GLOBAL DYNAMICS OF THE TOXICANT-TAXIS MODEL WITH ROBIN BOUNDARY CONDITIONS

HAI-YANG JIN, KING-YEUNG LAM, AND ZHI-AN WANG

Abstract. This paper is concerned with the following spatiotemporal population-toxicant model with toxicant-taxis in a bounded domain $\Omega \subset \mathbb{R}^n (n \geq 1)$ with inhomogeneous Robin boundary conditions

$$\begin{cases} u_t = d\Delta u + \chi\nabla\cdot(u\nabla w) + u(1-u) - \sigma uw, & x \in \Omega, t > 0 \\ w_t = \varepsilon\Delta w - \mu w - \lambda uw, & x \in \Omega, t > 0, \\ (d\nabla u + \chi u\nabla w)\cdot\nu = 0, \quad \nabla w\cdot\nu = \xi(h(x,t) - w), & x \in \partial\Omega, t > 0 \\ u(x,0) = u_0(x), \quad w(x,0) = w_0(x), & x \in \Omega, \end{cases}$$

where $u = u(x,t)$ and $w = w(x,t)$ denote the population density and toxicant concentration at location x and time t, respectively. Here the toxicant enters the environment through the boundary with a temporally and spatially heterogeneous ambient toxicant density $h(x,t)$. Under suitable assumptions on $h(x,t)$, we first establish the global existence of classical solutions in two-dimensional spaces ($n = 2$). Moreover, we show that every solution (u,w) converges to $(1,0)$ uniformly if $h(x,t)$ decays to zero as $t \to \infty$ with a mild rate satisfying

$$\lim_{t\to\infty} \int_t^{t+1} \|h(\cdot,\tau)\|_{L^1(\partial\Omega)} d\tau = 0.$$

If $h(x,t) \equiv h(x) \gneqq 0$ with $0 < h_0 = \sup_{x\in\partial\Omega} h(x)$, we establish the existence of non-constant positive steady states in all dimensional spaces ($n \geq 1$) under the condition

$$0 < h_0 < h^* := \min\left\{\frac{1}{\sigma}, \frac{d}{\chi}\right\}.$$

We further show that this non-constant steady state is unique and globally asymptotically stable if h_0 is sufficiently small. On the other hand, we prove that the species u is uniformly persistent if $\sigma < 1/h_0$, while the toxicant-only steady state is globally asymptotically stable if $\sigma > 1/M_h$ with some constant $M_h > 0$ smaller than h_0.

1. Introduction and main results

Due to anthropogenic activities such as industrial effluents and increased urbanization in recent decades, a great deal of toxicants and pollutants have been discharged into lakes and rivers. This seriously threatens the living organisms in these aquatic ecosystems. Toxicant increase in aquatic ecosystems has adverse effects on biospecies behavior, population growth, community structure and ecosystem integrity (see review articles [1, 4, 30]). It is therefore of paramount importance to understand the deleterious effects of toxicants on aquatic population dynamics and identifying the key factors determining the persistence or extinction so that suitable water quality standards and regulatory measures can be enacted to protect aquatic species and maintain ecosystem diversity. Towards this goal, various mathematical models describing the population-toxicant interactions were proposed such as the ordinary differential equation models [12–15, 18, 19], matrix population models [11, 16, 33, 34], reaction-advection-diffusion equations [36, 38, 39] and so on. These existing models were focused on the influence of toxicants on the population growth rate or on the environmental carrying capacity, without considering the spatial movement such as dispersion, transport and spatial avoidance of toxicants, etc. In fact, individuals may exhibit various toxicant-induced behavioral changes including spatial

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movement (cf. [3, 5, 30]). In the literature, reaction-advection-diffusion equations in one dimension have been used to describe the motion and transport process of river pollutants alone (cf. [27, 31]). The first reaction-advection-diffusion model considering the population-toxicant interactions with dynamical toxicants in a polluted river was proposed in a paper [39] with Danckwerts boundary conditions, where sufficient conditions for the population persistence or extinction were found based on the eigenvalue theory. The model proposed in [39] assumed that species and toxicants only undertook random diffusion. In reality, many aquatic species can detect and avoid toxicants (i.e. spatial avoidance) [2, 35]. Taking into account this essential factor, a spatiotemporal population-toxicant system with (negative) toxicant-taxis model was proposed in [10] as follows:

$$\begin{cases} u_t = d\Delta u + \chi\nabla\cdot(u\nabla w) + u(1-u) - \sigma uw, & x\in\Omega,\ t>0,\\ w_t = \varepsilon\Delta w + h(x) - \mu w - \lambda uw, & x\in\Omega,\ t>0,\\ \nabla u\cdot\nu = \nabla w\cdot\nu = 0, & x\in\partial\Omega,\ t>0, \end{cases} \tag{1.1}$$

where $u(x,t)$ and $w(x,t)$ represent the density of species and toxicant at location $x\in\Omega$ and time t, and Ω denotes the bounded habitat in $\mathbb{R}^n (n\geq 1)$ with smooth boundary $\partial\Omega$. All parameters are positive, where d and ε denote the random diffusion coefficients of the species and the toxicant, respectively. The toxicant-taxis term $\chi\nabla\cdot(u\nabla w)$ entails that the species can evade toxicants (i.e., the movement of individuals away from the gradient of toxicant concentration). The term $u(1-u)-\sigma uw$ accounts for the population growth under the influence of toxicants where σ is the toxicant-induced death rate. In the second equation of (1.1), $h(x)$ is the toxicant input rate, μ is the decay rate of toxicant due to the detoxification and λ denotes the uptake rate of toxicant by the aquatic species. The global existence of classical solutions of (1.1) in two dimensions ($n=2$) with $h(x)\in C(\bar{\Omega})$ was established in [10]. When $h(x)=h_0$ is a positive constant, the global stability of constant steady states and spatial patterns of (1.1) were further studied in [10] showing that the value of h_0 is critical for the persistence of species while the toxicant-taxis may introduce spatial patterns. Recently, the existence of non-constant positive steady states of (1.1) under certain conditions was established in [9] by the Leray-Schauder degree theorem when $h(x)$ is a positive constant.

The model (1.1) is based on the following assumptions: (a) the aquatic system under consideration is a closed environment where both the species and the toxicant cannot cross the habitat boundary due to homogeneous Neumann boundary conditions, which particularly implies that toxicants are not discharged into the aquatic system through the habitat boundary, but through other ways like rainfall mixed with toxic emissions (e.g., acid rain); (b) the toxicant input rate h is independent of time. However, in reality toxicants may enter aquatic systems (lake or river) through the boundaries such as industrial/agricultural runoff or polluted surface water. In addition, both anthropogenic activities and environmental changes vary seasonally. Clearly, these situations violate the assumptions (a) and (b) and are not described by the model (1.1). To this end, we update the model with toxicants permeating through the habitat boundary via a Robin-type boundary condition

$$\begin{cases} u_t = d\Delta u + \chi\nabla\cdot(u\nabla w) + u(1-u) - \sigma uw, & x\in\Omega, t>0,\\ w_t = \varepsilon\Delta w - \mu w - \lambda uw, & x\in\Omega, t>0,\\ (d\nabla u + \chi u\nabla w)\cdot\nu = 0,\quad \nabla w\cdot\nu = \xi(h(x,t)-w), & x\in\partial\Omega, t>0,\\ u(x,0)=u_0(x),\quad w(x,0)=w_0(x), & x\in\Omega. \end{cases} \tag{1.2}$$

The boundary conditions in (1.2) indicate that no aquatic species can cross the habitat boundary, while the toxicant with ambient density $h(x,t)$ enters or leaves the habitat through the boundary with an exchange coefficient $\xi>0$.

The main results of this paper include the global well-posedness (global boundedness and stabilization of solutions), as well as existence and globally asymptotic stability of non-constant steady states of (1.2). The global boundedness and stabilization of solutions to (1.2) are asserted in the following theorem.

Theorem 1.1 (Global boundedness and stabilization)**.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary. Assume that the initial data $(u_0, w_0) \in [W^{1,\infty}(\Omega)]^2$ with $u_0, w_0 \gneqq 0$ and the following assumption on $h(x,t)$ holds:*

(H_0) *$h(x,t) \in C^\infty(\partial\Omega \times [0,\infty))$ is a nonnegative bounded function satisfying*

$$\|h_t(\cdot,t)\|_{L^\infty(\partial\Omega)} \le C \quad \text{for all t} > 0,$$

where $C > 0$ is a constant independent of t.

Then the system (1.2) with $\chi \ge 0$ has a unique nonnegative global classical solution $(u,w) \in [C^0([0,\infty)\times\bar{\Omega}) \cap C^{2,1}((0,\infty)\times\bar{\Omega})]^2$, such that

$$\|u(\cdot,t)\|_{L^\infty} + \|w(\cdot,t)\|_{W^{1,\infty}} \le M, \quad \text{for all} \quad t > 0,$$

where $M > 0$ is a constant independent of t and σ. Furthermore, if $h(x,t)$ satisfies

$$\lim_{t\to\infty} \int_t^{t+1} \|h(\cdot,\tau)\|_{L^1(\partial\Omega)} d\tau = 0, \tag{1.3}$$

then the solution (u,w) converges to $(1,0)$ uniformly as $t \to \infty$.

