int_0^{S'} \varphi(g_t\gamma_1)dt - \int_0^{S'} \varphi(g_t\gamma_2')dt| < 2V/3$, and so $|\int_0^S \varphi(g_t\gamma_1)dt - \int_0^S \varphi(g_t\gamma_2)dt| < V$. □

6. Thermodynamic Formalism and Large Deviations

We now prove the applications of weak specification for CAT(−1) geodesic flows summarized in Theorem 1.2. First, we now have all the ingredients required to apply results from the literature to show that any Hölder continuous potential on GX has a unique equilibrium measure satisfying the Gibbs property. We show that these results can be used to prove the equidistribution of weighted periodic orbits for a Hölder continuous potential. We then prove the upper and lower bounds of the large deviations principle (Proposition 6.6). For the upper bound, the problem can be reduced unproblematically to the discrete time case where the required result follows from work of Pfister and Sullivan. The lower bound follows a standard method of proof but requires care in the continuous-time setting. We end the paper by presenting it in detail. As part of this argument we establish entropy density of ergodic measures for the geodesic flow on a CAT(-1) space (Proposition 6.7), which is of interest in its own right.

6.1. **Unique equilibrium states and Bowen-Margulis measure as unique measure of maximal entropy.** We refer to Walters [Wal82] as a standard reference for equilibrium states in discrete time, and the article by Bowen and Ruelle [BR75] for flows. Given a potential function φ, we study the question of whether there is a unique invariant measure which maximises the quantity $h_\mu + \int \varphi\, d\mu$, where h_μ is the measure-theoretic entropy. More precisely, given a flow $\mathcal{F}$ on a compact metric space X, and a continuous function $\varphi : X \to \mathbb{R}$ (called the *potential*), we define the *topological pressure* to be

$$P(\varphi) = \sup\{h_\mu + \int \varphi\, d\mu \mid \mu \text{ is an } \mathcal{F}\text{-invariant probability measure}\},$$

and an *equilibrium state* for φ to be a measure achieving this supremum. See §6.2 for an equivalent formulation of $P(\varphi)$ as an exponential growth rate of the number of distinct orbits for the system, weighted by φ.

An equilibrium measure for the constant function $\varphi = 0$ is called a *measure of maximal entropy*. In the setting of hyperbolic dynamics, there is a unique measure of maximal entropy which is often called the *Bowen-Margulis* measure, reflecting two of the classic constructions of this measure: Bowen's construction based on periodic orbits, and Margulis' construction as a local weighted product on stable and unstable leaves). In the case of geodesic flows on compact negative curvature Riemannian manifolds, the Bowen-Margulis measure is also given by the Patterson-Sullivan construction of a measure on the sphere at infinity. The equivalence of the Bowen-Margulis and Patterson-Sullivan constructions was obtained by Kaimanovich in this setting [Kai90, Kai91]. For geodesic flow in CAT(-1), the measure μ_{BM} known as the Bowen-Margulis measure is the one obtained through the Patterson-Sullivan construction, see e.g. Roblin [Rob03]. The terminology is justified by the result that the periodic orbits equidistribute to μ_{BM}, and it follows that μ_{BM} is a measure of maximal entropy. We further develop the analogy with hyperbolic dynamics by showing that μ_{BM} is indeed the *unique* measure of maximal entropy.

For an expansive flow, there exists an equilibrium state for every continuous potential. However, uniqueness can be a subtle question, and one which leads to further results about the measure. We will apply a suitable continuous time version of Bowen's classic theorem on uniqueness of equilibrium states in our setting:

Theorem 6.1. *Let $\mathcal{F}$ be a continuous flow on a compact metric space. Suppose that $\mathcal{F}$ is expansive and has the weak specification property. Then, for every potential φ with the Bowen property, there exists a unique equilibrium state μ_φ. Every such measure μ_φ satisfies the Gibbs property for φ.*

For flows with the strong version of specification, this result was proved by Franco [Fra77], generalizing Bowen's original discrete time argument [Bow75] to the flow case. Franco's argument could be modified to apply to weak specification, although there are some non-trivial extra complications involved since weak specification does not allow us to use periodic orbits in the construction of the unique equilibrium state. Formally, the statement for weak specification is a corollary of recent work by Climenhaga and the third named author [CT15], although that work is designed to apply much more generally in settings which do not have any global form of the specification property.

Here, the *Gibbs property for* φ is the property that for all $\rho > 0$, there is a constant $Q = Q(\rho) > 1$ such that for every $x \in X$ and $t \in \mathbb{R}$, we have

$$Q^{-1} e^{-tP(\varphi)+\Phi(x,t)} \le \mu(B_t(x,\rho)) \le Q e^{-tP(\varphi)+\Phi(x,t)}, \tag{6.1}$$

where $\Phi(x,t) = \int_0^t \varphi(f_s x)\, ds$ and $B_t(x,\rho) = \{y : d(f_s x, f_s y) < \rho \text{ for all } s \in [0,t]\}$. In particular, a measure of maximal entropy has the Gibbs property if for all $\rho > 0$, there is a constant $Q = Q(\rho) > 1$ such that for every $x \in X$ and $t \in \mathbb{R}$, we have

$$Q^{-1} e^{-th} \le \mu(B_t(x,\rho)) \le Q e^{-th}, \tag{6.2}$$

It follows immediately from Theorem 6.1, Theorem 1.1, Proposition 5.2 and Proposition 5.4 that

Proposition 6.2. *Every Hölder continuous function* φ *on* GX *has a unique equilibrium state. In particular, the Bowen-Margulis measure is the unique measure of maximal entropy. Furthermore, these measures satisfy the Gibbs property.*

6.2. **Equidistribution of weighted periodic orbits.** Let $\mathrm{Per}(t)$ denote the set of closed orbits for $\{g_s\}$ of least period at most t, and let φ be a continuous function. We define the *Gurevic pressure* to be

$$P_G(\varphi) = \limsup_{t\to\infty} \frac{1}{t} \log \sum_{\gamma\in\mathrm{Per}(t)} e^{\Phi(\gamma)}, \tag{6.3}$$

where $\Phi(\gamma)$ is the value given by integrating φ around the periodic orbit. It is easy to verify that in (6.3) we can instead sum over the set of periodic orbits of length between t and $t+\delta$, for any fixed $\delta > 0$. The pigeonhole principle yields the same upper exponential growth rate as in (6.3).

For $\gamma \in \mathrm{Per}(t)$, let $\mu(\gamma)$ be the natural measure around the orbit. That is, if γ has period t, then

$$\mu_\gamma := \frac{1}{t}\int_0^t \delta_{g_s\gamma} ds.$$

We say the *periodic orbits weighted by* φ *equidistribute* to a measure μ if

$$\frac{1}{C(t)} \sum_{\gamma\in\mathrm{Per}(t)} e^{\Phi(\gamma)} \mu_\gamma \to \mu, \tag{6.4}$$

where $C(t)$ is the normalizing constant $\sum_{\gamma\in\mathrm{Per}(t)} e^{\Phi(\gamma)}\mu_\gamma(GX)$. Equidistribution of weighted periodic orbits for equilibrium states was first investigated in a uniformly hyperbolic setting by Parry [Par88]. For CAT(-1) spaces, it is known that in the case $\varphi = 0$, periodic orbits equidistribute to the Bowen-Margulis measure [Rob03, Theorem 5.1.1], but the weighted case has not been considered and seems to require different techniques from those used in [PPS15, Rob03].

