A Semilinear Equation with Large Advection in Population Dynamics

A dissertation
submitted to the faculty of the graduate school
of the university of minnesota
by

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In partial fulfillment of the requirements
for the degree of
doctor of philosophy

Wei-Ming Ni, Advisor

May 2011

Acknowledgements

First of all I would like to thank God. I have been blessed in so many ways, at different points of my life, by things that are beyond my control. I thank Him for guiding my life and I trust that he will lead me safely home.

I would like to express my sincere gratitude to my advisor, Prof. Wei-Ming Ni, who has taught me so much with his enthusiasm, penetrating insight and unique taste for mathematics. I would not have made it without his constant support and encouragement. I also want to thank Prof. McGehee, Prof. Safonov and Prof. Sverak for reviewing my thesis and serving on my committee of my final defense. My special thank goes to Prof. Adams who was willing to serve on my committee at the last moment.

I am grateful to all of my friends here in Minnesota, whose friendship and care has kept me going for the past few years. I am especially grateful to Glenn Kenadjian for many meaningful conversations at Hong Kong Noodles.

I wish to thank my mother Kam-Ying Tsang, my father Chok-Kung Lam and my brother King-Lun Lam for their unconditional love and support in my entire life.

Finally, I want to thank my fiancée Wendy Xu, for keeping my life balanced and for being patient when I am "phased out". I cannot wait to start our life long journey with you, knowing that I will always have your love, support and understanding along the way.

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Abstract

A system of semilinear equations with large advection term arising from theoretical ecology is studied. The $2X2$ system of equations models two theoretical species competing for a common resources with density $m(x)$ that is spatially unevenly distributed. The two species are identical except for their modes of dispersal: one of them disperses completely randomly, while the other one, in addition to random diffusion, has a tendency to move up the gradient of the resource $m(x)$. It is proved in [Cantrell et. al. Proc. R. Soc. Edinb. 137A (2007), pp. 497-518.] that under mild condition on the resource density $m(x)$, the two species always coexist stably whenever the strength of the directed movement is large, regardless of initial conditions. In this paper we show that every equilibrium densities of the two species approaches a common limiting profile exhibiting concentration phenomena. As a result, the mechanism of coexistence of the two species are better understood and a recent conjecture of Cantrell, Cosner and Lou is resolved under mild conditions.

Contents

Chapter 1

Introduction

In this paper we study the shape of coexistence states of a reaction-diffusion-advection system from theoretical ecology. Consider

$$\begin{cases} U_t = \nabla \cdot (d_1 \nabla U - \alpha U \nabla m) + U(m(x) - U - V) & \text{in } \Omega \times (0, \infty), \\ V_t = d_2 \Delta V + V(m(x) - U - V) & \text{in } \Omega \times (0, \infty), \\ d_1 \frac{\partial U}{\partial \nu} - \alpha U \frac{\partial m}{\partial \nu} = \frac{\partial V}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0, \infty), \\ U(x, 0) = U_0(x) \geq 0 \quad \text{and} \quad V(x, 0) = V_0(x) \geq 0 & \text{in } \Omega \end{cases} \tag{1.1}$$

where Ω is a bounded smooth domain in $\mathbf{R}^N$ with boundary $\partial\Omega$; ∇ is the gradient operator; $\nabla\cdot$ is the divergence operator and $\Delta = \sum_{i=1}^N d^2/dx_i^2$ is the Laplace operator; U and V representing the population densities of two competing species with random dispersal rates d_1, d_2 respectively, are therefore non-negative; $m(x) \in C^2(\bar{\Omega})$ is a non-constant function representing the local intrinsic growth rate; $\alpha \geq 0$ is a parameter.

The system (1.1) originates from the diffusive Lotka-Voletrra model for two randomly-moving competitors in a closed, spatially varying environment. (See [Lo2] and the references therein.) In reality, it is very plausible that besides random dispersal, species could track the local resource gradient and move upward along it. (See, e.g. [BC, BL, CC].) And (1.1) was proposed in [CCL1] to study the joint effects of random diffusion and directed movement on population dynamics.

More precisely, we view the local intrinsic growth rate $m(x)$ as describing the quality and quantity of resources available at the point x. The two species which are competing for a common resource are identical except for their dispersal strategies: the species with density V disperses only by random diffusion, while the species with density U disperses by diffusion combined with directed movement up the gradient of m. The dispersal of the two competitors can be described in terms of the fluxes $J_V = -d_2\nabla V$ and $J_U = -d_1\nabla U + \alpha(\nabla m)U$. (See [CCL1] for a derivation of (1.1) and [Mu] for a discussion of how advection-diffusion equations can be derived in terms of fluxes.) Also, we assume $\alpha \geq 0$ to capture the hypothesis that the first competitor has a tendency to move up the gradient of m. The no-flux boundary conditions reflects the assumption that individuals do not cross the boundary $\partial\Omega$.

To assess whether or not directed movement confers an advantage for either competitor, it suffices to study the existence and stability of steady-states, which determines a significant amount of the dynamics of the competition system (1.1) ([CC, H]). For instance, see Theorem 1.5.

System (1.1) has attracted considerable attention recently. When $\alpha \geq 0$ is small and the diffusion rate d_1 of U is less than the diffusion rate d_2 of V, it is proved in [DHMP, CCL1] that the slower diffuser U always wipes out its faster moving competitor V regardless of initial conditions. In other words, $(\tilde{u}, 0)$ is globally asymptotically stable, where $\tilde{u}$ is the unique positive solution to

$$\begin{cases} \nabla\cdot(d_1\nabla\tilde{u} - \alpha\tilde{u}\nabla m) + \tilde{u}(m(x) - \tilde{u}) = 0 & \text{in } \Omega, \\ d_1\frac{\partial\tilde{u}}{\partial\nu} - \alpha\tilde{u}\frac{\partial m}{\partial\nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

As α increases, the species U has a stronger tendency to move towards more favorable regions and it is expected to continue to win the competition. It is rather surprising that U and V always coexist for α sufficiently large! More precisely, for a generic m, (1.1) has a *stable* coexistence steady-state $(\tilde{U}, \tilde{V})$ ($\tilde{U} > 0$ and $\tilde{V} > 0$) for all α sufficiently large. This so-called "Advection-mediated Coexistence" was discovered in [CCL2] and generalized in [CL]. It was further argued that *as α becomes large, the "smarter" competitor moves toward and concentrates in places with the most favorable local environments, leaving room in region with less resources for the other species to survive.* Furthermore, the above formal argument is justified mathematically in some special cases. For instance, the following is proved:

Theorem 1.1. [CL] *Assume $\int_\Omega m > 0$. If m has a single critical point x_0 which is a global maximum point,* $\det D^2m(x_0) \neq 0$ *and $\frac{\partial m}{\partial\nu} \leq 0$ on $\partial\Omega$, then for any positive steady-state $(\tilde{U}, \tilde{V})$ of* (1.1)*, as $\alpha \to \infty$,*

$$\tilde{V}(x) \to \theta_{d_2}(x) \textit{ in } C^{1,\beta}(\bar{\Omega}), \textit{ for any } \beta \in (0,1), \textit{ and}$$

$$\tilde{U}(x)e^{\alpha[\max_{\bar{\Omega}} m - m(x)]/d_1} \to 2^{N/2}[m(x_0) - \theta_{d_2}(x_0)] \textit{ uniformly in } \Omega.$$

where θ_d is the unique positive solution to

$$\begin{cases} d\Delta\theta + \theta(m-\theta) = 0 & \text{in } \Omega, \\ \frac{\partial \theta}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

Remark 1.2. Here the factor $2^{N/2}$ comes from the profile of $\tilde{U} \sim \tilde{U}(x_0)e^{\alpha[m(x)-\max m]/d_1}$ together with the integral constraint $\int_\Omega \tilde{U}(m(x) - \tilde{U} - \tilde{V})dx = 0$ obtained by integrating the equation over Ω.

In general, we have the following

Conjecture 1.3. [CCL2, CL] For general $m(x)$ (with multiple local maximum points in Ω), every positive steady-state $(\tilde{U}, \tilde{V})$ of (1.1) concentrates at every local maximum point of m as $\alpha \to \infty$.

Under mild conditions on m, the above conjecture was resolved in [LN] when $\Omega = (-1, 1)$, and in [L2] for higher dimensions. It turns out that $\tilde{U}$ *concentrates precisely at the local maximum points of* m *where* $m - \theta_{d_2}$ *is positive.* In this paper we are going to resolve the conjecture for all dimensions under mild conditions on m and determine the limiting profile of $(\tilde{U}, \tilde{V})$. In addition, to better understand the different roles played by the advection term and the reaction term, we are going to treat the following more general system

$$\begin{cases} U_t = \nabla \cdot (d_1 \nabla U - \alpha U \nabla m) + U(p(x) - U - V) & \text{in } \Omega \times (0, \infty), \\ V_t = d_2 \Delta V + V(p(x) - U - V) & \text{in } \Omega \times (0, \infty), \\ d_1 \frac{\partial U}{\partial \nu} - \alpha U \frac{\partial m}{\partial \nu} = \frac{\partial V}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0, \infty), \end{cases} \tag{1.4}$$

where $m = m(x)$ is not necessarily equal to $p = p(x)$. Define $\mathfrak{M}$ to be the set of all local maximum points of m. First we state the assumptions on m and p.

(H1) All local maximum points of m are non-degenerate and $\mathfrak{M} \subset \operatorname{int} \Omega$ and $\frac{\partial m}{\partial \nu} \leq 0$ on $\partial\Omega$.

(H2) If $x_0 \notin \mathfrak{M}$ is a critical point of m, then $\Delta m(x_0) > 0$.

(H3) $p = \chi(m) \in C^\beta(\bar{\Omega})$ for some $\beta \in (0, 1)$ and χ is strictly increasing and $\int_\Omega p\, dx \geq 0$.

The following result determines the limiting profile of $(\tilde{U}, \tilde{V})$.

Theorem 1.4. *Assume* **(H1)**, **(H2)** *and* **(H3)**. *Then, for all α sufficiently large,* (1.4) *has at least one stable coexistence steady-state. Moreover, if $(\tilde{U}, \tilde{V})$ is any coexistence steady-state of* (1.4), *then as $\alpha \to \infty$,*

(i) $\tilde{V}(x) \to \bar{\theta}_{d_2}(x)$ *in* $C^{1,\beta}(\bar{\Omega})$, *for any* $\beta \in (0,1)$;

(ii) for all $r > 0$, $\tilde{U}(x) \to 0$ *in* $\Omega \setminus [\cup_{x_0 \in \mathfrak{M}} B_r(x_0)]$ *uniformly and exponentially;*

(iii) for each $x_0 \in \mathfrak{M}$ *and each* $r > 0$ *small,*

$$\tilde{U}(x) - 2^{N/2} \max\{p(x_0) - \bar{\theta}_{d_2}(x_0), 0\} e^{\alpha[m(x)-m(x_0)]/d_1} \to 0 \text{ uniformly in } B_r(x_0).$$

Here $\mathfrak{M}$ denotes the set of all local maximum points of m and $\bar{\theta}_d$ is the unique positive solution to

$$\begin{cases} d\Delta\bar{\theta} + \bar{\theta}(p - \bar{\theta}) = 0 & \text{in } \Omega, \\ \frac{\partial \bar{\theta}}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.5}$$

Note that when $p \equiv m$, then $\bar{\theta}_{d_2} = \theta_{d_2}$ and this establishes Conjecture 1.3. More importantly, by the theory of monotone dynamical system (see, e.g. [H]) we have

Theorem 1.5. *Assume* **(H1)**, **(H2)** *and* **(H3)**. *Then for all α sufficiently large, there exists two coexistence steady-states $(\tilde{U}_i, \tilde{V}_i)$, $i = 1, 2$ such that $U_1 \geq U_2$ and $V_1 \leq V_2$, and the set $\{(U,V) \in X : \tilde{U}_1 \geq U \geq \tilde{U}_2$ and $\tilde{V}_1 \leq V \leq \tilde{V}_2\}$ is globally attracting among all non-trivial solutions of* (1.4).

Therefore Theorem 1.4 actually describes all possible outcomes of the competition between U and V when α is large, i.e. U and V always coexists with a unique limiting population density.

Remark 1.6. (i) It is proved in Appendix A of [L1] that if x_0 is a local maximum point of p, when d_2 is sufficiently small, $p(x_0) - \bar{\theta}_{d_2}(x_0) > 0$ if and only if $p(x_0) > 0$. On the other hand, when d_2 is large and p has more than one local maximum points, then $p(x_0) - \bar{\theta}_{d_2}(x_0)$ can sometimes be negative, even when $p(x_0) > 0$. In this case, Theorem 1.4 (iii) says that local maximum points of m can be a "trap" for U there. See Figure 1.1 for a one-dimensional picture.

(ii) The existence and stability of $(\tilde{U}, \tilde{V})$ follows from arguments in [CL, CCL2] and are proved in Section 2.

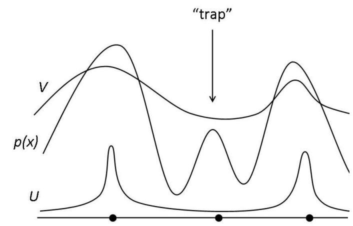

Figure 1.1: A trap for U.

By way of proving Theorem 1.4, we consider the following closely related single equation.

$$\begin{cases} u_t = \nabla \cdot (d\nabla u - \alpha u \nabla m) + u(p(x) - u) & \text{in } \Omega \times (0,\infty), \\ d\frac{\partial u}{\partial \nu} - \alpha u \frac{\partial m}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,\infty), \end{cases} \tag{1.6}$$

(1.6) was proposed in [BC] (when $p \equiv m$) to model the population dynamics of a single species with directed movement in a heterogeneous environment. It was proved in [BC] that if $\int_\Omega m\,dx > 0$ and $p \equiv m$, then (1.6) has a unique positive steady-state $\tilde{u}$ for all $\alpha \geq 0$. Moreover, $\tilde{u}$ is globally asymptotically stable. Similarly, in [CCL2, Lo1] it was conjectured that if $p \equiv m$, then as $\alpha \to \infty$, $\tilde{u}$ concentrates precisely on the set of all local maximum points of m. This conjecture was resolved in [L1] under mild conditions.

We shall determine the limiting profile of $\tilde{u}$ when Ω is in any dimensions and p is not necessarily equal to m. For the single equation (1.6), we can relax the assumption on p

(H4) $p \in C^\beta(\bar{\Omega})$ for some $\beta \in (0,1)$ and $\{x \in \bar{\Omega} : m(x) = \sup_\Omega m\} \subseteq \{x \in \bar{\Omega} : p(x) > 0\}$.

Now we have

Theorem 1.7. *Assume* **(H1)**, **(H2)** *and* **(H4)**. *Then for all α sufficiently large,* (1.6) *has a unique positive steady-state $\tilde{u}$ which is globally asymptotically stable. Moreover, for all small $r > 0$,*

$$\tilde{u}(x) \to 0 \quad \textit{uniformly and exponentially in } \Omega \setminus [\cup_{x_0 \in \mathfrak{M}} B_r(x_0)].$$

And for each $x_0 \in \mathfrak{M}$,

$$\tilde{u}(x) - 2^{N/2} \max\{p(x_0), 0\} e^{\alpha[m(x) - m(x_0)]/d} \to 0 \quad \textit{uniformly in } B_r(x_0).$$

Remark 1.8. The existence, uniqueness and stability of $\tilde{u}$ follows from arguments in [BC] and are proved in Section 2.

The main ingredients in the proof of Theorem 7.1 are the L^∞ estimate in Section 3, and the following Liouville-type result which seems to be new. It determines the limiting profile of $\tilde{u}$ and $\tilde{U}$ at each $x_0 \in \mathfrak{M}$ and is proved in the Appendix.

Proposition 1.9. *Let B be a symmetric positive definite $N \times N$ matrix and $0 < \sigma \in L^\infty_{loc}(\boldsymbol{R}^N)$ such that for some $R_0 > 0$,*

$$\sigma^2 = e^{-x^T Bx} \qquad \text{for all } x \in \boldsymbol{R}^N \setminus B_{R_0}(0),$$

then every nonnegative weak solution $w \in W^{1,2}_{loc}(\boldsymbol{R}^N)$ to

$$\nabla \cdot (\sigma^2 \nabla w) = 0 \qquad \text{in } \boldsymbol{R}^N \tag{1.7}$$

is a constant.

Remark 1.10. In general, some kind of asymptotic behavior is needed for this kind of result to hold; e.g. it is proved in [BCN] that for any $0 < \sigma \in L^\infty_{loc}(\mathbf{R}^N)$, a non-negative weak solution of (1.7) is a constant if there exists $C > 0$ such that $\int_{B_R} \sigma^2 w^2 \leq CR^2$ for all large $R > 0$. (See also [GG].)

In the situation we discussed so far, the directed movement to most favorable regions has not given U much advantage in its competition with V. A main reason behind that is that there are not "a lot" of most favorable regions. Mathematically, the set of local maximum points of U, which is denoted by $\mathfrak{M}$ is assumed to be of measure zero in the results presented above. In all those cases, we can see that U occupies only the region $\mathfrak{M}$, which is of measure zero, and does not really compete with V for resources.

Consider now the case when m assumes its global maximum on a set of positive measure (i.e. $|\mathfrak{M}| > 0$). Will U, being a resource-specialist of $\mathfrak{M}$, consume all the resources present at $\mathfrak{M}$, and be able to drive V to extinction? The following results says that, in some cases, U can wipe out V.

Proposition 1.11. *Let $\Omega = (-1, 1)$. If m satisfies*

$$m(x) = 1 \text{ in } [-1/2, 1/2], \quad m(x) < 1 \text{ and } m'(x) \neq 0 \text{ in } (-1, -1/2) \cup (1/2, 1). \tag{1.8}$$

and $\int_{-1}^{1} m < 1$, *then there exists* $\bar{d}_2 > 0$ *such that for any* $d_2 > \bar{d}_2$, $(\tilde{u}, 0)$ *is globally asymptotically stable for all* α *sufficiently large.*

This will be proved in Chapter 5.

Chapter 2

Existence, Uniqueness and Stability of $\tilde{u}$

In this chapter, we present the existence and stability results for positive steady-states of (1.6). The arguments are analogous to those in [BC] where the case $p \equiv m$ was treated and are presented here for completeness' sake. For later purposes, we shall study the positive solutions of the following slightly more general equation

$$\begin{cases} u_t = \nabla \cdot (d\nabla u - \alpha u \nabla m) + u(p_\alpha(x) - u) & \text{in } \Omega \times (0,\infty), \\ d\frac{\partial u}{\partial \nu} - \alpha u \frac{\partial m}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{2.1}$$

where

$$p_\alpha(x) \in C^\beta(\bar{\Omega}) \text{ for some } \beta \in (0,1), \text{ and } \lim_{\alpha\to\infty} p_\alpha = p \text{ in } C^\beta(\bar{\Omega}). \tag{2.2}$$

In particular, we will state sufficient condition on p to establish existence of $\tilde{u}$.

Theorem 2.1. *If* **(H4)** *holds, then for α sufficiently large, there exists a unique positive steady-state $\tilde{u} \in C^2(\bar{\Omega})$ of* (2.1) *which is globally asymptotically stable.*

Proof. By a transformation $v = e^{-\alpha m/d}u$, the steady-state equation of (2.1) is equivalent to the following

$$\begin{cases} \tilde{L}v = \nabla \cdot (de^{\alpha m/d}\nabla v) + e^{\alpha m/d}v(p_\alpha - e^{\alpha m/d}v) = 0 & \text{in } \Omega, \\ \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

Fix $\alpha \geq 0$ so that $\int_\Omega e^{\alpha m/d}p_\alpha dx > 0$, which is guaranteed for all large α by **(H4)**. For each such α, we shall construct a pair of upper and lower solutions to show the existence of at least one positive solution for (2.3) (and hence for (2.1)). (See e.g. [S].) First, take $\bar{v} = M$ for some large constant M, then,

$$\begin{cases} \tilde{L}\bar{v} = e^{\alpha m/d} M(p_\alpha - e^{\alpha m/d} M) < 0 & \text{in } \Omega, \\ \frac{\partial \bar{v}}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.4}$$

That means $\bar{v}$ is an upper solution of (2.3). For the lower solution, consider the following eigenvalue problem for λ:

$$\begin{cases} \nabla \cdot (d e^{\alpha m/d} \nabla \phi) + \lambda e^{\alpha m/d} p_\alpha \phi = 0 & \text{in } \Omega, \\ \frac{\partial \phi}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.5}$$

It is well-known that the principal eigenvalue λ_1 of (2.5) is negative if and only if $\int_\Omega e^{\alpha m/d} p_\alpha > 0$ (See e.g. Lemma 2.16 in [CC]), which is satisfied by our choice of α. So $\lambda_1 < 0$. Next, for each $\lambda \in \mathbf{R}$, consider the following eigenvalue problem for $\mu = \mu(\lambda)$:

$$\begin{cases} \nabla \cdot (d e^{\alpha m/d} \nabla \psi) + \lambda e^{\alpha m/d} p_\alpha \psi + \mu e^{\alpha m/d} \psi = 0 & \text{in } \Omega, \\ \frac{\partial \phi}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.6}$$

Now the principal eigenvalue $\mu_1(\lambda)$ of (2.6) is given by

$$\mu_1(\lambda) = \inf_{\psi \in H^1} \left\{ \frac{\int_\Omega e^{\alpha m/d} (d|\nabla \psi|^2 - \lambda p_\alpha \psi^2)\, dx}{\int_\Omega e^{\alpha m/d} \psi^2\, dx} \right\}$$

Observe that $\mu_1(\lambda)$ is a strictly concave function of λ, and $\mu_1(0) = \mu_1(\lambda_1) = 0$ (where λ_1 is the principal eigenvalue of (2.5)). Since $\lambda_1 < 0$, we have $\mu_1(1) < 0$. Denote the eigenfunction corresponding to $\mu_1(1) < 0$ by ψ_1. We can assume $\psi_1 > 0$ and $|\psi_1|_{L^\infty(\Omega)} = 1$. Then $\underline{v} = \epsilon \psi_1$ satisfies, for $\epsilon > 0$ sufficiently small,

$$\begin{cases} \tilde{L}\underline{v} = e^{\alpha m/d} \epsilon \psi_1 (-\mu_1(1) - e^{\alpha m/d} \epsilon \psi_1) > 0 & \text{in } \Omega, \\ \frac{\partial \bar{v}}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.7}$$

Thus $\underline{v} = \epsilon \psi_1 > 0$ is a lower solution of (2.3). By the method of upper and lower solutions, (2.3) has at least one positive solution $\tilde{v}$.

The uniqueness of $\tilde{v}$ can be proved in a standard fashion and we include the proof here for the sake of completeness. Suppose that there exist two positive solutions to (2.3), say v_1 and v_2. If the sign of $v_1 - v_2$ does not change, without loss of generality we assume $v_1 \geq v_2$ and $v_1 \not\equiv v_2$, then

$$\begin{aligned}
0 &= \int_\Omega [\nabla \cdot (de^{\alpha m/d} \nabla v_1) + e^{\alpha m/d} v_1 (p_\alpha - e^{\alpha m/d} v_1)] v_2 \, dx \\
&= \int_\Omega [v_1 \nabla \cdot (de^{\alpha m/d} \nabla v_2) + e^{\alpha m/d} v_1 v_2 (p_\alpha - e^{\alpha m/d} v_1)] \, dx \\
&= \int_\Omega [-e^{\alpha m/d} v_1 v_2 (p_\alpha - e^{\alpha m/d} v_2) + e^{\alpha m/d} v_1 v_2 (p_\alpha - e^{\alpha m/d} v_1)] \, dx \\
&= \int_\Omega e^{2\alpha m/d} v_1 v_2 (v_2 - v_1) < 0
\end{aligned}$$

This is a contradiction.

Otherwise we set $\underline{v}_1 = \max\{v_1, v_2\}$, then $\underline{v}_1$ is a weak lower solution of (2.3) and $\underline{v}_1 \geq v_2$, $\underline{v}_1 \not\equiv v_2$. Moreover $\bar{v} \equiv M$ is an upper solution of (2.3) and $\underline{v}_1 \leq M$ provided M large. Then there exists another solution $\tilde{v}_1$ satisfying $\underline{v}_1 \leq \tilde{v}_1$. Repeat the previous argument we can derive that

$$\tilde{v}_1 \equiv v_2$$

which is a contradiction. Thus problem (2.3) has a unique positive solution $\tilde{v}$. It remains to show that $\tilde{v}$ is globally asymptotically stable. For any positive initial data v_0, choose ϵ small, M large such that

$$\epsilon\psi_1 \leq v_0 \leq M,$$

and $v^+ \equiv M$, $v^- = \epsilon\psi_1$ are the upper and lower solutions of (2.3) respectively. Then on the one hand, it follows that

$$v(x,t;\epsilon\psi_1) \leq v(x,t;v_0) \leq v(x,t;M) \qquad \text{for any } x \in \Omega \text{ and } t \in (0,\infty),$$

where $v(x,t;v_0)$ denotes the unique solution of (2.3) with initial condition v_0. On the other hand, it is easy to see that

$$v(\cdot,t;\epsilon\psi_1) \to \tilde{v}, \qquad v(\cdot,t;M) \to \tilde{v}, \text{ as } t \to \infty.$$

By the parabolic maximum principle, we derive that

$$v(\cdot,t;v_0) \to \tilde{v}.$$

This proves the global asymptotic stability of $\tilde{v}$ with respect to (2.3), which is equivalent to the global asymptotic stability of $\tilde{u} = e^{\alpha m/d}\tilde{v}$ with respect to (2.1).

Finally, $\tilde{v}$ and hence $\tilde{u} = e^{\alpha m/d}\tilde{v}$ is in $C^2(\bar{\Omega})$ by standard elliptic regularity theory. This proves Theorem 2.1. $\square$

Chapter 3

Upper and Lower Estimates of $\tilde{u}$

In this section we derive qualitative properties of the unique steady-state $\tilde{u}$ of the single equation (2.1) which will become useful when we treat the 2×2 system later. Hereafter we denote the set of local maximum points of $m(x)$ by $\mathfrak{M}$.

3.1 Upper estimates of $\tilde{u}$

In one space dimension, i.e. $\Omega = (-1, 1)$, we have the following result

Theorem 3.1. *If $m(x) \in C^2([-1,1])$ is nonconstant and $xm'(x) \leq 0$ at ± 1 then $\tilde{u} \to 0$ in any compact subset of $\{x \in [-1,1] : m'(x) \neq 0\}$.*

The result is an improvement of Theorem 1.7(ii) in [CCL2]. Our main contribution here is to remove the assumption that $\mathfrak{M}$ has to be finite. This generalization will become useful as we treat the case when m assumes its global maximum on a set instead of at a single point in Chapter 5.

If we impose an assumption on Δm at saddle points, we have a much better result for general space dimensions. Namely, we can prove that $\tilde{u} \to 0$ exponentially in compact subsets of $\Omega \setminus \mathfrak{M}$.

Theorem 3.2. *Let $\{m(x) : x \in \mathfrak{M}\}$ be a finite set ($\mathfrak{M}$ not necessarily has measure zero.) and suppose* **(H2)**. *Then given any compact subset K of $\bar{\Omega} \setminus \mathfrak{M}$, there exists $\gamma > 0$ such that $|\tilde{u}|_{L^\infty(K)} \leq e^{-\gamma\alpha}$ for all α large.*

The proof of which is contained in [L1] and is omitted. Instead we are going to present a stronger result (Theorem 3.3 below) which applies to the case when every local maximum points of m is non-degenerate.

The following theorem is first proved in [L1, L2], but the proof we present here is the result of a discussion with X. Chen which can generalize to the case for any α large and, $d > 0$.

Theorem 3.3. *Assume* **(H1)** *and* **(H2)***. Then given a small $r > 0$, there exists positive constants C, γ and α such that for any $\alpha \geq \alpha_0$, and any $d > 0$*

$$\tilde{u} \leq \begin{cases} Ce^{\gamma\alpha[m(x)-m(x_0)]/d} & \text{in } \cup_{x_0 \in \mathfrak{M}} B_r(x_0), \\ e^{-\gamma\alpha/d} & \text{in } \Omega \setminus \cup_{x_0 \in \mathfrak{M}} B_r(x_0). \end{cases}$$

In proving Theorem 3.3, we have the following useful

Lemma 3.4. *Under the assumptions of Theorem 3.3, $|\tilde{u}|_{L^p(\Omega)} \to 0$ for any $p \geq 1$.*

Proof of Theorem 3.1.