Remark 1.1. The conditions (H_0) and (1.3) can be fulfilled by a wide range of functions which decay slowly in time, such as $h(x,t) = (1+t)^{-p}\zeta(x)$ for any $p > 0$ or non-monotone function like $h(x,t) = e^{-\lambda t}(\sin t + 1)\zeta(x)$ with $\lambda > 0$, where $\zeta(x)$ is a bounded nonnegative function.

Theorem 1.1 states that if the ambient toxicant density $h(x,t)$ depends on time and decays to zero as $t \to \infty$, the global classical solution will converge to $(1,0)$ uniformly as $t \to \infty$. Below we aim to consider the asymptotic behavior of solutions if $h(x,t)$ is stationary in time. That is, we consider $h(x,t) \equiv h(x) \gneqq 0$ (i.e. $h(x)$ is nonnegative but not identical to zero) and is smooth on $\partial\Omega$. Clearly the global existence and boundedness of classical solutions established in Theorem 1.1 hold true. We are interested in the existence and global stability of non-constant steady states of (1.2) which satisfy

$$\begin{cases} 0 = d\Delta U + \nabla\cdot(\chi U \nabla W) + U(1-U) - \sigma UW, & x \in \Omega, \\ 0 = \varepsilon\Delta W - \mu W - \lambda UW, & x \in \Omega, \\ (d\nabla U + \chi U\nabla W)\cdot\nu = 0, \quad \nabla W\cdot\nu = \xi(h(x) - W), & x \in \partial\Omega. \end{cases} \tag{1.4}$$

In the sequel, we denote

$$\sup_{x\in\partial\Omega} h(x) := h_0 > 0. \tag{1.5}$$

It is straightforward to check that if (1.4) admits a solution, then it must be non-constant. Clearly the system (1.4) has a toxicant-only steady state $(0, w_*)$, where $w_* = w_*(x)$ is the unique non-constant positive solution of the following system

$$\begin{cases} 0 = \varepsilon\Delta w_* - \mu w_*, & x \in \Omega, \\ \nabla w_*\cdot\nu = \xi(h(x) - w_*), & x \in \partial\Omega. \end{cases} \tag{1.6}$$

We observe that the solution of (1.6) must be non-constant if exists. Also, the existence of solutions to system (1.6) can be obtained by the method of upper-lower solutions in view of that 0 and h_0 are a sub-solution and a super-solution, respectively. The uniqueness is a consequence of the strong maximum principle and Hopf boundary point lemma.

Apart from the toxicant-only semi-trivial steady state $(0, w_*)$, we can show that if $h_0 > 0$ is suitably small, then (1.6) admits a unique non-constant solution which is globally asymptotic stable, as given in the following theorem.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ be a bounded domain with smooth boundary. Assume h is non-trivial, nonnegative and stationary, i.e. $h = h(x) \gneqq 0$, and that $h_0 = \sup_{x\in\partial\Omega} h(x)$ satisfies*

$$0 < h_0 < h^* := \min\Big\{\frac{1}{\sigma}, \frac{d}{\chi}\Big\}.$$

Then the system (1.2) admits a positive non-constant classical steady state solution $(U(x), W(x)) \in C^{2+\alpha}(\bar\Omega) \times C^{2+\alpha}(\bar\Omega)$ satisfying (1.4) with

$$\ell_* \leq e^{\frac{\chi}{d}W(x)}U(x) \leq \ell^*, \ \ 0 < W(x) \leq h_0, \tag{1.7}$$

where $\ell_* = \min\limits_{0\leq z\leq h_0} \ell(z)$ and $\ell^* = \max\limits_{0\leq z\leq h_0} \ell(z)$ with $\ell(z) = (1-\sigma z)e^{\frac{\chi}{d}z}$. Moreover, (U, W) is unique and, if $n = 2$ and if $h_0 > 0$ is sufficiently small, then it is globally exponentially stable.

Remark 1.2. The condition $\sigma h_0 < 1$ is imposed to obtain the positivity of non-constant steady states. The condition $\chi h_0 < d$ is used to prove the continuous dependence of the mapping $V \to W[V]$ in Lemma 3.2, but it can be removed if μ is large or λ is small (see Remark 3.1). Particularly if $\chi = 0$, the mere condition $\sigma h_0 < 1$ suffices to warrant the existence of (U, W).

The results of Theorem 1.2 do not address the global dynamics of (1.2) when $h_0 > 0$ is not sufficiently small. The following theorem will partly elucidate this question.

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary. Assume h is non-trivial, nonnegative and stationary and denote $h_0 = \sup_{x\in\partial\Omega} h(x) > 0$. Let (u, w) be the solution of the time-dependent problem (1.2) obtained in Theorem 1.1, then the following results hold.*

(1) *If $\sigma h_0 < 1$, then $u > 0$ is uniformly persistent, namely there is a constant $\delta_0 > 0$ independent of initial data such that $\liminf\limits_{t\to\infty} \inf\limits_{x\in\Omega} u \geq \delta_0$.*
(2) *If $\sigma M_h > 1$, then (u, w) converges to $(0, w_*)$ uniformly and exponentially as $t \to \infty$, where $M_h := \min\limits_{x\in\bar\Omega} \tilde w_*(x)$ and $0 < \tilde w_*(x) \leq h_0$ satisfies*

$$\begin{cases} 0 = \varepsilon\Delta \tilde w_* - (\mu + \lambda M)\tilde w_*, & x \in \Omega \\ \nabla \tilde w_* \cdot \nu = \xi(h(x) - \tilde w_*), & x \in \partial\Omega, \end{cases} \tag{1.8}$$

and $M > 0$ is the constant given in Theorem 1.1 and $M_h > 0$ is independent of σ.

Remark 1.3. Since $M_h > 0$ is independent of σ, the condition $\sigma M_h > 1$ is non-empty as long as $\sigma > 0$ is large enough. Theorem 1.3(2) implies that, for fixed parameters d, ε, μ, χ, λ, ambient toxicant density $h(x)$ and initial data, the solution (u, w) will exponentially converge to $(0, w_*)$ (i.e., the aquatic species will go extinction) provided that the toxicant's lethality σ is strong. The global dynamics of system (1.2) with $h(x, t) = h(x) \gneqq 0$ remains open for the case $1/h_0 \leq \sigma \leq 1/M_h$.

1.1. **Discussion and biological interpretations.** In this paper, we analyze a spatiotemporal population-toxicant model with toxicant-taxis in a bounded domain. In previous work, toxicants are introduced into the model at a positive rate inside the domain. Here we study the system under the assumption that the toxicant enters the model only via the boundary, which can be more realistic in many situations when the domain represents a lake and pollutants are introduced into the lake due to human activities in surrounding areas. Under the assumption (H_0), we first establish the well-posedness of the time-dependent problem in Theorem 1.1. In case the ambient toxicant density is independent of time (i.e. $h = h(x)$), we demonstrate in Theorem 1.2 the existence of non-constant equilibrium solutions $(U(x), W(x))$. Furthermore, we also prove the global attractivity of such nonconstant equilibrium solutions when $\|h\|_\infty$ is sufficiently small. This means that the long-time dynamics of the system, and particularly the population level of the organism, can be estimated using the solution to the stationary problem. Finally, we inquire the situation when $\|h\|_\infty$ is not necessarily small, and obtain a sufficient condition which says that (i) the organism population persists when $\|h\|_\infty$ is small while (ii) a large ambient toxicant density h (in some appropriate sense according to Theorem 1.3) leads to the extinction of the organism.

Sketch of proof ideas. The inhomogeneous Robin boundary conditions incapacitate the direct L^2-estimate method as in [10] to obtain the global existence of solutions to system (1.2). Instead, inspired by some ideas in [7, 24], we first use change of variables to reformulate system (1.2) with homogenized boundary conditions. Then we study the reformulated problem based on subtle energy estimates and semigroup theory to derive the global boundedness of solutions by frequently switching between the estimates of the original and changed variables (see Section 2). To study the global stability of the constant steady state $(1, 0)$, we first use the condition (1.3) to derive that $\lim_{t\to\infty} \|w(\cdot, t)\|_{L^\infty} = 0$. Then we show $\|u(\cdot, t) - 1\|_{L^\infty} \to 0$ as $t \to \infty$ by constructing a Lyapunov energy function based on the generalized relative entropy inequality associated with $\int_\Omega (u - 1 - \ln u)$. To achieve this, we use Harnack's inequality to show that $\inf_{x\in\Omega} u(x, t)$ has a positive lower bound for large time, by which we show that the entropy energy $\int_\Omega (u - 1 - \ln u)$ is equivalent to the L^2-energy $\int_\Omega (u - 1)^2$. With this crucial finding, we employ the dissipation of Lyapunov energy function to show that $\lim_{t\to\infty} \|u(\cdot, t) - 1\|_{L^2} = 0$. By deriving the boundedness of $\|u\|_{C^\theta}$ for $t > 1$, we finally obtain that $u \to 1$ uniformly as $t \to \infty$ and complete the proof of Theorem 1.1.

The existence of non-constant positive solutions determined by (1.4) was proved in virtue of Schauder's fixed point theorem. To this end, we first transfer the no-flux boundary condition of U into the homogeneous Neumann boundary condition and split the system (1.4) into two subsystems to construct a solution map. Based on suitable estimates for the solutions of two subsystems, we show that this solution map is continuous and relatively compact and hence yields a fixed point by the Schauder fixed point theorem. With the method of energy estimates, we further prove the solution of (1.4) is unique and globally asymptotically stable if $h_0 > 0$ is small, which proves Theorem 1.2.