We recall that the topological pressure for an expansive flow is defined to be

$$P(\varphi) = \lim_{t\to\infty} \frac{1}{t} \log\sup\left\{ \sum_{x\in E} e^{\int_0^t \varphi(g_t x)} \mid E \text{ is a } (t,\epsilon)\text{-separated set}\right\},$$

where ϵ is an expansivity constant for the flow, and a set E is (t,ϵ)-separated if for every distinct $x, y \in E$ we have $y \notin \overline{B}_t(x,\epsilon)$.

The proof of the Variational Principle [Wal82, Theorem 9.10] shows that if $P_G(\varphi) = P(\varphi)$, then since $\mathrm{Per}(t)$ is a sequence of (t,ϵ)-separated sets (for any expansivity constant ϵ) whose growth rate well approximates the topological pressure, then any weak* limit of $\frac{1}{C(T)}\sum_{\gamma\in\mathrm{Per}(T)} e^{\Phi(\gamma)}\mu_\gamma$ is an equilibrium state for φ. See Remark 3 of [GS14]. Thus if we know that $P_G(\varphi) = P(\varphi)$, and that φ has a unique equilibrium state μ, it follows immediately that the periodic orbits weighted by φ equidistribute to μ.

To prove that $P_G(\varphi) = P(\varphi)$ for a Hölder continuous φ, we first require a closing lemma for our setting. The idea is that for the suspension flow over a shift of finite type, an orbit segment can always be approximated by a periodic orbit. Using ideas similar to those used earlier in the paper, we show that this property passes to GX using the orbit semi-equivalence.

Lemma 6.3. *For all $\epsilon > 0$, there exists $R > 0$ so that for any orbit segment (γ, t) for (GX, g_t), there exists $\gamma^* \in \mathrm{Per}(t+R)$ so that $d_t(\gamma,\gamma^*) < \epsilon$.*

Proof. The proof uses many of the same ideas as the proof of Theorem 3.3. Let $\epsilon > 0$ be given and fix an orbit segment (γ, t) for (GX, g_t). Let $h : Y \to GX$ be the orbit semi-equivalence provided by Lemma 2.8, where $(Y, \phi_t) = \mathrm{Susp}(\Sigma, T)$ and Σ is a topologically transitive shift of finite type. Let K and $T = T(\epsilon)$ be the constants from Lemma 2.11 and Lemma 2.12 respectively, and let $\delta > 0$ be so small that $d_Y(y_1, y_2) < \delta$ implies $d_{GX}(h(y_1), h(y_2)) < \epsilon/3K$.

Fix a lift $(y, \hat{t})$ under h of $(g_{-T}\gamma, t + 2\epsilon + 2T)$, so

$$\{h(\phi_s y) : s \in [0, \hat{t}]\} = \{g_s\gamma : s \in [-T, t + T + 2\epsilon]\}.$$

On the suspension flow, it is easy to check that we can close orbit segments to periodic orbits. That is, for all $\delta > 0$, there exists $\hat{R}$ so that for all $(y, \hat{t})$, there exists y' so that $d_t(y, y') < \delta$ and y' is periodic with period at most $\hat{t} + \hat{R}$. This property follows from the corresponding fact for Σ. We take such a point y' for the orbit segment (y, t) and $\delta > 0$ under consideration. Then for all $s \in [0, \hat{t}]$,

$$d_{GX}(h(\phi_s y'), h(\phi_s y)) < \epsilon/3K.$$

Thus, writing $\gamma' := h(y')$ and reparameterizing, we see there is a time change ρ so that for all $s \in [0, t + 2\epsilon + 2T]$,

$$d_{GX}(g_{\rho(s)}\gamma', g_s(g_{-T}\gamma)) < \epsilon/3K.$$

Using Lemma 2.11, we see that for all $s \in [0, t + 2\epsilon + 2T]$,

$$d_X(\gamma'(\rho(s)), g_{-T}\gamma(s)) < \epsilon/3.$$

Now we apply Proposition 3.2 to obtain that for all $s \in [0, t + 2T]$

$$d_X(\gamma'(s), g_{-T}\gamma(s)) < \epsilon.$$

Now we apply Lemma 2.12 to obtain that for all $s \in [T, t + T]$,

$$d_{GX}(g_s\gamma', g_s(g_{-T}\gamma)) < 2\epsilon,$$

and thus for all $s \in [0, t]$, $d_{GX}(g_s(g_T\gamma'), g_s(\gamma)) < 2\epsilon$. We let $\gamma^* = g_T\gamma'$, and we have shown that $d_t(\gamma^*, \gamma) < 2\epsilon$.

Now it is clear that γ^* is a periodic orbit, so it only remains to show that its period is controlled. Let t^* be the period of γ^*. We observe that the orbit segment $(g_t\gamma^*, t^* - t)$ is a subset of the image under h of the orbit segment $(\phi_{\hat{t}}y', R')$. So we let R be a value so that for all $y \in Y$, the image of an orbit segment (y, R') under the orbit equivalence h is contained in the orbit segment $(h(y), R)$. This is possible by the compactness argument given in §7.2. Thus, the period of γ^* is at most $t+R$, so at scale 2ϵ, we have verified the property that we need. □

Lemma 6.4. *For any Hölder continuous function* $\varphi : GX \to \mathbb{R}$*, we have* $P_G(\varphi) = P(\varphi)$.

Proof. Let 2ϵ be an expansivity constant. Since $\mathrm{Per}(t)$ is $(t,2\epsilon)$-separated, it is clear that $P_G(\varphi) \leq P(\varphi)$. For the other inequality, take a sequence of $(t,2\epsilon)$-separated sets E_t so that

$$\frac{1}{t}\log \sum_{x\in E_t} e^{\int_0^t \varphi(g_s x)} \to P(\varphi).$$

Then by Lemma 6.3, for each $x \in E_t$, there exists a periodic orbit $\gamma(x)$ with $d_t(x,\gamma(x)) < \epsilon$ and $\{\gamma(x) \mid x \in E_t\} \subset \mathrm{Per}(T+R)$. Since E_t is $(t,2\epsilon)$-separated, if $x \neq y$ then $\gamma(x) \neq \gamma(y)$. Note that

$$\left|\Phi(\gamma(x)) - \int_0^t \varphi(g_s x)\right| \leq \left|\int_0^t \varphi(g_s\gamma(x)) - \int_0^t \varphi(g_s x)\right| + R\|\varphi\| \leq V + R\|\varphi\|,$$

where V is the constant appearing in the Bowen constant for φ. Thus,

$$\sum_{\gamma\in \mathrm{Per}(t+R)} e^{\Phi(\gamma)} \geq \sum_{\{\gamma(x)\mid x\in E_t\}} e^{\Phi(\gamma)} \geq e^{-V-R\|\varphi\|}\sum_{x\in E_t} e^{\int_0^t \varphi(g_s x)},$$

and so

$$\frac{1}{t+R}\log \sum_{\gamma\in\mathrm{Per}(t+R)} e^{\Phi(\gamma)} \geq \frac{t}{t+R}\left(\frac{1}{t}\log\sum_{x\in E_t} e^{\int_0^t \varphi(g_s x)}\right) - \frac{V+R\|\varphi\|}{t+R}.$$