Lemma 3.5 (Lemma 3.3 of [CCL2])**.** *Suppose that $\frac{\partial m}{\partial \nu} \leq 0$ on $\partial\Omega$, then*

$$|\tilde{u}|_{L^\infty(\Omega)} \leq |m|_{L^\infty(\Omega)} + \alpha|\Delta m|_{L^\infty(\Omega)}$$

Proof. The result follows from a direct application of the maximum principle. □

Lemma 3.6 (Lemma 3.4 of [CCL2])**.** *Suppose that $\frac{\partial m}{\partial \nu} \leq 0$ on $\partial\Omega$. There then exists some constant C, independent of α, such that*

$$\int_\Omega \tilde{u}|\nabla m|^2 \leq \frac{C}{\alpha}.$$

Proof. Multiply the equation by m and integrating in Ω, we have

$$-\int_\Omega \nabla m \cdot [d\nabla\tilde{u} - \alpha\tilde{u}\nabla m] + \int_\Omega m\tilde{u}(p_\alpha - \tilde{u}) = 0$$

Since

$$\int_\Omega \nabla\tilde{u} \cdot \nabla m = -\int_\Omega \tilde{u}\Delta m + \int_{\partial\Omega} \tilde{u}\frac{\partial m}{\partial \nu} \leq -\int_\Omega \tilde{u}\Delta m,$$

we have

$$\int_\Omega \tilde{u}|\nabla m|^2 \leq \frac{1}{\alpha}\int_\Omega [\tilde{u}(-d\Delta m - mp_\alpha) + m\tilde{u}^2].$$

The result follows from the boundedness of $\tilde{u}$ in $L^2(\Omega)$. □

Define $I_\delta = (-1,1) \setminus \{x \in (-1,1) : \exists z,\ m'(z) = 0 \text{ and } \operatorname{dist}(x,z) \le \delta\}$.

We can easily see that I_δ consists of finitely many open intervals, as $\{x \in (-1,1) : \exists z\, m'(z) = 0 \text{ and } \operatorname{dist}(x,z) < \delta\}$ consists of disjoint union of intervals with length more than $\delta > 0$ and there can only be finitely many of them in $(-1,1)$.

We also note that for any $\delta > 0$, there exists $0 < \delta_1 < \delta_2$ such that $\{x \in (-1,1) : |m'(x)| > \delta_2\} \subset I_\delta \subset \{x \in (-1,1) : |m'(x)| > \delta_1\}$. Also $\delta_i \to 0$ for $i = 1,2$ as $\delta \to 0$.

Taking the above two observations into account, Theorem 3.1 follows from exactly the same lines as in the proof of [CCL,Thm 1.7 (ii)]. We present them here for completeness only.

It suffices to prove that for any given $\delta > 0$ small, $\tilde{u} \to 0$ uniformly in I_δ. Now fix $\delta > 0$ small, we observe as above that $I_\delta = \cup_{k=1}^{K}(a_k, b_k)$ for some $K \in \mathbf{N}$. Let $x_\alpha \in \bar{I}_\delta$ be such that $\tilde{u}(x_\alpha) = \max_{\bar{I}_\delta} \tilde{u}$.

As first step, we shall prove that for each $\delta > 0$ small, $\tilde{u}$ is bounded in I_δ independent of α. Assume to the contrary, assume $\tilde{u}(x_\alpha) \to \infty$ as $\alpha \to \infty$. Passing to some subsequence if necessary, we may assume that $x_\alpha \to x^*$ and $x_\alpha, x^* \in [a_i, b_i]$ for some $1 \le i \le K$.

Set $x = x_\alpha + y/\alpha$, and define

$$w_\alpha(y) = \frac{\tilde{u}(x_\alpha + y/\alpha)}{\tilde{(}x_\alpha)}.$$

Hence, w_α satisfies $w_\alpha(0) = 1$, $0 < w_\alpha(y) \le 1$, and

$$\frac{d}{dy}\left[d\frac{dw_\alpha}{dy} - m'\left(x_\alpha + \frac{y}{\alpha}\right)w_\alpha\right] + \frac{1}{\alpha^2}w_\alpha\left[p_\alpha\left(x_\alpha + \frac{y}{\alpha}\right) - \tilde{u}(x_\alpha)w_\alpha\right] = 0$$

in $J_\alpha := (-\alpha(x_\alpha - a_i), \alpha(b_i - x_\alpha))$. As $\alpha \to \infty$, passing to a sequence if necessary, J_α converges to some interval J, where J contains one of the following: $(-\infty, +\infty)$, $[0, +\infty)$ or $(-\infty, 0]$.

Claim 3.7. *Given any compact subset K of J, $|w_\alpha|_{C^2(K)}$ is bounded for sufficiently large α.*

To establish our assertion, we first observe that both w_α and $\tilde{u}(x_\alpha)/\alpha$ (Lemma 3.5) are uniformly bounded for large α. Integrating the equation of $\tilde{u}$ from $x = -1$

to $x = x_\alpha$, we have

$$d\tilde{u}'(x_\alpha) - \alpha m'(x_\alpha)\tilde{u}(x_\alpha) + \int_{-1}^{x_\alpha} \tilde{u}(p_\alpha - \tilde{u}) = 0.$$

Hence, $\tilde{u}'(x_\alpha)/(\alpha\tilde{u}(x_\alpha))$ is uniformly bounded for large α. Note that in this proof it suffices to assume that $\tilde{u}(x_\alpha) = \max_{\bar{I}_\delta} \tilde{u}$ is uniformly bounded from below by some positive constant. This implies that $w_\alpha'(0)$ is uniformly bounded since $w_\alpha'(0) = \tilde{u}'(x_\alpha)/(\alpha\tilde{u}(x_\alpha))$. Now integrating the equation of w_α from 0 to y, we find that

$$dw_\alpha'(y) - m'\left(x_\alpha + \frac{y}{\alpha}\right) w_\alpha(y) - dw_\alpha'(0) + m'(x_\alpha)w_\alpha(0)$$
$$+\frac{1}{\alpha^2}\int_{-1}^{y} w_\alpha \left[p_\alpha\left(x_\alpha + \frac{y}{\alpha}\right) - \tilde{u}(x_\alpha)w_\alpha\right] dy = 0$$

Therefore, $|w_\alpha|_{C^1(K)}$ is uniformly bounded for large α. By the equation of w_α, we see that $|w_\alpha|_{C^2(K)}$ is uniformly bounded. This proves our assertion.

By our assertion and a standard diagonal process, passing to a sequence if necessary, we see that $w_\alpha \to w^*$ in $C^1(K)$, where K is any compact subset of J. By the equation of w_α, $w_\alpha \to w^*$ in $C^2(K)$. Hence, w^* satisfies $w^*(0) = 1$ and $0 \leq w^* \leq 1$.

By Lemma 3.6, we have

$$\int_{-1}^{1} \tilde{u}(x)[m'(x)]^2\, dx \leq \frac{C}{\alpha}.$$

Since $|m'| \geq \delta_1$ in $(a_i, b_i) \subset I_\delta$, we have

$$\int_{a_i}^{b_i} \tilde{u}(x)dx \leq \frac{C}{\delta_1^2\alpha}.$$

By the change of variable $x = x_\alpha + y/\alpha$ and the definition of w_α, we obtain

$$\int_{J_\alpha} w_\alpha(y)\, dy \leq \frac{C}{\delta_1^2\tilde{u}(x_\alpha)}.$$

In particular,

$$\int_{J_\alpha\cap(-1,1)} w_\alpha(y)\, dy \leq \frac{C}{\delta_1^2\tilde{u}(x_\alpha)}. \tag{3.1}$$

Passing to the limit in (3.1), by $\tilde{u}(x_\alpha) \to \infty$ we have

$$\int_{J\cap(-1,1)} w^*(y)\,dy \leq 0.$$

This implies that $w^* \equiv 0$ in $J \cap (-1,1)$, which contradicts $w^*(0) = 1$ since $0 \in J \cap (-1,1)$. This proves the fact that for each small $\delta > 0$, $|\tilde{u}|_{L^\infty(I_\delta)}$ is bounded independent of α.

Next we prove that for any $\delta > 0$, $u(x_\alpha) = |\tilde{u}|_{L^\infty(I_\delta)} \to 0$ as $\alpha \to \infty$. Passing to a sequence if necessary, we assume that there exists $\delta > 0$ and $\eta > 0$ such that $\tilde{u}(x_\alpha) \geq \eta$ for sufficiently large α. Set $x = x_\alpha + y/\alpha$ and define $w_\alpha = \tilde{u}(x_\alpha + y/\alpha)$. Hence $w_\alpha(0) \geq \eta$. Passing to a subsequence if necessary, we may assume that $x_\alpha \to x^* \in \bar{I}_\delta$ as $\alpha \to \infty$. We may also assume that $x_\alpha, x^* \in [a_i, b_i]$ for some $1 \leq i \leq K$. By assumption, $m'(0) \geq 0 \geq m'(1)$, so there are only three possibilities:$-1 < a_i, b_i < 1$, $-1 = a_i < b_i < 1$ or $-1 < a_i < b_i = 1$.

We first consider the case when $-1 < a_i < b_i < 1$. For this case, we can find some interval $(c_i, d_i) \in I_{\delta/2}$ such that $[a_i, b_i] \subset (c_i, d_i)$. Then w_α satisfies

$$\frac{d}{dy}\left[d\frac{dw_\alpha}{dy} - m'\left(x_\alpha + \frac{y}{\alpha}\right) w_\alpha\right] + \frac{1}{\alpha^2}\left[p_\alpha\left(x_\alpha + \frac{y}{\alpha}\right) - w_\alpha\right] = 0$$

in $J_\alpha := (-\alpha(x_\alpha - c_i), \alpha(d_i - x_\alpha))$. Since $x_\alpha \in [a_i, b_i] \subset (c_i, d_i)$, we see that J_α converges to $(-\infty, +\infty)$ as $\alpha \to \infty$. By the boundedness of $|\tilde{u}|_{L^\infty(I_{\delta/2})}$ independent of α, w_α is uniformly bounded in J_α. Analogous to the proof of the boundedness of $|\tilde{u}|_{L^\infty(I_{\delta/2})}$, passing to some sequence if necessary, we may assume that $w_\alpha \to w^*$ in $C^2(K)$, where K is any compact subset of $(-\infty, +\infty)$. Hence w^* satisfies $w^*(0) \geq \eta$, $0 \leq w^*(y) \leq C$ in $(-\infty, +\infty)$ and

$$d\frac{d^2w^*}{dy^2} - m'(x^*)\frac{dw^*}{dy} = 0 \quad \text{in } (-\infty, +\infty).$$

Hence, $w^* = c_1 + c_2 e^{m'(x^*)y/d}$ for some constants c_1 and c_2. Since w^* is bounded in $(-\infty, +\infty)$, we see that $c_2 = 0$. This together with $w^*(0) \geq \eta$ implies that $w^* \equiv w^*(0)$ in $(-\infty, +\infty)$.

By Lemma 3.6, we have

$$\int_{c_i}^{d_i} \tilde{u}(x)[m'(x)]^2\,dx \leq \frac{C}{\alpha}.$$

Since for some $\delta_3 > 0$, $|m'| \geq \delta_3$ in $(c_i, d_i) \subseteq I_{\delta/2}$, by the change of variable $x = x_\alpha + y/\alpha$ and the definition of w_α, we obtain

$$\int_{J_\alpha} w_\alpha(y)\, dy \leq \frac{4C}{(\delta_3)^2}.$$

For any $L > 0$, $[-L, L] \subset J_\alpha$ for sufficiently large α. Hence,

$$\int_{-L}^{L} w_\alpha(y)\, dy \leq \frac{4C}{\delta_3^2}.$$

Passing to the limit we find that

$$\int_{-L}^{L} w^*(y)\, dy \leq \frac{4C}{\delta_3^2}.$$

i.e. $2L\eta \leq 4C/\delta^2$ since $w^* \geq \eta$. This is a contradiction, since $L > 0$ is arbitrary.

Next we consider the case when $a_i = -1$ and $b_i < 1$. For this case, if $x^* > -1$, then we can use the same proof as above to reach a contradiction. ($J_\alpha \to (-\infty, +\infty)$ and so w^* is equal to some positive constant.) It remains to consider the case when $x^* = -1$. Since $|m'(x_\alpha)| \geq \delta_1 > 0$ and $x_\alpha \to x^* = -1$, we see that $|m'(-1)| \geq \delta_1$. Since we can assume that $m'(-1) \geq 0$, we have $m'(-1) > 0$. By the same argument as before, we can assume that $w_\alpha \to w^*$ as $\alpha \to \infty$, $w^*(0) \geq \eta$, $0 \leq w^* \leq C$, and w^* satisfies

$$d\frac{d^2 w^*}{dy^2} - m'(-1)\frac{dw^*}{dy} = 0$$

in some interval J which contains $[0, +\infty)$. Hence, $w^* = c_1 + c_2 \exp\{m'(-1)y/d\}$ in $[0, +\infty)$. Since $m'(0) > 0$, $w^*(0) \geq \eta$, $0 \leq w^* \leq C$, the only possibility is that $w^* \equiv w^*(0) \geq \eta$ in $[0, +\infty)$. Then, as in the case when $-1 < a_i < b_i < 1$ (with $[-L, L]$ being replaced by $[0, L]$) and as in the previous case, we can apply Lemma 3.6 to reach a contradiction.

The case when $a_i > -1$ and $b_i = 1$ can be treated similarly. This completes the proof. $\square$

Proof of Theorem 3.3. It suffices to prove Lemma 3.4 and the following lemma.

Lemma 3.8. *Assume* **(H1)** *and* **(H2)**. *Then given a small* $r > 0$, *there exists* *positive constants* C, γ *such that*

$$\tilde{u} \leq \begin{cases} CMe^{\gamma\alpha[m(x)-m(x_0)]/d} & in\ \cup_{x_0\in\mathfrak{M}} B_r(x_0), \\ e^{-\gamma\alpha/d} & in\ \Omega \setminus \cup_{x_0\in\mathfrak{M}} B_r(x_0), \end{cases}$$

where $M = |\tilde{u}|_{L^\infty(\Omega)}$.

By the assumption, all critical points of m are non-degenerate ($\det D^2 m \neq 0$) and the set of local maximum points $\mathfrak{M} \subset \Omega$ lies in the interior, hence

$$\mathfrak{M} = \{x \in \Omega : \nabla m(x) = 0,\ \det D^2 m(x) < 0\} \subset \operatorname{int}\Omega \tag{3.2}$$

Moreover, $\Delta m > 0$ on $\mathfrak{M}_0$ and that $\partial_\nu m \leq 0$ on $\partial\Omega$, where $\mathfrak{M}_0 = \{x \in \bar{\Omega} : |\nabla m| = 0\} \setminus \mathfrak{M}$. Consequently, $\mathfrak{M}$ is finite. Denote

$$\{m(x) : x \in \mathfrak{M}\} = \{m_1, m_2, ..., m_n\},\ m_1 < ... < m_n,\ \mathfrak{M}_i = \{x \in \mathfrak{M} : m(x) = m_i\} \tag{3.3}$$

The non-degeneracy implies that there exists $r > 0$, $K > 0$ such that

$$\frac{1}{K}|z-x|^2 \leq m(z) - m(x) \leq K|\nabla m(x)|^2 \leq K^2|z-x|^2 \quad \forall x \in B_r(z), \forall z \in \mathfrak{M}. \tag{3.4}$$

Set

$$m_0 = \min_{\overline{\Omega}} m, \quad \eta = \min_{1\leq i\leq n}\{m_i - m_{i-1}, r^2/K\}, \tag{3.5}$$

and fix $0 < \delta_1 < 1$, and define recursively

$$\delta_{i+1} = \frac{\delta_i \eta}{m_{i+1} - m_i + \eta}\ ,\ i = 1, 2, ..., n-1 \tag{3.6}$$

Then we have

$$1 > \delta_1 > \delta_2 > ... > \delta_n \equiv \delta^* = \delta_1 \prod_{i=1}^{n-1} \frac{\eta}{m_{i+1} - m_i + \eta} > 0.$$

By possibly enlarging K, defined in (7.2), we also can assume

$$\frac{\delta^*\alpha}{d}|\nabla m|^2 + \Delta m > 0 \quad \text{in } \overline{\Omega \setminus D_1},\ D_1 = \cup_{z\in\mathfrak{M}} \overline{B_{K\sqrt{\frac{d}{\alpha}}}(z)} \tag{3.7}$$

Define

$$\Omega_1 = \Omega, \quad \Omega_{i+1} = \{x \in \Omega : m(x) > m_i - \eta\} \setminus \cup_{z\in\mathfrak{M}_i} \overline{B_r(z)}$$

Note that $\Omega_{i+1} \subset\subset \Omega_i$ from definition of η.

Define

$$M = \sup_{\Omega} \tilde{u}, \quad L = \frac{1}{2}|D^2 m|_\infty K^2, \quad \phi_i = Me^L e^{\frac{\alpha\delta_i}{d}(m(x)-m_i)}. \tag{3.8}$$

Then using (3.7), $\Omega \setminus D_1$ for $i = 1, ..., n$.

$$\begin{aligned} N[\phi] :=& -\nabla \cdot (d\nabla\phi - \alpha\phi\nabla m) - \phi(p - \tilde{u}) \\ \geq& -\nabla \cdot (d\nabla\phi - \alpha\phi\nabla m) - \phi p \\ N[\phi_i] =& \phi_i[\alpha(1-\delta_i)(\frac{\delta_i\alpha}{d}|\nabla m|^2 + \Delta m) - p] \\ \geq& \phi_i[\alpha(1-\delta_i)(\frac{\delta_*\alpha}{d}|\nabla m|^2 + \Delta m) - p] \geq 0 \end{aligned}$$

$$N[\phi_i] \geq 0 \quad \text{in } \Omega \setminus D_1 \qquad \text{and} \qquad d\partial_\nu\phi_i - \alpha\phi_i\partial_\nu m \geq 0 \qquad \text{on } \partial\Omega \tag{3.9}$$

Whereas when $x \in D_1 = \cup_{z\in\mathfrak{M}} B_{K\sqrt{\frac{d}{\alpha}}}(z)$ and by (7.2)

$$m(x) - m_i \geq -\frac{1}{2}|D^2 m|_\infty (K\sqrt{\frac{d}{\alpha}})^2$$

$$\frac{\delta_i\alpha}{d}(m(x) - m_i) \geq -\frac{\alpha}{d}(\frac{1}{2}|D^2 m|_\infty K^2 \frac{d}{\alpha}) = -\frac{1}{2}|D^2 m|_\infty K^2 = -L$$

Hence

$$\phi_i(x) = Me^L e^{\frac{\delta_i\alpha}{d}(m(x)-m_i)} \geq Me^L e^{-L} = M \geq \tilde{u} \text{ in } D_1. \tag{3.10}$$

Hence by (3.9), (3.10) and comparison, $\tilde{u}(x) \leq \phi_1(x)$ in $\Omega_1 \setminus D_1 = \Omega \setminus D_1$. But we also have $\tilde{u}(x) \leq M \leq \phi_1(x)$ in D_1 by (3.10). Hence,

$$\tilde{u}(x) \leq \phi_1(x) = Me^L e^{\frac{\delta_1\alpha}{d}(m(x)-m_1)} \qquad \text{in } \overline{\Omega}.$$

Next we consider ϕ_2 on Ω_2. On $\partial\Omega_2 \setminus \partial\Omega$, we have $m(x) \geq m_1 - \eta$. We either have

$$x \in \cup_{z\in\mathfrak{M}_1} \partial B_r(z) \quad \text{or} \quad (x \notin \cup_{z\in\mathfrak{M}_1} \partial B_r(z) \text{ and } m(x) = m_1 - \eta)$$

Since on $\cup_{z\in\mathfrak{M}_1}\partial B_r(z)$,

$$m(x) \le m_1 - \frac{1}{K}|x-z|^2 = m_1 - \frac{r^2}{K} < m_1 - \eta,$$

we must have

$$x \notin \cup_{z\in\mathfrak{M}_1}\partial B_r(z) \text{ and } m(x) = m_1 - \eta$$

Consequently on $\partial\Omega_2 \setminus \partial\Omega$, by definition of δ_i,

$$\frac{\phi_2}{\phi_1} = \exp\{\frac{\delta_2\alpha}{d}(m(x)-m_2) - \frac{\delta_1\alpha}{d}(m(x)-m_1)\} = 1$$

Hence $\phi_2 = \phi_1 \ge \tilde{u}$ on $\partial\Omega_2 \setminus \partial\Omega$. By (3.9), (3.10) and comparison, much as before we can conclude that $\phi_2 \ge \tilde{u}$ in Ω_2.

Suppose $\phi_i \ge \tilde{u}$ on Ω_i. Then on $\partial\Omega_{i+1} \setminus \partial\Omega$, as before, we must have $m(x) = m_i - \eta$ and $x \notin \cup_{z\in\mathfrak{M}_i} B_r(z)$

$$\frac{\phi_{i+1}}{\phi_i} = \exp\{\frac{\delta_{i+1}\alpha}{d}(m(x)-m_{i+1}) - \frac{\delta_i\alpha}{d}(m(x)-m_i)\} = 1$$

$$\phi_{i+1} = \phi_i \ge \tilde{u} \text{ on } \partial\Omega_{i+1} \setminus \partial\Omega.$$

By(3.9) and (3.10), as before, we conclude that $\phi_{i+1} \ge \tilde{u}$ on Ω_{i+1}.

In conclusion, $\phi_i \ge \tilde{u}$ on Ω_i, $i = 1, ..., n$. Hence

$$\tilde{u} \le Me^L e^{\frac{\delta^*\alpha}{d}(m(x)-m_i)} \quad \text{in } \cup_{z\in\mathfrak{M}_i} B_r(z) \tag{3.11}$$

$$\tilde{u} \le Me^L e^{-\frac{\delta^*\alpha}{dK}r^2} \text{ in } \Omega \setminus \cup_{z\in\mathfrak{M}_i} B_r(z) \tag{3.12}$$

This proves Lemma 3.8. Next we prove Lemma 3.4.

If $\frac{\alpha}{d}$ stays bounded, then consider $w(x) = e^{-\frac{\alpha m(x)}{d}}\tilde{u}$ which satisfies

$$\begin{cases} d\nabla\cdot(e^{\frac{\alpha m}{d}}\nabla w) + e^{\frac{\alpha m}{d}} w(p(x) - e^{\frac{\alpha m}{d}} w) = 0 & \text{in } \Omega, \\ \partial_\nu w = 0 & \text{on } \partial\Omega. \end{cases}$$

Let $x_0 \in \bar{\Omega}$ be such that $w(x_0) = \sup_\Omega w$, then by Hopf boundary lemma and maximum principle, $e^{\frac{\alpha m(x_0)}{d}} w(x_0) \le p(x_0)$

$$\sup_\Omega \tilde{u} \le (\sup_\Omega e^{\frac{\alpha m}{d}})(\sup_\Omega w) \le (\sup_\Omega e^{\frac{\alpha m}{d}}) e^{-\frac{\alpha m(x_0)}{d}} p(x_0) < \infty.$$

Now suppose $\frac{\alpha}{d} \to \infty$. By Theorem 3.3, for some $z \in \Sigma$ and $R > 0$, $M = |\tilde{u}|_{L^\infty(\Omega)}$ is assumed in $B_{R\sqrt{d/\alpha}}(z)$ for all α large. Rescale $x = z + y\sqrt{\frac{d}{\alpha}}$,

$$d(\frac{\alpha}{d}\Delta_y \tilde{u}) - \alpha \nabla_x m(\sqrt{\frac{\alpha}{d}}\nabla_y \tilde{u}) + \tilde{u}(p - \tilde{u} - \alpha\Delta m) = 0$$

Divide by α,

$$\Delta_y \tilde{u} - (\sqrt{\frac{\alpha}{d}}\nabla_x m(z + y\sqrt{\frac{d}{\alpha}})) \cdot \nabla_y \tilde{u} + (\frac{p - \tilde{u} - \alpha\Delta m}{\alpha})\tilde{u} = 0 \qquad \text{in } B_R(0)$$

Hence for each $R > 0$ the coefficients are bounded in $L^\infty(B_R(0))$. (The middle term tends to $D^2 m(z) y$.) So we apply the Harnack's inequality to get a constant $c > 0$ such that

$$cM^2(\frac{d}{\alpha})^{N/2} \le \int_{B_{R\sqrt{\frac{d}{\alpha}}}} \tilde{u}^2 \le \int_\Omega \tilde{u}^2 \qquad \text{in } B_R(0)$$

On the other hand,

$$\int_\Omega \tilde{u}^2 = \int_\Omega \tilde{u} m \le C \int_\Omega \tilde{u} \le CM(\frac{d}{\alpha})^{N/2},$$

by (3.11) and (3.12), this means that M has to be bounded uniformly in $\frac{\alpha}{d} \to \infty$.

Finally, the L^p estimates follows directly from (3.11) and (3.12) and the uniform boundedness of $|\tilde{u}|_{L^\infty(\Omega)}$ in α. This proves Lemma 3.4 and concludes the proof of Theorem 3.3.

□

3.2 Lower estimates of $\tilde{u}$

Now we prove results indicating that the unique steady-state $\tilde{u}$ of (1.6) (resp. $(\tilde{U}, \tilde{V})$ of (1.4)) are nontrivial in the vicinity of $\mathfrak{M}$.

Theorem 3.9. *Assume that $\tilde{u}$ is the unique positive steady-state of* (1.6)*. If there* *exists a closed set* $\Omega_0 \subset\subset \Omega$ *and a positive constant* $\epsilon_0 > 0$ *such that*

$$m(x) = \begin{cases} m_0 & \text{if } x \in \Omega_0, \\ < m_0 & \text{if } x \in \Omega^{\epsilon_0} \setminus \Omega_0. \end{cases}$$

where $\Omega^\epsilon := \{x \in \Omega : \mathrm{dist}(x, \Omega_0) < \epsilon\}$. *Then, for any* $0 < \epsilon < \epsilon_0$

$$\liminf_{\alpha \to \infty} \int_{\Omega_\epsilon} \tilde{u} \geq \int_{\Omega_\epsilon} p(x_0). \tag{3.13}$$

In particular, if x_0 is a strict maximum point (i.e. $\Omega_0 = \{x_0\}$), then we have

Corollary 3.10. *If* $x_0 \in \bar{\Omega}$ *is a strict local maximum point of* p *and* m, *then for any ball* B *centered at* x_0,

$$\liminf_{\alpha \to \infty} \sup_B u \geq p(x_0). \tag{3.14}$$

For the system (1.4), we also have the following

Theorem 3.11. *Under the same assumption on* $m(x)$ *as in Theorem* 3.9. *Let* $x_0 \in \bar{\Omega}$ *be a strict local maximum point of* p *and* m *(which coincides by* **(H(3))***). Assume that* $(\tilde{U}, \tilde{V})$ *is any coexistence state of* (1.4).

$$\liminf_{\alpha \to \infty} \sup_{\Omega_\epsilon} \tilde{U} \geq p(x_0) - \limsup_{\alpha \to \infty} \tilde{V}(x_0). \tag{3.15}$$

The next result is first proved in [BL] for the case when $\mathfrak{M}$ is finite and that $D^2 m$ is invertible at each $x_0 \in \mathfrak{M}$. In the following we make the generalization to the case when $\mathfrak{M}$ is any higher dimensional set (e.g. a curve). In particular, $D^2 m$ is not necessarily invertible at each $x_0 \in \mathfrak{M}$.