To show the persistence result asserted in Theorem 1.3-(1), we derive an inequality

$$\frac{d}{dt}\int_\Omega u = \int_\Omega u(1 - u - \sigma w) \geq \delta \int_\Omega u,$$

form some small constant $\delta > 0$ under the condition $\sigma h_0 < 1$. Then the persistence is obtained by repeatedly using Harnack's inequality. To study the global asymptotic stability of toxicant-only state $(0, w_*)$ with large σ, the key is to show that $\inf_{x\in\Omega} w(x, t)$ has a positive lower bound independent of σ as time is large. In fact, using the comparison principle and energy estimates, we find $\sigma \inf_{x\in\Omega} w(x, t) \geq \frac{\sigma M_h - 1}{2}$ for large time and hence

$$\frac{d}{dt}\int_\Omega u = \int_\Omega u(1 - u - \sigma w) \leq \frac{1 - \sigma M_h}{2}\int_\Omega u, \tag{1.9}$$

which implies $\|u(\cdot, t)\|_{L^1} \to 0$ exponentially $t \to \infty$ if $\sigma M_h > 1$. Then $\|u(\cdot, t)\|_{L^\infty} \to 0$ as $t \to \infty$ follows by the interpolation inequality and boundedness of $\|u(\cdot, t)\|_{C^\theta}$ for $t > 1$. Finally using the energy estimates, we derive $\|w(\cdot, t) - w_*\|_{L^\infty} \to 0$ exponentially as $t \to \infty$, which completes the proof of Theorem 1.3-(2).

The rest of this paper is arranged as follows. In Section 2, we reformulate the problem (1.2) into a problem with homogenous Neumann boundary conditions and establish the global boundedness of solutions (i.e. Theorem 1.1) by the delicate bootstrap argument. The existence and stability of non-constant steady states asserted in Theorem 1.2 are proved in Section 3. Finally, we prove Theorem 1.3 in Section 4.

2. Boundedness and Stabilization: Proof of Theorem 1.1

In this section, we shall prove the global boundedness of solutions to the system (1.2), which consists of the local existence of solutions and the global *a priori* estimate of solutions. To this end, we first homogenize the boundary conditions by introducing some transformations and establish the local existence of solution for the transformed system, from which the local existence of solution for the original system (1.2) follows.

2.1. **Reformulation of the problem with homogenized boundary conditions.** To prove the boundedness of solutions, we first introduce a transformation to homogenize the boundary conditions. Noting the assumptions on $h(x,t)$ in (H_0), and using [24, Theorem 9.4 in Chap. 1], we can find some bounded functions $g_1 \in C^\infty(\bar\Omega)$ and $g_2 \in C^\infty(\bar\Omega \times [0,\infty))$ satisfying

$$\|g_1\|_{L^\infty} + \|\nabla g_1\|_{L^\infty} + \|\Delta g_1\|_{L^\infty} \le \gamma_1 \quad \text{in } \Omega, \tag{2.1}$$

and

$$\|g_2\|_{L^\infty} + \|\nabla g_2\|_{L^\infty} + \|\Delta g_2\|_{L^\infty} + \|g_{2t}\|_{L^\infty} \le \gamma_2 \quad \text{in } \Omega \times (0,\infty), \tag{2.2}$$

with some positive constants $\gamma_i (i = 1, 2)$ such that $\nu \cdot \nabla g_1(x) = \xi$ on $\partial\Omega$ and

$$g_2(x,t) = h(x,t) \;\text{ and }\; \nabla g_2 \cdot \nu = 0, \;\text{on } \partial\Omega \times (0,\infty).$$

Motivated by an idea of [7], we introduce the following transformation

$$\tilde u = u e^{\frac{\chi}{d} w} \;\text{ and }\; \tilde w = e^{g_1}(g_2 - w), \tag{2.3}$$

and then using the facts $(d\nabla u + \chi u \nabla w)\cdot \nu = 0$ and $\nabla w \cdot \nu = \xi(h - w)$ on $\partial\Omega$, we have

$$\nabla \tilde u \cdot \nu = \nabla \tilde w \cdot \nu = 0, \quad \text{on } \partial\Omega \times (0,\infty). \tag{2.4}$$

Using the transformation (2.3), we have

$$\tilde u_t = e^{\frac{\chi}{d} w} u_t + \frac{\chi}{d} u e^{\frac{\chi}{d} w} w_t =: I_1 + I_2. \tag{2.5}$$

On the other hand, with some simple calculations, we have

$$d\nabla u + \chi u \nabla w = d e^{-\frac{\chi}{d} w} \nabla \tilde u,$$

which implies that

$$d\Delta u + \nabla\cdot(\chi u \nabla w) = d e^{-\frac{\chi}{d} w}\Delta \tilde u - \chi e^{-\frac{\chi}{d} w} \nabla w \cdot \nabla \tilde u.$$

Then we can rewrite I_1 as follows

$$\begin{aligned} I_1 = e^{\frac{\chi}{d} w} u_t =& e^{\frac{\chi}{d} w}[d\Delta u + \nabla\cdot(\chi u \nabla w)] + e^{\frac{\chi}{d} w} u(1-u) - \sigma e^{\frac{\chi}{d} w} u w \\ =& d\Delta \tilde u - \chi \nabla w \cdot \nabla \tilde u + \tilde u (1 - e^{-\frac{\chi}{d} w}\tilde u) - \sigma \tilde u w. \end{aligned} \tag{2.6}$$

Similarly, we can rewrite I_2 as follows:

$$\begin{aligned} I_2 = \frac{\chi}{d} u e^{\frac{\chi}{d} w} w_t =& \frac{\chi}{d} u e^{\frac{\chi}{d} w}[\varepsilon \Delta w - \mu w - \lambda u w] \\ =& \frac{\chi\varepsilon}{d}\nabla\cdot(\tilde u \nabla w) - \frac{\chi\varepsilon}{d}\nabla w \cdot \nabla \tilde u - \frac{\chi\mu}{d}\tilde u w - \frac{\chi\lambda}{d} e^{-\frac{\chi}{d} w} w \tilde u^2. \end{aligned} \tag{2.7}$$

Substituting (2.6) and (2.7) into (2.5), and using the fact $w = g_2 - e^{-g_1}\tilde w =: \gamma(\tilde w)$, we obtain

$$\begin{cases} \tilde u_t = d\Delta \tilde u + \frac{\chi\varepsilon}{d}\nabla\cdot[\tilde u \nabla \gamma(\tilde w)] - \frac{\chi(\varepsilon+d)}{d}\nabla\gamma(\tilde w)\cdot\nabla\tilde u + F_1(\tilde u, \tilde w), & x \in \Omega, t > 0, \\ \tilde w_t = \varepsilon\Delta\tilde w - \mu \tilde w - 2\varepsilon \nabla g_1 \cdot \nabla \tilde w + F_2(\tilde u, \tilde w), & x \in \Omega, t > 0, \\ \frac{\partial \tilde u}{\partial \nu} = \frac{\partial \tilde w}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ \tilde u(x,0) = \tilde u_0(x), \;\tilde w(x,0) = \tilde w_0(x), & x \in \Omega, \end{cases} \tag{2.8}$$

where

$$F_1(\tilde u, \tilde w) = \tilde u - \frac{\chi\mu + d\sigma}{d}\tilde u \gamma(\tilde w) - \left(1 + \frac{\chi\lambda}{d}\gamma(\tilde w)\right) e^{-\frac{\chi}{d}\gamma(\tilde w)}\tilde u^2$$

and

$$F_2(\tilde u, \tilde w) = \varepsilon(|\nabla g_1|^2 - \Delta g_1)\tilde w + e^{g_1}[g_{2t} - \varepsilon\Delta g_2 + \mu g_2 + \lambda e^{-\frac{\chi}{d}\gamma(\tilde w)}\tilde u \gamma(\tilde w)],$$

as well as

$$\tilde u_0 = u_0 e^{\frac{\chi}{d} w_0} \;\text{ and }\; \tilde w_0 = e^{g_1}(g_2(\cdot,0) - w_0).$$

Then for the transformed system (2.8), we can invoke the semigroup estimates method as in [7, Proposition 2.6] to establish the local existence of classical solutions with the fixed-point theorem. We skip the proof details for brevity and state the local existence result for the original system (1.2) as follows.

Lemma 2.1 (Local existence)**.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary and the hypotheses (H_0) hold. Assume $(u_0, w_0) \in [W^{1,\infty}(\Omega)]^2$ with $u_0, w_0 \gneqq 0$. Then there exists $T_{max} > 0$ such that the problem (1.2) has a unique classical solution $(u, w) \in [C^0([0, T_{max}) \times \bar{\Omega}) \cap C^{2,1}((0, T_{max}) \times \bar{\Omega})]^2$ satisfying $u, w > 0$ for all $t > 0$. Moreover*

$$if\ T_{max} < \infty,\ then\ \|u(\cdot,t)\|_{L^\infty} + \|w(\cdot,t)\|_{W^{1,\infty}} \to \infty \quad as \quad t \nearrow T_{max}.$$

Moreover, we can show the solution (u, w) of (1.2) has the following basic estimates.