Taking a limit as $t\to\infty$, we obtain $P_G(\varphi) \geq P(\varphi)$, which completes the proof. □

In summary, for any Hölder continuous $\varphi : GX \to \mathbb{R}$, since $P_G(\varphi) = P(\varphi)$ and φ has a unique equilibrium state μ_φ, it follows that the periodic orbits weighted by φ are equidistributed in the sense that

$$\frac{1}{C(t)}\sum_{\gamma\in\mathrm{Per}(t)} e^{\Phi(\gamma)}\mu_\gamma \to \mu_\varphi.$$

This result holds true by the same proof if $\varphi : GX \to \mathbb{R}$ has the Bowen property.

6.3. **Large Deviations Principle for the Bowen-Margulis measure and other equilibrium states.** We obtain the large deviations principle for all the measures considered in this section, in particular the Bowen-Margulis measure. The large deviations principle is a statement which describes the decay rate of the measure of points whose Birkhoff sums are experiencing a large deviation from their expected value (given by the Birkhoff ergodic theorem).

Definition 6.5. Let m be an equilibrium measure for a potential φ (with respect to $\mathcal{F}$). We say that m satisfies the *upper large deviations principle* if for any continuous observable ψ: $X \to \mathbb{R}$ and any $\epsilon > 0$,we have

$$\limsup_{t\to\infty} \frac{1}{t}\log m\left\{x : \left|\frac{1}{t}\int_0^t \psi(f_s x)\,ds - \int \psi\,dm\right| \geq \epsilon\right\} \leq -q(\epsilon), \tag{6.5}$$

where the rate function q is given by

$$q(\epsilon) := P(\varphi) - \sup_{\substack{\nu\in\mathcal{M}_{\mathcal{F}}(X)\\ |\int\psi\,dm - \int\psi\,d\nu|\geq\epsilon}} \left(h_\nu(f) + \int \varphi\,d\nu\right), \tag{6.6}$$

or $q(\epsilon) = \infty$ when $\{\nu \in \mathcal{M}_{\mathcal{F}}(X) : |\int \psi\, dm - \int \psi\, d\nu| \geq \epsilon\} = \varnothing$. We say that the *lower large deviations principle* holds if the above statement holds with $\geq$ in place of $\leq$, and lim inf in place of lim sup in (6.5). We say that m satisfies the *large deviations principle* if both upper and lower large deviations hold: that is, the above statement holds with equality in place of $\leq$ in (6.5), and the lim sup becomes a limit. For a continuous map f, we say the *lower large deviations principle* holds (and similarly for *upper*) if the above statement holds with t replaced by n and $\frac{1}{t}\int_0^t \varphi(f_s x)\, ds$ replaced by $\sum_{i=0}^{n-1} \varphi(f^i x)$ in (6.5), and $\mathcal{M}_{\mathcal{F}}(X)$ replaced by $\mathcal{M}_f(X)$ in (6.6).

For a given function, the statement above is known as the level-1 large deviations principle. However, when this result applies to every continuous function, as we ask for in the definition above, it is equivalent to the level-2 large deviations principle [CRL11, Yam09]. We have the following result.

Proposition 6.6. *For every Hölder continuous function φ on GX, the unique equilibrium state satisfies the large deviations principle. In particular, the Bowen-Margulis measure satisfies the large deviations principle for $\varphi = 0$.*

We now prove this result, treating the upper and lower large deviations bounds separately.

6.4. **Upper large deviations.** For the upper large deviations principle, we can reduce to considering the time-1 map of the flow. It is easy to see that the upper large deviations principle for the flow follows from the upper large deviations principle for the time-1 map. This follows because (6.5) can be verified for any continuous function ψ by applying the large deviations principle for the time-1 map to the continuous function $\psi_1 := \int_0^1 \psi(f_s x) ds$.

The Gibbs property (6.1) for the flow immediately yields the Gibbs property with respect to the time-1 map.

$$Q^{-1} e^{-tP(\varphi)+\sum_{i=0}^{n-1}\varphi_1(f^i x)} \leq \mu(B_n(x,\rho;f_1)) \leq Q e^{-tP(\varphi)+\sum_{i=0}^{n-1}\varphi_1(f^i x)},$$

where $B_n(x,\epsilon;f_1) = \{y : d_1(f_1^i x, f_1^i y) < \rho \text{ for all } i \in \{0,\dots,n-1\}\}$, and d_1 is the metric equivalent to d given by $d_1(x,y) = \sup_{t\in[0,1)} d(f_t x, f_t y)$. Note also that from the variational principle and flow invariance of the measure $P(\varphi_1, f_1) = P(\varphi, \mathcal{F})$.

It is well known that in the discrete time case the upper large deviations principle can be proved under the hypotheses of the upper Gibbs property and upper semi-continuity of the entropy map $\mu \to h_\mu$ (which follows from expansivity of the flow). This follows from Theorem 3.2 of [PS05], whose hypotheses are the existence of an *upper-energy function* and upper semi-continuity of the entropy map. The existence of an upper-energy function e_μ can easily be deduced from the upper bound in the Gibbs property (by setting $e_\mu := P(\phi_1, f_1) - \phi_1(x)$). See section 7.2 of [CT15] for this argument.

Thus, we have the upper large deviations for φ_1 for μ with respect to f_1, and thus the upper large deviations principle for φ with respect to the flow of (6.5).

6.5. **Lower bounds.** We now verify the lower large deviations principle. In the discrete time case, lower large deviations is proved as Theorem 3.1 of Pfister and Sullivan [PS05] under the following three hypotheses (see also Theorem 3.1 of [Yam09]):

(1) Upper semi-continuity of the entropy map;
(2) Existence of a "lower-energy function", which follows easily from the lower Gibbs property;

(3) Entropy density of ergodic measures in the space of invariant measures.

For a map f, the third hypothesis listed above, *entropy density of ergodic measures*, is the property that for any f-invariant measure μ, for any $\eta > 0$, we can find an ergodic measure ν such that $D(\mu, \nu) < \eta$ and $|h(\nu) - h(\mu)| < \eta$, where D is the standard metric on the space of measures on X that is compatible with the weak* topology (see section 6.1 of [Wal82]).

Entropy density is known to be true for maps with the almost product property [PS05], which is a weaker hypothesis than the specification property (the one with exact gaps). The basic argument was first proved for $\mathbb{Z}^d$-shifts with specification by Eizenberg, Kifer and Weiss [EKW94]. However, no reference is available for maps with weak specification, or for flows. In this section, we carefully prove entropy density for flows with weak specification. While this extension is expected, care must be taken in the argument, and dealing with the variable gap length is a non-trivial extension of the existing proofs.