Theorem 3.12. *If* $m(x) \in C^2(\bar{\Omega})$ *assumes a local maximum value* M *in a (closed) set* $\Omega_M \subset\subset \Omega$, *i.e. let* $\Omega_M^\epsilon := \{x \in \Omega : \mathrm{dist}(x, \Omega_M) < \epsilon\}$, *we have*

$$m(x) = \begin{cases} M & \text{in } \Omega_M \\ M/2 < \quad < M & \text{in } \Omega_M^\epsilon \setminus \Omega_M \end{cases}$$

Then for all α *large, the unique positive steady-state* $\tilde{u}$ *to* (1.6) *satisfies*

$$\tilde{u}(x) \geq \chi(M) e^{\alpha[m(x) - M]/d} \qquad \forall x \in \Omega_M^{\epsilon/2}$$

where χ *is defined in* **(H(3))**.

Proof of Theorem 3.9. Let $\tilde{u}$ be the unique solution to (1.6), and $\Omega_0, \Omega^\epsilon$ as defined in the statement of Theorem 3.9. Then $\tilde{u}$ is the principal eigenfunction of the following eigenvalue problem with principal eigenvalue 0:

$$\begin{cases} \nabla \cdot (d\nabla\phi - \alpha\phi\nabla m) + (p - \tilde{u})\phi + \lambda\phi = 0 & in \ \ \Omega, \\ d\frac{\partial\phi}{\partial\nu} - \alpha\phi\frac{\partial m}{\partial\nu} = 0 & on \ \ \partial\Omega. \end{cases} \tag{3.16}$$

Now by the transformation $\phi = e^{\alpha m/d}\psi$, (3.16) is equivalent to

$$\begin{cases} \nabla \cdot (de^{\alpha m}\nabla\psi) + (p - \tilde{u})\psi e^{\alpha m/d} + \lambda e^{\alpha m/d}\psi = 0 & in \ \ \Omega, \\ \frac{\partial\psi}{\partial\nu} = 0 & on \ \ \partial\Omega. \end{cases} \tag{3.17}$$

with principal eigenvalue equal to 0. The variational characterization of the principal eigenvalue of (3.17) implies

$$0 = \lambda = \inf_{\psi \in H^1} \left\{ \frac{\int e^{\alpha m/d}(d|\nabla\psi|^2 + (u - p)\psi^2)}{\int e^{\alpha m/d}\psi^2} \right\}$$

By assumption, for any small $\epsilon > 0$, $\max_{\partial\Omega^\epsilon} m < m_0$. For any δ such that $0 < \delta < \min\{m_0 - \max_{\partial\Omega^\epsilon} m, \chi(m_0) - \max_{\partial\Omega^\epsilon} p\}$, define

$$\begin{aligned} M_1 :=& m_0 - \frac{\delta}{3} > m_0 - \frac{2\delta}{3} := M_2, \\ U_1 :=& \{x \in \Omega^{\epsilon_0} : m(x) > m_0 - \frac{\delta}{3}\} \\ U_2 :=& \{x \in \Omega^{\epsilon_0} : m(x) > m_0 - \frac{2\delta}{3}\} \\ U_3 :=& \{x \in \Omega^{\epsilon_0} : m(x) > m_0 - \delta\}. \end{aligned}$$

Note that we have $U_1 \subset\subset U_2 \subset\subset U_3 \subset\subset \Omega^{\epsilon_0}$. Now take a smooth test function ψ such that,

$$\psi(x) = \begin{cases} 1 & \text{if } x \in U_2 \\ 0 & \text{if } x \in \Omega \setminus U_3 \end{cases} \qquad 0 \leq \psi(x) \leq 1 \qquad |\nabla\psi| \leq C(\delta)$$

Then,

$$\begin{aligned}
0 &\leq \frac{\int de^{\alpha m/d}|\nabla\psi|^2 + \int e^{\alpha m/d}(u-p)\psi^2}{\int e^{\alpha m/d}\psi^2} \\
&\leq \frac{\int_{U_3} de^{\alpha M_2/d}C(\delta)^2}{\int_{U_1} e^{\alpha M_1/d}} + \frac{\int_{U_3} e^{\alpha m/d}(u-p)\psi^2}{\int_{U_3} e^{\alpha m/d}\psi^2} \\
&\leq C'(\delta)e^{\alpha(M_2-M_1)} + \max_{\overline{U_3}}(u-p) \\
&\leq C'(\delta)e^{-\frac{\epsilon\alpha}{3}} + \max_{\overline{U_3}} u - \chi(m_0) + \delta.
\end{aligned}$$

For α sufficiently large, the first term in the last line will become less than δ, hence (3.13) follows.

□

Proof of Theorem 3.11. Follows from exactly the same argument as in the proof of Theorem 3.9 □

Proof of Theorem 3.12. Let $p(x) = \chi(m(x))$ and $\bar{\delta} > 0$ be chosen small such that $M - \bar{\delta}$ is a regular value of m and satisfies

$$\sup_{\partial\Omega_M^\epsilon} m < M - \bar{\delta} \qquad \chi(M - \bar{\delta}) > 0.$$

Since $M - \bar{\delta}$ is a regular value of m, consider

$$O_1 = \{x \in \Omega_M^\epsilon : m(x) > M - \bar{\delta}\}$$

then

$$\nabla m = \nu \cdot \frac{\partial m}{\partial \nu} \qquad \text{and} \qquad \frac{\partial m}{\partial \nu} < 0 \qquad \text{on } \partial O_1.$$

Therefore, there exists $\underline{\delta} \in (0, \bar{\delta})$ and $O_2 := \{x \in O_1 : M - \bar{\delta} < m(x) < M - \underline{\delta}\}$ such that

$$m'(x) \neq 0 \quad \text{ in } \bar{O}_2.$$

Define a smooth cut-off $\rho : \mathbf{R} \to \mathbf{R}$ by

$$\rho(t) = \begin{cases} 1 & t \geq M - \underline{\delta} \\ 0 & t \leq M - \bar{\delta} \end{cases}$$

such that

$$\rho, \rho' > 0 \text{ in } (M-\bar{\delta}, M-\underline{\delta}) \qquad \text{and} \qquad \rho'' > 0 \text{ in } \left(M-\bar{\delta}, \frac{2M-\bar{\delta}-\underline{\delta}}{2}\right).$$

And define

$$D(x) := \begin{cases} 1 & \text{in } O_1 \setminus O_2 \\ \rho(m(x)) & \text{in } O_2 \\ 0 & \text{in } \Omega \setminus O_1. \end{cases}$$

and

$$\underline{u}(x) := Me^{\alpha[m(x)-M]/d} D(x)$$

Now we calculate

$$\begin{aligned} & d\nabla \underline{u} - \alpha \underline{u} \nabla m \\ =& Me^{\alpha[m(x)-M]/d} \alpha \nabla m D + dMe^{\alpha[m(x)-M]/d} \nabla D - \alpha Me^{\alpha[m(x)-M]/d} D \nabla m \\ =& dMe^{\alpha[m(x)-M]/d} \nabla D \end{aligned}$$

then,

$$\begin{aligned} & \nabla \cdot (d\nabla \underline{u} - \alpha \underline{u} \nabla m) + \underline{u}(\chi(m) - \underline{u}) \\ =& \nabla \cdot (dMe^{\alpha[m(x)-M]/d} \nabla D) + Me^{\alpha[m(x)-M]/d} D(\chi(m) - \underline{u}) \\ =& \alpha Me^{\alpha[m(x)-M]/d} \nabla D \cdot \nabla m + dMe^{\alpha[m(x)-M]/d} \Delta D + Me^{\alpha[m(x)-M]/d} D(\chi(m) - \underline{u}) \\ =& Me^{\alpha[m(x)-M]/d} \{\alpha \nabla D \cdot \nabla m + d\Delta D + D(\chi(m) - \underline{u})\} \end{aligned}$$

In a neighborhood of the boundary $\partial\Omega$, $\underline{u}$ is identically zero and the boundary condition for lower solution is satisfied automatically. It suffices to show that

$$\alpha \nabla D \cdot \nabla m + d\Delta D + D(\chi(m) - \underline{u}) \geq 0. \tag{3.18}$$

First we notice that

$$g(m) = \frac{e^{\alpha m/d}}{\chi(m)} \qquad \text{with} \qquad g'(m) = e^{\alpha m/d} \left(\frac{\alpha}{d\chi(m)} - \frac{\chi'(m)}{\chi(m)^2} \right)$$

is increasing in $\chi - 1\{[M-\bar{\delta}, M]\}$ if $\alpha > d \sup_{[M-\bar{\delta}, M]} \chi'/\chi(M-\bar{\delta})$. Hence if $m(x) \in$

O_2, then

$$\frac{e^{\alpha m(x)/d}}{\chi(m(x))} \leq \frac{e^{\alpha M/d}}{\chi(M)}$$

which implies

$$p(x) = \chi(m(x)) \geq \chi(M)e^{\alpha[m(x)-M]/d} \geq \chi(M)e^{\alpha[m(x)-M]/d}D(x) = \underline{u}$$

Therefore (7.6) is satisfied automatically when $D \equiv 0, 1$. It suffices to show (7.6) in O_2. Now,

$$\alpha\nabla D \cdot \nabla m = \alpha\rho'|\nabla m|^2 \geq 0 \qquad \Delta D = \rho''|\nabla m|^2 + \rho'\Delta m$$

Choose $\delta' \in (\underline{\delta}, \bar{\delta})$ such that $\inf_{M-\delta' \leq m \leq M-\underline{\delta}} \rho > 1/2$ and

$$\sup_{M-\delta' \leq m \leq M-\underline{\delta}} \rho'|\Delta m| < \frac{\chi(M-\bar{\delta})}{6d} \qquad \sup_{M-\delta' \leq m \leq M-\underline{\delta}} |\rho''||\nabla m| < \frac{\chi(M-\bar{\delta})}{6d}$$

Then in $\{x \in O_2 : M - \delta' \leq m \leq M - \underline{\delta}\}$, since $0 \leq \frac{1}{2}\underline{u} \leq \frac{1}{2}Me^{-\alpha\underline{\delta}/d} \to 0$

$$\begin{aligned}
&\alpha\nabla D \cdot \nabla m + d\Delta D + D(\chi(m) - \underline{u}) \\
\geq& 0 + d\rho''|\nabla m|^2 + d\rho'\Delta m + \frac{1}{2}(\chi(M-\bar{\delta}) - \overline{u}) \\
\geq& \left[\frac{\chi(M-\bar{\delta})}{6} - d|\rho''||\nabla m|^2\right] + \left[\frac{\chi(M-\bar{\delta})}{6} - d\rho'|\Delta m|\right] + \left[\frac{\chi(M-\bar{\delta})}{6} - \frac{1}{2}\underline{u}\right] \\
\geq& 0
\end{aligned}$$

for all α large.

For $\{x \in O_2 : M - \bar{\delta} \leq m \leq M - \delta'\}$, we have

$$\frac{\alpha\rho'}{2} + d\rho'' \geq 0 \qquad \text{and} \qquad \frac{\alpha|\nabla m|^2}{2} + d\Delta m \geq 0$$

for α large. Then

$$\begin{aligned}
&\alpha\nabla D\cdot\nabla m + d\Delta D + D(m-\underline{u})\\
\geq&\alpha\rho'|\nabla m|^2 + d\rho''|\nabla m|^2 + d\rho'\Delta m + 0\\
=&|\nabla m|^2\left(\frac{\alpha\rho'}{2} + d\rho''\right) + \rho'\left(\frac{\alpha}{2}|\nabla m|^2 + d\Delta m\right)\\
\geq&0
\end{aligned}$$

for α large. $\square$

Chapter 4

Existence and Stability Properties of Coexistence Steady-State $(\tilde{U}, \tilde{V})$

Consider the steady-state system of (1.4).

$$\begin{cases} \nabla \cdot (d_1 \nabla U - \alpha \nabla m) + U(p - U - V) = 0 & \text{in } \Omega, \\ d_2 \Delta V + V(p - U - V) = 0 & \text{in } \Omega, \\ d_1 \frac{\partial U}{\partial \nu} - \alpha U \frac{\partial m}{\partial \nu} = \frac{\partial V}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

We call a solution $(\tilde{U}, \tilde{V})$ of (4.1) a coexistence state if $\tilde{U}(x) > 0$ and $\tilde{V}(x) > 0$ in $\bar{\Omega}$. In this chapter we are going to prove that there exists at least one coexistence state for (4.1), which follows from the arguments in [CCL2]. We present the proof here for the sake of completeness. Denote by $(U(x, t; U_0), V(x, t; V_0))$ a solution of (4.1) with initial condition $(U_0(x), V_0(x))$.

Theorem 4.1. *Assume* **(H1)**, **(H2)** *and* **(H3)** *are satisfied, then for* α *sufficiently large, there exists coexistence states* $(\tilde{U}_i, \tilde{V}_i)$, $i = 1, 2$ *of* (4.1) *such that* $U_1 \leq U_2$, $V_1 \geq V_2$ *and the set* $X_0 := \{(U, V) : U_1 \leq U \leq U_2 \text{ and } V_1 \geq V \geq V_2\}$ *is globally attracting, i.e. given any initial condition* $(U_0(x), V_0(x))$,

$$\operatorname{dist}\big((U(x, t; U_0), V(x, t; V_0)), X_0\big) \to 0 \qquad \text{as } t \to \infty.$$

Moreover, there exists at least one stable coexistence state $(\tilde{U}, \tilde{V}) \in X_0$ *of* (4.1).

Remark 4.2. (i) Alternatively, the assumptions **(H1)** and **(H2)** can be replaced by "The set of critical points of m has measure zero".

(ii) In Chapter 5 and Chapter 7 we are going to prove that $(\tilde{U}_i, \tilde{V}_i)$, $i = 1, 2$ have a common limiting profile as $\alpha \to \infty$. Moreover, in the special case when m is constant on $\mathfrak{M}$, we are going to prove in Chapter 8 that $(\tilde{U}_1, \tilde{V}_1) = (\tilde{U}_2, \tilde{V}_2)$. That is, there exists a globally asymptotically stable coexistence steady-state for (1.4).

Proof of Theorem 4.1. By the transformation $W(x) = e^{-\alpha m(x)/d_1} U(x)$, (1.4) becomes

$$\begin{cases} W_t = e^{-\alpha m(x)/d_1} \nabla \cdot (d_1 e^{\alpha m(x)/d_1} \nabla W) + W(p - e^{\alpha m(x)/d_1} W - V) & \text{in } \Omega \times (0, \infty), \\ V_t = d_2 \Delta V + V(p - e^{\alpha m(x)/d_1} W - V) & \text{in } \Omega \times (0, \infty), \\ \frac{\partial W}{\partial \nu} = \frac{\partial V}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0, \infty), \end{cases} \tag{4.2}$$

which is a monotone dynamical system. By the theory of monotone dynamical system [H], it suffices to show the instability of the semitrivial steady-states $(e^{-\alpha m(x)/d_1} \tilde{u}, 0)$ and $(0, \bar{\theta}_{d_2})$ of (4.2) which is equivalent to the instability of $(\tilde{u}, 0)$ and $(0, \bar{\theta}_{d_2})$ of (1.4). (Here $\tilde{u}$ is the unique positive steady-state of (1.6) whose existence is proved in Theorem 2.1, and $\bar{\theta}_{d_2}$ is the unique positive solution of (1.5).)

First we consider the linear instability of $(\tilde{u}, 0)$, determined by the following eigenvalue problem:

$$\begin{cases} \nabla \cdot (d_1 \nabla \phi - \alpha \phi \nabla m) + (p - 2\tilde{u})\phi - \tilde{u}\psi + \lambda \phi = 0 & \text{in } \Omega, \\ d_2 \Delta \psi + (p - \tilde{u})\psi + \lambda \psi = 0 & \text{in } \Omega, \\ d_1 \frac{\partial \phi}{\partial \nu} - \alpha \phi \frac{\partial m}{\partial \nu} = \frac{\partial \psi}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.3}$$

Since (4.3) decouples, and $\nabla \cdot (d_1 \nabla - \alpha \nabla m) + (p - 2\tilde{u})$ is invertible, it suffices to show that the principal eigenvalue σ_1 of the second equation of (4.3)

$$\begin{cases} d_2 \Delta \psi_1 + (p - \tilde{u})\psi_1 + \sigma_1 \psi_1 = 0 & \text{in } \Omega, \\ \frac{\partial \psi_1}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.4}$$

is negative, where ψ_1 is the corresponding eigenfunction. But if we divide the equation by ψ_1 and integrate over Ω, we have

$$\sigma_1 = -d_2 \int_\Omega \frac{|\psi_1|^2}{\psi^2}\, dx - \int_\Omega p - \tilde{u}\, dx < -\int_\Omega p - \tilde{u}\, dx \to -\int_\Omega p\, dx < 0$$

as $\alpha \to \infty$. Since $\int_\Omega \tilde{u}\, dx \to 0$ as $\alpha \to \infty$ by Lemma 3.4. Therefore $(\tilde{u}, 0)$ is unstable for α large.

Next we linearize (1.4) at $(0, \bar{\theta}_{d_2})$ and consider the following eigenvalue problem:

$$\begin{cases} \nabla(d_1 \nabla\phi - \alpha\phi\nabla m) + (p - \bar{\theta}_{d_2})\phi + \lambda\phi = 0 & \text{in } \Omega, \\ d_2\Delta\psi - \bar{\theta}_{d_2}\phi + (p - 2\bar{\theta}_{d_2})\psi + \lambda\psi = 0 & \text{in } \Omega, \\ d_1 \frac{\partial\phi}{\partial\nu} - \alpha\phi\frac{\partial m}{\partial\nu} = \frac{\partial\psi}{\partial\nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.5}$$

To show that $(0, \bar{\theta}_{d_2})$ is unstable, again since $d_2\Delta + (p - 2\bar{\theta}_{d_2})$ is invertible, it suffices to show that the principal eigenvalue μ_1 of the first equation of (4.5)

$$\begin{cases} \nabla(d_1 \nabla\phi - \alpha\phi\nabla m) + (p - \bar{\theta}_{d_2})\phi + \mu\phi = 0 & \text{in } \Omega, \\ d_1 \frac{\partial\phi}{\partial\nu} - \alpha\phi\frac{\partial m}{\partial\nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.6}$$

is negative. By the transformation $\varphi = e^{-\alpha m/d_1}\phi$, (4.6) becomes

$$\begin{cases} \nabla(d_1 e^{\alpha m/d_1} \nabla\varphi) + e^{\alpha m/d_1}(p - \bar{\theta}_{d_2})\varphi + \mu e^{\alpha m/d_1}\varphi = 0 & \text{in } \Omega, \\ \frac{\partial\varphi}{\partial\nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.7}$$

Therefore by variational characterization,

$$\mu_1 = \inf_{\varphi \in H^1(\Omega)} \left\{ \frac{\int_\Omega e^{\alpha m/d_1}[d_1|\nabla\varphi|^2 + (\bar{\theta}_{d_2} - p)\varphi^2]\,dx}{\int_\Omega e^{\alpha m/d_1}\varphi^2\,dx} \right\}.$$

By the maximum principle, we have $|p|_{L^\infty(\Omega)} > |\bar{\theta}_{d_2}|_{L^\infty(\Omega)} + 3\delta$ for some small positive constants δ. Now take a smooth cut-off function φ such that $0 \le \varphi \le 1$ and

$$\varphi = \begin{cases} 1 & \text{in } \{x \in \Omega : m(x) \ge |m|_{L^\infty(\Omega)} - 2\delta \text{ and } p(x) \ge |p|_{L^\infty(\Omega)} - 2\delta\}, \\ 0 & \text{in } \{x \in \Omega : m(x) \le |m|_{L^\infty(\Omega)} - 3\delta \text{ or } p(x) \le |p|_{L^\infty(\Omega)} - 3\delta\}. \end{cases}$$

Then

$$\begin{aligned} \mu_1 &\le \frac{\int_\Omega e^{\alpha m/d_1}[d_1|\nabla\varphi|^2 + (\bar{\theta}_{d_2} - p)\varphi^2]\,dx}{\int_\Omega e^{\alpha m/d_1}\varphi^2\,dx} \\ &\le C\frac{e^{\alpha(m(x_0)-2\delta)/d_1}}{e^{\alpha(m(x_0)-\delta)/d_1}} + \sup_{\operatorname{supp}\varphi}(\bar{\theta}_{d_2} - p) \\ &\le Ce^{-\delta\alpha/d_1} + |\bar{\theta}_{d_2}|_{L^\infty(\Omega)} - |p|_{L^\infty(\Omega)} + 3\delta \\ &\to |\bar{\theta}_{d_2}|_{L^\infty(\Omega)} - (|p|_{L^\infty(\Omega)} - 3\delta) < 0 \qquad \text{as } \alpha \to \infty. \end{aligned}$$

Therefore the principal eigenvalue μ_1 of (4.7) and hence of (4.6) is negative for all α

large. Hence $(0, \bar{\theta}_{d_2})$ is unstable for α large. This completes the proof.

$\square$

Chapter 5

Limiting Profile: One-Dimensional Case

In this chapter we shall treat (1.1) in one space dimension (i.e. $\Omega = (-1, 1)$):

$$\begin{cases} U_t = (d_1 U' - \alpha U m')' + U(m - U - V) & \text{in } (-1,1) \times (0,\infty), \\ V_t = d_2 V'' + V(m - U - V) & \text{in } (-1,1) \times (0,\infty), \\ d_1 U' - \alpha U m' = 0 = V' & \text{on } \{-1,1\} \times (0,\infty). \end{cases} \tag{5.1}$$

The steady-states of (5.1) satisfies

$$\begin{cases} (d_1 U' - \alpha U m')' + U(m - U - V) = 0 & \text{in } (-1,1), \\ d_2 V'' + V(m - U - V) = 0 & \text{in } (-1,1), \\ d_1 U' - \alpha U m' = 0 = V' & \text{at } x = \pm 1. \end{cases} \tag{5.2}$$

This chapter is self-contained and part of it is published in [LN]. In this chapter we will not only treat the generic case when m has discrete local maximum points, we are also going to consider the case when m assumes its maximum over an interval. It is revealed in some cases that *U can actually drive V to extinction*. The methods and arguments involved are elementary, but most of them cannot be generalized in an obvious manner to treat the multi-dimensional case. The rest of the thesis is independent of this chapter.

5.1 $\mathfrak{M}_+$ is discrete

Let $\mathfrak{M}_+ = \{x \in \bar{\Omega} : x$ is a local maximum point of m and $m(x) > 0\}$. We will assume throughout the rest of section 5.1 that $\Omega = (-1, 1)$ and that $m(x)$ satisfies the following conditions:

(M1) $m(x) \in C^3([-1, 1])$ and $xm'(x) \le 0$ at $x = \pm 1$.

(M2) $\mathfrak{M}_+ \subseteq (-1, 1)$ and all critical points of m are nondegenerate.

(M3) $\int_\Omega m > 0$.

Note that **(M2)** implies $m(x)$ has only a finite number of local maximum points.By Theorem 4.1, (5.2) has at least one coexistence state $(\tilde{U}, \tilde{V})$. Our main result for (5.2) now reads as follows.

Theorem 5.1. *Let* $(\tilde{U}, \tilde{V})$ *be a positive solution of* (5.2). *Then, as* $\alpha \to \infty$, *it holds that*

(i) $\tilde{V} \to \theta_{d_2}$ *in* $C^{1,\beta}$*;*

(ii) for any $x_0 \in \mathfrak{M}_+$ *and any* $r > 0$ *small,*

$$|\tilde{U}(x) - \max\{\sqrt{2}\,[m(x_0) - \theta_{d_2}(x_0)]\,, 0\}e^{\alpha[m(x)-m(x_0)]/d_1}|_{L^\infty(x_0-r,x_0+r)} \to 0;$$

(iii) for any neighborhood $\mathfrak{N}$ *of* $\mathfrak{M}_+$, $\tilde{U} \to 0$ *in* $(-1, 1)\backslash\mathfrak{N}$ *uniformly and exponentially.*

From Theorem 5.1 we see that not only the peaks of U are located, the profiles of U for large α near its concentrations are also determined. In particular, we have proved that $|\tilde{U}|_{L^\infty}$ remains uniformly bounded in α. It is noteworthy that the L^∞ bound for the higher dimensional case is proved along the same spirit as the one-dimensional case. However, it is not a direct generalization and several non-trivial issues has to be taken care of.

By way of proving Theorem 5.1, we first consider the following closely related single equation which was proposed in [BC] to model the population dynamics of a single species

$$\begin{cases} u_t = (du' - \alpha u m')' + u(m - u) = 0 & \text{in } (-1, 1) \times (0, \infty), \\ du' - \alpha u m' = 0 & \text{on } \{-1, 1\} \times (0, \infty), \end{cases} \tag{5.3}$$

whose steady-state equation is given by

$$\begin{cases} (du' - \alpha um')' + u(m-u) = 0 & \text{in } (-1,1), \\ du' - \alpha um' = 0 & \text{at } x = \pm 1, \end{cases} \tag{5.4}$$

By Theorem 2.1, there exists a unique positive steady-state $\tilde{u}$ of (5.3). We have the following limiting profile of $\tilde{u}$.

Theorem 5.2. *Let $\tilde{u}$ be the unique positive steady-state of* (5.4). *Then, for any $r > 0$ small and any $x_0 \in \mathfrak{M}_+$, as $\alpha \to \infty$,*

(i) $\tilde{u} \to 0$ uniformly and exponentially in $(-1,1) \setminus \cup_{x_0 \in \mathfrak{M}_+} (x_0 - r, x_0 + r)$,

(ii) $|\tilde{u} - \sqrt{2} m(x_0) e^{\alpha[m(x) - m(x_0)]/d}|_{L^\infty(x_0 - r, x_0 + r)} \to 0$.

5.1.1 Proof of Theorem 5.2.

In this section, we will prove Theorem 5.2. First, we recall the following facts about (5.4).

Theorem 5.3. *Suppose that m satisfies* **(M1),(M2) and (M3)**. *Then the following statements hold.*

(i) $\tilde{u} \to 0$ in $L^2(-1,1)$ as $\alpha \to \infty$.

(ii) $|\tilde{u}|_{L^\infty} \le |m|_{L^\infty} + \alpha |\Delta m|_{L^\infty}$.

(iii) For each $x_0 \in \mathfrak{M}_+$ and any $r > 0$,

$$\liminf_{\alpha \to \infty} \left(\max_{B_r(x_0)} \tilde{u} \right) \ge m(x_0).$$

(iv) For each neighborhood $\mathfrak{N}$ of $\mathfrak{M}_+$, there exists $b > 0$ such that $0 \le \tilde{u} \le e^{-b\alpha}$ in $(-1,1) \setminus \mathfrak{N}$.

Proof. Parts (i) and (ii) are proved in Theorem 3.5 and Lemma 3.3 of [CCL2]. Parts (iii) and (iv) are established in Theorems 1.4 and 1.5 of [L1]. □

To analyze (5.4), we first integrate (5.4) from -1 to x,

$$d\tilde{u}'(x) - \alpha \tilde{u}(x) m'(x) + \int_{-1}^{x} \tilde{u}(m - \tilde{u}) = 0,$$

i.e.

$$(d\ln\tilde{u})' = d\frac{\tilde{u}'}{\tilde{u}} = \alpha m' - \frac{1}{\tilde{u}}\int_{-1}^{x}\tilde{u}(m-\tilde{u}). \tag{5.5}$$

Hence, for any x, $x_\alpha \in (-1,1)$ we have

$$\ln\tilde{u}(x) - \ln\tilde{u}(x_\alpha) = \alpha[m(x)-m(x_\alpha)]/d - \int_{x_\alpha}^{x}\frac{1}{d\tilde{u}(z)}\left(\int_{-1}^{z}\tilde{u}(m-\tilde{u})\right)dz,$$

and we have derived the following basic formula which we will use repeatedly in this section:

$$\frac{\tilde{u}(x)}{\tilde{u}(x_\alpha)} = \exp\left\{\alpha\left[m(x)-m(x_\alpha)\right]/d - \int_{x_\alpha}^{x}\frac{1}{d\tilde{u}(z)}\left(\int_{-1}^{z}\tilde{u}(m-\tilde{u})\right)dz\right\}. \tag{5.6}$$

We first estimate the integral in (5.6).

Lemma 5.4. *There exists a constant $C>0$ independent of α, such that*

$$\left|\int_{-1}^{z}\tilde{u}(m-\tilde{u})\right| \le C|\tilde{u}|_{L^2(-1,1)}$$

for all $z\in(-1,1)$ whenever $\tilde{u}$ exists.