Lemma 2.2. *Let (u, w) be the solution of (1.2) obtained in Lemma 2.1. Then for all $t \in (0, T_{max})$, it holds that*

$$\|u(\cdot,t)\|_{L^1} \le M_0 := \int_\Omega u_0 + |\Omega|, \tag{2.9}$$

and

$$\|w(\cdot,t)\|_{L^\infty} \le M_1 := \max\Big\{\|w_0\|_{L^\infty}, \|h(x,t)\|_{L^\infty(\partial\Omega\times(0,\infty))}\Big\}. \tag{2.10}$$

Proof. Integrating the first equation of (1.2) by parts, we have

$$\frac{d}{dt}\int_\Omega u + \int_\Omega u = 2\int_\Omega u - \int_\Omega u^2 - \sigma\int_\Omega uw = -\int_\Omega (u-1)^2 - \sigma\int_\Omega uw + |\Omega| \le |\Omega|,$$

which gives (2.9) by Grönwall's inequality. Moreover, (2.10) follows directly from the comparison principle. □

Below, we recall two basic results.

Lemma 2.3 ([26])**.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary and $u \in W^{1,2}(\Omega)$. Then for any $\epsilon > 0$, there exists a constant $C_\epsilon > 0$ such that*

$$\|u\|_{L^3}^3 \le \epsilon\|\nabla u\|_{L^2}^2\|u\ln|u|\|_{L^1} + C_\epsilon(\|u\|_{L^1}^2\|u\ln|u|\|_{L^1} + \|u\|_{L^1}).$$

Lemma 2.4 ([37])**.** *Let $y(t) \in C^1([0,\infty))$ and $g(t) \in C^0([0,\infty))$ be nonnegative functions satisfying*

$$y(t) \le y(0)e^{-\Lambda t} + \int_0^t e^{-\Lambda(t-s)}g(s)ds, \quad \text{for all} \quad t > 0,$$

with some $\Lambda > 0$. Then if $g(t)$ is bounded on $[0,\infty)$ and satisfies

$$\int_t^{t+1} g(s)ds \to 0 \quad \text{as} \quad t \to \infty,$$

we have

$$y(t) \to 0 \quad \text{as} \quad t \to \infty.$$

2.2. *A priori* **estimates.** In this subsection, we derive some *a priori* estimates of solutions based on the coupled energy estimate method. Noting the no-flux boundary condition on u and the fact $\|w(\cdot,t)\|_{W^{1,\infty}} \le c_1\|\tilde{w}(\cdot,t)\|_{W^{1,\infty}} + c_2$, we are motivated to establish the *a priori* estimates of solutions for the following system:

$$\begin{cases} u_t = d\Delta u + \chi\nabla\cdot(u\nabla w) + u(1-u) - \sigma uw, & x \in \Omega, t > 0,\\ \tilde{w}_t = \varepsilon\Delta\tilde{w} - \mu\tilde{w} - 2\varepsilon\nabla g_1\cdot\nabla\tilde{w} + F_3(u,\tilde{w}), & x \in \Omega, t > 0,\\ (d\nabla u + \chi u\nabla w)\cdot\nu = 0, \quad \nabla\tilde{w}\cdot\nu = 0, & x \in \partial\Omega, t > 0,\\ u(x,0) = u_0(x), \quad \tilde{w}(x,0) = \tilde{w}_0(x) = e^{g_1}(g_2(\cdot,0) - w_0), & x \in \Omega, \end{cases} \tag{2.11}$$

with $\tilde{w} = e^{g_1}(g_2 - w)$ and

$$F_3(u,\tilde{w}) = \varepsilon(|\nabla g_1|^2 - \Delta g_1)\tilde{w} + e^{g_1}(g_{2t} - \varepsilon\Delta g_2 + \mu g_2 + \lambda uw). \tag{2.12}$$

Lemma 2.5. *Let $(u, \tilde{w})$ be the solution of* (2.11). *Then it holds that*

$$\frac{d}{dt}\int_\Omega u\ln u \le K_1\int_\Omega |\nabla\tilde{w}|^4 - \int_\Omega u^2\ln u + \int_\Omega u\ln u + K_2, \tag{2.13}$$

where K_1 and K_2 are positive constants independent of t and σ.

Proof. Multiplying the first equation of (2.11) by $\ln u + 1$ and integrating the resulting equation by parts, along with the fact $-u\ln u \le \frac{1}{e}$ for all $u \ge 0$, and Young's inequality

$$-\chi\int_\Omega \nabla w\cdot\nabla u \le d\int_\Omega \frac{|\nabla u|^2}{u} + \frac{\chi^2}{4d}\int_\Omega u|\nabla w|^2,$$

we derive

$$\begin{aligned}\frac{d}{dt}\int_\Omega u\ln u =& -d\int_\Omega \frac{|\nabla u|^2}{u} - \chi\int_\Omega \nabla w\cdot\nabla u - \int_\Omega (\ln u+1)u^2\\ &+ \int_\Omega(\ln u+1)u - \sigma\int_\Omega u(\ln u+1)w\\ \le& \frac{\chi^2}{4d}\int_\Omega u|\nabla w|^2 - \int_\Omega(\ln u+1)u^2 + \int_\Omega u\ln u + \int_\Omega u + \frac{\sigma}{e}\int_\Omega w\end{aligned}$$

which, together with facts $\int_\Omega u \le M_0$ in (2.9) and $\|w(\cdot,t)\|_{L^\infty} \le M_1$ in (2.10), gives

$$\frac{d}{dt}\int_\Omega u\ln u \le \frac{\chi^2}{4d}\int_\Omega u|\nabla w|^2 - \int_\Omega u^2\ln u - \int_\Omega u^2 + \int_\Omega u\ln u + c_1, \tag{2.14}$$

where $c_1 := M_0 + \frac{\sigma M_1|\Omega|}{e}$. Using Young's inequality, one has

$$\frac{\chi^2}{4d}\int_\Omega u|\nabla w|^2 \le \int_\Omega u^2 + \frac{\chi^4}{64d^2}\int_\Omega |\nabla w|^4. \tag{2.15}$$

Substituting (2.15) into (2.14) gives

$$\frac{d}{dt}\int_\Omega u\ln u \le \frac{\chi^4}{64d^2}\int_\Omega |\nabla w|^4 - \int_\Omega u^2\ln u + \int_\Omega u\ln u + c_1. \tag{2.16}$$

Using (2.1), (2.2) and noting $\|w(\cdot,t)\|_{L^\infty} \le M_1$, one can derive that

$$\|\tilde{w}(\cdot,t)\|_{L^\infty} = \|e^{g_1}(g_2 - w)\|_{L^\infty} \le \|e^{g_1}\|_{L^\infty}(\|g_2\|_{L^\infty} + \|w\|_{L^\infty}) \le e^{\gamma_1}(\gamma_2 + M_1) =: \gamma_3, \tag{2.17}$$

and hence

$$\begin{aligned}|\nabla w| = |\nabla g_2 - e^{-g_1}\nabla\tilde{w} + \tilde{w}e^{-g_1}\nabla g_1| &\le |\nabla g_2| + e^{|g_1|}|\nabla\tilde{w}| + e^{|g_1|}|\nabla g_1||\tilde{w}|\\ &\le e^{\gamma_1}|\nabla\tilde{w}| + \gamma_2 + \gamma_1\gamma_3 e^{\gamma_1},\end{aligned} \tag{2.18}$$

which gives

$$\begin{aligned}\frac{\chi^4}{64d^2}\int_\Omega |\nabla w|^4 &\le \frac{\chi^4}{64d^2}\int_\Omega [e^{\gamma_1}|\nabla\tilde{w}| + \gamma_2 + \gamma_1 e^{\gamma_1}]^4\\ &\le \frac{\chi^4 e^{4\gamma_1}}{d^2}\int_\Omega |\nabla\tilde{w}|^4 + \frac{\chi^4(\gamma_2 + \gamma_1\gamma_3 e^{\gamma_1})^4|\Omega|}{d^2},\end{aligned}$$

which substituted into (2.16) gives (2.13) and thus completes the proof of Lemma 2.5. □

Lemma 2.6. *There are some positive constants K_3, K_4 and K_5 independent of t and σ such that the solution of* (2.11) *satisfies*

$$\frac{d}{dt}\int_\Omega |\nabla\tilde{w}|^2 + 2\mu\int_\Omega |\nabla\tilde{w}|^2 + K_3\int_\Omega |\nabla\tilde{w}|^4 \le K_4\int_\Omega u^2 + K_5. \tag{2.19}$$

Proof. Integrating the second equation of (2.11) multiplied by $-\Delta\tilde{w}$ and using the homogeneous Neumann boundary condition $\nabla\tilde{w}\cdot\nu = 0$ on $\partial\Omega$, we find that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla\tilde{w}|^2 + \varepsilon\int_\Omega |\Delta\tilde{w}|^2 + \mu\int_\Omega |\nabla\tilde{w}|^2 = 2\varepsilon\int_\Omega \nabla\tilde{w}\cdot\nabla g_1\Delta\tilde{w} - \int_\Omega F_3(u,\tilde{w})\Delta\tilde{w}. \tag{2.20}$$

With (2.1) and Young's inequality, we have

$$2\varepsilon\int_\Omega \nabla\tilde{w}\cdot\nabla g_1\Delta\tilde{w} \le \frac{\varepsilon}{4}\int_\Omega |\Delta\tilde{w}|^2 + 4\varepsilon\int_\Omega |\nabla g_1|^2|\nabla\tilde{w}|^2 \le \frac{\varepsilon}{4}\int_\Omega |\Delta\tilde{w}|^2 + 4\varepsilon\gamma_1^2\int_\Omega |\nabla\tilde{w}|^2. \tag{2.21}$$