Remark. The time-1 map f_1 of a flow with weak specification may not satisfy the entropy density condition: consider a suspension flow with roof function constant height 1. Each ergodic measure for f_1 is supported on a single height, i.e on $X \times \{h\}$ for some $h \in [0, 1)$. Take an f_1-invariant measure given by a convex combination of an ergodic measure on $X \times \{0\}$, and an ergodic measure on $X \times \{\frac{1}{2}\}$. This measure can clearly not be approximated weak* by an ergodic f_1-invariant measure. Thus, for our lower large deviations argument, it is advantageous to work at the level of the flow rather than try to recover the result from the discrete time results.

We prove that for a flow $F = \{f_t\}$ with weak specification and expansivity, the ergodic measures are entropy dense in the space of F-invariant measures.

Proposition 6.7. *Let F be an expansive flow with the weak specification property. Let μ be an F-invariant probability measure. Then for any $\eta > 0$, we can find an F-invariant ergodic measure ν such that $D(\mu, \nu) < \eta$ and $|h(\nu) - h(\mu)| < \eta$.*

The strategy is to construct a closed F-invariant set $Y \subset X$ such that every invariant measure supported on Y is weak*-close to μ, and such that the topological entropy of Y is close to $h(\mu)$. For $x \in X$ and $t \in \mathbb{R}$, define

$$\mathcal{E}_t(x) := \frac{1}{t} \int_0^t \delta_{f_s x} ds.$$

The measures $\mathcal{E}_t(x)$ are sometimes called the *empirical measures* for the flow. Given a set $U \subset \mathcal{M}_{\mathcal{F}}(X)$, let

$$X_{t,U} := \{x \in X \mid \mathcal{E}_t(x) \in U\}.$$

From now on, we fix $\eta > 0$, and let $\mathcal{B} := B(\mu, 5\eta)$ and for $m \geq 1$, let

$$Y_m := \{x \mid f_s x \in X_{m,\overline{\mathcal{B}}} \text{ for all } s \geq 0\}. \tag{6.7}$$

Each Y_m is closed and forward invariant, so we can consider the dynamics of the semi-flow $F^+ = \{f_t : t \geq 0\}$ on Y_m. We could modify the definition of Y_m by replacing "$s \geq 0$" with "$s \in \mathbb{R}$" to get a flow-invariant set, but we avoid this to simplify the book-keeping of arguments that appear later in our proof. It is unproblematic to work with a set which is only forward invariant because measures which are invariant for $F^+|_{Y_m}$ can easily be shown to be invariant for $\mathcal{F}$. More precisely, consider $\nu \in \mathcal{M}_{F^+}(Y_m)$. Thinking of ν as a measure on X, then for each $t \geq 0$,

$\nu \in \mathcal{M}_{f_t}(X)$. Since f_t is invertible, then ν is f_{-t} invariant. Thus $\nu \in \mathcal{M}_F(X)$. We prove the following lemma.

Lemma 6.8. *For any $m \geq 1$, if $\nu \in \mathcal{M}_{F^+}(Y_m)$, then $D(\mu, \nu) \leq 6\eta$.*

Proof. Assume that $\nu \in \mathcal{M}_{F^+}(Y_m)$ is ergodic. Since ν is ergodic, there exists a generic point $x \in Y_m$, that is so $\mathcal{E}_t(x)$ converges to ν. For a large value of t, we chop the orbit (x,t) into segments of length m (and a remainder), and use that for each i, $f_{im}x \in X_{m,\overline{\mathcal{B}}}$. More precisely, for $t \in \mathbb{R}$, write $t = sm + q$ where s is an integer and $0 \leq q < m$. Then

$$D(\mathcal{E}_t(x), \mu) \leq \sum_{i=0}^{s-1} \frac{m}{t} D(\mathcal{E}_m(f_{im}x), \mu) + \frac{q}{t} D(\mathcal{E}_q(f_{sm}x), \mu).$$

Since $D(\mathcal{E}_m(f_{im}x), \mu) \leq 5\eta$, we have $\sum_{i=0}^{s-1} \frac{m}{t} D(\mathcal{E}_m(f_{im}x), \mu) \leq 5\eta$. For the remaining error term, writing M for the diameter of $\mathcal{M}(X)$, let t be large enough so that $mM/t < \eta$. Then $D(\mathcal{E}_t(x), \mu) < 6\eta$. Thus, taking $t \to \infty$, we have the lemma for ν ergodic. The result for ν non-ergodic follows from ergodic decomposition. □

We will let $Y := Y_{Kn}$ for values of K and n to be chosen shortly. By expansivity, the entropy map $\mu \to h(\mu)$ is upper semi-continuous. So by the variational principle and the fact that measures in Y are weak*-close to μ, then the topological entropy of Y cannot be much larger than $h(\mu)$; by choosing η small enough, we can guarantee that $h(Y) < h(\mu) + \gamma$. To show that Y has entropy close to $h(\mu)$, we use our specification property to build a large number of (t,ϵ)-separated points inside Y for arbitrarily large t, thus giving a lower bound on the topological entropy of Y.

We rely on the following result, whose proof is a general argument based on the definition of entropy and the Birkhoff ergodic theorem. In the discrete time case, it is a corollary of Proposition 2.1 of [PS05] (see also Proposition 2.5 of [Yam09]).

Proposition 6.9. *Let μ be ergodic and $h < h(\mu)$. Then there exists $\epsilon > 0$ such that for any neighborhood U of μ, there exists T so that for any $t \geq T$ there exists a (t,ϵ)-separated set $\Gamma \subset X_{t,U}$ such that $\#\Gamma \geq e^{th}$.*

Now use the ergodic decomposition of μ to find $\lambda = \sum_{i=1}^{p} a_i \mu_i$ such that the μ_i are ergodic, the $a_i \in (0,1)$ such that $\sum_{i=1}^{p} a_i = 1$, $D(\mu, \lambda) \leq \eta$, and $h(\lambda) > h(\mu) - \eta$. See [You90] for a proof that this is possible.

Let $h_i = 0$ when $h(\mu_i) = 0$, and $\max(0, h(\mu_i) - \eta) < h_i < h(\mu_i)$ otherwise. Take $3\epsilon_i$ and T_i so that the conclusion of Proposition 6.9 holds for μ_i and h_i, and let ϵ' be the minimum of the ϵ_i, and T be the maximum of the T_i. Let

$$\mathrm{Var}(D, \epsilon) := \sup\{D(\delta_x, \delta_y) \mid d(x,y) < \epsilon\}.$$

Note that since the map $x \to \delta_x$ is continuous, we have $\mathrm{Var}(D,\epsilon) \to 0$ as $\epsilon \to 0$. Choose $\epsilon < \epsilon'$ so that $\mathrm{Var}(D,\epsilon) < \eta$. Choose t such that letting $t_i := a_i t$, then $t_i \geq T$ for every i. Note that $t = \sum_{i=1}^{p} t_i$. We are free to choose t as large as we like relative to p, and $\tau(\epsilon)$, the maximum transition time provided by the weak specification property for F at scale ϵ. We will specify how large t needed to be chosen later in the proof.