Proof. By integrating (5.4), we have $\int_{-1}^{1}\tilde{u}(m-\tilde{u})dx = 0$ and hence $|\tilde{u}|_{L^2} \le |m|_{L^2}$. Now,

$$\left|\int_{-1}^{z}\tilde{u}(m-\tilde{u})\right| \le |m|_{L^2}|\tilde{u}|_{L^2} + |\tilde{u}|_{L^2}^2 \le (2|m|_{L^2})|\tilde{u}|_{L^2}.$$

□

As m has only a finite number of nondegenerate interior local maximum points, there exist a small positive constant ϵ_0 and a positive constant C_0 such that $m'' < -C_0$ and $m>0$ on

$$\mathfrak{N} \equiv \cup_{x_0\in\mathfrak{M}_+}(x_0-\epsilon_0, x_0+\epsilon_0).$$

From Part (iv) of Theorem 5.3, we have $\tilde{u} \le e^{-b\alpha}$ on $\Omega\backslash\mathfrak{N}$ for some constant $b>0$. Hence, if we set $\delta_1 = \alpha^{-\frac{17}{32}}$ and $\delta_2 = \alpha^{-\frac{1}{4}}$, we have, for $i=1,2$,

$$I'_{\delta_i} \equiv \{x\in\Omega \mid \tilde{u}(x) > \delta_i\} \subseteq \mathfrak{N}$$

for all large α. Note that $I'_{\delta_1} \supseteq I'_{\delta_2}$.

Proposition 5.5. *For each* $x_0 \in \mathfrak{M}_+$*, and* $i = 1, 2$*,* $I'_{\delta_i} \cap (x_0 - \epsilon_0, x_0 + \epsilon_0)$ *is nonempty and connected, for* α *large. In other words,* I'_{δ_i} *consists of exactly* $\#\mathfrak{M}_+$ *disjoint intervals for* α *large.*

Proof. To prove the connectedness by contradiction, suppose that there are at least two connected components of $I'_{\delta_i} \cap (x_0 - \epsilon_0, x_0 + \epsilon_0)$. Then, there exists a local minimum point $\bar{x} \in (x_0 - \epsilon_0, x_0 + \epsilon_0)$ such that $\tilde{u}(\bar{x}) \leq \delta_i$, $\tilde{u}'(\bar{x}) = 0$ and $\tilde{u}''(\bar{x}) \geq 0$. Writing (5.4) as

$$d\tilde{u}'' - \alpha m' \tilde{u}' + \tilde{u}(m - \tilde{u} - \alpha m'') = 0,$$

we see that,

$$u(\bar{x}) \geq m(\bar{x}) - \alpha m''(\bar{x}) \geq \inf_{\Omega} m + \alpha C_0 \geq 1 > \delta_i,$$

for α sufficiently large, a contradiction. $I'_{\delta_i} \cap (x_0 - \epsilon_0, x_0 + \epsilon_0)$ being nonempty is a consequence of Theorem 5.3 (iii). $\square$

Now fix $x_0 \in \mathfrak{C}_+$ and let $x_\alpha \in I'_{\delta_i} \cap (x_0 - \epsilon_0, x_0 + \epsilon_0)$ be a maximum point of $\tilde{u}$ in $I'_{\delta_i}(x_0) \equiv I'_{\delta_i} \cap (x_0 - \epsilon_0, x_0 + \epsilon_0)$; i.e.

$$\tilde{u}(x_\alpha) = \max\{\tilde{u}(x) \mid x \in I'_{\delta_i} \cap (x_0 - \epsilon_0, x_0 + \epsilon_0)\} \tag{5.7}$$

Observe that x_α does not depend on $i = 1, 2$, by Part (iii) of Theorem 5.3. To estimate the location of x_α, we deduce by (5.5) that for α large,

$$\begin{aligned} |m'(x_\alpha)| &= \frac{1}{\alpha \tilde{u}(x_\alpha)} \left| \int_{-1}^{x_\alpha} \tilde{u}(m - \tilde{u}) \right| \\ &\leq \frac{C}{\alpha} |\tilde{u}|_{L^2} \end{aligned}$$

by Lemma 5.4 and Theorem 5.3 (iii). Since

$$x_\alpha \in (x_0 - \epsilon_0, x_0 + \epsilon_0),\ m''(x_0) < -C_0,\ m'(x_0) = 0,$$

Mean Value Theorem implies that

$$C_0 |x_\alpha - x_0| \leq m'(x_\alpha) \leq \frac{C}{\alpha} |\tilde{u}|_{L^2} = o(\frac{1}{\alpha}) \tag{5.8}$$

by Theorem 5.3 (ii).

Next, we turn to estimating $|I'_{\delta_2}|$.

Lemma 5.6. *For any $M > 0$ and any $x_0 \in \mathfrak{M}_+$, both $(x_0 - \frac{M}{\sqrt{\alpha}}, x_0 + \frac{M}{\sqrt{\alpha}})$ and $(x_\alpha - \frac{M}{\sqrt{\alpha}}, x_\alpha + \frac{M}{\sqrt{\alpha}})$ are contained in I'_{δ_2} for α large.*

Proof. To prove this by contradiction, suppose that there exist $M_0 > 0$ and a sequence $\alpha_j \to \infty$ with $z_{\alpha_j} \in \partial I'_{\delta_2}$, such that

$$|z_{\alpha_j} - x_{\alpha_j}| \leq \frac{M_0}{\sqrt{\alpha_j}}.$$

then (5.8) implies that

$$|z_{\alpha_j} - x_0| \leq \frac{(1 + M_0)}{\sqrt{\alpha_j}}.$$

From (5.6) and (5.8) it follows that

$$\begin{aligned}
\frac{\tilde{u}(z_{\alpha_j})}{\tilde{u}(x_{\alpha_j})} &= \exp\left\{\alpha_j[m(z_{\alpha_j}) - m(x_{\alpha_j})]/d - \int_{x_{\alpha_j}}^{z_{\alpha_j}} \frac{1}{d\tilde{u}(z)} \left(\int_{-1}^{z} \tilde{u}(m - \tilde{u})\right) dz\right\} \\
&\geq \exp\left\{\alpha_j \left[m(z_{\alpha_j}) - m(x_0) + m(x_0) - m(x_{\alpha_j})\right]/d - \int_{x_{\alpha_j}}^{z_{\alpha_j}} \frac{C}{d\tilde{u}(z)} |\tilde{u}|_{L^2}\right\} \\
&\geq \exp\left\{\alpha_j[m(z_{\alpha_j}) - m(x_0) + O(|x_{\alpha_j} - x_0|^2)]/d - \frac{C}{d\delta_2} |z_{\alpha_j} - x_{\alpha_j}||\tilde{u}|_{L^2}\right\} \\
&= \exp\left\{\alpha_j[\frac{1}{2}m''(x_0)|z_{\alpha_j} - x_0|^2 + o(\frac{1}{\alpha_j^2})]/d - C\alpha_j^{\frac{1}{4d}} \frac{M_0}{d\sqrt{\alpha_j}} |\tilde{u}|_{L^2}\right\} \\
&\geq \exp\left\{\alpha_j \left[\frac{1}{2}m''(x_0)\frac{(M_0 + 1)}{d\alpha_j}\right] - o(\alpha_j^{-\frac{1}{4}})\right\} \\
&\to \exp[\frac{1}{2}m''(x_0)(M_0 + 1)^2/d] > 0
\end{aligned}$$

as $\alpha_j \to \infty$.

On the other hand, $\frac{\tilde{u}(z_{\alpha_j})}{\tilde{u}(x_{\alpha_j})} \to 0$ as $\alpha_j \to \infty$ since $\tilde{u}(x_{\alpha_j}) \geq \frac{m(x_0)}{2} > 0$ for α large and $\tilde{u}(z_{\alpha_j}) = \delta_2 \to 0$, a contradiction. Thus $(x_\alpha - \frac{M}{\sqrt{\alpha}}, x_\alpha + \frac{M}{\sqrt{\alpha}}) \subseteq I'_{\delta_2}$. The fact that $(x_0 - \frac{M}{\sqrt{\alpha}},\ x_0 + \frac{M}{\sqrt{\alpha}}) \subseteq I'_{\delta_2}$ for α large now follows from (5.8). □

Now we come to the upper estimate of $|I'_{\delta_2}|$.

Proposition 5.7. *For α large $|I'_{\delta_2}| = o(\frac{1}{\alpha^c})$ for any $0 < c < \frac{1}{2}$. In particular, $|I'_{\delta_2}| = o(\frac{1}{\alpha^{1/3}})$.*

Proof. Fix $\frac{1}{4} < c < \frac{1}{2}$. Suppose the assertion is false. Then for some $x_0 \in \mathfrak{M}_+$ there is a sequence $\alpha_j \to \infty$ such that for each j, there exists $z_{\alpha_j} \in I'_{\delta_2}(x_0)$ with

$|z_{\alpha_j} - x_0| = \frac{k_1}{\alpha_j^c}$, for some constant $k_1 > 0$. From (5.6) and (5.8) it follows that (for simplicity we suppress the subindex j).

$$\begin{aligned}\frac{\tilde{u}(z_\alpha)}{\tilde{u}(x_\alpha)} &\le \exp\{\alpha[m(z_\alpha) - m(x_0) + m(x_0) - m(x_\alpha)]/d \\ &\qquad + \frac{C}{\delta_2}|z_\alpha - x_\alpha||\tilde{u}|_{L^2}/d\} \\ &\le \exp[-\alpha k_2 |z_\alpha - x_0|^2/d + o(1) + Ck_1\alpha^{\frac{1}{4d}-c}|\tilde{u}|_{L^2}] \\ &\le \exp(-k_3\alpha^{1-2c}/d)\end{aligned}$$

for α large, where k_2, k_3 are two positive constants.

On the other hand, from Theorem 5.3 (ii) we have

$$\frac{\tilde{u}(z_\alpha)}{\tilde{u}(x_\alpha)} \ge \frac{\delta_2}{|\tilde{u}|_{L^\infty}} \ge \frac{\delta_2}{|m|_{L^\infty} + \alpha|\Delta m|_{L^\infty}} \ge \frac{k_4}{\alpha^{5/4}}$$

for some constant $k_4 > 0$, a contradiction. □

Theorem 5.8.

$$\frac{\tilde{u}(x)}{\tilde{u}(x_\alpha)} \exp\left[-\frac{\alpha}{2d} m''(x_0)(x - x_0)^2\right] \to 1$$

uniformly in $I'_{\delta_2}(x_0)$ *for each* $x_0 \in \mathfrak{M}_+$ *as* $\alpha \to \infty$. *In particular,*

$$\frac{1}{2}\tilde{u}(x_\alpha)e^{\frac{\alpha}{2d}m''(x_0)(x-x_0)^2} \le \tilde{u}(x) \le 2\tilde{u}(x_\alpha)e^{\frac{\alpha}{2d}m''(x_0)(x-x_0)^2}$$

in $I'_{\delta_2}(x_0)$ *for all* α *large.*

Proof. By (5.6) again we have, for $x \in I'_{\delta_2}(x_0)$,

$$\begin{aligned}&\left|\frac{\tilde{u}(x)}{\tilde{u}(x_\alpha)} \exp\left[-\frac{\alpha}{2d} m''(x_0)(x - x_0)^2\right] - 1\right| \\ &= \left|\exp\left\{\alpha[m(x) - m(x_\alpha)]/d - \int_{x_\alpha}^{x} \frac{\int_1^z \tilde{u}(m - \tilde{u})}{d\tilde{u}(z)} dz - \frac{\alpha}{2d} m''(x_0)(x - x_0)^2\right\} - 1\right| \\ &= |g_1(x) - g_2(x)| \exp \xi(x)\end{aligned}$$

where

$$g_1(x) = \alpha[m(x) - m(x_\alpha)]/d - \frac{\alpha}{2d} m''(x_0)(x - x_0)^2 \tag{5.9}$$

$$g_2(x) = \int_{x_\alpha}^{x} \frac{1}{d\tilde{u}(z)} \left(\int_{-1}^{z} \tilde{u}(m - \tilde{u})\right) dz \tag{5.10}$$

and $\xi(x)$ lies in between 0 and $g_1(x) - g_2(x)$. Now, our assertion follows from the following observations:

$$\begin{aligned} |g_1(x)| &\le \frac{\alpha}{d}\left|m(x) - m(x_0) - \frac{1}{2}m''(x_0)(x-x_0)^2\right| + \frac{\alpha}{d}\left|m(x_0) - m(x_\alpha)\right| \\ &\le \alpha \cdot O(|x-x_0|^3) + \alpha \cdot O(|x_0 - x_\alpha|^2) \to 0. \end{aligned}$$

by (5.8) and Proposition 5.7, and

$$\begin{aligned} |g_2(x)| &\le \frac{C}{\delta_2}|x - x_\alpha||\tilde{u}|_{L^2} \\ &\le C\alpha^{\frac{1}{4}}|I'_{\delta_2}| \cdot |\tilde{u}|_{L^2} \to 0. \end{aligned}$$

by Proposition 5.7 and Theorem 5.3 (i). □

Eventually we will show that $|\tilde{u}|_{L^\infty}$ is uniformly bounded for all α large. The following is the first step.

Lemma 5.9. $|\tilde{u}|^2_{L^\infty} \le C\sqrt{\alpha}\int_\Omega \tilde{u}^2$ *for α large. In particular,* $|\tilde{u}|_{L^\infty} = o(\alpha^{\frac{1}{4}})$ *for α large.*

Proof.

$$\begin{aligned} \int_\Omega \tilde{u}^2 &\ge \int_{I'_{\delta_2}(x_0)} \tilde{u}^2 \\ &\ge \frac{1}{4}\tilde{u}^2(x_\alpha)\int_{I'_{\delta_2}(x_0)} \exp[\alpha m''(x_0)(x-x_0)^2/d]dx \\ &\ge \frac{1}{4}\tilde{u}^2(x_\alpha)\int_{-M}^{M} \exp(m''(x_0)y^2/d)dy \cdot \frac{1}{\sqrt{\alpha}} \\ &\ge \frac{C}{\sqrt{\alpha}}\tilde{u}^2(x_\alpha). \end{aligned}$$

for any $M > 0$, by Theorem 5.8 and Lemma 5.6, where $y = \sqrt{\alpha}(x - x_0)$. Our assertion now follows from Theorem 5.3 (i). □

Lemma 5.10. $\int_\Omega \tilde{u}^2 = O(\alpha^{-\frac{1}{4}})$ *for α large.*

Proof. From (5.4) and Theorem 5.8 we have

$$\begin{aligned}
\int_\Omega \tilde{u}^2 &= \int_\Omega m\tilde{u} \le C \int_\Omega \tilde{u} \\
&= C\left(\int_{[\tilde{u}\le\delta_2]} \tilde{u} + \int_{[\tilde{u}>\delta_2]} \tilde{u}\right) \\
&= C\left(\int_{[\tilde{u}\le\delta_2]} \tilde{u} + \sum_{x_0\in\mathfrak{M}_+} \int_{I_{\delta_2(x_0)}} \tilde{u}\right) \\
&\le C|\Omega|\delta_2 + \sum_{x_0\in\mathfrak{M}_+} 2\tilde{u}(x_\alpha) \int_{I'_{\delta_2}(x_0)} \exp\left[\frac{\alpha}{2d} m''(x_0)(x-x_0)^2\right] \\
&\le C\alpha^{-\frac{1}{4}} + o(\alpha^{\frac{1}{4}}) \sum_{x_0\in\mathfrak{M}_+} \frac{1}{\sqrt{\alpha}} \int_{\mathbf{R}} \exp\left[\frac{1}{2d} m''(x_0)y^2\right] dy \\
&= O(\alpha^{-\frac{1}{4}}).
\end{aligned}$$

□

To estimate I'_{δ_1}, we begin with the following counterpart of Proposition 5.7.

Proposition 5.11. *For α large, $|I'_{\delta_1}| = o(\frac{1}{\alpha^c})$ for any $0 < c < \frac{1}{2}$. In particular, $|I'_{\delta_1}| = o(\frac{1}{\alpha^{13/32}})$.*

Proof. Fix $\frac{7}{16} < c < \frac{1}{2}$. Suppose that the assertion is false. Then for some $x_0 \in \mathfrak{M}_+$ there is a sequence $\alpha_j \to \infty$ such that for each j, there exists $z_{\alpha_j} \in I'_{\delta_1}(x_0)$, with $|z_{\alpha_j} - x_0| = \frac{k_1}{\alpha^c}$, for some constant $k_1 > 0$. From (5.6), (5.8) and Lemma 5.10, it follows that (again we suppress the subindex j, for simplicity)

$$\begin{aligned}
\frac{\tilde{u}(z_\alpha)}{\tilde{u}(x_\alpha)} &\le \exp\left\{\alpha[m(z_\alpha) - m(x_0)]/d + \alpha[m(x_0) - m(x_\alpha)]/d + \frac{1}{d\delta_1} C|z_\alpha - x_\alpha||\tilde{u}|_{L^2}\right\} \\
&\le \exp\left[-\alpha k_2 |z_\alpha - x_0|^2/d + o(1) + C\alpha^{\frac{17}{32}-c}|\tilde{u}|_{L^2}/d\right] \\
&\le \exp\left[-k_3\alpha^{1-2c}/d + o(1) + C\alpha^{\frac{17}{32}-c-\frac{1}{8}}/d\right] \\
&\le \exp\left[-k_4\alpha^{1-2c}/d\right]
\end{aligned}$$

for α large, where k_2, k_3, k_4 are positive constants. On the other hand, from Theorem 5.3 (ii) we have

$$\frac{\tilde{u}(z_\alpha)}{\tilde{u}(x_\alpha)} \ge \frac{\delta_1}{|\tilde{u}|_{L^\infty} + \alpha|\Delta m|_{L^\infty}} \ge \frac{k_5}{\alpha^{\frac{49}{32}}},$$

a contradiction. □

Now we have the counterpart of Theorem 5.8 for I'_{δ_1}.

Theorem 5.12.
$$\frac{\tilde{u}(x)}{\tilde{u}(x_\alpha)} \exp\left[-\frac{\alpha}{2d} m''(x_0)(x-x_0)^2\right] \to 1 \qquad (5.11)$$

uniformly in $I_{\delta_1(x_0)}$ *for each* $x_0 \in \mathfrak{M}_+$ *as* $\alpha \to \infty$. *In particular, we have, for each* $\epsilon > 0$,

$$(1-\epsilon)\tilde{u}(x_\alpha)e^{\frac{\alpha}{2d}m''(x_0)(x-x_0)^2} \le \tilde{u}(x) \le (1+\epsilon)\tilde{u}(x_\alpha)e^{\frac{\alpha}{2d}m''(x_0)(x-x_0)^2}, \qquad (5.12)$$

and

$$(1-\epsilon)\tilde{u}(x_\alpha)e^{\alpha[m(x)-m(x_0)]/d} \le \tilde{u}(x) \le (1+\epsilon)\tilde{u}(x_\alpha)e^{\alpha[m(x)-m(x_0)]/d} \qquad (5.13)$$

uniformly in I'_{δ_1}, *for* α *large.*

Proof. As in the proof of Theorem 5.8, we have, for $x \in I'_{\delta_1}(x_0)$,

$$\left|\frac{\tilde{u}(x)}{\tilde{u}(x_\alpha)} \exp\left[-\frac{\alpha}{2d} m''(x_0)(x-x_0)^2\right] - 1\right| = |g_1(x) - g_2(x)| \exp \xi(x)$$

where g_1 and g_2 are given in (5.9) and (5.10) respectively, and $\xi(x)$ lies in between 0 and $g_1(x) - g_2(x)$. $g_1(x)$ and $g_2(x)$ can be estimated in a similar fashion as in the proof of Theorem 5.8:

$$\begin{aligned}|g_1(x)| &\le \left|\alpha[m(x) - m(x_0) - \frac{1}{2}m''(x_0)(x-x_0)^2]/d\right| + \alpha|m(x_0) - m(x_\alpha)|/d \\ &\le \alpha\left[O(|x-x_0|^3) + O(|x_0 - x_\alpha|^2)\right] \to 0\end{aligned}$$

in view of Proposition 5.11 and (5.8). Similarly,

$$\begin{aligned}|g_2(x)| &\le C\frac{1}{\delta_1}|x - x_\alpha||u|_{L^2} \\ &\le o\left(\alpha^{\frac{17}{32}-\frac{13}{32}-\frac{1}{8}}\right) \to 0\end{aligned}$$

by Lemma 5.10 and Proposition 5.11. Thus (5.11) and (5.12) hold. (5.13) follows from the fact that

$$
\begin{aligned}
&\exp\left[-\frac{\alpha}{2d}m''(x_0)(x-x_0)^2\right]\exp\{\alpha[m(x)-m(x_0)]/d\}\\
=&\exp\left[\alpha O(|x-x_0|^3)\right]\to 1
\end{aligned}
$$

for $x \in I'_{\delta_1}$, by Proposition 5.11. □

Next we show that $|\tilde{u}|_{L^\infty}$ is uniformly bounded in α large.

Theorem 5.13. *$|\tilde{u}|_{L^\infty}$ is uniformly bounded for all α large.*

Proof. Let $\tilde{u}(x_\alpha) = |u|_{L^\infty}$, Lemma 5.9 and (5.12) imply that

$$
\begin{aligned}
\tilde{u}^2(x_\alpha) &\le C\sqrt{\alpha}\int_\Omega \tilde{u}^2 = C\sqrt{\alpha}\int_\Omega m\tilde{u} \le C\sqrt{\alpha}\int_\Omega \tilde{u}\\
&= C\sqrt{\alpha}\left(\int_{[\tilde{u}\le\delta_1]}\tilde{u} + \int_{[\tilde{u}>\delta_1]}\tilde{u}\right)\\
&\le C\sqrt{\alpha}\left(|\Omega|\delta_1 + \sum_{x_0\in\mathfrak{M}_+} C\tilde{u}(x_\alpha)\int_{I_{\delta_1(x_0)}}\exp\left[\frac{\alpha}{2d}m''(x_0)(x-x_0)^2\right]dx\right)\\
&\le C|\Omega|\alpha^{-\frac{17}{32}+\frac{1}{2}} + \sum_{x_0\in\mathfrak{M}_+} C\tilde{u}(x_\alpha)\sqrt{\alpha}\int_{\mathbf{R}}\exp\left[\frac{1}{2d}m''(x_0)y^2\right]\frac{dy}{\sqrt{\alpha}}
\end{aligned}
$$

Therefore,

$$
|\tilde{u}|^2_{L^\infty} \le C(1+|\tilde{u}|_{L^\infty}).
$$

Since C is independent of α, $|\tilde{u}|_{L^\infty}$ must be uniformly bounded for all α large. □

Theorem 5.14. *For each $x_0 \in \mathfrak{M}_+$, we have*

$$
\lim_{\alpha\to\infty}\tilde{u}(x_\alpha) = \sqrt{2}m(x_0)
$$

where x_α is given by (5.7).

Proof. Integrating the equation (5.4) from $x_0-\epsilon_0$ to $x_0+\epsilon_0$ gives

$$
(d\tilde{u}' - \alpha\tilde{u}m')\Big|_{x_0-\epsilon_0}^{x_0+\epsilon_0} + \int_{x_0-\epsilon_0}^{x_0+\epsilon_0} m\tilde{u} = \int_{x_0-\epsilon_0}^{x_0+\epsilon_0}\tilde{u}^2 \tag{5.14}
$$

First, we claim that for some $\tilde{b} > 0$,

$$(d\tilde{u}' - \alpha\tilde{u}m')\Big|_{x_0-\epsilon_0}^{x_0+\epsilon_0} \le e^{-\tilde{b}\alpha}, \tag{5.15}$$

for all α large. By Theorem 5.3 (iv), at $x = x_0 \pm \epsilon_0$, $|\alpha\tilde{u}m'| \le C\alpha e^{-b\alpha} \le Ce^{-\tilde{b}\alpha}$ for some $\tilde{b} > 0$. Thus, to show (5.15) it suffices to prove

$$d\tilde{u}'(x_0 \pm \epsilon_0) \le Ce^{-\tilde{b}\alpha}. \tag{5.16}$$

We will prove (5.16) only for the case $x_0 - \epsilon_0$, as the other case can be handled in a similar fashion.

Case 1. $x_0 = \min \mathfrak{M}_+$.

Then, integrating the equation (5.4) from -1 to $x_0 - \epsilon_0$, we obtain

$$d\tilde{u}'(x_0 - \epsilon_0) = \alpha(\tilde{u}m')(x_0 - \epsilon_0) - \int_{-1}^{x_0-\epsilon_0} \tilde{u}(m - \tilde{u}) \tag{5.17}$$

by the no-flux boundary condition at -1. Now every term on the right-hand side of (5.17) is bounded by $C\alpha e^{-b\alpha}$ or $Ce^{-b\alpha}$, therefore our assertion follows.

Case 2. $x_0 > \min \mathfrak{M}_+$.

Without loss of generality we may assume that there exists $x_1 \in \mathfrak{M}_+$ such that $\mathfrak{M}_+$ has no other points in the interval (x_1, x_0). Then, by Theorem 5.3 (iii),(iv), there exists $\tilde{x} \in (x_1, x_0)$ such that $\tilde{u}'(\tilde{x}) = 0$ and $\tilde{u} < e^{-b\alpha}$ in between $\tilde{x}$ and $x_0 - \epsilon_0$.

Now, we integrate (5.4) from $\tilde{x}$ to $x_0 - \epsilon_0$,

$$\begin{aligned}
(d\tilde{u}' - \alpha\tilde{u}m')\Big|_{\tilde{x}}^{x_0-\epsilon_0} &= -\int_{\tilde{x}}^{x_0-\epsilon_0} \tilde{u}(m - \tilde{u}) \\
d\tilde{u}'(x_0 - \epsilon_0) &= \alpha\tilde{u}m'\Big|_{\tilde{x}}^{x_0-\epsilon_0} - \int_{\tilde{x}}^{x_0-\epsilon_0} \tilde{u}(m - \tilde{u}) \\
&< C\alpha e^{-b\alpha} \le e^{-\tilde{b}\alpha}
\end{aligned}$$

for α large, where $\tilde{b}$ is another positive constant and (5.16) is established.

From (5.14) and (5.15) we obtain

$$\sqrt{\alpha}\int_{x_0-\epsilon_0}^{x_0+\epsilon_0} m\tilde{u} = \sqrt{\alpha}\int_{x_0-\epsilon_0}^{x_0+\epsilon_0} \tilde{u}^2 + O(\sqrt{\alpha}e^{-\tilde{b}\alpha}). \tag{5.18}$$

Next, we need the following technical lemma.

Lemma 5.15.

$$\begin{aligned}
\limsup_{\alpha\to\infty}\sqrt{\alpha}\int_{x_0-\epsilon_0}^{x_0+\epsilon_0} m\tilde{u} &= m(x_0)\left(\int_{\mathbf{R}} e^{\frac{1}{2d}m''(x_0)y^2}dy\right)\limsup_{\alpha\to\infty}\tilde{u}(x_\alpha),\\
\liminf_{\alpha\to\infty}\sqrt{\alpha}\int_{x_0-\epsilon_0}^{x_0+\epsilon_0} m\tilde{u} &= m(x_0)\left(\int_{\mathbf{R}} e^{\frac{1}{2d}m''(x_0)y^2}dy\right)\liminf_{\alpha\to\infty}\tilde{u}(x_\alpha),\\
\limsup_{\alpha\to\infty}\sqrt{\alpha}\int_{x_0-\epsilon_0}^{x_0+\epsilon_0} \tilde{u}^2 &= \left(\int_{\mathbf{R}} e^{m''(x_0)y^2/d}dy\right)\limsup_{\alpha\to\infty}\tilde{u}^2(x_\alpha)\\
\liminf_{\alpha\to\infty}\sqrt{\alpha}\int_{x_0-\epsilon_0}^{x_0+\epsilon_0} \tilde{u}^2 &= \left(\int_{\mathbf{R}} e^{m''(x_0)y^2/d}dy\right)\liminf_{\alpha\to\infty}\tilde{u}^2(x_\alpha).
\end{aligned}$$

We postpone the proof of Lemma 5.15 and continue to prove Theorem 5.14. Taking limsup and liminf respectively as $\alpha\to\infty$ on both sides of (5.18) we have

$$\left(\limsup_{\alpha\to\infty}\tilde{u}(x_\alpha)\right)m(x_0)\int_{\mathbf{R}} e^{\frac{1}{2d}m''(x_0)y^2}dy = \left(\limsup_{\alpha\to\infty}\tilde{u}(x_\alpha)\right)^2\int_{\mathbf{R}} e^{m''(x_0)y^2/d}dy,$$

and

$$\left(\liminf_{\alpha\to\infty}\tilde{u}(x_\alpha)\right)m(x_0)\int_{\mathbf{R}} e^{\frac{1}{2d}m''(x_0)y^2}dy = \left(\liminf_{\alpha\to\infty}\tilde{u}(x_\alpha)\right)^2\int_{\mathbf{R}} e^{m''(x_0)y^2/d}dy.$$

Since $0 < m(x_0) \le \liminf_{\alpha\to\infty} u(x_\alpha) \le \limsup_{\alpha\to\infty} u(x_\alpha) < \infty$, we obtain

$$\limsup_{\alpha\to\infty}\tilde{u}(x_\alpha) = \liminf_{\alpha\to\infty}\tilde{u}(x_\alpha) = \sqrt{2}m(x_0)$$

and our conclusion follows. □

It remains to prove Lemma 5.15. We will only show the first equality as the rest are similar.