Using the definition of $F_3(u,\tilde{w})$ in (2.12), and the estimates in (2.1), (2.2) and (2.17), one has

$$\begin{aligned} |F_3(u,\tilde{w})| \le& \varepsilon(|\nabla g_1|^2 + |\Delta g_1|)|\tilde{w}| + e^{|g_1|}(|g_{2t}| + \varepsilon|\Delta g_2| + \mu|g_2| + \lambda uw) \\ \le& \varepsilon(\gamma_1+1)\gamma_1\gamma_3 + e^{\gamma_1}(\gamma_2 + \varepsilon\gamma_2 + \mu\gamma_2 + \lambda M_1 u) \\ \le& \gamma_4(1+u), \end{aligned} \tag{2.22}$$

with $\gamma_4 := \varepsilon(\gamma_1+1)\gamma_1\gamma_3 + e^{\gamma_1}(\gamma_2 + \varepsilon\gamma_2 + \mu\gamma_2 + \lambda M_1)$. Then using Young's inequality, one can derive that

$$\begin{aligned} -\int_\Omega F_3(u,\tilde{w})\Delta\tilde{w} \le \int_\Omega |F_3(u,\tilde{w})||\Delta\tilde{w}| \le \gamma_4\int_\Omega (1+u)|\Delta\tilde{w}| \\ \le \frac{\varepsilon}{4}\int_\Omega |\Delta\tilde{w}|^2 + \frac{2\gamma_4^2}{\varepsilon}\int_\Omega u^2 + \frac{2\gamma_4^2}{\varepsilon}|\Omega|. \end{aligned} \tag{2.23}$$

Then substituting (2.21) and (2.23) into (2.20), we obtain

$$\frac{d}{dt}\int_\Omega |\nabla\tilde{w}|^2 + \varepsilon\int_\Omega |\Delta\tilde{w}|^2 + 2\mu\int_\Omega |\nabla\tilde{w}|^2 \le 8\varepsilon\gamma_1^2\int_\Omega |\nabla\tilde{w}|^2 + \frac{4\gamma_4^2}{\varepsilon}\int_\Omega u^2 + \frac{4\gamma_4^2}{\varepsilon}|\Omega|. \tag{2.24}$$

Using the Gagliardo-Nirenberg inequality and the fact $\|\tilde{w}(\cdot,t)\|_{L^\infty} \le \gamma_3$ in (2.17) again, we have

$$\int_\Omega |\nabla\tilde{w}|^4 = \|\nabla\tilde{w}\|_{L^4}^4 \le c_2(\|\Delta\tilde{w}\|_{L^2}^2\|\tilde{w}\|_{L^\infty}^2 + \|\tilde{w}\|_{L^\infty}^4) \le c_2\gamma_3^2\|\Delta\tilde{w}\|_{L^2}^2 + c_2\gamma_3^4,$$

which gives

$$\varepsilon\int_\Omega |\Delta\tilde{w}|^2 \ge \frac{\varepsilon}{c_2\gamma_3^2}\int_\Omega |\nabla\tilde{w}|^4 - \varepsilon\gamma_3^2. \tag{2.25}$$

Substituting (2.25) into (2.24), alongside Young's inequality, one has

$$\begin{aligned} &\frac{d}{dt}\int_\Omega |\nabla\tilde{w}|^2 + 2\mu\int_\Omega |\nabla\tilde{w}|^2 + \frac{\varepsilon}{c_2\gamma_3^2}\int_\Omega |\nabla\tilde{w}|^4 \\ &\le 8\varepsilon\gamma_1^2\int_\Omega |\nabla\tilde{w}|^2 + \frac{4\gamma_4^2}{\varepsilon}\int_\Omega u^2 + \frac{4\gamma_4^2}{\varepsilon}|\Omega| + \varepsilon\gamma_3^2 \\ &\le \frac{\varepsilon}{2c_2\gamma_3^2}\int_\Omega |\nabla\tilde{w}|^4 + \frac{4\gamma_4^2}{\varepsilon}\int_\Omega u^2 + 32\varepsilon c_2\gamma_1^4\gamma_3^2|\Omega| + \frac{4\gamma_4^2}{\varepsilon}|\Omega| + \varepsilon\gamma_3^2, \end{aligned}$$

which gives (2.19). Then we complete the proof of Lemma 2.6. □

Lemma 2.7. *Let* $(u,\tilde{w})$ *be a solution of* (2.11)*. Then we have*

$$\|u\ln u\|_{L^1} + \|\nabla\tilde{w}\|_{L^2} \le K_6, \tag{2.26}$$

where $K_6 > 0$ *is a constant independent of* t *and* σ*.*

Proof. Multiplying (2.19) by $\frac{K_1}{K_3}$, and combining it with (2.13), we obtain

$$\begin{aligned} &\frac{d}{dt}\left(\int_\Omega u\ln u + \frac{K_1}{K_3}\int_\Omega |\nabla\tilde{w}|^2\right) + 2\mu\left(\int_\Omega u\ln u + \frac{K_1}{K_3}\int_\Omega |\nabla\tilde{w}|^2\right) \\ &\le \int_\Omega \left(-u^2\ln u + \frac{K_1K_4}{K_3}u^2 + (1+2\mu)\,u\ln u\right) + K_2 + \frac{K_1K_5}{K_3}. \end{aligned} \tag{2.27}$$

Define $\mathcal{F}(u) := -u^2 \ln u + \frac{K_1 K_4}{K_3} u^2 + (1+2\mu)\, u \ln u$. Then $\mathcal{F}(u)$ is a continuous function in $[0,\infty)$ such that

$$\lim_{u\to 0} \mathcal{F}(u) = 0 \quad\text{and}\quad \lim_{u\to\infty} \mathcal{F}(u) = -\infty,$$

which implies that there exists a constant $c_1 > 0$ such that

$$\int_\Omega \left(-u^2 \ln u + \frac{K_4 K_1}{K_3} u^2 + (1+2\mu)\, u \ln u\right) = \int_\Omega \mathcal{F}(u) \le c_1. \tag{2.28}$$

Substituting (2.28) into (2.27) and letting $c_2 := c_1 + K_2 + \frac{K_1 K_5}{K_3}$, we have

$$\frac{d}{dt}\left(\int_\Omega u \ln u + \frac{K_1}{K_3}\int_\Omega |\nabla \tilde{w}|^2\right) + 2\mu \left(\int_\Omega u \ln u + \frac{K_1}{K_3}\int_\Omega |\nabla \tilde{w}|^2\right) \le c_2,$$

which, together with Grönwall's inequality, gives

$$\int_\Omega u \ln u + \frac{K_1}{K_3}\int_\Omega |\nabla \tilde{w}|^2 \le c_3. \tag{2.29}$$

From (2.29), one derives

$$\int_\Omega |\nabla \tilde{w}|^2 \le \frac{K_3 c_3}{K_1}. \tag{2.30}$$

Noting the fact $-u\ln u \le \frac{1}{e}$ for all $u \ge 0$, from (2.29) we have $\int_\Omega u \ln u \le c_3$ and hence

$$\int_\Omega |u\ln u| = \int_\Omega \left|u \ln u + \frac{1}{e} - \frac{1}{e}\right| \le \int_\Omega \left(u \ln u + \frac{1}{e}\right) + \int_\Omega \frac{1}{e} \le c_3 + \frac{2|\Omega|}{e}. \tag{2.31}$$

Then the combination of (2.30) and (2.31) gives (2.26). The proof of Lemma 2.7 is finished. □

Next, we shall use the coupled energy estimate method to establish the boundedness of $\|u(\cdot,t)\|_{L^2}$. To this end, we first show the following results.

Lemma 2.8. *Let $(u,\tilde{w})$ be a solution of (2.11). Then it holds that*

$$\frac{d}{dt}\int_\Omega u^2 + \int_\Omega u^2 + d\int_\Omega |\nabla u|^2 \le K_7 \int_\Omega u^2 |\nabla \tilde{w}|^2 + K_7, \tag{2.32}$$

where $K_7 > 0$ is a constant independent of t and σ.