Let $U_i = B(\mu_i, \eta)$. Take $(t_i, 3\epsilon)$-separated sets $\Gamma_i \subset X_{t_i,U_i}$ such that $\#\Gamma_i \geq e^{t_i h_i}$.

Now we use the weak specification property for the flow at scale ϵ to define a map

$$\Phi : \prod_{i=1}^{\infty} (\Gamma_1 \times \cdots \times \Gamma_p) \to X.$$

That is, given $(x_{11},\dots x_{1p},x_{21},\dots,x_{2p},\dots)$, where $x_{ij}\in\Gamma_j$, we find a point $y\in X$ which ϵ-shadows (x_{11},t_1), then after a transition period of time at most τ, ϵ-shadows (x_{12},t_2), and so on. Such a y can be found by the infinitary version of the weak specification property, see Lemma 7.1.

We will show that the image of Φ is a subset of Y, and then use Φ to construct (t,ϵ)-separated sets for large t which satisfy cardinality estimates that yield the estimate we require on $h(Y)$.

First we show that the image of Φ belongs to Y. The construction was chosen so that each time a portion of the orbit of y approximates a sequence of orbit segments in $\Gamma_1\times\cdots\times\Gamma_p$, the orbit has spent exactly the right amount of time approximating each of $\mu_1,\dots,\mu_p$ so that the appropriate empirical measure for y is close to μ. Thus, in what follows, we show that the empirical measures of y are close to μ along a subsequence corresponding to the times when y approximates a sequence in $\prod_{i=1}^k(\Gamma_1\times\cdots\times\Gamma_p)$. From there we bootstrap to all sufficiently large times.

Fix a point y in the image of Φ, so $y=\Phi(x_{11},\dots x_{1p},x_{21},\dots,x_{2p},\dots)$, where $x_{ij}\in\Gamma_j$ for all $i\geq 1, j\in\{1,\dots,p\}$. Let $\tau_{ij}(y)$ be the length of the transition time in the specification property that occurs immediately after approximating the orbit segment (x_{ij},t_j). Let $c=\sum_{i=1}^p t_i+(p-1)\tau$: this is the upper bound on the total time taken to approximate a sequence of orbits in $\Gamma_1\times\cdots\times\Gamma_p$. The precise time to approximate such a sequence of orbits for a point y is given by $c_k(y)=\sum_{i=1}^p t_i+\sum_{i=1}^{p-1}\tau_{ki}(y)$. Let $b_k(y)=\sum_{i=1}^k(c_i(y)+\tau_{ip}(y))$, and $b_0=0$. Then $b_k(y)$ is the total time that y spends approximating a sequence of orbits in $\prod_{i=1}^k(\Gamma_1\times\cdots\times\Gamma_p)$.

Lemma 6.10. *For all $k\geq 0$, we have $D(\mathcal{E}_c(f_{b_k(y)}y),\mu)\leq 5\eta$.*

Proof. Fix $k\geq 1$, and write $y'=f_{b_{k-1}(y)}y$, $\tau_j=\tau_{kj}(y)$, and $s_i=\sum_{j=1}^i t_j+\sum_{j=1}^{i-1}\tau_j$, so s_i is the total time that y' initially spends approximating the corresponding sequence in $\Gamma_1\times\cdots\times\Gamma_i$. Then, writing M for the diameter of $M_{\mathcal{F}}(X)$ in the metric D, we remove the 'uncontrolled' portion of the orbit of y from consideration by using the estimate

$$D\left(\mathcal{E}_c(y'),\sum_{i=1}^p\frac{t_i}{c}\mathcal{E}_{t_i}(f_{s_{i-1}+\tau_{i-1}}y')\right)\leq\frac{p}{c}\tau M.$$

Now since $d_{t_i}(f_{s_{i-1}+\tau_{i-1}}y',x_{ki})<\epsilon$, for each i, we have

$$D\left(\mathcal{E}_{t_i}(f_{s_{i-1}+\tau_{i-1}}y'),\mathcal{E}_{t_i}(x_{ki})\right)<t_i\operatorname{Var}(D,\epsilon)<t_i\eta.$$

Thus, by chosing t, and hence c, so large that $\frac{p}{c}\tau M<\eta$, we have

$$D\left(\mathcal{E}_c(y'),\sum_{i=1}^p\frac{t_i}{c}\mathcal{E}_{t_i}(x_{ki})\right)<\frac{p}{c}\tau M+\sum_{i=1}^p\frac{t_i}{c}\eta<2\eta.$$

Now since for each i, $x_{ki}\in X_{t_i,U_i}$, we have

$$D\left(\sum_{i=1}^p\frac{t_i}{c}\mathcal{E}_{t_i}(x_{ki}),\sum_{i=1}^p\frac{t_i}{c}\mu_i\right)\leq\sum_{i=1}^p\frac{t_i}{c}\eta<\eta.$$

Furthermore, we have $t\leq c=\sum_{i=1}^p t_i+(p-1)\tau\leq t+p\tau$, so if t is chosen to be much larger than $p\tau$ then t_i/c is close to $t_i/t=a_i$ and we can ensure that

$$D\left(\sum_{i=1}^p\frac{t_i}{c}\mathcal{E}_{t_i}(x_{ki}),\sum_{i=1}^p a_i\mu_i\right)<\eta.$$

Putting all this together, we have

$$D(\mathcal{E}_c(f_{b_{k-1}}y),\mu) \le D\left(\mathcal{E}_c(y'),\sum_{i=1}^{p}\frac{t_i}{c}\mathcal{E}_{n_i}(x_{ki})\right) + D\left(\sum_{i=1}^{p}\frac{t_i}{c}\mathcal{E}_{t_i}(x_{ki}),\sum_{i=1}^{p}\frac{t_i}{c}\mu_i\right)$$
$$+ D\left(\sum_{i=1}^{p}\frac{t_i}{c}\mu_i,\sum_{i=1}^{p}a_i\mu_i\right) + D\left(\sum_{i=1}^{p}a_i\mu_i,\mu\right) < 5\eta.$$

□

The previous lemma was where we required that t is large relative to τ and p. In the next lemma, we specify how large K needs to be chosen. The idea is that an orbit segment of y of length $K(c+\tau)$ will comprise of $K-2$ sub-segments of length c where Lemma 6.10 applies and so the empirical measures along the subsegments are close to μ. Additional deviation of the empirical measure along the whole orbit segment is made arbitrarily small by choosing K large. This is the strategy for the proof of the following lemma.