Proof of Lemma 5.15. First, observe that

$$\left|\sqrt{\alpha}\int_{(x_0-\epsilon_0,x_0+\epsilon_0)\setminus I'_{\delta_1}} m(x)\tilde{u}(x)\,dx\right| \le \sqrt{\alpha}\delta_1|m|_{L^\infty}\cdot 2\epsilon_0 \to 0 \tag{5.19}$$

as $\alpha \to \infty$. Now for any $\epsilon > 0$, (5.12) implies that

$$\sqrt{\alpha}(1-\epsilon)\tilde{u}(x_\alpha) \int_{I'_{\delta_1}(x_0)} m(x) e^{\frac{\alpha}{2d} m''(x_0)(x-x_0)^2} dx \le \sqrt{\alpha} \int_{I'_{\delta_1}(x_0)} m\tilde{u}\, dx \\ \le \sqrt{\alpha}(1+\epsilon)\tilde{u}(x_\alpha) \int_{I'_{\delta_1}(x_0)} m(x) e^{\frac{\alpha}{2d} m''(x_0)(x-x_0)^2} dx. \tag{5.20}$$

We compute, for any constant $M > 0$, by Lemma 5.6,

$$\int_{-M}^{M} m_\alpha(y) e^{\frac{1}{2d} m''(x_0) y^2} dy \le \sqrt{\alpha} \int_{I'_{\delta_1}(x_0)} m(x) e^{\frac{\alpha}{2d} m''(x_0)(x-x_0)^2} dx \\ \le \int_{\mathbf{R}} m_\alpha(y) e^{\frac{1}{2d} m''(x_0) y^2} dy. \tag{5.21}$$

where $y = \sqrt{\alpha}(x - x_0)$ and $m_\alpha(y) = m(x)$. For $-M \le y \le M$, we have

$$x_0 - \frac{M}{\sqrt{\alpha}} \le x \le x_0 + \frac{M}{\sqrt{\alpha}}$$

and thus for α large, $|m_\alpha(y) - m(x_0)| \to 0$ as $\alpha \to \infty$. This implies that

$$\left| \int_{-M}^{M} m_\alpha(y) e^{\frac{1}{2d} m''(x_0) y^2} dy - m(x_0) \int_{-M}^{M} e^{\frac{1}{2d} m''(x_0) y^2} dy \right| \to 0$$

as $\alpha \to \infty$. On the other hand, (extending $m_\alpha(y)$ trivially outside $\sqrt{\alpha}\{(-1,1) - x_0\}$)

$$\int_{\mathbf{R} \setminus (-M,M)} (|m_\alpha(y)| + m(x_0)) e^{\frac{1}{2d} m''(x_0) y^2} dy \to 0$$

as $M \to \infty$, since $m''(x_0) < 0$. Hence,

$$\left| \int_{\mathbf{R}} m_\alpha(y) e^{\frac{1}{2d} m''(x_0) y^2} dy - \int_{\mathbf{R}} m(x_0) e^{\frac{1}{2d} m''(x_0) y^2} dy \right| \to 0$$

as $\alpha \to \infty$, and (5.21) becomes, for any $M > 0$,

$$m(x_0) \int_{-M}^{M} e^{\frac{1}{2d} m''(x_0) y^2} dy + o(1) \\ \le \sqrt{\alpha} \int_{I'_{\delta_1}(x_0)} m(x) e^{\frac{\alpha}{2d} m''(x_0)(x-x_0)^2} dx \le m(x_0) \int_{\mathbf{R}} e^{\frac{1}{2d} m''(x_0) y^2} dy$$

holds for α large. Thus

$$\lim_{\alpha \to \infty} \sqrt{\alpha} \int_{I'_{\delta_1}(x_0)} m(x) e^{\frac{\alpha}{2d} m''(x_0)(x-x_0)^2} dx = m(x_0) \int_{\mathbf{R}} e^{\frac{1}{2d} m''(x_0) y^2} dy$$

since M can be arbitrarily large. Now from (5.20) we conclude that

$$(1-\epsilon)\left[\limsup_{\alpha\to\infty}\tilde{u}(x_\alpha)\right] m(x_0)\int_{\mathbf{R}} e^{\frac{1}{2d}m''(x_0)y^2}dy$$

$$\le \limsup_{\alpha\to\infty}\sqrt{\alpha}\int_{I'_{\delta_1}(x_0)} m\tilde{u} \le (1+\epsilon)\left[\limsup_{\alpha\to\infty}\tilde{u}(x_\alpha)\right] m(x_0)\int_{\mathbf{R}} e^{\frac{1}{2d}m''(x_0)y^2}dy$$

Combining (5.19) and the inequality above we have

$$\limsup_{\alpha\to\infty}\sqrt{\alpha}\int_{x_0-\epsilon_0}^{x_0+\epsilon_0} m\tilde{u} = m(x_0)\left(\int_{\mathbf{R}} e^{\frac{1}{2d}m''(x_0)y^2}dy\right)\limsup_{\alpha\to\infty}\tilde{u}(x_\alpha)$$

This finishes the proof. □

Theorem 5.2 now follows from Proposition 5.5, Theorems 5.12 and 5.14.

5.1.2 Proof of Theorem 5.1.

As before, in this section we assume $\Omega = (-1,1)$ and $m(x) \in C^2([-1,1])$ satisfies **(M1),(M2),(M3)**. Let $(\tilde{U},\tilde{V})$ be a coexistence state of (5.2), whose existence for large α is established in [CCL2] and generalized in [CL]. Again, in this section, the sub-indices d_2 and α will be suppressed when there is no confusion.

Lemma 5.16. *$0 \le \tilde{U} \le \tilde{u}$ and $0 \le \tilde{V} \le \theta_{d_2}$ in $(-1,1)$ where $\tilde{u}$ is the unique positive solution of* (5.4) *and θ_{d_2} is the unique positive solution to*

$$\begin{cases} d_2\theta'' + \theta(m-\theta) = 0 & \text{in } (-1,1) \\ \theta' = 0 & \text{at } x = -1, 1. \end{cases} \tag{5.22}$$

Proof. The existence and uniqueness of θ_{d_2} is standard. (See, e.g. Lemma 7.1 in [CL]). By (5.2), $\tilde{U}$ satisfies,

$$\begin{cases} (d_1\tilde{U}' - \alpha\tilde{U}m')' + \tilde{U}(m-\tilde{U}) = \tilde{U}\tilde{V} \ge 0 & \text{in } (-1,1) \\ d_1\tilde{U}' - \alpha\tilde{U}m' = 0 & \text{at } \ x = -1, 1. \end{cases} \tag{5.23}$$

and $\tilde{V}$ satisfies

$$\begin{cases} d_2\tilde{V}'' + \tilde{V}(m-\tilde{V}) = U\tilde{V} \ge 0 & \text{in } \ (-1,1) \\ \tilde{V}' = 0 & \text{at } \ x = -1, 1. \end{cases} \tag{5.24}$$

It follows that $\tilde{U}$ and $\tilde{V}$ are lower solutions of (5.4) and (6.1) respectively. Since $\tilde{u}, \theta_{d_2}$ are the unique positive steady-states of (5.4) and (6.1) respectively which are globally asymptotically stable, the inequalities follow from standard upper and lower solutions arguments. □

Lemma 5.17. *$\tilde{V} \to \theta_{d_2}$ in $C^{1,\beta}([-1,1])$ for any $0 < \beta < 1$.*

Proof. By Lemma 5.16, Theorem 5.13, $\{\tilde{U}, \tilde{V}\}_\alpha$ is bounded in $L^\infty(-1,1)$ uniformly. Hence by (6.4), $\{\tilde{V}\}$ is bounded in $C^2([-1,1])$ uniformly and is therefore relatively compact in $C^{1,\beta}([-1,1])$ for any $0 < \beta < 1$.

Next, take an arbitrary subsequence $\{V_{\alpha_i}\}_i$ such that $V_{\alpha_i} \to V_0$ in $C^{1,\beta}([-1,1])$ for some $V_0 \in C^{1,\beta}([-1,1])$. Then V_0 satisfies $d_2 V_0'' + V_0(m - V_0) = 0$ weakly, i.e. for any $\psi \in H^1(-1,1)$,

$$-d_2 \int_{-1}^{1} V_0' \psi' + \int_{-1}^{1} \psi V_0 (m - V_0) = 0.$$

Take, for $x_0 \in [-1, 1)$

$$\psi_{\epsilon, x_0} = \begin{cases} 1 & x < x_0 \\ \frac{x_0 + \epsilon - x}{\epsilon} & x_0 \le x < x_0 + \epsilon \\ 0 & x \ge x_0 + \epsilon \end{cases}$$

$$\psi_{\epsilon, 1} = \begin{cases} 1 & x < 1 - \epsilon \\ \frac{1-x}{\epsilon} & 1 - \epsilon \le x \le 1. \end{cases}$$

Now, letting $\epsilon \to 0_+$, we have

$$d_2 V_0'(x_0) + \int_{-1}^{x_0} V_0(m - V_0) = 0, \qquad \forall x_0 \in [-1, 1].$$

We then have $V_0' \in C^1([-1,1])$, i.e. $V_0 \in C^2([-1,1])$ and so V satisfies (6.1) in the classical sense. Hence $V_0 \equiv \theta_{d_2}$ by uniqueness. Thus, $\tilde{V} \to \theta_{d_2}$ in $C^{1,\beta}([-1,1])$ for any $0 < \beta < 1$. □

The following result is contained in Theorem 1.8 of [L1].

Lemma 5.18. *For any $r > 0$ and $x_0 \in \mathfrak{M}_+$,*

$$\liminf_{\alpha \to \infty} \max_{B_r(x_0)} \tilde{U} \ge m(x_0) - \theta(x_0).$$

Lemma 5.19. *Suppose that* $\liminf_\alpha \sup_{[x_0-\epsilon_0,x_0+\epsilon_0]} \tilde{U} > 0$, *then*

$$\frac{\tilde{U}(x)}{\tilde{U}(x_\alpha)} \exp\{\alpha[m(x_0)-m(x)]/d_1\} \to 1 \tag{5.25}$$

uniformly in $[x_0 - \frac{M}{\sqrt{\alpha}}, x_0 + \frac{M}{\sqrt{\alpha}}]$, *for any* $M > 0$ *as* $\alpha \to \infty$, *where* $\tilde{U}(x_\alpha) = \sup_{[x_0-\epsilon_0,x_0+\epsilon_0]}\tilde{U}$ *and* $x_\alpha \in (x_0 - \frac{M}{\sqrt{\alpha}}, x_0 + \frac{M}{\sqrt{\alpha}})$ *for* α *large.*

Proof. . The existence of x_α follows from Lemma 5.16, Theorems 5.12 and 5.14. Also, by (5.8) and its proof, $\alpha m'(x_\alpha) = o(1)$ and $\alpha[m(x_0) - m(x_\alpha)] = o(1)$. Now, let

$$w(x) = \tilde{U}(x)\exp\{\alpha[m(x_0)-m(x)]/d_1\}.$$

By Lemma 5.16, Theorems 5.12 and 5.14, w is bounded in $L^\infty([x_0 - \frac{M}{\sqrt{\alpha}}, x_0 + \frac{M}{\sqrt{\alpha}}])$ uniformly in α. Moreover, it satisfies

$$\begin{cases} (d_1 \exp\{\alpha[m(x)-m(x_0)]/d_1\}w')' + F_\alpha = 0 & \text{in } [x_0 - \frac{M}{\sqrt{\alpha}}, x_0 + \frac{M}{\sqrt{\alpha}}] \\ w(x_\alpha) = \tilde{U}(x_\alpha)\exp\{\alpha[m(x_0)-m(x_\alpha)]/d_1\} \\ w'(x_\alpha) = -w(x_\alpha)\alpha m'(x_\alpha)/d_1 \end{cases} \tag{5.26}$$

where $F_\alpha = \tilde{U}(m - \tilde{U} - \tilde{V})$ is bounded in $L^\infty([x_0 - \frac{M}{\sqrt{\alpha}}, x_0 + \frac{M}{\sqrt{\alpha}}])$ uniformly in α. Thus,

$$d_1 \exp\{\alpha[m(x)-m(x_0)]\}w'(x) = -w(x_\alpha)\alpha m'(x_\alpha)\exp\{\alpha[m(x_\alpha)-m(x_0)]\} - \int_{x_\alpha}^x F_\alpha,$$

and

$$w(x) - w(x_\alpha) = \frac{1}{d_1}\int_{x_\alpha}^x \Big[\exp\{\alpha[m(x_0)-m(y)]/d_1\}\times$$
$$(-w(x_\alpha)\alpha m'(x_\alpha)\exp\{\alpha[m(x_\alpha)-m(x_0)]/d_1\} - \int_{x_\alpha}^y F_\alpha)\Big]dy.$$

It is not hard to see that the integrand on the right-hand side is bounded in $L^\infty([x_0 - \frac{M}{\sqrt{\alpha}}, x_) + \frac{M}{\sqrt{\alpha}}])$ for each M. Therefore, since $|x - x_\alpha| \le \frac{2M}{\sqrt{\alpha}} \to 0$, we see that $|w(x) - w(x_\alpha)| \to 0$ uniformly in $[x_0 - \frac{M}{\sqrt{\alpha}}, x_0 + \frac{M}{\sqrt{\alpha}}]$. Now $\tilde{U}(x_\alpha)$ is bounded away from 0 as $\alpha \to \infty$, and $\alpha[m(x_0) - m(x_\alpha)] = o(1)$. Therefore (5.25) is proved. □

Lemma 5.20. *If* $m(x_0) - \theta(x_0) > 0$*, then* (5.25) *holds and*

$$\lim_{\alpha\to\infty} \tilde{U}(x_\alpha) = \sqrt{2}(m(x_0) - \theta(x_0)).$$

Proof. By Lemma 5.18, the assumption of Lemma 5.19 is satisfied. Therefore (5.25) holds. Now we proceed to evaluate $\lim_{\alpha\to\infty} \tilde{U}(x_\alpha)$. We first claim that for some small constant $\epsilon_0 > 0$, and some $\tilde{b} > 0$,

$$\int_{x_0-\epsilon_0}^{x_0+\epsilon_0} \tilde{U}(m - \tilde{U} - \tilde{V})dx = O(e^{-\tilde{b}\alpha}). \tag{5.27}$$

By integrating from $x_0 - \epsilon_0$ to $x_0 + \epsilon_0$, we obtain

$$\int_{x_0-\epsilon_0}^{x_0+\epsilon_0} \tilde{U}(m - \tilde{U} - \tilde{V})dx = -(d_1\tilde{U}' - \alpha\tilde{U}m')\Big|_{x_0-\epsilon_0}^{x_0+\epsilon_0}.$$

By Lemma 5.16 and Theorem 5.3 (iv), it suffices to show that $d_1\tilde{U}'(x_0 \pm \epsilon_0) = O(e^{-\tilde{b}\alpha})$. We shall only estimate $\tilde{U}'(x_0 + \epsilon_0)$, as the other can be handled in a similar fashion.

Case 1. $x_0 = \max \mathfrak{M}_+$.

Integrating the equation from $x_0 + \epsilon_0$ to 1, we obtain

$$d_1\tilde{U}'(x_0 + \epsilon_0) = \alpha(\tilde{U}m')(x_0 + \epsilon_0) + \int_{x_0+\epsilon_0}^{1} \tilde{U}(m - \tilde{U} - \tilde{V})dx$$

by the no-flux boundary condition at 1. Now every term on the right hand side is bounded by $C\alpha e^{-b\alpha}$, therefore our assertion follows.

Case 2. $x_0 < \max \mathfrak{M}_+$.

At least one of the following holds:

(i) $\tilde{U}(x) \leq O(e^{-b\alpha})$ in $[x_0 + \epsilon_0, 1)$;

(ii) there exists $\tilde{x} \in (x_0, 1)$ such that $\tilde{U}'(\tilde{x}) = 0$ and $\tilde{U}(x) \leq O(e^{-b\alpha})$ in the closed interval between $x_0 + \epsilon_0$ and $\tilde{x}$.

The assertion follows as in *Case* 1 if (i) holds. If (ii) holds, integrate from $\tilde{x}$ to

$x_0 + \epsilon_0$. Then

$$|\tilde{U}'(x_0 + \epsilon_0)| \le \left|\alpha(\tilde{U}m')\Big|_{\tilde{x}}^{x_0+\epsilon_0}\right| + \left|\int_{\tilde{x}}^{x_0+\epsilon_0} \tilde{U}(m - \tilde{U} - \tilde{V})dx\right|$$

and the assertion holds. Hence (6.5) is proved.

By changing coordinates $y = \sqrt{\alpha}(x - x_0)$ in (6.5),

$$\begin{aligned}\left|\int_{-M}^{M} \tilde{U}(m - \tilde{U} - \tilde{V})dy\right| &\le \alpha^{\frac{1}{2}} \left|\int_{(x_0-\epsilon_0,x_0+\epsilon_0)\setminus[x_0-\frac{M}{\sqrt{\alpha}},x_0+\frac{M}{\sqrt{\alpha}}]} \tilde{U}(m - \tilde{U} - \tilde{V})\right| \\ &\quad + O(\alpha^{\frac{1}{2}}e^{-\tilde{b}\alpha}) \\ &\le C\int_{\mathbf{R}\setminus[-M,M]} e^{\frac{1}{2d_1}m''(x_0)y^2}dy + O(\alpha^{\frac{-1}{32}}) + O(\alpha^{\frac{1}{2}}e^{-\tilde{b}\alpha})\end{aligned}$$

by Lemma 5.16, Theorem 5.12 and Theorem 5.14.

By taking $\alpha_i \to \infty$ such that $\tilde{U}(x_{\alpha_i}) \to \limsup_{\alpha\to\infty} \tilde{U}(x_\alpha)$, making use of Lemmas 5.17 and 5.19, we have

$$\begin{aligned}&\Big|(\limsup \tilde{U}(x_\alpha))(m(x_0) - \theta(x_0))\int_{-M}^{M} e^{\frac{1}{2d_1}m''(x_0)y^2}dy \\ &\qquad -(\limsup \tilde{U}(x_\alpha))^2 \int_{-M}^{M} e^{m''(x_0)y^2/d_1}dy\Big| \\ &\le C\int_{\mathbf{R}\setminus[-M,M]} e^{\frac{1}{2d_1}m''(x_0)y^2}dy.\end{aligned}$$

Take $M \to +\infty$, we have

$$P(\limsup \tilde{U}(x_\alpha)) = 0 \quad \text{where } P(s) = \sqrt{2}(m(x_0) - \theta(x_0))s - s^2. \tag{5.28}$$

Similarly, we have

$$P(\liminf \tilde{U}(x_\alpha)) = 0. \tag{5.29}$$

Now if $m(x_0) - \theta(x_0) > 0$, then by Lemmas 5.16 and 5.18,

$$+\infty > \limsup_{\alpha\to\infty} \tilde{U}(x_\alpha) \ge \liminf_{\alpha\to\infty} \tilde{U}(x_\alpha) \ge m(x_0) - \theta(x_0) > 0.$$

By (5.28) and (5.29), $\limsup_{\alpha\to\infty} \tilde{U}(x_\alpha) = \liminf_{\alpha\to\infty} \tilde{U}(x_\alpha) = \sqrt{2}(m(x_0) - \theta(x_0))$. □

Lemma 5.21. *If $m(x_0) - \theta(x_0) \le 0$, then for each small $r > 0$, $\tilde{U} \to 0$ uniformly in $(x_0 - r, x_0 + r)$.*

Proof. . Suppose to the contrary that there exists a sequence $\alpha_i \to \infty$, such that $\lim_{\alpha_i \to \infty} \left[\sup_{(x_0-\epsilon_0, x_0+\epsilon_0)} \tilde{U} \right] > 0$. Then by the same arguments in the proof of (5.28),

$$P(\lim_{\alpha_i \to \infty} \tilde{U}(x_{\alpha_i})) = 0 \text{ where } P(s) = \sqrt{2}(m(x_0) - \theta(x_0))s - s^2,$$

a contradiction, because P does not have any positive roots. □

We now prove Theorem 5.1.

Proof of Theorem 5.1. Part (i) follows from Lemma 5.17. Part (ii) is a consequence of Lemmas 5.19, 5.20 and 5.21. Finally, part (iii) follows from Lemma 5.16 and Theorem 5.3(iv). □

5.2 $\mathfrak{M}_+$ is non-discrete

Next we consider the case when m has a "plateau-like" local maximum. More precisely, we assume

$$m(x) = 1 \text{ in } [-1/2, 1/2], \quad m(x) < 1 \text{ and } m'(x) \neq 0 \text{ in } (-1, -1/2) \cup (1/2, 1). \quad (5.30)$$

Then we are going to show

Theorem 5.22. *For each α large,* (5.1) *has at least one stable coexistence state. Moreover, if $(\tilde{U}, \tilde{V})$ is any coexistence state of* (5.1)*, then for any sequence $\alpha_k \to \infty$, there exists a subsequence $\alpha_{k'}$ such that $\tilde{U} \to U_0$ in $C^2([-1/2, 1/2])$ and $\tilde{V} \to V_0$ in $C^2([-1, 1])$, where (U_0, V_0) are positive solution to*

$$\begin{cases} d_1 U'' + U(m(x) - U - V) = 0 & \textit{in } (-1/2, 1/2) \\ d_2 V'' + V(m(x) - U - V) = 0 & \textit{in } (-1, 1) \\ U' = 0 \quad \textit{at } x = \pm 1/2 \quad \textit{and} \quad m' = V' = 0 \quad \textit{at } x = \pm 1. \end{cases} \quad (5.31)$$

Note that here U has a significant competition with V. This is different from the previous case when $u \to 0$ in $L^p(\Omega)$ for any $p \ge 1$. An application of the result in this section is Proposition 1.11 announced in the introduction, which says that if m satisfies (5.30), then in some cases, U always wipe out V.

We will first treat the single equation (5.3) in the next subsection and then prove Theorem 5.22 in the next.

5.2.1 Qualitative properties of $\tilde{u}$

Let $\tilde{u}$ be the unique positive solution to (5.4), we have

Theorem 5.23. *$|\tilde{u}|_{L^\infty(-1,1)}$ is bounded independent of α large.*

The proof of Theorem 5.23 is presented at the end of this section.

Corollary 5.24. *If $(\tilde{U}, \tilde{V})$ is any coexistence state of (1.6), then $\tilde{U} < \tilde{u}$ in $(-1, 1)$ and hence $|\tilde{U}|_{L^\infty}$ is bounded independent of α large.*

Lemma 5.25. *$\underline{u} = e^{\alpha[m(x)-1]/d}$ is a lower solution to (5.4), hence $\tilde{u} \geq 1$ in $[-1/2, 1/2]$ for all α large.*

Proof. It is a particular case of Theorem 3.12. It can also be verified directly without using the cut-off argument.

$$(d\underline{u}' - \alpha \underline{u} m')' + \underline{u}(m - \underline{u}) = \underline{u}(m - \underline{u}) \geq 0$$

provided α is large enough. The boundary inequality is also satisfied. □

Lemma 5.26. *$\displaystyle\int_{-1}^{1} |\tilde{u}(m - \tilde{u})| dx \to 0$ as $\alpha \to \infty$.*

Proof. First observe that $\int_{-1}^{1} \tilde{u}(m - \tilde{u}) dx = 0$ and so

$$\int_{\{x \in (-1,1): \tilde{u} < m\}} \tilde{u}(m - \tilde{u}) dx = -\int_{\{x \in \Omega: \tilde{u} \geq m\}} \tilde{u}(m - \tilde{u}) dx \geq 0$$

But then Lemma 5.25 implies that $\{x : \tilde{u} < m\} \in [(-1, -1/2) \cup (1/2, -1)]$. Moreover, $\tilde{u} \to 0$ pointwise in $(-1, -1/2) \cup (1/2, -1)$. Therefore Theorem 3.1 and bounded convergence theorem implies that

$$\int_{\{x: \tilde{u} < m\}} \tilde{u}(m - \tilde{u}) dx = o(1).$$

□

Lemma 5.27. $\tilde{u} \to 1$ *uniformly in* $[-1/2, 1/2]$ *as* $\alpha \to \infty$.

Proof. Lemma 5.25 implies that $\tilde{u} > m$ in $[-1/2, 1/2]$, therefore by the equation $\tilde{u}$ is subharmonic ($\tilde{u}'' \geq 0$) in $[-1/2, 1/2]$. By Lemma 5.26,

$$\int_{-1/2}^{1/2} |\tilde{u} - m| dx = \int_{-1/2}^{1/2} 1 \cdot |\tilde{u} - m| dx \leq \int_{\{x:\tilde{u} \geq m\}} \tilde{u}(\tilde{u} - m) dx = o(1)$$

This implies $\tilde{u} \to 1$ in L^1 and $\tilde{u} \geq 1$ in $[-1/2, 1/2]$. This, subharmonicity, implies that $\tilde{u} \to 1$ in compact subsets of $(-1/2, 1/2)$. This, and that

$$d|u'| = |\int_{-1}^{x} u(u - m)| \leq 2 \int_{-1}^{1} m^2$$

being bounded uniformly proves the lemma. □

Lemma 5.28. $\lim_{\alpha \to \infty} |\tilde{U}|_{L^\infty(-1,1)} \geq 1 - |\theta_{d_2}|_{L^\infty(-1,1)} > 0$

This follows from Theorem 3.11. The last strict inequality holds by maximum principle.

Proof of Theorem 5.23. Let $x_\alpha \in [-1, 1]$ be defined such that $\tilde{u}(x_\alpha) = |\tilde{u}|_{L^\infty(-1,1)}$. Then $x_\alpha \in (-1, 1)$ by Hopf boundary lemma and Neumann b.c. Hence

$$\tilde{u}'(x_\alpha) = 0 \tag{5.32}$$

Also, by Lemma 5.25,

$$\text{There exists } c_0 > 0 \text{ such that } \tilde{u}(\alpha) \geq c_0 \text{ for all } \alpha \text{ large.} \tag{5.33}$$

Now use the first identity

$$\tilde{u}'(x) - \alpha \tilde{u}(x) m'(x) = -\int_{-1}^{x} \tilde{u}(m - \tilde{u}) dz \tag{5.34}$$

then (5.32), (5.33) and Lemma 5.26 implies that

$$\alpha m'(x_\alpha) = o(1) \tag{5.35}$$

In particular, $x_\alpha \to [-1/2, 1/2]$ can be quantified. Now WLOG suppose $x_\alpha < 0$, then for all x between x_α and $-1/2$,

$$\begin{aligned}\tilde{u}'(x) &= \alpha\tilde{u}m'(x) + \int_{-1}^{x} \tilde{u}(\tilde{u}-m)dz \\ &\geq \int_{-1}^{x} \tilde{u}(\tilde{u}-m)dz \geq C \\ \tilde{u}(x_\alpha) &\leq \tilde{u}(-1/2) - \int_{x_\alpha}^{-1/2} \tilde{u}' \\ &\leq \tilde{u}(-1/2) + C|x_\alpha + 1/2|\end{aligned}$$

Here $\tilde{u}(-1/2)$ is uniformly bounded in α by Lemma 5.27. □

5.2.2 Proof of Theorem 5.22

Proof of Theorem 5.22. Firstly, $\tilde{V}$, $\tilde{V}''$ is bounded in $L^\infty(\Omega)$, that means by passing to a subsequence we may assume $\tilde{V} \to V_0$ in $C^{1,\beta}$ for any β, where V_0 satisfies the same equation.