Proof. Multiplying the first equation of (2.11) by $2u$, integrating the result with respect to x over Ω, one has

$$\begin{aligned}\frac{d}{dt}\int_\Omega u^2 + 2d\int_\Omega |\nabla u|^2 &= -2\chi \int_\Omega u\nabla u \cdot \nabla w + 2\int_\Omega u^2(1-u) - 2\sigma \int_\Omega u^2 w \\ &\le d\int_\Omega |\nabla u|^2 + \frac{\chi^2}{d}\int_\Omega u^2 |\nabla w|^2 + 2\int_\Omega u^2 - 2\int_\Omega u^3.\end{aligned} \tag{2.33}$$

Using (2.18), we can derive that

$$\begin{aligned}\frac{\chi^2}{d}\int_\Omega u^2|\nabla w|^2 &\le \frac{\chi^2}{d}\int_\Omega u^2 [e^{\gamma_1}|\nabla \tilde{w}| + \gamma_2 + \gamma_1\gamma_3 e^{\gamma_1}]^2 \\ &\le \frac{2\chi^2 e^{2\gamma_1}}{d}\int_\Omega u^2 |\nabla \tilde{w}|^2 + \frac{2\chi^2(\gamma_2 + \gamma_1\gamma_3 e^{\gamma_1})^2}{d}\int_\Omega u^2.\end{aligned} \tag{2.34}$$

Substituting (2.34) into (2.33), we can find a constant $c_1 := \frac{2\chi^2(\gamma_2+\gamma_1\gamma_3 e^{\gamma_1})^2 + 2d}{d}$ such that

$$\frac{d}{dt}\int_\Omega u^2 + d\int_\Omega |\nabla u|^2 + 2\int_\Omega u^3 \le \frac{2\chi^2 e^{2\gamma_1}}{d}\int_\Omega u^2|\nabla\tilde{w}|^2 + c_1 \int_\Omega u^2. \tag{2.35}$$

Using the Young's inequality, we have

$$(c_1 + 1)\int_\Omega u^2 \le 2\int_\Omega u^3 + c_2,$$

which, substituted into (2.35), gives

$$\frac{d}{dt}\int_\Omega u^2+\int_\Omega u^2+d\int_\Omega |\nabla u|^2 \le \frac{2\chi^2 e^{2\gamma_1}}{d}\int_\Omega u^2|\nabla \tilde{w}|^2+c_2,$$

which yields (2.32). □

Lemma 2.9. *Let $(u,\tilde{w})$ be a solution of (2.11). Then there exists a constant $K_8>0$ independent of t and σ such that*

$$\frac{d}{dt}\int_\Omega |\nabla \tilde{w}|^4+\int_\Omega |\nabla \tilde{w}|^4+\varepsilon\int_\Omega |\nabla|\nabla \tilde{w}|^2|^2+\varepsilon\int_\Omega |\nabla \tilde{w}|^2|D^2\tilde{w}|^2 \le K_8\int_\Omega u^2|\nabla \tilde{w}|^2+K_8. \tag{2.36}$$

Proof. From the second equation of (2.11), we have

$$\begin{aligned}(|\nabla \tilde{w}|^2)_t &= 2\varepsilon\nabla\tilde{w}\cdot\nabla\Delta\tilde{w}-2\mu|\nabla\tilde{w}|^2-4\varepsilon\nabla\tilde{w}\cdot\nabla(\nabla g_1\cdot\nabla w)+2\nabla\tilde{w}\cdot\nabla F_2(u,\tilde{w})\\ &=\varepsilon\Delta|\nabla\tilde{w}|^2-2\varepsilon|D^2\tilde{w}|^2-2\mu|\nabla\tilde{w}|^2-4\varepsilon\nabla\tilde{w}\cdot\nabla(\nabla g_1\cdot\nabla w)+2\nabla\tilde{w}\cdot\nabla F_3(u,\tilde{w}),\end{aligned} \tag{2.37}$$

where we have used the identity $2\nabla\tilde{w}\cdot\nabla\Delta\tilde{w}=\Delta|\nabla\tilde{w}|^2-2|D^2\tilde{w}|^2$. Then multiplying (2.37) by $2|\nabla\tilde{w}|^2$ and integrating the results by parts along with the boundary conditions, we have

$$\begin{aligned}&\frac{d}{dt}\int_\Omega|\nabla\tilde{w}|^4+4\mu\int_\Omega|\nabla\tilde{w}|^4+2\varepsilon\int_\Omega|\nabla|\nabla\tilde{w}|^2|^2+4\varepsilon\int_\Omega|\nabla\tilde{w}|^2|D^2\tilde{w}|^2\\ &=2\varepsilon\int_{\partial\Omega}|\nabla\tilde{w}|^2\frac{\partial|\nabla\tilde{w}|^2}{\partial\nu}dS-4\varepsilon\int_\Omega|\nabla\tilde{w}|^2\nabla\tilde{w}\cdot\nabla(\nabla g_1\cdot\nabla\tilde{w})\\ &\quad+4\int_\Omega|\nabla\tilde{w}|^2\nabla\tilde{w}\cdot\nabla F_3(u,\tilde{w})\\ &=2\varepsilon\int_{\partial\Omega}|\nabla\tilde{w}|^2\frac{\partial|\nabla\tilde{w}|^2}{\partial\nu}dS-4\varepsilon\int_\Omega\Delta\tilde{w}|\nabla\tilde{w}|^2\nabla\tilde{w}\cdot\nabla g_1-4\varepsilon\int_\Omega\Delta g_1|\nabla\tilde{w}|^4\\ &\quad-4\int_\Omega(\nabla|\nabla\tilde{w}|^2\cdot\nabla\tilde{w})F_3(u,\tilde{w})-4\int_\Omega\Delta\tilde{w}|\nabla\tilde{w}|^2F_3(u,\tilde{w}).\end{aligned} \tag{2.38}$$

Noting the fact $\nabla\tilde{w}\cdot\nu|_{\partial\Omega}=0$, from [25, Lemma 4.2], we have $\frac{\partial|\nabla\tilde{w}|^2}{\partial\nu}\le C_\Omega|\nabla\tilde{w}|^2$ on $\partial\Omega$, for some constant $C_\Omega>0$, which, combined with the following trace inequality [29, Remark 52.7]

$$\|f\|_{L^2(\partial\Omega)}\le\delta\|\nabla f\|_{L^2(\Omega)}+c_\delta\|f\|_{L^2(\Omega)},\quad\text{for some constant }\delta>0$$

enables us to estimate the first term on the right hand of (2.38) as follows:

$$2\varepsilon\int_{\partial\Omega}|\nabla\tilde{w}|^2\frac{\partial|\nabla\tilde{w}|^2}{\partial\nu}dS\le2\varepsilon C_\Omega\||\nabla\tilde{w}|^2\|^2_{L^2(\partial\Omega)}\le\frac{\varepsilon}{3}\int_\Omega|\nabla|\nabla\tilde{w}|^2|^2+c_1\int_\Omega|\nabla\tilde{w}|^4. \tag{2.39}$$

Moreover, we can use Young's inequality, (2.1) and the fact $|\Delta\tilde{w}|\le\sqrt{2}|D^2\tilde{w}|$ to derive

$$\begin{aligned}-4\varepsilon\int_\Omega\Delta\tilde{w}|\nabla\tilde{w}|^2\nabla\tilde{w}\cdot\nabla g_1&\le4\sqrt{2}\varepsilon\gamma_1\int_\Omega|D^2\tilde{w}||\nabla\tilde{w}|^3\\ &\le\varepsilon\int_\Omega|D^2\tilde{w}|^2|\nabla\tilde{w}|^2+8\varepsilon\gamma_1^2\int_\Omega|\nabla\tilde{w}|^4\end{aligned} \tag{2.40}$$

and

$$-4\varepsilon\int_\Omega\Delta g_1|\nabla\tilde{w}|^4\le4\varepsilon\|\Delta g_1\|_{L^\infty}\int_\Omega|\nabla\tilde{w}|^4\le4\varepsilon\gamma_1\int_\Omega|\nabla\tilde{w}|^4. \tag{2.41}$$

Using (2.22) and Young's inequality again, one has

$$\begin{aligned}
&-4\int_\Omega \nabla|\nabla\tilde{w}|^2\cdot\nabla\tilde{w}F_3(u,\tilde{w})-4\int_\Omega \Delta\tilde{w}|\nabla\tilde{w}|^2F_3(u,\tilde{w})\\
&\le 4\gamma_4\int_\Omega |\nabla|\nabla\tilde{w}|^2||\nabla\tilde{w}|(1+u)+4\sqrt{2}\gamma_4\int_\Omega |D^2\tilde{w}||\nabla\tilde{w}|^2(1+u)\\
&\le \frac{\varepsilon}{3}\int_\Omega |\nabla|\nabla\tilde{w}|^2|^2+\varepsilon\int_\Omega |D^2\tilde{w}|^2|\nabla\tilde{w}|^2+\frac{20\gamma_4^2}{\varepsilon}\int_\Omega |\nabla\tilde{w}|^2(1+u)^2\\
&\le \frac{\varepsilon}{3}\int_\Omega |\nabla|\nabla\tilde{w}|^2|^2+\varepsilon\int_\Omega |D^2\tilde{w}|^2|\nabla\tilde{w}|^2+\frac{40\gamma_4^2}{\varepsilon}\int_\Omega |\nabla\tilde{w}|^2+\frac{40\gamma_4^2}{\varepsilon}\int_\Omega u^2|\nabla\tilde{w}|^2\\
&\le \frac{\varepsilon}{3}\int_\Omega |\nabla|\nabla\tilde{w}|^2|^2+\varepsilon\int_\Omega |D^2\tilde{w}|^2|\nabla\tilde{w}|^2+4\mu\int_\Omega |\nabla\tilde{w}|^4+\frac{40\gamma_4^2}{\varepsilon}\int_\Omega u^2|\nabla\tilde{w}|^2+\frac{10\gamma_4^4|\Omega|}{\mu\varepsilon^2}.
\end{aligned}\tag{2.42}$$