Lemma 6.11. *If y is a point in the image of Φ, then $y \in Y$.*

Proof. Given $s \ge 0$, we need to show that $f_s y \in X_{Kt,\overline{\mathcal{B}}}$ for a suitably chosen K. The idea is that taking the unique m so that $b_m < s \le b_{m+1}$, we have

$$\mathcal{E}_{Kt}(f_s y) = \sum_{i=1}^{K-2}\frac{c}{Kt}\mathcal{E}_c(f_{b_{m+i}}y) + \text{ error.}$$

The error term has two sources. First, there are at most K segments of y's orbit, each of length at most τ, used as the transition segments in the application of the specification property in the construction of Φ. Second, there is a run of length at most $t+\tau$ at both the start and end of the orbit segment $(f_s y, Kt)$. More precisely, using Lemma 6.10, we have

$$D(\mathcal{E}_{Kt}(f_s y),\mu) \le \frac{c(K-2)}{Kt}5\eta + \frac{\tau K}{Kt}M + \frac{2M(t+\tau)}{Kt} \le \frac{c}{t}5\eta + \frac{\tau M}{t} + \frac{2M}{K} + \frac{2M\tau}{Kt}.$$

We see that if K and t are large enough, then the right hand side is arbitrarily small. Thus $y \in Y_{Kt} = Y$. □

Now we prove our entropy estimates. We use Φ to define a map

$$\Phi_m : \prod_{i=1}^{m}(\Gamma_1 \times \cdots \times \Gamma_p) \to Y.$$

For each $\underline{x} \in \prod_{i=1}^{m}(\Gamma_1 \times \cdots \times \Gamma_p)$, we make a choice of $\underline{y} \in \prod_{i=1}^{\infty}(\Gamma_1 \times \cdots \times \Gamma_p)$ with $y_{ij} = x_{ij}$ for $i \in \{1,\dots,m\}$, $j \in \{1,\dots,p\}$, and we define $\Phi_m(\underline{x}) := \Phi(\underline{y})$. By Lemma 6.11, the image of Φ_m belongs to Y. For $j \in \{1,\dots,mp-1\}$, let $\tau_j(\underline{x}) \in [0,\tau]$ denote the jth transition time that occurs when applying the specification property in the definition of $\Phi_m(\underline{x})$.

Lemma 6.12. *There exists a constant C so that for all m, the image of Φ_m contains a $(b_m,\epsilon/2)$-separated set E_m with $\#E_m \ge C^{-m}\#\prod_{i=1}^{m}(\Gamma_1 \times \cdots \times \Gamma_p)$.*

Proof. Let $k \in \mathbb{N}$ be large enough so that, writing $\zeta := \tau/k$, we have $d(x, f_s x) < \epsilon/2$ for every $x \in X$ and $s \in (-\zeta,\zeta)$. We partition the interval $[0, mp\tau]$ into kmp subintervals $I_1,\dots,I_{kmp}$ of length ζ, denoting this partition as P.

Given $\underline{x} \in \prod_{i=1}^{m}(\Gamma_1 \times \cdots \times \Gamma_p)$, take the sequence $n_1,\dots,n_k$ so that

$$\tau_1(\underline{x}) + \cdots + \tau_i(\underline{x}) \in I_{n_i} \text{ for every } 1 \le i \le mp-1.$$

Now let $l_1 = n_1$ and $l_{i+1} = n_{i+1} - n_i$ for $1 \le i \le k-2$, and let $l(\underline{x}) := (l_1, \dots, l_{k-1})$. Since $\tau_{i+1}(\underline{x}) \in [0, \tau]$, we have $n_i \le n_{i+1} \le n_i + k$ for each i, so $l(\underline{x}) \in \{0, \dots, k-1\}^{mp-1}$.

Now given $\bar{l} \in \{0, \dots, k-1\}^{mp-1}$, let $\Gamma^{\bar{l}} \subset \prod_{i=1}^m (\Gamma_1 \times \dots \times \Gamma_p)$ be the set of all $\underline{x}$ such that $l(\underline{x}) = \bar{l}$. Note that if $\underline{x}, \underline{x}' \in \Gamma^{\bar{l}}$ and $i \in \{1, \dots, k-1\}$, then by construction, $\tau_1(\underline{x}) + \dots + \tau_i(\underline{x})$ and $\tau_1'(\underline{x}) + \dots + \tau_i'(\underline{x})$ belong to the same element of the partition P.

We show that Φ_m is 1-1 on each $\Gamma^{\bar{l}}$. Fix $\bar{l}$ and let $\underline{x}, \underline{x}' \in \Gamma^{\bar{l}}$ be distinct. Let j be the smallest index such that $x_j \neq x_j'$. Write $\tau_i = \tau_i(\underline{x})$ and $\tau_i' = \tau_i(\underline{x}')$. Let $r = \sum_{i=1}^j (t_i + \tau_i)$ and $r' = \sum_{i=1}^j (t_i + \tau_i')$. Since $\sum_{i=1}^j \tau_i$ and $\sum_{i=1}^j \tau_i$ belong to the same element of P, then $|r - r'| = |\sum_{i=1}^j \tau_i - \sum_{i=1}^j \tau_i'| < \zeta$.

Because $x_j \neq x_j' \in \Gamma_i$ for some $i \in \{i, \dots, p\}$ and Γ_i is $(t_i, 3\epsilon)$-separated, we have $d_{t_i}(x_j, x_j') > 3\epsilon$. Now we have

$$d_{b_m}(\Phi_m \underline{x}, \Phi_m \underline{x}') \ge d_{t_i}(f_r \Phi_m \underline{x}, f_r \Phi_m \underline{x}') > d_{t_i}(f_r \Phi_m \underline{x}, f_{r'} \Phi_m \underline{x}') - \epsilon/2,$$

where the $\epsilon/2$ term comes from the fact that $d_{t_i}(f_r \Phi_m \underline{x}', f_{r'} \Phi_m \underline{x}') \le \epsilon/2$ by our choice of ζ. For the first term, observe that

$$d_{t_i}(f_r \Phi_m \underline{x}, f_{r'} \Phi_m \underline{x}') \ge d_{t_i}(x_j, x_j') - d_{t_i}(x_j, f_r \Phi_m \underline{x}) - d_{t_i}(f_{r'} \Phi_m \underline{x}', x_j') > d_{t_i}(x_j, x_j') - 2\epsilon.$$

It follows that $d_{b_m}(\Phi_m \underline{x}, \Phi_m \underline{x}') > \epsilon/2$. Thus, Φ_m is 1-1 on $\Gamma^{\bar{l}}$ and $\Phi_m(\Gamma^{\bar{l}})$ is $(b_m, \epsilon/2)$-separated. There are k^{mp-1} choices for $\bar{l}$, so letting $C = k^p$, by the pigeon hole principle, there exists $\bar{l}$ so that $\#\Gamma^{\bar{l}} \ge C^{-m} \#(\prod_{i=1}^m \Gamma_1 \times \dots \times \Gamma_p)$. For this $\bar{l}$, we let $E_m := \Phi_m(\Gamma^{\bar{l}})$. $\square$

We have that

$$\#E_m \ge C^{-m} \left(\prod_{i=1}^p \#\Gamma_i\right)^m \ge C^{-m} e^{m \sum_{i=1}^p t_i h_i} \ge C^{-m} e^{mt \sum_{i=1}^p a_i h_i} e^{-m \sum_{i=1}^p h_i}$$

Thus, $\frac{1}{tm} \log \#E_m > h(\mu) - \eta + \frac{1}{t}(\sum_{i=1}^p h_i + C)$. Taking a limit as $m \to \infty$, and observing that $\frac{b_m}{tm} \to 1$, this shows that $h(Y) > h(\mu) - \eta$.