Secondly,

$$\tilde{U}'(-1/2) - \alpha\tilde{U}(-1/2)m'(-1/2) = -\int_{-1}^{-1/2} \tilde{U}(m - \tilde{U} - \tilde{V})$$

which implies

$$|\tilde{U}'(-1/2) - 0| \leq C\int_{-1}^{-1/2} \tilde{U} = o(1)$$

by bounded convergence again. Therefore $\tilde{U}'(-1/2) \to 0$. Similarly, $\tilde{U}'(1/2) \to 0$. And by the boundedness of $\tilde{U}$, in $L^\infty([-1/2, 1/2])$ and so is $\tilde{U}''$ by the equation, we also have compactness and pass to a subsequence that $\tilde{U} \to U_0$ in $C^2([-1/2, 1/2])$ satisfying the desired equation. □

5.2.3 Proof of Proposition 1.11

Proof of Proposition 1.11. The linearized stability of $(\tilde{u}, 0)$ is given by the following eigenvalue problem:

$$\begin{cases} (d_1\phi' - \alpha\phi m')' + (m - 2\tilde{u})\phi - \tilde{u}\psi + \lambda\phi = 0 & \text{in } (-1,1), \\ d_2\psi'' + (m - \tilde{u})\psi + \lambda\psi = 0 & \text{in } (-1,1), \\ d_1\phi' - \alpha\phi m' = \psi' = 0 & \text{at } \pm 1. \end{cases}$$

Since the above system decouples, the principal eigenvalue is given by the following simpler eigenvalue problem:

$$\begin{cases} d_2\psi'' + (m - \tilde{u})\psi + \sigma_1\psi_1 = 0 & \text{in } (-1,1), \\ \psi' = 0 & \text{at } \pm 1, \end{cases}$$

Now $\tilde{u} \to \chi_{(-1/2,1/2)}$ in L^2, therefore $m - \tilde{u} \to m - \chi_{(-1/2,1/2)}$ whose integral over Ω is negative. Hence, the local stability of $(\tilde{u}, 0)$ follows from the following standard property of eigenvalue problems with indefinite weight.

Lemma 5.29. *Let Ω be a bounded smooth domain in $\mathbf{R}^N$, g be a function of x, and let λ_1 be the principal eigenvalue of the following problem*

$$\begin{cases} \Delta\phi + \lambda g\phi = 0 & \text{in } \Omega, \\ \frac{\partial\phi}{\partial\nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

Then

$$\lambda_1 = \begin{cases} > 0 & \text{if } \int_\Omega g < 0, \\ = 0 & \text{if } \int_\Omega g = 0, \\ < 0 & \text{if } \int_\Omega g > 0. \end{cases}$$

Moreover, define $\mu_1(d, g)$ to be the principal eigenvalue of the following problem

$$\begin{cases} \Delta\varphi + g\varphi + \mu_1\varphi = 0 & \text{in } \Omega, \\ \frac{\partial\varphi}{\partial\nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

Then $\mu_1(d, g) < 0$ for all $d > 0$ if $\lambda_1 \leq 0$. On the other hand, if $\lambda_1 > 0$, then

$$\mu_1(d, g) = \begin{cases} > 0 & \text{if } d > 1/\lambda_1, \\ = 0 & \text{if } d = 1/\lambda_1, \\ < 0 & \text{if } d < 1/\lambda_1. \end{cases}$$

Now it suffices to show that there are no coexistence states for (5.1). Since then by the local stability of $(\tilde{u}, 0)$ and the theory of monotone semi-flow, there exists a connecting orbit from $(0, \theta_{d_2})$ to $(\tilde{u}, 0)$. And the global stability of $(\tilde{u}, 0)$ can be proved by comparison methods. Now suppose $(\tilde{U}, \tilde{V})$ is a coexistence state of (5.1).

Then for each α, $\tilde{U}$ is the unique positive steady-state of

$$\begin{cases} (d_1 U' - \alpha U m') + U(m - \tilde{V} - U) = 0 & \text{in } (-1,1), \\ d_1 U' - \alpha U m' = 0 & \text{at } \pm 1. \end{cases} \tag{5.36}$$

We first claim that $\tilde{U} > 1$ in $[-1/2, 1/2]$. .... But then $\square$

Chapter 6

A Liouville-Type Theorem of $\mathbf{div}(e^{y^T By}\nabla w) = 0$

Here we prove Proposition 6.1. By an orthogonal change of coordinates, it suffices to show the following.

Proposition 6.1. *Let* $0 < \lambda_1 \leq ... \leq \lambda_N$ *and* $0 < \sigma \in L^\infty_{loc}(\boldsymbol{R}^N)$ *such that for some* $R_0 > 0$, $\sigma^2 = e^{-\sum_{i=1}^N \lambda_i x_i^2}$ *for all* $x \in \boldsymbol{R}^N \setminus B_{R_0}(0)$. *Then every nonnegative weak solution* $w \in W^{1,2}_{loc}(\boldsymbol{R}^N)$ *to*

$$\nabla \cdot (\sigma^2 \nabla w) = 0 \qquad \text{in } \boldsymbol{R}^N, \tag{6.1}$$

is a constant.

Note that (6.1) implies

$$\Delta w - \sum_{i=1}^N \lambda_i x_i D_i w = 0 \qquad \text{in } \mathbf{R}^N \setminus B_{R_0}(0). \tag{6.2}$$

First we note that by local elliptic L^p estimates, w is smooth in $\{x \in \mathbf{R}^N : |x| > R_0\}$. (i.e. when σ is smooth.) We will need the following classical Harnack inequality. (See Theorem 8.20 in [GT] and a remark after it.)

Theorem 6.2. [GT] *If* $w \in W^{1,2}(\Omega)$ *satisfies*

$$\begin{cases} D_i(a_{ij} D_j w) + b_i D_i w + cw = 0 & \text{in } \Omega \\ w \geq 0 & \text{in } \Omega. \end{cases}$$

Then for any ball $B_{4R}(y) \in \Omega$, we have

$$\sup_{B_R(y)} w \le C \inf_{B_R(y)} w,$$

where $C \le C_0^{K \log K}$, $C_0 = C_0(N)$, $K = \Lambda/\lambda + \nu R$, $\lambda|\xi|^2 \le a_{ij}\xi_i\xi_j \le \Lambda|\xi|^2$ *and* $\nu^2 = (|b|_{L^\infty(B_{4R})}/\lambda)^2 + |c|_{L^\infty(B_{4R})}/\lambda$.

We shall also make use of a general result due to [BCN].

Theorem 6.3. [*BCN*] *If for some positive* $\sigma^2 \in L^\infty_{loc}(\boldsymbol{R}^N)$ *and constant* $C > 0$, $w \in W^{1,2}_{loc}(\boldsymbol{R}^N)$ *satisfies*

$$\begin{cases} w\,\nabla\cdot(\sigma^2\nabla w) \ge 0 \text{ in } \boldsymbol{R}^N \text{ locally,} \\ \int_{B_R} \sigma^2 w^2 \le CR^2, \end{cases}$$

then w is a constant.

In particular, a sufficient condition for the solution w to (6.1) to be a constant is that $e^{-\sum_{i=1}^N \lambda_i x_i^2} w^2(x)$ being integrable over $\mathbf{R}^N$.

Corollary 6.4. *Assume* $w \in W^{1,2}_{loc}(\boldsymbol{R}^N)$ *satisfies*

$$\begin{cases} \nabla\cdot(\sigma^2\nabla w) = 0 & \text{in } \boldsymbol{R}^N \text{ locally,} \\ 0 \le w(x) \le e^{c\sum_{i=1}^N \lambda_i x_i^2} & \text{for some } 0 < c < 1/2, \end{cases}$$

where σ is as in Proposition 6.1, *then w is a constant.*

We start with some notations concerning the level sets of $e^{-\sum_{i=1}^N \lambda_i x_i^2}$. Define

$$\Sigma_1 := \left\{ y \in \mathbf{R}^N : \sum_{i=1}^N \lambda_i y_i^2 = 1 \right\} \quad \text{and} \quad \Sigma_R := \left\{ x \in \mathbf{R}^N : \sum_{i=1}^N \lambda_i x_i^2 = R^2 \right\}.$$

For each $y \in \Sigma_1$ and $R > 0$, define $\gamma = \gamma(y, R)$ by $\sum_{i=1}^N \lambda_i y_i^2 e^{2\lambda_i\gamma} = R^2$. ($\gamma$ is well-defined since for each $y \in \Sigma_1$, $\gamma \mapsto \sum_{i=1}^N \lambda_i y_i^2 e^{2\lambda_i\gamma}$ is a diffeomorphism from $\mathbf{R}$ to $(0, \infty)$.) Next, we define

$$\Phi(R) = \int_{\Sigma_R} \|(\lambda_i x_i)_i\| w(x)\, dS_x. \tag{6.3}$$

Here $(z_i)_i$ is understood as $(z_1, ..., z_N) \in \mathbf{R}^N$, $\|\cdot\|$ is the usual Euclidean norm in $\mathbf{R}^N$ and dS_y, dS_x are the area elements for the manifolds Σ_1 and Σ_R respectively.

We are going to prove a differential inequality of Φ that describes the growth of w.

Lemma 6.5.
$$\frac{\sum_{i=1}^N \lambda_i}{\lambda_N R}\Phi(R) \le \Phi'(R) \le \frac{\sum_{i=1}^N \lambda_i}{\lambda_1 R}\Phi(R)$$

Lemma 6.5 implies $\frac{d}{dR}\big[R^{-\frac{\sum \lambda_i}{\lambda_1}}\Phi(R)\big] \le 0 \le \frac{d}{dR}\big[R^{-\frac{\sum \lambda_i}{\lambda_N}}\Phi(R)\big]$. In particular,

$$(R/R_0)^{\frac{\sum \lambda_i}{\lambda_N}}\Phi(R_0) \le \Phi(R) \le (R/R_0)^{\frac{\sum \lambda_i}{\lambda_1}}\Phi(R_0) \qquad \text{for all } R \ge R_0. \tag{6.4}$$

Remark 6.6. When $\lambda_i = \lambda$ for all i, then the equation possesses radial symmetry. In that case, this Lemma follows immediately from the observation that $\bar{w}$, the spherical mean of w, which solves an ODE,

$$\begin{cases} \bar{w}_{rr} + \frac{N-1}{r}\bar{w}_r - \lambda r \bar{w}_r = 0, \\ \bar{w}_r(0) = 0. \end{cases}$$

must be a constant.

Before we prove Lemma 6.5, we first express $\Phi(R)$ as an integral over Σ_1.

Lemma 6.7.
$$\Phi(R) = \int_{\Sigma_1} e^{\gamma \sum_i \lambda_i} w((y_i e^{\lambda_i \gamma})_i) \|(\lambda_i y_i)_i\| \, dS_y$$

Lemma 6.7 can be obtained by a change of variables and is a direct consequence of Lemma 6.8 below.

Lemma 6.8. *Let $\phi : \Sigma_1 \to \Sigma_R$ be a diffeomorphism defined by $(y_1, ..., y_N) \mapsto (x_1, ..., x_N) = (y_1 e^{\lambda_1 \gamma}, ..., y_N e^{\lambda_N \gamma})$, where $\gamma = \gamma(y, R)$. Then the Jacobian $J\phi(y)$ is given by*
$$J\phi(y) = \frac{e^{\gamma \sum \lambda_i}\|(\lambda_i y_i)_i\|}{\|(\lambda_i y_i e^{\lambda_i \gamma})_i\|}$$

The proof of Lemma 6.8 is postponed till the end of the section. Also, we have

Lemma 6.9.
$$\frac{d\gamma}{dR}(y_1, ..., y_N) = \frac{R}{\|(\lambda_i y_i e^{\lambda_i \gamma})_i\|^2}$$

Proof. Differentiating $\sum \lambda_i y_i^2 e^{2\lambda_i \gamma} = R^2$ with respect to R, we have

$$2\big(\sum \lambda_i^2 y_i^2 e^{2\lambda_i\gamma}\big)\frac{d\gamma}{dR} = 2R$$

Hence,

$$\frac{d\gamma}{dR} = \frac{R}{\|(\lambda_i y_i e^{\lambda_i\gamma})_i\|^2} = \frac{R}{\|(\lambda_i x_i)_i\|^2},$$

where $\gamma = \gamma(y, R)$ and $x_i = y_i e^{\lambda_i \gamma}$, $i = 1, ..., N$. $\square$

Proof of Lemma 6.5. By Lemma 6.7, for any $R > R_0$,

$$\begin{aligned}
\Phi(R) &= \int_{\Sigma_1} w(y_i e^{\lambda_i\gamma}) e^{\gamma\sum\lambda_i}\|(\lambda_i y_i)_i\|\, dS_y \\
\Phi'(R) &= \int_{\Sigma_1} [\nabla w(y_i e^{\lambda_i\gamma})\cdot(\lambda_i y_i e^{\lambda_i\gamma})_i]\frac{d\gamma}{dR} e^{\gamma\sum\lambda_i}\|(\lambda_i y_i)_i\|\, dS_y \\
&\quad + \int_{\Sigma_1} w(y_i e^{\lambda_i\gamma})\frac{d\gamma}{dR}\big(\sum\lambda_i\big) e^{\gamma\sum\lambda_i}\|(\lambda_i y_i)_i\|\, dS_y \\
&= \int_{\Sigma_1} \nabla w(y_i e^{\lambda_i\gamma})\cdot\frac{(\lambda_i x_i)_i}{\|(\lambda_i x_i)_i\|}\cdot\frac{Re^{\gamma\sum\lambda_i}\|(\lambda_i y_i)_i\|}{\|(\lambda_i x_i)_i\|}\, dS_y \\
&\quad + \int_{\Sigma_1} w(y_i e^{\lambda_i\gamma})\big(\sum\lambda_i\big)\frac{R}{\|(\lambda_i x_i)_i\|^2} e^{\gamma\sum\lambda_i}\|(\lambda_i y_i)_i\|\, dS_y \\
&= Re^{R^2}\int_{\Sigma_R} e^{-\sum\lambda_i x_i^2}\frac{\partial w}{\partial\nu}\, dS_x \\
&\quad + \int_{\Sigma_1} w(y_i e^{\lambda_i\gamma})\frac{(\sum\lambda_i)Re^{\gamma\sum\lambda_i}\|(\lambda_i y_i)_i\|}{\|(\lambda_i x_i)_i\|^2}\, dS_y \\
&= \int_{\Sigma_1} w(y_i e^{\lambda_i\gamma})\frac{(\sum\lambda_i)Re^{\gamma\sum\lambda_i}\|(\lambda_i y_i)_i\|}{\|(\lambda_i x_i)_i\|^2}\, dS_y
\end{aligned}$$

where we have made use of Lemmas 6.8 and 6.9 as well as the fact that $e^{-\sum\lambda_i x_i^2} = e^{-R^2}$ on Σ_R for the second last equality. The last equality is a consequence of (6.1). Hence

$$\begin{aligned}
\frac{R\sum\lambda_i}{\max_{\Sigma_R}\|(\lambda_i x_i)_i\|^2}\Phi(R) \le \Phi'(R) &\le \frac{R\sum\lambda_i}{\min_{\Sigma_R}\|(\lambda_i x_i)_i\|^2}\Phi(R) \\
\frac{R\sum\lambda_i}{\max_{\Sigma_R}\sum\lambda_i^2 x_i^2}\Phi(R) \le \Phi'(R) &\le \frac{R\sum\lambda_i}{\min_{\Sigma_R}\sum\lambda_i^2 x_i^2}\Phi(R) \\
\frac{\sum\lambda_i}{R\lambda_N}\Phi(R) \le \Phi'(R) &\le \frac{\sum\lambda_i}{R\lambda_1}\Phi(R)
\end{aligned}$$

The last line is due to $\sum_{i=1}^N \lambda_i x_i^2 = R^2$ on Σ_R, and $0 < \lambda_1 \le ... \le \lambda_N$. $\square$

By virtue of Corrollary 6.4, Proposition 6.1 is a consequence of the following Lemma.

Lemma 6.10. *For all $\epsilon > 0$, there exists $K(\epsilon) > 0$ such that $w(x) \le K(\epsilon) e^{\epsilon \sum \lambda_i x_i^2}$ in $\boldsymbol{R}^N$.*

Proof. Assume to the contrary that there exists $\epsilon_0 > 0$, $R_k \to \infty$ and $z_k = (z_{k,i})_{i=1}^N \in \Sigma_{R_k}$ such that $w(z_k) \ge e^{\epsilon_0 \sum \lambda_i z_{k,i}^2} = e^{\epsilon_0 R_k^2}$. Then apply Theorem 6.2 to $B_4(z_k)$, (with w satisfying (6.2), we have $\Lambda = \lambda = 1$, $\nu = O(R_k)$),

$$w(x) \ge C_1^{-R_k \log R_k} w(z_k) \ge C_1^{-R_k \log R_k} e^{\epsilon_0 R_k^2} \ge e^{\epsilon_1 R_k^2}$$

whenever $|x - z_k| < 1$, for some $C_1 = C(N)$ and $0 < \epsilon_1 < \epsilon_0$ and for all k large. Then

$$\Phi(R_k) = \int_{\Sigma_{R_k}} w(x) \|(\lambda_i x_i)_i\| dS_x \ge C R_k e^{\epsilon_1 R_k^2}.$$

This contradicts the power-like growth obtained in Lemma 6.5. $\square$

Finally, we supply the proof of Lemma 6.8.

Proof of Lemma 6.8. Fix $R > 0$. Let $\bar{y} = (\bar{y}_i)_{i=1}^N \in \Sigma_1$ and $\phi(y) = (\phi_i(y))_{i=1}^N = (y_i e^{\lambda_i \gamma})_{i=1}^N$. (Here $\gamma = \gamma(y)$.) Denote the tangent plane of $\Sigma_1 \subset \mathbf{R}^N$ at $\bar{y}$ after translation to the origin by $T_{\bar{y}}(\Sigma_1)$. Given $\bar{y}' \in T_{\bar{y}}(\Sigma_1)$. To evaluate $[\nabla \phi(\bar{y})](\bar{y}')$, let $y(t) = (y_i(t))_{i=1}^N$ be a smooth curve on Σ_1 such that $y(0) = \bar{y}$ and $y'(0) = \bar{y}' = (\bar{y}_i')_i$. Then by definition of a tangent plane,

$$\begin{aligned}
[\nabla \phi(\bar{y})](\bar{y}') &= \frac{d}{dt}\bigg|_{t=0} \phi(y(t)) \\
&= \frac{d}{dt}\bigg|_{t=0} (y_i(t) e^{\lambda_i \gamma(y(t))})_i \\
&= (\bar{y}_i' e^{\lambda_i \gamma(y(0))})_i + (\lambda_i \bar{y}_i e^{\lambda_i \gamma(y(0))})_i \frac{d\gamma}{dt}(y(0)) \\
&= (\bar{y}_i' e^{\lambda_i \gamma(\bar{y})})_i + (\lambda_i \bar{y}_i e^{\lambda_i \gamma(\bar{y})})_i \frac{d\gamma}{dt}(y(0)) \\
&= P\Big((\bar{y}_i' e^{\lambda_i \gamma(\bar{y})})_i \Big) = P\Big(\Psi\big|_{T_{\bar{y}}(\Sigma_1)}(\bar{y}') \Big)
\end{aligned}$$

where P is the orthogonal projection from $\mathbf{R}^N$ onto $T_{\phi(\bar{y})}(\Sigma_R)$, and $\Psi : \mathbf{R}^N \to \mathbf{R}^N$ is the linear map given by $(y_i)_i \mapsto (y_i e^{\lambda_i \gamma(\bar{y})})_i$. (Since $(\lambda_i \bar{y}_i e^{\lambda_i \gamma(\bar{y})})_i \perp T_{\phi(\bar{y})}(\Sigma_R)$ and $[\nabla \phi(\bar{y})](\bar{y}') \in T_{\phi(\bar{y})}(\Sigma_R)$.)

$$\Psi((\lambda_i \bar{y}_i)_i) = (\lambda_i \bar{y}_i e^{\lambda_i \gamma(\bar{y})})_i = (\lambda_i \phi_i(\bar{y}))_i. \tag{6.5}$$

That is, the normal to $\bar{y}$ with respect to Σ_1 is mapped under Ψ to the normal to $\phi(\bar{y})$ with respect to Σ_R. Now let $\{e_i\}_{i=1}^N$ and $\{\tilde{e}_i\}_{i=1}^N$ be two orthonormal bases such that

$$\begin{cases} \text{span}\{e_1, e_2, ..., e_{N-1}\} = T_{\bar{y}}(\Sigma_1), & e_N = \frac{(\lambda_i \bar{y}_i)_i}{\|(\lambda_i \bar{y}_i)_i\|}; \\ \text{span}\{\tilde{e}_1, \tilde{e}_2, ..., \tilde{e}_{N-1}\} = T_{\phi(\bar{y})}(\Sigma_1), & \tilde{e}_N = \frac{(\lambda_i \phi_i(\bar{y}))_i}{\|(\lambda_i \phi_i(\bar{y}))_i\|}. \end{cases}$$

Then by (6.5), Ψ can be represented by the following matrix

$$\Psi = \begin{bmatrix} & & & 0 \\ & P(\Psi|_{T_{\bar{x}_0}(\Sigma_1)}) & & \vdots \\ & & & 0 \\ a_{N,1} & \dots & a_{N,N-1} & a_{N,N} \end{bmatrix}$$

where $a_{N,N} = \frac{\|(\lambda_i \phi_i(\bar{y}))_i\|}{\|(\lambda_i \bar{y}_i)_i\|}$. Hence,

$$\begin{aligned} \det \Psi &= a_{N,N} \cdot \det(P(\Psi|_{T_{\bar{x}_0}(\Sigma_1)})) \\ &= a_{N,N} \cdot J\phi(\bar{y}) \end{aligned}$$

And so

$$J\phi(\bar{y}) = \frac{\det\Psi}{a_{N,N}} = \frac{e^{\gamma \sum \lambda_i} \|(\lambda_i \bar{y}_i)_i\|}{\|(\lambda_i \phi_i(\bar{y}))_i\|}$$

□

Chapter 7

Limiting Profile: Higher-Dimensional Case

Let $\tilde{u}$ be the unique positive steady-state of (1.6) and $(\tilde{U}, \tilde{V})$ be a coexistence steady-state of (1.4). In this chapter we study the limiting profile of $\tilde{u}$ and $(\tilde{U}, \tilde{V})$ as the strength of the advection term α goes to ∞. As a byproduct, Conjecture 1.3 is resolved under mild condition on m.

For the case when the set of local maximum points of m, $\mathfrak{M}$, is finite, we have the following results. The first one concerns the unique positive steady-states single equation (1.6).

Theorem 7.1. *Assume* **(H1)**, **(H2)** *and* **(H4)**. *Then for all α sufficiently large,* (1.6) *has a unique positive steady-state $\tilde{u}$ which is globally asymptotically stable. Moreover, for all small $r > 0$,*

$$\tilde{u}(x) \to 0 \quad \textit{uniformly and exponentially in } \Omega \setminus [\cup_{x_0 \in \mathfrak{M}} B_r(x_0)].$$

And for each $x_0 \in \mathfrak{M}$,

$$\tilde{u}(x) - 2^{N/2} \max\{p(x_0), 0\} e^{\alpha[m(x)-m(x_0)]/d} \to 0 \quad \textit{uniformly in } B_r(x_0).$$

For the coexistence steady-states $(\tilde{U}, \tilde{V})$ of the system (1.4), we have

Theorem 7.2. *Assume* **(H1)**, **(H2)** *and* **(H3)**. *Then, for all α sufficiently large,* (1.4) *has at least one stable coexistence steady-state. Moreover, if $(\tilde{U}, \tilde{V})$ is any coexistence steady-state of* (1.4), *then as $\alpha \to \infty$,* *(i)* $\tilde{V}(x) \to \bar{\theta}_{d_2}(x)$ *in* $C^{1,\beta}(\bar{\Omega})$*, for any* $\beta \in (0,1)$*;*

(ii) for all $r > 0$*,* $\tilde{U}(x) \to 0$ *in* $\Omega \setminus [\cup_{x_0 \in \mathfrak{M}} B_r(x_0)]$ *uniformly and exponentially;*

(iii) for each $x_0 \in \mathfrak{M}$ *and each* $r > 0$ *small,*

$$\tilde{U}(x) - 2^{N/2} \max\{p(x_0) - \bar{\theta}_{d_2}(x_0), 0\} e^{\alpha[m(x)-m(x_0)]/d_1} \to 0 \text{ uniformly in } B_r(x_0).$$

Here $\mathfrak{M}$ denotes the set of all local maximum points of m and $\bar{\theta}_d$ is the unique positive solution to (1.5).

7.1 Proof of Theorem 7.1

Here we prove Theorem 7.1. In fact, for later purposes, we are going to establish the result for the more general equation (2.1):

Theorem 7.3. *In addition to the assumptions of Theorem* 7.1*, assume* (2.2)*. Then for all* α *sufficiently large,* (2.1) *has a unique positive solution* $\tilde{u}$*.* $\tilde{u}$ *is globally asymptotically stable, and for all small* $r > 0$*,* $\tilde{u}(x) \to 0$ *uniformly and exponentially in* $\Omega \setminus \cup_{x_0 \in \mathfrak{M}} B_r(x_0)$*. Moreover, for each* $x_0 \in \mathfrak{M}$*,*

$$\tilde{u}(x) - 2^{N/2} \max\{p(x_0), 0\} e^{\alpha[m(x)-m(x_0)]/d} \to 0 \tag{7.1}$$

uniformly in $B_r(x_0)$*.*

It is easy to see that Theorem 7.1 is a consequence of Theorem 7.3. We first recall the following useful notation.

$$\frac{1}{K}|z-x|^2 \le m(z) - m(x) \le K|\nabla m(x)|^2 \le K^2|z-x|^2 \quad \forall x \in B_r(z), \forall z \in \mathfrak{M}. \tag{7.2}$$

We first apply Proposition 1.9 to obtain the limiting profile of $\tilde{u}$.