Then substituting (2.39), (2.40), (2.41) and (2.42) into (2.38), it follows that

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega |\nabla\tilde{w}|^4+\int_\Omega |\nabla\tilde{w}|^4+\frac{4\varepsilon}{3}\int_\Omega |\nabla|\nabla\tilde{w}|^2|^2+2\varepsilon\int_\Omega |\nabla\tilde{w}|^2|D^2\tilde{w}|^2\\
&\le (1+c_1+8\varepsilon\gamma_1^2+4\varepsilon\gamma_1)\int_\Omega |\nabla\tilde{w}|^4+\frac{40\gamma_4^2}{\varepsilon}\int_\Omega u^2|\nabla\tilde{w}|^2+\frac{10\gamma_4^4|\Omega|}{\mu\varepsilon^2}.
\end{aligned}\tag{2.43}$$

Using the Gagliardo-Nirenberg inequality along with the fact $\||\nabla\tilde{w}|^2\|_{L^1}=\|\nabla\tilde{w}\|_{L^2}^2\le K_6^2$ from (2.26), we have

$$\begin{aligned}
(1+c_1+8\varepsilon\gamma_1^2+4\varepsilon\gamma_1)\int_\Omega |\nabla\tilde{w}|^4&=(1+c_1+8\varepsilon\gamma_1^2+4\varepsilon\gamma_1)\||\nabla\tilde{w}|^2\|_{L^2}^2\\
&\le c_2\|\nabla|\nabla\tilde{w}|^2\|_{L^2}\||\nabla\tilde{w}|^2\|_{L^1}+c_2\||\nabla\tilde{w}|^2\|_{L^1}^2\\
&\le \frac{\varepsilon}{3}\int_\Omega |\nabla|\nabla\tilde{w}|^2|^2+c_3.
\end{aligned}\tag{2.44}$$

Substituting (2.44) into (2.43), we obtain (2.36) directly. The proof of Lemma 2.9 is complete. □

Lemma 2.10. *Suppose $(u,\tilde{w})$ is a solution of* (2.11). *Then it holds that*

$$\|u(\cdot,t)\|_{L^2}+\|\nabla\tilde{w}(\cdot,t)\|_{L^4}\le K_9,\tag{2.45}$$

where $K_9>0$ is a constant indenpendent of t and σ.

Proof. Combining (2.32) and (2.36), we can find two positive constants c_1 and c_2 such that

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega (u^2+|\nabla\tilde{w}|^4)+\int_\Omega (u^2+|\nabla\tilde{w}|^4)+d\int_\Omega |\nabla u|^2\\
&\quad+\varepsilon\int_\Omega |\nabla|\nabla\tilde{w}|^2|^2+\varepsilon\int_\Omega |\nabla\tilde{w}|^2|D^2\tilde{w}|^2\le c_1\int_\Omega u^2|\nabla\tilde{w}|^2+c_2.
\end{aligned}\tag{2.46}$$

Using the Hölder inequality and Young's inequality, one can find a positive constant κ_1 small enough (which will be chosen later) such that

$$c_1\int_\Omega u^2|\nabla\tilde{w}|^2\le c_1\|u\|_{L^3}^2\|\nabla\tilde{w}\|_{L^6}^2\le \kappa_1\|\nabla\tilde{w}\|_{L^6}^6+c_3\|u\|_{L^3}^3.\tag{2.47}$$

Noting the facts $\|u\ln u\|_{L^1}\le K_6$ and $\|u\|_{L^1}\le M_0$, and using Lemma 2.3, one has

$$c_3\|u\|_{L^3}^3\le d\|\nabla u\|_{L^2}^2+c_4.\tag{2.48}$$

Then substituting (2.47) and (2.48) into (2.46), it holds that

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega (u^2+|\nabla\tilde{w}|^4)+\int_\Omega (u^2+|\nabla\tilde{w}|^4)+\varepsilon\int_\Omega |\nabla|\nabla\tilde{w}|^2|^2+\varepsilon\int_\Omega |\nabla\tilde{w}|^2|D^2\tilde{w}|^2\\
&\le \kappa_1\int_\Omega |\nabla\tilde{w}|^6+c_2+c_4.
\end{aligned}\tag{2.49}$$

Using the Gagliardo-Nirenberg inequality and the fact $\|\nabla\tilde{w}\|_{L^2} \le K_6$ in (2.26), we obtain

$$\begin{aligned}\int_\Omega |\nabla\tilde{w}|^6 = \||\nabla w|^2\|_{L^3}^3 &\le c_5\left(\|\nabla|\nabla\tilde{w}|^2\|_{L^2}^2\||\nabla w|^2\|_{L^1} + \||\nabla\tilde{w}|^2\|_{L^1}^3\right)\\ &\le c_5K_6^2\|\nabla|\nabla\tilde{w}|^2\|_{L^2}^2 + c_5K_6^6.\end{aligned} \tag{2.50}$$

Substituting (2.50) into (2.49), and choosing $\kappa_1 = \frac{\varepsilon}{c_5K_6^2}$, we can derive that

$$\frac{d}{dt}\int_\Omega \left(u^2 + |\nabla\tilde{w}|^4\right) + \int_\Omega \left(u^2 + |\nabla\tilde{w}|^4\right) \le c_6,$$

which gives (2.45) directly by using Grönwall's inequality. Then we complete the proof of Lemma 2.10. $\square$

Lemma 2.11. *Let $(u, \tilde{w})$ be a solution of the system* (2.11)*. Then we can find a positive constant K_{10} independent of t and σ such that*

$$\|u(\cdot,t)\|_{L^4} \le K_{10}. \tag{2.51}$$

Proof. We multiply the first equation of (2.11) with u^3 and integrate it by part over Ω to have

$$\begin{aligned}\frac{1}{4}\frac{d}{dt}\int_\Omega u^4 &= -3d\int_\Omega u^2|\nabla u|^2 - 3\chi\int_\Omega u^3\nabla u\cdot\nabla w + \int_\Omega u^4 - \int_\Omega u^5 - \sigma\int_\Omega u^4 w\\ &\le -d\int_\Omega u^2|\nabla u|^2 + \frac{9\chi^2}{8d}\int_\Omega u^4|\nabla w|^2 + \int_\Omega u^4 - \int_\Omega u^5,\end{aligned}$$

which, together with the fact $|\nabla w| \le e^{\gamma_1}|\nabla\tilde{w}| + \gamma_2 + \gamma_1\gamma_3 e^{\gamma_1}$ in (2.18), gives

$$\begin{aligned}\frac{d}{dt}\int_\Omega u^4 + \int_\Omega u^4 + d\int_\Omega |\nabla u^2|^2 \le& \frac{9\chi^2}{2d}\int_\Omega u^4|\nabla w|^2 + 5\int_\Omega u^4 - 4\int_\Omega u^5\\ \le& c_1\int_\Omega u^4|\nabla\tilde{w}|^2 + c_2\int_\Omega u^4 - 4\int_\Omega u^5,\end{aligned} \tag{2.52}$$

where $c_1 := \frac{9\chi^2 e^{2\gamma_1}}{d}$ and $c_2 := \frac{9\chi^2(\gamma_2+\gamma_1\gamma_3e^{\gamma_1})^2+5d}{d}$. From (2.45), we have $\|u^2\|_{L^1} = \|u\|_{L^2}^2 \le K_9^2$ and $\|\nabla\tilde{w}\|_{L^4} \le K_9$. Then using the Gagliardo-Nirenberg inequality and Young's inequality, we can derive that

$$\begin{aligned}c_1\int_\Omega u^4|\nabla\tilde{w}|^2 &\le c_1\left(\int_\Omega u^8\right)^{\frac{1}{2}}\left(\int_\Omega |\nabla\tilde{w}|^4\right)^{\frac{1}{2}}\\ &\le c_3\left(\|\nabla u^2\|_{L^2}^{\frac{3}{2}}\|u^2\|_{L^1}^{\frac{1}{2}} + \|u^2\|_{L^1}^2\right)\|\nabla\tilde{w}\|_{L^4}^2\\ &\le c_3K_9^3\|\nabla u^2\|_{L^2}^{\frac{3}{2}} + c_3K_9^6\\ &\le d\|\nabla u^2\|_{L^2}^2 + c_4.\end{aligned} \tag{2.53}$$

Again, using Young's inequality, we have

$$c_2\int_\Omega u^4 \le 4\int_\Omega u^5 + c_5. \tag{2.54}$$

Substituting (2.53) and (2.54) into (2.52), we have

$$\frac{d}{dt}\int_\Omega u^4 + \int_\Omega u^4 \le c_4 + c_5,$$

which together with Grönwall's inequality gives (2.51). Then the proof of Lemma 2.11 is finished. $\square$

Lemma 2.12. *The solution $(u, \tilde{w})$ of* (2.11) *satisfies*

$$\|\nabla\tilde{w}(\cdot,t)\|_{L^\infty} \le K_{11}, \tag{2.55}$$

where $K_{11} > 0$ is a constant independent of t and σ.