Since $h(Y) = \sup\{h(\nu) : \nu \text{ is ergodic and } \nu \in \mathcal{M}_{F^+}(Y)\}$, we can find an ergodic measure ν supported on Y with $h(\nu) > h(\mu) - \eta$. The discussion preceding Lemma 6.8 shows that $\nu \in \mathcal{M}_F(X)$. Thus ν satisfies the conclusion of Proposition 6.7.

Completing the proof of lower large deviations. Now that we have entropy density of ergodic measures, the rest of the argument is standard. Nevertheless, we do not know of a convenient reference in continuous time, so we sketch the proof. First observe that it is clear that entropy density of ergodic measures means that it is possible to consider only ergodic measures in the expression

$$\sup\left\{h_\nu(f) + \int \varphi d\nu : \left|\int \psi\, dm - \int \psi\, d\nu\right| \ge \epsilon\right\}.$$

Thus, for the lower large deviations, it will suffice to show is that for any ergodic μ with $|\int \psi\, dm - \int \psi\, d\nu| > \epsilon$ and $\delta > 0$ sufficiently small that

$$\lim_{t \to \infty} \frac{1}{t} \log m \left\{x : \left|\frac{1}{t} \int_0^t \psi(f_s x)\, ds - \int \psi\, d\mu\right| \le \delta\right\} \ge P(\varphi) - (h(\mu) + \int \varphi d\mu). \tag{6.8}$$

This is achieved by a combination of the Gibbs property for m, and basic cardinality estimates for μ. A sketch goes as follows.

- For a suitable small $\eta > 0$, from the Katok entropy formula, and the Birkhoff ergodic theorem, we can find a sequence of (t, η) separated sets with $\#E_t > e^{t(h(\mu)-\eta)}$ so that for $\phi \in \{\varphi, \psi\}$ we have
$$\sup_{y \in B_t(x,\eta), x \in E_t} \left| \frac{1}{t} \int_0^t \phi(f_s x)\, ds - \int \phi\, d\mu \right| \le \delta$$
- Then
$$m\left\{ x : \left| \frac{1}{t} \int_0^t \psi(f_s x)\, ds - \int \psi\, d\mu \right| \le \delta \right\} \ge \sum_{x \in E_t} m(B_t(x, \eta))$$
- By the Gibbs property, $m(B_t(x,\eta)) \ge C^{-1} e^{-tP(\varphi) + \int_0^t \varphi(f_s x) ds}$, and since $x \in E_t$, $\int_0^t \psi(f_s x) ds \ge \int t\psi d\mu - t\delta$.
- Thus
$$\begin{aligned} m\left\{ x : \left| \frac{1}{t} \int_0^t \psi(f_s x)\, ds - \int \psi\, d\mu \right| \le \delta \right\} &\ge Q^{-1} \#E_t e^{-tP(\varphi) + t\psi d\mu - t\delta} \\ &\ge Q^{-1} e^{-t(P(\varphi) - (h(\mu) + \int \varphi d\mu) + \eta + \delta)} \end{aligned}$$

  The proof of the lower large deviations principle follows.

7. Some technical results

We provide details of some technicals results that we used earlier in the paper.

7.1. Finitary to infinitary specification.

Lemma 7.1. *Let $\mathcal{F}$ be a flow on a compact metric space X, and assume that $\mathcal{F}$ satisfies the weak specification property. Then the conclusion of the specification property also holds for any countably infinite sequence of orbit segments.*

Proof. Let $\delta > 0$ be the scale, and $\tau > 0$ the maximum gap size for the scale $\delta/3$ provided by the weak specification property for $\mathcal{F}$. It is clear from the definition that τ can also serve as a maximum gap size for the scale δ.

Now let $\{(x_i, t_i)\}_{i \in \mathbb{N}}$ be a countably infinite sequence of orbit segments. For each $j \in \mathbb{N}$, we use the weak specification on the first j orbit segments $\{(x_i, t_i)\}_{i=1}^j$ to produce a point $y_j \in X$ and corresponding transition times $\tau_i^{(j)}$ $(1 \le i \le j)$, so that appropriate iterates of y_j $(\delta/3)$-shadow the prescribed orbit segments. Since the space X is compact, one can choose an accumulation point for the sequence $\{y_j\}_{j \in \mathbb{N}}$, call it y. Passing to a subsequence, we may assume that $y_j \to y$.

We now want to verify that y has the desired property. To do this, we need to produce a countable collection τ_i of transition times, and check the corresponding specification property. First, look at the sequence $\{\tau_1^{(j)}\}_{j \in \mathbb{N}} \subset [0, \tau]$. Passing to a subsequence if necessary, we may assume $\{\tau_1^{(j)}\}_{j \in \mathbb{N}}$ converges to $\tau_1 \in [0, \tau]$. Next consider the sequence $\{\tau_2^{(j)}\}_{j \ge 2, j \in \mathbb{N}} \subset [0, \tau]$. Again, passing to a subsequence, we can choose a limiting $\tau_2 \in [0, \tau]$. Continuing in this manner, we obtain a sequence of transition times $\{\tau_i\}_{i \in \mathbb{N}}$.

Having defined the transition times, we now verify the infinitary specification condition. Given $k \in \mathbb{N}$, we consider the finitely many orbit segments $\{(x_i, t_i)\}_{i=1}^k$. By compactness, there is an $\epsilon > 0$ with the property that, for any pair of points satisfying $d(z, z') \le \epsilon$, we have $d_{s_k}(z, z') < \delta/3$. By continuity of the flow, there is also an $\epsilon' > 0$ so that for all $x \in X$, $|t - t'| < \epsilon'$, and $1 \le i \le k$, we have $d_{t_i}(f_t(x), f_{t'}(x)) < \delta/3$.

We now choose a $y' := y_N$ from the approximating sequence having the following two properties: (i) $d(y', y) < \epsilon$, and (ii) each $|\tau_i^{(N)} - \tau_i| < \epsilon'/k$, for $1 \le i \le k$.