Proposition 7.4. *For each* $R > 0$ *and each* $x_0 \in \mathfrak{M}$*,*

$$|\tilde{u}(x)e^{\alpha[m(x_0)-m(x)]/d} - \tilde{u}(x_0)|_{L^\infty(B_{R/\sqrt{\alpha}}(x_0))} \to 0 \qquad \text{as } \alpha \to \infty. \tag{7.3}$$

Proof. Since $\alpha[m(x_0+\sqrt{d/\alpha}y)-m(x_0)] \to \frac{1}{2}y^TD^2m(x_0)y$ uniformly on compact subsets of $\mathbf{R}^N$, it suffice to show that for each $x_0 \in \mathfrak{M}$ and $y \in \{y' \in \mathbf{R}^N : x_0+\sqrt{\frac{d}{\alpha}}y' \in \Omega\}$,

$$\tilde{u}(x_0+\sqrt{\frac{d}{\alpha}}y)e^{-\frac{1}{2}y^TD^2m(x_0)y} - \tilde{u}(x_0) \to 0, \qquad \text{as } \alpha \to \infty,$$

uniformly in every compact subset of $\mathbf{R}^N$. Now let

$$w_\alpha(y) = \tilde{u}(x_0+\sqrt{\frac{d}{\alpha}}y)e^{-\frac{1}{2}y^TD^2m(x_0)y}.$$

Then w_α satisfies the equation

$$\Delta_y w + \tilde{P}\cdot\nabla_y w + \tilde{Q}w = 0 \quad \text{in } \sqrt{\frac{\alpha}{d}}(\Omega - x_0),$$

where

$$\begin{aligned}
\tilde{P} =& 2y^TD_x^2m(x_0) - \sqrt{\frac{\alpha}{d}}\nabla_x m(x_0+\sqrt{\frac{d}{\alpha}}y),\\
\tilde{Q} =& \Delta_x m(x_0) - \Delta_x m(x_0+\sqrt{\frac{d}{\alpha}}y) + |D_x^2m(x_0)y|^2 - y^TD_x^2m(x_0)\cdot\sqrt{\frac{\alpha}{d}}\nabla_x m(x_0+\sqrt{\frac{d}{\alpha}}y)\\
&- [\tilde{u}(x_0+\sqrt{\frac{d}{\alpha}}y) - p_\alpha(x_0+\sqrt{\frac{d}{\alpha}}y)]/\alpha.
\end{aligned}$$

By Theorem 3.3 and the fact that $\sqrt{\frac{\alpha}{d}}\nabla_x m(x_0+\sqrt{\frac{d}{\alpha}}y) \to D_x^2m(x_0)y$ uniformly on compact sets of $\mathbf{R}^N$, we have $\lim_{\alpha\to\infty}\tilde{P} = y^TD_x^2m(x_0)$ and $\lim_{\alpha\to\infty}\tilde{Q} = 0$ uniformly in compact subsets of $\mathbf{R}^N$ as $\alpha \to \infty$. Hence by elliptic L^p estimates (by Theorem 3.3 w_α is bounded in $L^\infty(K)$ uniformly in α for each compact subset K in $\mathbf{R}^N$), after passing to a subsequence if necessary, w_α converges to some limit w_0 uniformly in every compact subset of $\mathbf{R}^N$. This w_0 satisfy

$$\left\{\ \nabla\cdot(e^{\frac{1}{2}y^TD_x^2m(x_0)y}\nabla w_0) = 0 \text{ in } \mathbf{R}^N, w_0(0) < \infty, w_0(y) \geq 0 \text{ in } \mathbf{R}^N,\right.$$

which must be a constant by Proposition 1.9. Now if for some subsequence $\alpha_k \to \infty$, $u_{\alpha_k}(x_0) = w_{\alpha_k}(0)$ converges as $k \to \infty$, then $w_{\alpha_k}(x) - u_{\alpha_k}(x_0) \to 0$ uniformly on compact subsets of $\mathbf{R}^N$. The conclusion now follows from the uniqueness of the limit. □

To get the complete profile of $\tilde{u}$, it suffices to calculate the exact limit of $\tilde{u}(x_0)$. We shall prove the following in a series of lemmas.

Proposition 7.5. *For each* $x_0 \in \mathfrak{M}$, $\lim\limits_{\alpha\to\infty} \tilde{u}(x_0) = \max\{2^{N/2}p(x_0), 0\}$. *where* $p(x) = \lim_{\alpha\to\infty} p_\alpha(x)$ *for all* $x \in \Omega$.

Lemma 7.6. *Given* $\delta > 0$ *small,*

$$\left|\sum_{x_0\in\mathfrak{M}} \int_{B_R(0)} \tilde{u}^2(x_0 + \sqrt{\frac{d}{\alpha}}y) - p_\alpha(x_0 + \sqrt{\frac{d}{\alpha}}y)\tilde{u}(x_0 + \sqrt{\frac{d}{\alpha}}y)dy\right| < \delta$$

for all R, α *sufficiently large.*

Proof. By Theorem 3.3, given $r_1 > 0$, there exist $C, \gamma > 0$ such that

$$\tilde{u} \le \begin{cases} Ce^{\gamma\alpha[m(x)-m(x_0)]/d} & \text{in } \cup_{x_0\in\mathfrak{M}} B_{r_1}(x_0) \\ e^{-\gamma\alpha} & \text{in } \Omega \setminus \cup_{x_0\in\mathfrak{M}} B_{r_1}(x_0) \end{cases}$$

By integrating (2.1) over Ω, we have $\int_\Omega \tilde{u}^2 - p_\alpha \tilde{u}\, dx = 0$, so

$$\left|\sum_{x_0\in\mathfrak{M}} \int_{B_{R\sqrt{d/\alpha}}(x_0)} \tilde{u}^2 - p_\alpha\tilde{u}\, dx\right| \le \sum_{x_0\in\mathfrak{M}} \int_{B_{r_1}(x_0)\setminus B_{R\sqrt{d/\alpha}}(x_0)} |\tilde{u}^2 - p_\alpha\tilde{u}|\, dx + O(e^{-\gamma\alpha})$$

Multiply by $\alpha^{N/2}$ and changing coordinates $x = x_0 + \sqrt{\frac{d}{\alpha}}y$, we have, for α sufficiently large,

$$\begin{aligned} &\left|\sum_{x_0\in\mathfrak{M}} \int_{B_R(0)} \tilde{u}^2(x_0 + \sqrt{\frac{d}{\alpha}}y) - p_\alpha(x_0 + \sqrt{\frac{d}{\alpha}}y)\tilde{u}(x_0 + \sqrt{\frac{d}{\alpha}}y)dy\right| \\ &\le C(\alpha/d)^{N/2} \sum_{x_0\in\mathfrak{M}} \int_{B_{r_1}(x_0)\setminus B_{R\sqrt{d/\alpha}}(x_0)} |\tilde{u}^2 - p_\alpha\tilde{u}|\, dx + O(\alpha^{N/2}e^{-\epsilon'\alpha}) \\ &\le C(\alpha/d)^{N/2} \sum_{x_0\in\mathfrak{M}} \int_{B_{r_1}(x_0)\setminus B_{R\sqrt{d/\alpha}}(x_0)} e^{\epsilon\alpha[m(x)-m(x_0)]/d}dx + O(\alpha^{N/2}e^{-\epsilon'\alpha}) \\ &\le C\int_{\mathbf{R}^N\setminus B_R(0)} e^{-\epsilon|y|^2/K}dy + O(\alpha^{N/2}e^{-\epsilon'\alpha}) \\ &< \delta \end{aligned}$$

if α, R are sufficiently large. The third inequality from (7.2).

□

Lemma 7.7.

$$\lim_{\alpha\to\infty}\sum_{x_0\in\mathfrak{M}}\int_{\boldsymbol{R}^N} e^{\frac{1}{2}y^T D^2 m(x_0) y}dy\left[\tilde{u}^2(x_0)-2^{N/2}p_\alpha(x_0)\tilde{u}(x_0)\right]=0.$$

Proof. Since $|\tilde{u}|_{L^\infty(\Omega)}$ is uniformly bounded, by compactness of bounded sequences in $\mathbf{R}$ it suffices to show that for any $\delta>0$ and for any sequence $\alpha_k\to\infty$ such that the associated $\lim_{k\to\infty}u_{\alpha_k}(x_0)$ converges at each $x_0\in\mathfrak{M}$, it holds that (writing $\alpha=\alpha_k$)

$$|\sum_{x_0\in\mathfrak{M}}\int_{\mathbf{R}^N} e^{\frac{1}{2}y^T D^2 m(x_0)y}dy[2^{-N/2}\tilde{u}^2(x_0)-p_\alpha(x_0)\tilde{u}(x_0)]\big|<3\delta,$$

for all $\alpha=\alpha_k$ large. Now,

$$\begin{aligned}
&|\sum_{x_0\in\mathfrak{M}}\int_{\mathbf{R}^N} e^{\frac{1}{2}y^T D^2 m(x_0)y}dy[2^{-N/2}\tilde{u}^2(x_0)-p_\alpha(x_0)\tilde{u}(x_0)]\big|\\
=&|\sum_{x_0\in\mathfrak{M}}\int_{\mathbf{R}^N}\left[e^{y^T D^2 m(x_0)y}\tilde{u}^2(x_0)-e^{\frac{1}{2}y^T D^2 m(x_0)y}p_\alpha(x_0)\tilde{u}(x_0)\right]dy\big|\\
\le&|\sum_{x_0\in\mathfrak{M}}\int_{B_R(0)}\left[e^{y^T D^2 m(x_0)y}\tilde{u}^2(x_0)-e^{\frac{1}{2}y^T D^2 m(x_0)y}p_\alpha(x_0)\tilde{u}(x_0)\right]dy\big|+\delta\\
\le&|\sum_{x_0\in\mathfrak{M}}\int_{B_R(0)}\left[\tilde{u}^2(x_0+\sqrt{\frac{d}{\alpha}}y)-p_\alpha(x_0)\tilde{u}(x_0+\sqrt{\frac{d}{\alpha}}y)\right]dy\big|\\
&+\sum_{x_0\in\mathfrak{M}}\int_{B_R}\Bigg[|\tilde{u}^2(x_0)e^{y^T D^2 m(x_0)y}-\tilde{u}^2(x_0+\sqrt{\frac{d}{\alpha}}y)|\\
&+|p_\alpha(x_0)||\tilde{u}(x_0+\sqrt{\frac{d}{\alpha}}y)-\tilde{u}(x_0)e^{\frac{1}{2}y^T D^2 m(x_0)y}|\Bigg]dy+\delta\\
<&3\delta.
\end{aligned}$$

The first inequality holds by fixing $R=R(\delta)$ large and the strict inequality follows from Proposition 7.4 and Lemma 7.6. □

Lemma 7.8. *For each* $x_0\in\mathfrak{M}$, $\liminf_{\alpha\to\infty}\tilde{u}(x_0)\ge 2^{N/2}\max\{p(x_0),0\}$.

Proof. If $p(x_0)\le 0$ there is nothing to prove. Now let $p(x_0)>0$, $\tilde{u}$ be the unique solution to (2.1), and fix $x_0\in\mathfrak{M}$. For each α large, $\tilde{u}$ is the principal eigenfunction of the following eigenvalue problem with principal eigenvalue $\mu_1 = 0$.

$$\begin{cases} \nabla \cdot (d\nabla\phi - \alpha\phi\nabla m) + (p_\alpha - \tilde{u})\phi + \mu\phi = 0 & \text{in } \Omega, \\ d\frac{\partial\phi}{\partial\nu} - \alpha\phi\frac{\partial m}{\partial\nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{7.4}$$

By the transformation $\phi = e^{\alpha m/d}\psi$, (7.4) is equivalent to the self-adjoint problem

$$\begin{cases} \nabla \cdot (de^{\alpha m/d}\nabla\psi) + (p_\alpha - \tilde{u})e^{\alpha m/d}\psi + \mu e^{\alpha m/d}\psi = 0 & \text{in } \Omega, \\ \frac{\partial\psi}{\partial\nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{7.5}$$

with principal eigenvalue $\mu_1 = 0$. The variational characterization of problem (7.5) implies

$$0 = \inf_{\psi\in H^1(\Omega)} \left\{ \frac{\int_\Omega e^{\alpha m/d}[d|\nabla\psi|^2 + (u_\alpha - p_\alpha)\psi^2]dx}{\int_\Omega e^{\alpha m/d}\psi^2 dx} \right\}.$$

For any $B_r(x_0)$ with $r > 0$ small and $0 < \zeta < 1/2$, by the nondegeneracy of $m(x)$ at x_0, we have

$$m(x_0) > \max_{\bar{B}_r(x_0)\setminus B_{(1-\zeta)r}(x_0)} m := M_1.$$

Now take $\zeta > 0$ even smaller such that $M_2 := \min_{\bar{B}_{\zeta r}(x_0)} m > M_1$, which is possible by (7.2). Take a smooth test function ψ such that

$$\psi(x) = \begin{cases} 1, & \text{if } x \in B_{(1-\zeta)r}(x_0), \\ 0, & \text{if } x \in \mathbf{R}^N \setminus B_r(x_0), \end{cases} \quad 0 \le \psi \le 1,\ |D\psi| < \frac{2}{\zeta r}.$$

Then,

$$\begin{aligned} 0 &\le \frac{\int_\Omega e^{\alpha m/d}[d|\nabla\psi|^2 + (\tilde{u} - p_\alpha)\psi^2]dx}{\int_\Omega e^{\alpha m/d}\psi^2 dx} \\ &\le \frac{\int_{B_r(x_0)} de^{\alpha M_1/d}(\frac{2}{\zeta r})^2\, dx}{\int_{B_{\zeta r}(x_0)} e^{\alpha M_2/d}\, dx} + \frac{\int_{B_{\zeta r}(x_0)} e^{\alpha m/d}(\tilde{u} - p_\alpha)\psi^2\, dx}{\int_{B_r(x_0)} e^{\alpha m/d}\psi^2\, dx} \\ &\le \frac{|B_r|}{|B_{\zeta r}|}\frac{4d}{(\zeta r)^2}e^{\alpha(M_1 - M_2)/d} + \frac{\int_{B_r(x_0)} e^{\alpha[m - m(x_0)]/d}(\tilde{u} - p_\alpha)\, dx}{\int_{B_{(1-\zeta)r}(x_0)} e^{\alpha[m-m(x_0)]/d}\, dx}. \end{aligned}$$

This implies that

$$\liminf_{\alpha\to\infty} \frac{\int_{B_r(x_0)} e^{\alpha[m(x)-m(x_0)]/d}[\tilde{u}(x) - p_\alpha(x)]dx}{\int_{B_{(1-\zeta)r}(x_0)} e^{\alpha[m(x)-m(x_0)]/d}\, dx} \ge 0. \tag{7.6}$$

By Lebesgue's dominated convergence and (2.2),

$$\lim_{\alpha\to\infty} \frac{\int_{B_r(x_0)} e^{\alpha[m(x)-m(x_0)]/d} p_\alpha(x)\,dx}{\int_{B_{(1-\zeta)r}(x_0)} e^{\alpha[m(x)-m(x_0)]/d}\,dx} = p(x_0).$$

By Proposition 7.4 and Theorem 3.3, for each $R \geq R_0$ and $\eta > 0$, for all α large, we have

$$\tilde{u} \leq \begin{cases} (1+\eta)\tilde{u}(x_0) e^{\alpha[m(x)-m(x_0)]/d} & \text{in } B_{R\sqrt{d/\alpha}(x_0)}, \\ Ce^{\gamma\alpha[m(x)-m(x_0)]/d} & \text{in } B_{r_1}(x_0) \setminus B_{R\sqrt{d/\alpha}(x_0)}. \end{cases}$$

where γ is given in the statement of Theorem 3.3. Therefore, for any $\eta > 0$ small,

$$\begin{aligned}
&\lim_{\alpha\to\infty} \frac{\int_{B_r(x_0)} e^{\alpha[m-m(x_0)]/d} p_\alpha dx}{\int_{B_{(1-\zeta)r}(x_0)} e^{\alpha[m-m(x_0)]/d} dx} \\
&\leq \liminf_{\alpha\to\infty} \frac{\int_{B_r(x_0)} e^{\alpha[m-m(x_0)]/d} \tilde{u} dx}{\int_{B_{(1-\zeta)r}(x_0)} e^{\alpha[m-m(x_0)]/d} dx} \\
&\leq \liminf_{\alpha\to\infty} (1+\eta)\tilde{u}(x_0) \frac{\int_{B_{R\sqrt{d/\alpha}}(x_0)} e^{2\alpha[m-m(x_0)]/d} dx}{\int_{B_{(1-\zeta)r}(x_0)} e^{\alpha[m-m(x_0)]/d} dx} + C \frac{\int_{\Gamma_{\alpha,R}(x_0)} e^{\gamma\alpha[m-m(x_0)]/d} dx}{\int_{B_{(1-\zeta)r}(x_0)} e^{\alpha[m-m(x_0)]/d} dx} \\
&\leq (1+\eta)\big(\liminf_{\alpha\to\infty} \tilde{u}(x_0)\big)\Big(\lim_{\alpha\to\infty} \frac{\int_{B_{(1-\zeta)r}(x_0)} e^{2\alpha[m-m(x_0)]/d} dx}{\int_{B_{(1-\zeta)r}(x_0)} e^{\alpha[m-m(x_0)]/d} dx}\Big) + C \frac{\int_{\mathbf{R}^N\setminus B_R} e^{-\gamma|y|^2/K}\,dy}{\int_{B_1} e^{-K|y|^2}\,dy}
\end{aligned}$$

where K is given in (7.2) and the second inequality follows from (7.2). Since

$$\lim_{\alpha\to\infty} \frac{\int_{B_{\zeta(1-\zeta)r}(x_0)} e^{2\alpha[m-m(x_0)]/d} dx}{\int_{B_{(1-\zeta)r}(x_0)} e^{\alpha[m-m(x_0)]/d} dx} = 2^{-N/2},$$

taking $R \to \infty$ and $\eta \to 0^+$, we have

$$2^{-N/2} \liminf_{\alpha\to\infty} \tilde{u}(x_0) \geq p(x_0).$$

□

Proposition 7.5 follows from Lemmas 7.7 and 7.8.

Proof of Theorem 7.3. The existence, uniqueness and global stability is proved in

Theorem 2.1. By Theorem 3.3, we have $\tilde{u} \to 0$ uniformly and exponentially in any compact subset of $\Omega \setminus \mathfrak{M}$. Finally, (7.1) follows from Theorem 3.3, Propositions 7.4 and 7.5. $\square$

7.2 Proof of Theorem 1.4

Assume **(H1)**, **(H2)** and **(H3)**.

Lemma 7.9. *Let* $(\tilde{U}, \tilde{V})$ *be a coexistence steady-state of* (1.4)*, then*

$$0 < \tilde{U} \leq \tilde{u}, \quad 0 < \tilde{V} \leq \bar{\theta}_{d_2}$$

where $\tilde{u}$ *is the unique positive steady-state of* (1.6) *and* $\bar{\theta}_{d_2}$ *is the unique positive solution of* (1.5)*.*

Proof. $\tilde{U}$ satisfies

$$\begin{cases} \nabla \cdot (d_1 \nabla U - \alpha U \nabla m) + U(p - U) = U\tilde{V} \geq 0, & \text{in } \Omega, \\ d_1 \frac{\partial U}{\partial \nu} - \alpha U \frac{\partial m}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

and is a lower solution of (1.6). Hence $\tilde{U} \leq \tilde{u}$ by the global asymptotic stability of $\tilde{u}$. The inequality $\tilde{V} \leq \bar{\theta}_{d_2}$ holds for similar reasons. $\square$

Lemma 7.10. *Let* $(\tilde{U}, \tilde{V})$ *be a coexistence steady-state of* (1.4)*, then for any* $\beta \in (0, 1)$*,*

$$\lim_{\alpha \to \infty} |\tilde{V} - \bar{\theta}_{d_2}|_{C^{1,\beta}(\bar{\Omega})} = 0.$$

Proof. Let $\overline{U} \in C^\beta(\bar{\Omega})$, $\alpha_0 > 0$ be fixed such that $0 < \tilde{U} \leq \tilde{u} \leq \overline{U}$ for all $\alpha \geq \alpha_0$ and $\int_\Omega p - \overline{U} > 0$. (The existence of $\overline{U}$ and α_0 follows from Theorem 3.3 and that $\int_\Omega p > 0$ by **(H3)**.) Let $\underline{V}$ be the unique positive solution to

$$\begin{cases} d_2 \Delta V + V(p - \overline{U} - V) = 0 & \text{in } \Omega, \\ \frac{\partial V}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{7.7}$$

The existence is standard. (See e.g. Lemma 7.1 in [CL].) Then

$$d_2 \Delta \underline{V} + \underline{V}(p - \tilde{U} - \underline{V}) = \underline{V}(\overline{U} - \tilde{U}) \geq 0,$$

and so $\underline{V}$ is a lower solution of the single equation

$$\begin{cases} d_2\Delta V + V(p - \tilde{U} - V) = 0 \text{ in } \Omega, \\ \frac{\partial V}{\partial \nu} = 0 \text{ on } \partial\Omega. \end{cases} \tag{7.8}$$

of which $\tilde{V}$ is the globally asymptotically stable solution. Hence

$$0 < \underline{V} \leq \tilde{V}. \tag{7.9}$$

Now by Lemma 7.9, $|\tilde{U}|_{L^\infty(\Omega)}$ and $|\tilde{V}|_{L^\infty(\Omega)}$ are uniformly bounded in α. Therefore, by elliptic L^p estimates applied to the equation $d_2\Delta\tilde{V} + \tilde{V}(p - \tilde{U} - \tilde{V}) = 0$, $\{\tilde{V}\}_\alpha$ is bounded in $W^{2,p}(\Omega)$ for all $p \geq 1$. Therefore, by possibly passing to a subsequence, we can assume $\tilde{V} \to V_0$ for some $V_0 > 0$ satisfying (1.5). From (7.9) the fact that $\bar{\theta}_{d_2}$ is the unique positive solution of (1.5), for any $\beta \in (0,1)$, $\tilde{V} \to \bar{\theta}_{d_2}$ in $C^{1,\beta}(\overline{\Omega})$ as $\alpha \to \infty$. □

Proof of Theorem 1.4. The existence and stability of $(\tilde{U}, \tilde{V})$ are proved in Theorem 4.1. (i) is proved by Lemma 7.10. Whereas (ii) is a consequence of Theorem 3.3.

To finish the proof of Theorem 1.4. it remains to show that for each $x_0 \in \mathfrak{M}$,

$$\left| \tilde{U} - 2^{N/2} \max\{p(x_0) - \bar{\theta}_{d_2}(x_0), 0\} e^{\alpha[m(x) - m(x_0)]/d_1} \right|_{L^\infty(B_r(x_0))} \to 0.$$

But this follows from Theorem 7.3 and Lemma 7.10 by taking p_α to be $p - \tilde{V}$. □

Chapter 8

Uniqueness of Coexistence State $(\tilde{U}, \tilde{V})$

In this chapter we shall present a uniqueness result for (1.1) By Theorem 1.3 of [CL] (See also Theorem 4.1), for α sufficiently large, (1.1) has at least one stable coexistence steady state $(\tilde{U}, \tilde{V})$. Inspired by the arguments in [CL], we shall prove that under certain assumptions on $m(x)$, $(\tilde{U}, \tilde{V})$ is in fact **unique**! An immediate consequence of uniqueness is the globally asymptotic stability of , $(\tilde{U}, \tilde{V})$. We first state the assumptions on $m(x)$. Let $\mathfrak{M}$ be the set of all local maximum points of $m(x)$.

(A) $m \in C^2(\overline{\Omega})$, such that all critical points are non-degenerate, and $\mathfrak{M} \subset \Omega$. $\frac{\partial m}{\partial \nu} \leq 0$ on $\partial\Omega$ and $\Delta m(x_0) > 0$ at every local minimum or saddle points of m.

(B) $m \equiv m_0$ on $\mathfrak{M}$.

In particular, $\mathfrak{M}$ is finite and located in the interior of Ω. Our main result is

Theorem 8.1. *If $m(x)$ satisfies (A) and (B), then for all α sufficiently large, there exists a unique co-existence steady state $(\tilde{U}, \tilde{V})$ for* (1.1)*. Moreover, $(\tilde{U}, \tilde{V})$ is globally asymptotically stable.*

8.1 Proof of Theorem 8.1

We have the following theorem contained in [L1] concerning the limiting profile of $(\tilde{U}, \tilde{V})$. The proof of the theorem will be included here for completeness.

Theorem 8.2 ([L1]). *If $m(x)$ satisfies (A) and (B) then there exists $C > 0$ such that for any co-existence steady state $(\tilde{U}, \tilde{V})$,*

$$0 < \tilde{U} \leq Ce^{\alpha[m(x)-m_0]/d_1} \tag{8.1}$$

and for any $x_0 \in \mathfrak{M}$, and any $R > 0$,

$$\lim_{\alpha\to\infty} \|\tilde{U}e^{\alpha[m(x_0)-m(x)]/d_1} - 2^{N/2}[m(x_0) - \theta(x_0)]\|_{L^\infty(B_{R/\sqrt{\alpha}}(x_0))} = 0 \tag{8.2}$$

$$\lim_{\alpha\to\infty} \|\tilde{V} - \theta_{d_2}\|_{C^{1,\beta}(\overline{\Omega})} = 0 \text{ for all } \beta \in (0,1). \tag{8.3}$$

It suffices to show the following lemma and the rest follows from Chapter 7.

Lemma 8.3. *With the assumption of Theorem 8.2. Let $\tilde{u}$ be the unique positive steady-state of (1.6), then there exists $C > 0$ such that*

$$\tilde{u}(x) \leq Ce^{\alpha[m(x)-m_0]/d} \qquad \text{for all } x \in \Omega \text{ and all } \alpha \text{ large.} \tag{8.4}$$

Proof. Consider $w = e^{(-\alpha+\epsilon)m(x)/d}\tilde{u}(x)$. Then in Ω, w satisfies

$$d\Delta w + (\alpha - 2\epsilon)\nabla m \cdot \nabla w - \{\epsilon(\alpha-\epsilon)|\nabla m|^2/d + \epsilon\Delta m + u - m\}w = 0 \tag{8.5}$$

Let $z^* = z^*(\alpha) \in \overline{\Omega}$ be such that $w(z^*) = \max_{\overline{\Omega}} w$. Then, for $x \in \Omega$,

$$u(x) \leq u(z^*)e^{(-\alpha+\epsilon)[m(z^*)-m(x)]/d}. \tag{8.6}$$

We notice that on $\partial\Omega$,

$$\begin{aligned}\frac{\partial w}{\partial \nu} &= e^{(-\alpha+\epsilon)m/d}(d\frac{\partial \tilde{u}}{\partial \nu} + (-\alpha+\epsilon)\tilde{u}\frac{\partial m}{\partial \nu}) \\ &= e^{(-\alpha+\epsilon)m/d}(\alpha\tilde{u}\frac{\partial m}{\partial \nu} + (-\alpha+\epsilon)\tilde{u}\frac{\partial m}{\partial \nu}) \\ &= e^{(-\alpha+\epsilon)m/d}\epsilon\frac{\partial m}{\partial \nu} \leq 0.\end{aligned}$$

Therefore by the maximum principle, no matter $z^* \in \partial\Omega$ or Ω, $\nabla w(z^*) = 0$ and $\Delta w(z^*) \leq 0$. Hence, by (8.5)

$$\frac{\epsilon(\alpha-\epsilon)}{d}|\nabla m|^2 + \epsilon\Delta m + u \leq m \qquad \text{at } x = z^*, \tag{8.7}$$

and

$$u(z^*) \leq m(z^*) - \epsilon \Delta m(z^*). \tag{8.8}$$

Now take $\epsilon = \max_{x_0}\{\frac{m(x_0)}{\Delta m(x_0)}\}$, with the maximum taken over all positive saddle points and local minimum points x_0 of $m(x)$ such that $m(x_0) > 0$. (Take $\epsilon = 1$ if it is an empty set.) Notice that $\epsilon > 0$ by (A). Then by (8.7), we have

$$\frac{\epsilon(\alpha - \epsilon)|\nabla m|^2}{d} \leq m(z^*) - \epsilon \Delta m \leq |m|_\infty + \epsilon |\Delta m|_\infty,$$

which implies that $|\nabla m(z^*)| \to 0$ as $\alpha \to \infty$. Thus,

$$\operatorname{dist}(z^*, \{x \in \Omega : |\nabla m(x)| = 0\}) \to 0.$$

Next, we claim that in fact we have $\operatorname{dist}(z^*, \mathfrak{M}) \to 0$.

Assume to the contrary that there exists $\alpha_k \to \infty$, such that $z^*(\alpha_k) \to x_0$ as $k \to \infty$ where x_0 is a saddle point or a minimum point. Then by (8.8) and the choice of ϵ,

$$0 \leq u(z^*) \leq m(z^*) - \epsilon \Delta m(z^*) \to m(x_0) - \epsilon \Delta m(x_0) < 0,$$

which is a contradiction. Therefore, $\operatorname{dist}(z^*, \mathfrak{M}) \to 0$. Recalling that $m(x) \equiv m_0$ on $\mathfrak{M}$, by (7.2) we deduce that there exists $C > 0$ such that

$$m_0 - m(z^*) \leq C|\nabla m(z^*)|^2, \text{ for all } \alpha \text{ large},$$

since the inequality holds in a neighborhood of $\mathfrak{M}$, where z^* eventually enters. Hence by (8.7) again,

$$(\alpha - \epsilon)(m_0 - m(z^*)) \leq \frac{C(\alpha - \epsilon)|\nabla m(z^*)|^2}{d} \leq \frac{C}{d}\big(\frac{m(z^*)}{\epsilon} - \Delta m(z^*)\big).$$

Therefore,

$$(\alpha - \epsilon)(m_0 - m(z^*)) \leq \frac{C}{d}\big(\frac{m_1}{\epsilon} + \|\Delta m\|_\infty\big) \tag{8.9}$$

And for every $x \in \Omega$, from (8.6),

$$\begin{aligned} e^{-\alpha[m(x)-m_0]/d}\tilde{u}(x) &\leq e^{-\alpha[m(x)-m_0]/d}\tilde{u}(z^*)e^{(\alpha-\epsilon)[m(x)-m(z^*)]/d} \\ &= \tilde{u}(z^*)e^{\epsilon[m_0-m(x)]/d+(\alpha-\epsilon)[m_0-m(z^*)]/d} \\ &\leq (m_0+\epsilon\|\Delta m\|_\infty)e^{2\epsilon|m|_\infty/d+C(\frac{m_0}{\epsilon}+\|\Delta m\|_\infty)/d}, \end{aligned}$$

by (8.8) and (8.9). Since the right hand side is a constant independent of x and α, (8.4) is proved.

□

Following the approach in [Lo2], we will first show that every co-existence steady states is stable. Then the uniqueness and the global asymptotic stability follows from the existence of connecting orbit ([H]), and the stability/instability of the three steady states $(\tilde{u},0)$, $(0,\theta_{d_2})$ and $(\tilde{U},\tilde{V})$. (Here θ_{d_2} is the unique positive solution to (8.3).)

The instability of semitrivial states are proved is contained in the proof of Theorem 4.1.

Theorem 8.4. *For all α sufficiently large, $(\tilde{u},0)$, $(0,\theta_{d_2})$ are both unstable.*

Now let $(\tilde{U},\tilde{V})$ be any coexistence state of (1.1). To study the stability/instability of $(\tilde{U},\tilde{V})$, we linearize (1.1) at $(\tilde{U},\tilde{V})$.

$$\begin{cases} \nabla\cdot(d_1\nabla\tilde{\phi}-\alpha\tilde{\phi}\nabla m)+(m-2\tilde{U}-\tilde{V})\tilde{\phi}-\tilde{U}\tilde{\psi}=-\tilde{\lambda}\tilde{\phi} & \text{in } \Omega \\ d_2\Delta\tilde{\psi}-\tilde{V}\tilde{\phi}+(m-\tilde{U}-2\tilde{V})\tilde{\psi}=-\tilde{\lambda}\tilde{\psi} & \text{in } \Omega \\ d_1\frac{\partial\tilde{\phi}}{\partial\nu}-\alpha\tilde{\phi}\frac{\partial m}{\partial\nu}=\frac{\partial\tilde{\psi}}{\partial\nu}=0 & \text{on } \partial\Omega \end{cases} \tag{8.10}$$

By Krein-Rutman Theorem, there exists a principal eigenvalue $\tilde{\lambda}_1 \in \mathbf{R}$ of (8.10) with eigenfunction $(\tilde{\phi},\tilde{\psi})$ such that $\tilde{\phi} > 0 > \tilde{\psi}$. And if $\tilde{\lambda} \in \mathbb{C}$ is any eigenvalue of (8.10), then $\tilde{\lambda}_1 \leq \mathrm{Re}(\tilde{\lambda})$. In other words, the sign of $\tilde{\lambda}_1$ determines the linear stability of $(\tilde{U},\tilde{V})$. Thus, Theorem 8.1 follows from the following two results.

Theorem 8.5. *If m satisfies* (A) *and* (B)*, then $\tilde{\lambda}_1 > 0$ for all α sufficiently large.*

To prove Theorem 8.5, we first study the principal eigenvalue λ_1 of the following eigenvalue problem.

$$\begin{cases} \nabla \cdot (d_1 \nabla \varphi - \alpha \varphi \nabla m) + (m - 2\tilde{U} - \tilde{V})\varphi = -\lambda \varphi & \text{in } \Omega \\ d_1 \frac{\partial \varphi}{\partial \nu} - \alpha \varphi \frac{\partial m}{\partial \nu} = 0 & \text{on } \partial \Omega \end{cases} \tag{8.11}$$

Lemma 8.6. *If m satisfies (A), then* $\lambda_1 \to \min_{x_0 \in \mathfrak{M}} |m(x_0) - \theta_{d_2}(x_0)|$.

We will prove Lemma 8.6 in the next section. Assuming Lemma 8.6, we now prove Theorem 8.5.

Proof of Theorem 8.5. For each α large, let $\lambda_1 \in \mathbb{R}$ be the principal eigenvalue of (8.10) and $\tilde{\phi} > 0 > \tilde{\psi}$ be its eigenfunctions. Assume to the contrary that there exists a sequence $\alpha_j \to \infty$ such that $\lambda_1 = \lambda_1(\alpha_j) \le 0$, then we claim that

Claim 8.7. *For any $p > 1$, there exists $C > 0$ independent of α_j such that*

$$\|\tilde{\psi}\|_{W^{2,p}(\Omega)} \le C \|\tilde{V}\tilde{\phi}\|_{L^p(\Omega)} \le C \|\tilde{\phi}\|_{L^p(\Omega)} \tag{8.12}$$

for some constant $C > 0$ independent of α_j large by (8.3).

To prove the claim we observe that by the second equation of (8.10),

$$\begin{cases} d_2 \Delta \tilde{\psi} + (m - \tilde{U} - 2\tilde{V} + \tilde{\lambda}_1)\tilde{\psi} = \tilde{V}\tilde{\phi} & \text{in } \Omega \\ \frac{\partial \tilde{\psi}}{\partial \nu} = 0 & \text{on } \partial \Omega \end{cases}$$

By Theorem 8.2, $\tilde{U}$ and $\tilde{V}$ are bounded in $L^\infty(\Omega)$ uniformly in α. By Lemma 8.3, as $\alpha \to \infty$, $\tilde{U} \to 0$ in L^p for any $p \ge 1$ and $\tilde{V} \to \theta_{d_2}$ in $C^{1,\beta}(\bar{\Omega})$ for any $\beta \in (0, 1)$. Since $d_2 \Delta + (m - 2\theta_{d_2})$ is invertible and $\tilde{\lambda}_1 \le 0$, Claim 8.7 thus follows by elliptic estimates. Now take $p = 2$ in (8.12), and normalize $\|\tilde{\psi}\|_{W^{2,2}(\Omega)} = 1$. We shall see that $\|\tilde{\phi}\|_{L^2(\Omega)} \to 0$ as $\alpha \to \infty$ which is a contradiction to (8.12).

More precisely, let $f_\alpha = e^{\frac{\alpha}{2d_1}[m_0 - m(x)]}\tilde{U}\tilde{\psi}$ and $\tilde{w} = e^{\frac{\alpha}{2d_1}[m_0 - m(x)]}\tilde{\phi}$. Then

$$\begin{cases} d_1 \Delta \tilde{w} + \left(-\frac{\alpha^2}{4d_1}|\nabla m|^2 - \frac{\alpha}{2}\Delta m + m - 2\tilde{U} - \tilde{V} + \tilde{\lambda}_1\right)\tilde{w} = f_\alpha & \text{in } \Omega \\ d_1 \frac{\partial \tilde{w}}{\partial \nu} - \frac{\alpha}{2}\tilde{w}\frac{\partial m}{\partial \nu} = 0 & \text{on } \partial \Omega \end{cases} \tag{8.13}$$

Now,

$$\begin{cases} d_1 \Delta w + \left(-\frac{\alpha^2}{4d_1}|\nabla m|^2 - \frac{\alpha}{2}\Delta m + m - 2\tilde{U} - \tilde{V}\right)w = -\lambda w & \text{in } \Omega \\ d_1 \frac{\partial w}{\partial \nu} - \frac{\alpha}{2}w\frac{\partial m}{\partial \nu} = 0 & \text{on } \partial \Omega \end{cases} \tag{8.14}$$

is a self-adjoint eigenvalue problem with the spectrum $\{\lambda_i\} \subset \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) \geq \lambda_1\}$. And by Lemma 8.6 and (B), $\lambda_1 \to \min_{x_0 \in \mathfrak{M}} (m_0 - \theta_{d_2}(x_0)) > 0$. Hence

$$\|\tilde{w}\|_{L^2(\Omega)} \leq \frac{C}{\lambda_1 - \tilde{\lambda}_1} \|f_\alpha\|_{L^2(\Omega)} \leq \frac{C}{\lambda_1 - \tilde{\lambda}_1} \|\tilde{\psi}\|_{L^p} \|\tilde{U} e^{\frac{\alpha}{2}[m_0 - m(x)]}\|_{L^q} \to 0.$$

by Theorem 8.2, where p, q are chosen such that $p > 2$, $W^{2,2} \subset L^p$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$. Therefore

$$\|\tilde{\phi}\|_{L^2(\Omega)} = \|e^{\frac{\alpha}{2d_1}[m(x) - m_0]} \tilde{w}\|_{L^2(\Omega)} \leq \|\tilde{w}\|_{L^2(\Omega)} \to 0.$$

□

8.2 Proof of Lemma 8.6

By variational characterization,

$$\lambda_1 = \inf_{\int_\Omega e^{\alpha m/d_1} \varphi^2 \, dx = 1} \int_\Omega e^{\alpha m/d_1} [d_1 |\nabla \varphi|^2 - (m - 2\tilde{U} - \tilde{V}) \varphi^2] dx \tag{8.15}$$

$$= \inf_{\int_\Omega w^2 \, dx = 1} \int_\Omega [d_1 |\nabla w - \frac{\alpha}{2} w \nabla m|^2 - (m - 2\tilde{U} - \tilde{V}) w^2] dx \tag{8.16}$$

First we claim

Lemma 8.8. *If m satisfies (A), then* $\limsup_{\alpha \to \infty} \lambda_1 \leq \min_{x_0 \in \mathfrak{M}} |m(x_0) - \theta(x_0)|$,

Proof. Fix $x_0 \in \mathfrak{M}$. Let η be a smooth cut-off function such that

$$\eta = \begin{cases} 1 & \text{in } B_r(x_0) \\ 0 & \text{in } \Omega \setminus B_{2r}(x_0) \end{cases}, \quad 0 \leq \eta \leq 1, \text{ and } |\nabla \eta| \leq \frac{2}{r}$$

Then,

$$\lambda_1 \leq \frac{\int_{B_{2r}(x_0)} e^{\alpha m/d_1} (2\tilde{U} + \tilde{V} - m) dx}{\int_{B_r(x_0)} e^{\alpha m/d_1} dx} + \frac{d_1 \int_{B_{2r}(x_0) \setminus B_r(x_0)} e^{\alpha m/d_1} 2/r dx}{\int_{B_{\epsilon r}(x_0)} e^{\alpha m/d_1} dx} = \mathrm{I} + \mathrm{II}.$$

where $\epsilon > 0$ is chosen small enough so that $\min_{\overline{B}_{\epsilon r}} m > \max_{\overline{B}_{2r}(x_0) \setminus B_r(x_0)} m$. Taking $\alpha \to \infty$, by Lebesgue's Dominated Convergence and Theorem 8.2,

$$\lim_{\alpha\to\infty} \mathrm{I} = 2^{1+N/2}[m(x_0)-\theta_{d_2}(x_0)]_+ \frac{\int_{\mathbb{R}^N} e^{y^T D^2 m(x_0) y} dy}{\int_{\mathbb{R}^N} e^{\frac{1}{2} y^T D^2 m(x_0) y} dy} + [\theta_{d_2}(x_0) - m(x_0)]$$

$$= |m(x_0) - \theta_{d_2}(x_0)|, \quad \text{and}$$

$$\lim_{\alpha\to\infty} \mathrm{II} = 0 \quad \text{by the choice of } \epsilon,$$

where $[m(x_0)-\theta_{d_2}(x_0)]_+ = \max\{m(x_0)-\theta_{d_2}(x_0), 0\}$. Hence $\limsup_{\alpha\to\infty} \lambda_1 \leq |m(x_0) - \theta_{d_2}(x_0)|$ for any $x_0 \in \mathfrak{M}$. The lemma is proved after taking minimum over $x_0 \in \mathfrak{M}$, □

Next, it suffices to show that

Lemma 8.9. *If m satisfies (A), then* $\liminf_{\alpha\to\infty} \lambda_1 \geq \min_{x_0\in\mathfrak{M}} |m(x_0) - \theta(x_0)|$.

Proof. Observe that by (8.16) and the boundedness of $\tilde{U}$ and $\tilde{V}$, λ_1 is bounded from below uniformly in α. Therefore, $\liminf_{\alpha\to\infty} \lambda_1 > -\infty$. We first select a sequence $\alpha_j \to \infty$ so that along the sequence

$$\lambda_1 \to \liminf_{\alpha\to\infty} \lambda_1, \quad \tilde{w}^2 \to \mu \quad \text{(in meas.)}.$$

From (8.16) we obtain

$$\liminf_{\alpha\to\infty} \lambda_1 \geq \lim_{j\to\infty} \int_\Omega (2\tilde{U} + \tilde{V} - m) w^2\, dx = \int_\Omega (2\tilde{U} + \tilde{V} - m)\mu(dx) \geq -|m|_\infty. \tag{8.17}$$

In view of Lemma 8.8, roughly speaking, it suffices to show that μ is supported on $\mathfrak{M}$. In fact, we are going to show

Lemma 8.10. *If $m(x)$ satisfies (A), then*

(i) $\mu(\Omega \setminus \mathfrak{M}) = 0$.

(ii) *there exists $C > 0$ such that for each $x_0 \in \mathfrak{M}$ and $r > 0$ small,*

$$w(x) \leq C\alpha^{N/4} e^{-\sqrt{\alpha/d_1}|x-x_0|} \text{ in } B_r(x_0).$$

(iii) *if* $\mu(\{x_0\}) > 0$*, then*

$$(\frac{d_1}{\alpha})^{N/4} w(x_0 + \sqrt{\frac{d_1}{\alpha}} y) \to \sqrt{\frac{\mu(\{x_0\})}{\det(-D^2 m(x_0))\pi^N 2^{N/4}}} e^{\frac{1}{4} y^T D^2 m(x_0) y}$$

uniformly in compact subsets of $\mathbb{R}^N$*, where*

$$\det(-D^2 m(x_0))\pi^N 2^{N/4} = \int_{\mathbb{R}^N} e^{\frac{1}{2} y^T D^2 m(x_0) y} dy.$$

The proof of Lemma 8.10 follows largely from the arguments in [CL]. In particular, (i) follows from the same arguments in [CL] by observing that although c now changes with α, the following basic inequality

$$+\infty > c^* - c_* \geq \int_\Omega d_1 |\nabla w - \frac{\alpha}{2} w \nabla m|^2 \, dx$$

is still valid, where w is the principal eigenfunction and $c^* = \sup_{\alpha \geq 0, x \in \Omega} (2\tilde{U} + \tilde{V} - m)$ and $c_* = \inf_{\alpha \geq 0, x \in \Omega} (2\tilde{U} + \tilde{V} - m)$.

Here we prove Lemma 8.10 (ii). In the following Lemma 8.11 and Lemma 8.12 are the same as Lemma 5.1 and Lemma 5.2 in [CL]

Proof of Lemma 8.10 *(ii).* Let $\frac{1}{d_1} |\nabla m|^2_{L^\infty(\Omega)} + |\Delta m|_{L\infty(\Omega)} + c^* - c_* = L$.

Define $q_\alpha(x) = \frac{\alpha^2}{4d_1} |\nabla m|^2 + \frac{\alpha}{2} \Delta m + (2\tilde{U} + \tilde{V} - m) - \lambda_1$.

Lemma 8.11. *There exists* $M > 0$ *such that* $\|w\|_{L^\infty(\Omega)} \leq M\alpha^{N/2}$ *for all* $\alpha \geq 1$*.*

To prove the lemma, set $W(y) = w(y/\alpha)$ and $\Omega_\alpha = \{\alpha x : x \in \Omega\}$, then

$$\frac{|\Delta_y W|}{W} = \frac{|q_\alpha|}{d_1 \alpha^2} \leq \frac{L}{d_1} \quad \text{in } \Omega_\alpha, \quad \frac{|\frac{\partial W}{\partial \nu}|}{W} \leq \frac{|\nabla m|}{2d_1} \leq \frac{\sqrt{L}}{2d_1} \quad \text{on } \partial\Omega_\alpha$$

It then follows from local elliptic estimate that there exists a positive constant M such that for any $y \in \Omega_\alpha$,

$$W(y) \leq M \|W\|_{L^2(B(y,1) \cap \Omega_\alpha)} \leq M\alpha^{N/2} \|w\|_{L^2(\Omega)} = M\alpha^{N/2}.$$

The lemma thus follows.

Next, we prove the following lemma

Lemma 8.12. *Let k, r be positive constants and W be a C^2 function satisfying*

$$\Delta W = Q(x)W(x) > 0, \quad Q(x) \geq (N+1)k^2 \quad \text{for any } x \in B(0,r).$$

Then,

$$W(0) \leq e^{1-kr} \max\{W(x) : |x| = r\}.$$

Proof. Consider the function $\overline{W}(x) = W(x)e^{-\sqrt{1+|kx|^2}}$. It is easy to verify that $\overline{W}$ satisfies

$$\Delta\overline{W} + \frac{2k^2 x \cdot \nabla\overline{W}}{\sqrt{|kx|^2+1}} = \overline{W}\left\{Q(x) - \frac{k^2|kx|^2}{|kx|^2+1} - \frac{k^2}{\sqrt{|kx|^2+1}}\frac{(N-1)|kx|^2+N}{|kx|^2+1}\right\}$$

Thus, $\overline{W}$ cannot attain its maximum at any interior point. Therefore,

$$W(0) = e\overline{W}(0) \leq e \max_{\partial B(0,r)} \overline{W} = e^{1-\sqrt{1+|kr|^2}} \max_{\partial B(0,r)} W \leq e^{1-kr} \max_{\partial B(0,r)} W.$$

The assertion of the lemma thus follows. □

Lemma 8.13. *Assume that for some positive constants a and R,*

$$|\nabla m(x)|^2 \geq a|x-x_0|^2 \quad \text{for all } x \in B(x_0, 4R) \subset \Omega. \tag{8.18}$$

Then for every $\alpha \geq 1$,

$$w(x) \leq e^{1+\sqrt{4(L+N+1)/a}-\sqrt{\alpha/d_1}|x-x_0|} \max_{B(x_0,2|x-x_0|)} w \quad \text{for all } x \in B(x_0, 2R) \tag{8.19}$$

Proof. The assertion is trivially true if $\sqrt{\alpha/d_1}|x-x_0| \leq \sqrt{4(L+N+1)/a}$, that is to say, $x \in B(x_0, \sqrt{4(L+N+1)d_1/a\alpha})$. Hence, we consider $2R > \sqrt{4(L+N+1)d_1/a\alpha}$ and $x \in B(x_0, 2R) \setminus B(x_0, \sqrt{4(L+N+1)d_1/a\alpha})$.

For every $z \in B(x_0, 4R) \setminus B(x_0, \sqrt{4(L+N+1)d_1/a\alpha})$ we have

$$\frac{q_\alpha(z)}{d_1} \geq \frac{\alpha^2 a}{4d_1^2}|z-x_0|^2 - L\frac{\alpha}{d_1} \geq (N+1)\frac{\alpha}{d_1}.$$

Thus, applying Lemma 8.12 to $W(z) = w(x+z)$ in $B(0, |x-x_0| - \sqrt{4(L+N+1)d_1/a\alpha})$

with $k = \sqrt{\alpha/d_1}$, we obtain

$$\begin{aligned} w(x) &\leq e^{1-\sqrt{\alpha/d_1}(|x-x_0|-\sqrt{4(L+N+1)d_1/a\alpha})} \max_{B(0,|x-x_0|-\sqrt{4(L+N+1)/a\alpha})} w \\ &\leq e^{1+\sqrt{4(L+N+1)/a\alpha}-\sqrt{\alpha/d_1}|x-x_0|} \max_{B(x_0,2|x-x_0|)} w. \end{aligned}$$

The assertion of the lemma thus follows. □

Now let $x_0 \in \mathfrak{M}$, then there exists positive constants R, a such that (8.18) is satisfied. Hence (8.19) holds by Lemma 8.13. To finish the proof of Lemma 8.10(iii), we are first going to show that

$$\max_{B(x_0,2R)} w = O(\alpha^{N/4}). \tag{8.20}$$

Now for each $\alpha \geq 1$, consider the functions

$$W(x_0,\alpha;y) := (\frac{d_1}{\alpha})^{N/4} w(x_0 + \sqrt{\frac{d_1}{\alpha}}y),$$

$$\begin{aligned} Q(x_0,\alpha;y) := \frac{\alpha}{4d_1}|\nabla m(x_0+\sqrt{\frac{d_1}{\alpha}}y)|^2 + \frac{1}{2}\Delta m(x_0+\sqrt{\frac{d_1}{\alpha}}y) \\ + [2\tilde{U}(x_0+\sqrt{\frac{d_1}{\alpha}}y) + \tilde{V}(x_0+\sqrt{\frac{d_1}{\alpha}}y) - m(x_0+\sqrt{\frac{d_1}{\alpha}}y) - \tilde{\lambda}_1]/\alpha. \end{aligned}$$

Then we have the following:

$$\Delta_y W = Q(\alpha,y)W, \qquad \forall y \in B(0,\sqrt{\alpha/d_1}R),$$

$$\frac{a}{2}|y|^2 - L < |Q(\alpha,y)| \leq M^2|y|^2 + L, \qquad \forall y \in B(0,\sqrt{\alpha/d_1}R),$$

$$\int_{B(0,\sqrt{\alpha/d_1}R)} W^2(x_0,\alpha;y)\,dy = \int_{B(x_0,R)} w^2(x)\,dx,$$

where $M_2 = |D^2 m|^2_{L^\infty(\Omega)}$.

Let $y_0 \in \bar{B}(0,2\sqrt{\alpha/d_1}R)$ such that

$$W(x_0,\alpha;y_0) = \bar{M}(x_0,\alpha,R) := \max_{\bar{B}(0,2\sqrt{\alpha/d_1}R)} W.$$

Then in view of (8.19), we have

$$W(x_0, \alpha; y) \le \bar{M}(x_0, R, \alpha) e^{1+\sqrt{4(L+N+1)/a}-|y|}, \qquad \forall y \in B(0, \sqrt{\alpha} R).$$

To show (8.20), it remains to show that $\bar{M}(x_0, \alpha, R)$ is bounded, uniformly in α. Consider two cases.

(a) Suppose $y_0 \in \partial B(0, 2\sqrt{\alpha/d_1} R)$; then by (8.19) and Claim 8.11, we have

$$\begin{aligned} \bar{M} &= (\alpha/d_1)^{-N/4} w(\alpha, x_0 + y_0/\sqrt{\alpha/d_1}) \le \alpha^{-N/4} e^{1+\sqrt{4(L+N+1)/a}-|y_0|} M (\alpha/d_1)^{N/2} \\ &= M e^{1+\sqrt{4(L+N+1)/a}-2\sqrt{\alpha/d_1}R} (\alpha/d_1)^{N/4} \le M e^{1+\sqrt{4(L+N+1)/a}} R^{-N/2} \sup_{t>0} e^{-2t} t^{N/2} \\ &= M e^{1+\sqrt{4(L+N+1)/a}-N/2} (N/4)^{N/2} R^{-N/2}. \end{aligned}$$

(b) Suppose $y_0 \in B(0, 2\sqrt{\alpha/d_1} R)$. Then $\Delta_y W(x_0, \alpha; y_0) \le 0$. Consequently,

$$Q(x_0, \alpha; y_0) \le 0,$$

so that $y_0 \in B(0, \sqrt{2L/a})$. By the Harnack inequality [GT], there exists a constant $C = C(a, L, M_2)$ such that

$$\max_{B(0,\sqrt{2L/a})} W \le C \min_{B(0,\sqrt{2L/a})} W.$$

It then follows that

$$1 \ge \int_{B(0,\sqrt{2L/a})} W^2 \, dy \ge \frac{|B(0, \sqrt{2L/a})|}{C^2} \bar{M}^2,$$

that is,

$$\bar{M}(x_0, \alpha, R) \le \frac{C}{\sqrt{|B(0, \sqrt{2L/a})|}}.$$

Thus, $\bar{M}$ is bounded, uniformly in $\alpha \ge 1$.

Assume that $\displaystyle\int_{B_R} w^2 \, dx = \mu(B(x_0, R)) > 0$. Then there exist a sequence $\{\alpha_j\}$ and a function W^* such that

$$\lim_{j\to\infty} W(x_0, \alpha_j; y) = W^*(y) \qquad \text{locally uniformly in } \mathbf{R}^N,$$

$$\int_{\mathbf{R}^N} W^{*2}(y)\,dy = \lim_{j\to\infty}\int_{B(x_0,R)} w^2(\alpha_j,x)\,dx = \mu(B(x_0,R)) = \mu(\{x_0\}) > 0.$$

In addition, W^* solves the following equation:

$$\Delta W^*(y) = \{\frac{1}{4}|yD^2m(x_0)|^2 + \frac{1}{2}\Delta m(x_0)\}W^*(y) \qquad \text{in } \mathbf{R}^N,$$

$$0 < W^*(y) < Ce^{-|y|} \qquad \forall y\in\mathbf{R}^N.$$

Since $\det\big(d^2m(x_0)\big)\neq 0$, all eigenvalues of $D^2m(x_0)$ are non-zero. One can show that this equation has a solution if and only if $D^2m(x_0)$ is negative definite. In such a case, the solution is unique and is given by

$$W^*(y) = \sqrt{\frac{\mu(\{x_0\})}{\det\big(-D^2m(x_0)\big)\pi^N}}e^{\frac{1}{4}y^TD^2m(x_0)y}.$$

Hence Lemma 8.10 (iii) is proved. □

By (8.17), we have

$$\begin{aligned}\lambda_1 &\geq \int_\Omega (2\tilde{U}+\tilde{V}-m)w^2\,dx\\ &\geq \sum_{x_0\in\mathfrak{M}}\int_{B_{R/\sqrt{\alpha/d_1}}(x_0)}(2\tilde{U}+\tilde{V}-m)w^2\,dx\\ &= \sum_{x_0\in\mathfrak{M}}\int_{B_R(x_0)}[2\tilde{U}(x_0+\frac{y}{\sqrt{\alpha/d_1}})+\tilde{V}(x_0+\frac{y}{\sqrt{\alpha}})-m(x_0+\frac{y}{\sqrt{\alpha}})]\\ &\quad\times\alpha^{-N/2}w(x_0+\frac{y}{\sqrt{\alpha}})^2\,dy\end{aligned}$$

Now take $\alpha_j\to\infty$, we have

$$\begin{aligned}\liminf_{\alpha\to\infty}\lambda_1 &\geq \sum_{x_0\in\mathfrak{M}}\mu(\{x_0\})\Bigg\{\frac{\int_{B_R}2^{1+N/2}[m(x_0)-\theta_{d_2}(x_0)]_+e^{y^TD^2m(x_0)y}dy}{\int_{\mathbb{R}^N}e^{\frac{1}{2}y^TD^2m(x_0)y}dy}\\ &\qquad+[\theta_{d_2}(x_0)-m(x_0)]\Bigg\}\\ &= \sum_{x_0\in\mathfrak{M}}\mu(\{x_0\})|\theta_{d_2}(x_0)-m(x_0)|\\ &\geq \min_{\tilde{x}\in\mathfrak{M}}|\theta_{d_2}(\tilde{x})-m(\tilde{x})|\end{aligned}$$

Thus, Lemma 8.6 is proved.

□

Finally, I wish to remark that by Lemma 8.6, it is interesting to note that the support of μ is actually on $\tilde{\mathfrak{M}} = \{x_0 \in \mathfrak{M} : |m(x_0) - \theta_{d_2}(x_0)| = \min_{\tilde{x}\in\mathfrak{M}} |\theta_{d_2}(\tilde{x}) - m(\tilde{x})|\}$.

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