From property (i), we conclude that $d_{s_k}(y, y') < \delta/3$, and from property (ii), it follows immediately that $|(s_i^{(N)} + \tau_i^{(N)}) - (s_i + \tau_i)| < \epsilon'$ holds for all $1 \le i \le k$. We now have the estimate:

$$\begin{aligned} d_{t_i}(f_{s_{i-1}+\tau_{i-1}} y, x_i) &\le d_{t_i}(f_{s_{i-1}+\tau_{i-1}} y, f_{s_{i-1}+\tau_{i-1}} y') + d_{t_i}(f_{s_{i-1}+\tau_{i-1}} y', x_i) \\ &\le d_{s_k}(y, y') + d_{t_i}(f_{s_{i-1}+\tau_{i-1}} y', x_i) \\ &\le d_{s_k}(y, y') + d_{t_i}(f_{s_{i-1}+\tau_{i-1}} y', f_{s_{i-1}^{(N)}+\tau_{i-1}^{(N)}} y') + d_{t_i}(f_{s_{i-1}^{(N)}+\tau_{i-1}^{(N)}} y', x_i) \\ &\le \delta/3 + \delta/3 + \delta/3 = \delta. \end{aligned}$$

The first and third inequalities are just applications of the triangle inequality for the metric d_{t_i}. The second inequality comes from the definition of the metrics d_t, along with the fact that $s_{i-1} + \tau_{i-1} + t_i \le s_k$ for every $1 \le i \le k$. For the last inequality, the first term is controlled by property (i), while the second term is controlled by property (ii) and the choice of ϵ'. The last term is controlled by the specification property at scale $\delta/3$ for the point $y' = y_N$. This gives the desired estimate, and since this can be done for every $k \in \mathbb{N}$, completes the proof. □

7.2. **A technical lemma for orbit equivalence.** Let (X, ϕ_t) and (Y, ψ_t) be continuous flows on compact metric spaces. Let $h : X \to Y$ be a continuous orbit semi-equivalence between ϕ_t and ψ_t. By continuity of the orbit semi-equivalence, an orbit segment (x, t) for (X, ϕ_t) is mapped to an orbit segment $(h(x), \tau(x, t))$ for (Y, ψ_t). That is,

$$h(\{\phi_s(x) : s \in [0, t]\}) = \{\psi_s(h(x)) : s \in [0, \tau(x, t)]\},$$

and in particular, $h(\phi_t(x)) = \psi_{\tau(x,t)}(h(x))$.

Proposition 7.2. *Let (X, ϕ_t) and (Y, ψ_t) be continuous flows on compact metric spaces, and suppose that (Y, ψ_t) has no fixed points. Let $h : X \to Y$ be a continuous orbit semi-equivalence between ϕ_t and ψ_t. Then the function $\tau : X \times \mathbb{R}^+ \to \mathbb{R}^+$ defined as above is continuous.*

Proof. It is clear from continuity of the orbit semi-equivalence that as $s \to t$, $\tau(x, s) \to \tau(x, t)$, so it suffices to study the first coordinate and show that for a fixed t, if $x_k \to x$, then $\tau(x_k, t) \to \tau(x, t)$.

We fix $\epsilon > 0$. Since the flow (Y, ψ_t) has no fixed points, there exists $\delta > 0$ so that if $d(\psi_{s_1} y, \psi_{s_2} y) < \delta$, then $|s_1 - s_2| < \epsilon$. Let $\tau := \tau(x, t)$. Then, by continuity of the flow and h, $\psi_\tau(h(x_k)) \to \psi_\tau(h(x))$. Thus, for k large, we have

$$d(\psi_\tau(h(x_k)), \psi_\tau(h(x))) < \delta/2,$$

where d is the metric on Y. Now we consider the sequence $h(\phi_t x_k)$. By continuity, $h(\phi_t x_k) \to h(\phi_t x) = \psi_\tau(h(x))$. Thus, for k large, we have

$$d(h(\phi_t x_k), \psi_\tau(h(x))) < \delta/2,$$

and so we have $d(\psi_{\tau(x_k,t)}(h(x_k)), \psi_\tau(h(x_k)) = d(h(\phi_t x_k), \psi_\tau(h(x_k)) < \delta$, and these points are on the same orbit. Thus it follows that $|\tau(x_k, t) - \tau| < \epsilon$. It follows that $\tau(x_k, t) \to (x, t)$, and thus the function τ is continuous. □

Corollary 7.3. *Let* (X, ϕ_t)*,* (Y, ψ_t)*, and* $h : X \to Y$ *be as in Proposition 7.2. Then for all* t*, there exists* $K > 0$*, so that for all* $x \in X$*, the image of* (x, t) *under the orbit semi-equivalence* h *is contained in the orbit segment* $(h(x), K)$*. That is,*

$$h(\{\phi_s(x) : s \in [0,t]\}) \subseteq \{\psi_s(h(x)) : s \in [0,K]\}.$$

Proof. By continuity of τ, and compactness of $X \times \{t\}$, it follows that

$$\sup\{\tau(x,t) : x \in X\} < \infty,$$

which proves the corollary. □

7.3. **Topological transitivity and orbit equivalence.** The following Lemma allows us to assume the coding provided by Lemma 2.8 is topologically transitive.

Lemma 7.4. *Let* (Σ, T) *be a shift of finite type, and suppose that* $h : Susp(\Sigma, T) \to (X, \phi)$ *is a continuous, surjective orbit equivalence onto a space without isolated points. Suppose that the flow* ϕ *on* X *is topologically transitive. Then there exists a topologically transitive subshift* $(\Sigma', T|_{\Sigma'})$ *of* (Σ, T) *which is still of finite type, and such that* $Susp(\Sigma', T|_{\Sigma'})$ *maps surjectively onto* (X, ϕ) *via the restriction of* h*.*

Sketch of proof. Using the language $\mathcal{L}$ of the shift (Σ, T), we decompose its alphabet $\mathcal{A}$ as

$$\mathcal{A} = \{\text{non-essential symbols}\} \cup \bigcup_k E_k$$

where a symbol i is *non-essential* if $iwi \notin \mathcal{L}$ for any word w, and where the E_k are equivalence classes of the relation:

$$i \sim j \iff \text{there exist words } u, w \text{ such that } iuj, jwi \in \mathcal{L}$$

on the set of essential symbols. It is clear that for any element of Σ, all symbols of sufficiently large index lie in a single E_k. If we let $\mathcal{E}_k = \{\sigma \in \Sigma : \sigma(n) \in E_k \text{ for all } n\}$, then the subshift $(\mathcal{E}_k, T|_{\mathcal{E}_k})$ is clearly topologically transitive and also of finite type. It is easy to check that for any $\sigma \in \Sigma$, $d(T^n\sigma, \mathcal{E}_k) \to 0$ for some (unique) k.

Let x be a point of X whose orbit is dense in X. Since X has no isolated points, for any A, $\mathcal{O}(x, [A, \infty)) = \{\phi_t x : t \in [A, \infty)\}$ is also dense in X. Lift x to a point $(\sigma, s) \in Susp(\Sigma, T)$ and $\mathcal{O}(x, [A, \infty))$ to $\mathcal{O}((\sigma, s), [B(A), \infty))$. Since $Susp(\Sigma, T)$ is compact and h is continuous and surjective, it is closed. Hence, for arbitrarily large B, $h(\overline{\mathcal{O}((\sigma, s), [B, \infty))}) = X$.

Now, as $B \to \infty$, $\mathcal{O}((\sigma, s), [B, \infty))$ limits on the suspension of one of the subshifts $(\mathcal{E}_k, T)$. Using uniform continuity and closedness of h, it is now clear that $h|_{Susp(\mathcal{E}_k, T|_{\mathcal{E}_k})}$ is an orbit equivalence mapping surjectively onto (X, ϕ). □

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