Proof. Applying the variation-of-constants formula to the second equation of (2.11), we have

$$\tilde{w}(\cdot,t) = e^{(\varepsilon\Delta-\mu)t}\tilde{w}_0 - 2\varepsilon\int_0^t e^{(\varepsilon\Delta-\mu)(t-s)}\nabla g_1\cdot\nabla\tilde{w} + \int_0^t e^{(\varepsilon\Delta-\mu)(t-s)}F_3(u,\tilde{w}),$$

which gives

$$\nabla\tilde{w}(\cdot,t) = \nabla e^{(\varepsilon\Delta-\mu)t}\tilde{w}_0 - 2\varepsilon\int_0^t \nabla e^{(\varepsilon\Delta-\mu)(t-s)}\nabla g_1\cdot\nabla\tilde{w} + \int_0^t \nabla e^{(\varepsilon\Delta-\mu)(t-s)}F_3(u,\tilde{w}). \tag{2.56}$$

Applying the well-known semigroup estimate with homogeneous Neumann boundary conditions to (2.56), we can derive that

$$\begin{aligned}\|\nabla\tilde{w}(\cdot,t)\|_{L^\infty} \le & \|\nabla e^{(\varepsilon\Delta-\mu)t}\tilde{w}_0\|_{L^\infty} + 2\varepsilon\int_0^t \|\nabla e^{(\varepsilon\Delta-\mu)(t-s)}\nabla g_1\cdot\nabla\tilde{w}\|_{L^\infty}\\ & + \int_0^t \|\nabla e^{(\varepsilon\Delta-\mu)(t-s)}F_3(u,\tilde{w})\|_{L^\infty}\\ \le & c_1\|\nabla\tilde{w}_0\|_{L^\infty} + 2\varepsilon c_1\int_0^t (1+t^{-\frac{3}{4}})e^{-(\varepsilon\lambda_1+\mu)(t-s)}\|\nabla g_1\cdot\nabla\tilde{w}\|_{L^4}\\ & + c_1\int_0^t (1+t^{-\frac{3}{4}})e^{-(\varepsilon\lambda_1+\mu)(t-s)}\|F_3(u,\tilde{w})\|_{L^4},\end{aligned} \tag{2.57}$$

where λ_1 is the first nonzero eigenvalue of $-\Delta$ with homogeneous Neumann boundary condition. Using (2.1) and (2.45), we have

$$\|\nabla g_1\cdot\nabla\tilde{w}\|_{L^4} \le \gamma_1\|\nabla\tilde{w}\|_{L^4} \le \gamma_1 K_9. \tag{2.58}$$

On the other hand, from (2.22) and (2.51), one can derive that

$$\|F_3(u,\tilde{w})\|_{L^4} \le \gamma_4\|1+u\|_{L^4} \le c_2. \tag{2.59}$$

Substituting (2.58) and (2.59) into (2.57), and using the fact $\int_0^\infty (1+t^{-\frac{3}{4}})e^{-(\varepsilon\lambda_1+\mu)(t-s)} < \infty$, we have

$$\|\nabla\tilde{w}(\cdot,t)\|_{L^\infty} \le c_1\|\nabla\tilde{w}_0\|_{L^\infty} + c_1(2\varepsilon\gamma_1 K_9 + c_2)\int_0^\infty (1+t^{-\frac{3}{4}})e^{-(\varepsilon\lambda_1+\mu)(t-s)} \le c_3,$$

which gives (2.55). □

Proof of Theorem 1.1 (global existence). From Lemma 2.12, we have $\|\nabla\tilde{w}(\cdot,t)\|_{L^\infty} \le c_1$, which together with the fact $w = g_2 - e^{-g_1}\tilde{w}$ gives

$$\begin{aligned}\|\nabla w(\cdot,t)\|_{L^\infty} = \|\nabla g_2 - e^{-g_1}\nabla\tilde{w} + \tilde{w}e^{-g_1}\nabla g_1\|_{L^\infty} &\le \gamma_2 + \gamma_1\gamma_3 e^{\gamma_1} + e^{\gamma_1}\|\nabla\tilde{w}\|_{L^\infty}\\ &\le \gamma_2 + \gamma_1\gamma_3 e^{\gamma_1} + e^{\gamma_1}c_1.\end{aligned} \tag{2.60}$$

With (2.60) and the Moser iteration (cf. see [17, Lemma 1]), from the first equation of (1.2), we can derive the boundedness of $\|u(\cdot,t)\|_{L^\infty}$. Then the existence of global classical solutions follows from the extensibility criterion in Lemma 2.1. □

2.3. **Global stabilization.** In this subsection, we show that if $h(x,t)$ satisfies (1.3), then every solution (u,w) of (1.2) converges to $(1,0)$ uniformly in $\bar{\Omega}$ as $t\to\infty$. Before embarking on this, we first improve the regularity of u and w by the standard parabolic regularity theorem.

Lemma 2.13. *Let (u,w) be the nonnegative global classical solution of* (1.2) *obtained in Theorem 1.1. Then there exist $\theta\in(0,1)$ and $C>0$ such that*

$$\|u\|_{C^{\theta,\frac{\theta}{2}}(\bar{\Omega}\times[t,t+1])} \le C \quad \text{for all} \quad t\ge 1 \tag{2.61}$$

and

$$\|w\|_{C^{2+\theta,1+\frac{\theta}{2}}(\bar{\Omega}\times[t,t+1])} \le C \quad \text{for all} \quad t\ge 1. \tag{2.62}$$

Proof. We can rewrite the first equation of (1.2) as follows

$$u_t = \nabla \cdot A(x,t,u,\nabla u) + B(x,t,u)$$

with

$$A(x,t,u,\nabla u) = d\nabla u + \chi u \nabla w, \quad B(x,t,u) = u(1-u) - \sigma u w.$$

From the boundedness results obtained in Theorem 1.1, we know that there exist two positive constants c_1 and c_2 such that $\|u(\cdot,t)\|_{L^\infty} \le c_1$ and $\|w(\cdot,t)\|_{W^{1,\infty}} \le c_2$. Then we can check that

$$\begin{aligned} A(x,t,u,\nabla u)\cdot \nabla u &= (d\nabla u + \chi u\nabla w)\cdot \nabla u \\ &\ge d|\nabla u|^2 - \chi u|\nabla u||\nabla w| \\ &\ge \frac{d}{2}|\nabla u|^2 - \frac{\chi^2}{2d}u^2|\nabla w|^2 \\ &\ge \frac{d}{2}|\nabla u|^2 - \frac{\chi^2 c_1^2 c_2^2}{2d} \end{aligned} \tag{2.63}$$

and

$$|A(x,t,u,\nabla u)| = |d\nabla u + \chi u \nabla w| \le d|\nabla u| + \chi|u||\nabla w| \le d|\nabla u| + c_1 c_2 \chi, \tag{2.64}$$

as well as

$$|B(x,t,u)| = |u(1-u) - \sigma u w| \le c_1(1 + c_1 + \sigma c_2). \tag{2.65}$$

Then applying [28, Theorem 1.3] and using (2.63)-(2.65), we obtain (2.61). Moreover, we can use the standard parabolic regularity with (2.61) to derive (2.62) directly. □

Then we have the following results on the global convergence of w.

Lemma 2.14. *Let (u,w) be the solution obtained in Theorem 1.1, and assume* (1.3) *holds. Then it follows that*

$$\lim_{t\to\infty} \|w(\cdot,t)\|_{L^\infty} = 0. \tag{2.66}$$

Proof. We integrate the second equation of (1.2) alongside boundary condition $\nabla w\cdot \nu + \xi w = \xi h(x,t)$ to have

$$\begin{aligned} \frac{d}{dt}\int_\Omega w &= \varepsilon \int_\Omega \Delta w - \mu \int_\Omega w - \lambda \int_\Omega uw \\ &= -\varepsilon\xi \int_{\partial\Omega} w + \varepsilon\xi \int_{\partial\Omega} h(x,t) - \mu\int_\Omega w - \lambda \int_\Omega uw, \end{aligned}$$

which gives

$$\frac{d}{dt}\int_\Omega w + \mu \int_\Omega w \le \varepsilon\xi \int_{\partial\Omega} h(x,t) = \varepsilon\xi \|h(\cdot,t)\|_{L^1(\partial\Omega)},$$

and hence

$$\int_\Omega w \le e^{-\mu t}\int_\Omega w_0 + \varepsilon\xi \int_0^t e^{-\mu(t-s)}\|h(\cdot,t)\|_{L^1(\partial\Omega)} ds. \tag{2.67}$$

Then with (1.3), and applying Lemma 2.4, we can derive from (2.67) that

$$\lim_{t\to\infty} \|w(\cdot,t)\|_{L^1} = 0. \tag{2.68}$$

On the other hand, from Theorem 1.1, we know $\|\nabla w(\cdot,t)\|_{L^\infty} \le c_1$. Then using the Gagliardo-Nirenberg inequality, we can derive that

$$\|w\|_{L^\infty} \le c_2(\|\nabla w\|_{L^\infty}^{\frac{2}{3}}\|w\|_{L^1}^{\frac{1}{3}} + \|w\|_{L^1}) \le c_2 c_1^{\frac{2}{3}}\|w\|_{L^1}^{\frac{1}{3}} + c_2\|w\|_{L^1},$$

which together with (2.68) gives (2.66).

□

Next, we show $\|u(\cdot,t)-1\|_{L^\infty}\to 0$ as $t\to\infty$. To this end, we first apply the Harnack's inequality to show that $u(\cdot,t)$ has a lower bound for large t, which will be essentially used later. More precisely, we have the following results:

Lemma 2.15. *Let (u,w) be the solution obtained in Theorem 1.1, and assume that (1.3) holds. Then there exists a $T_0>0$ such that for all $t\geq T_0$*

$$\inf_{x\in\Omega} u(x,t)\geq \zeta_1, \tag{2.69}$$

where $\zeta_1>0$ is a constant independent of t.

Proof. Using the transformation $\tilde{u}=ue^{\frac{\chi}{d}w}$, from the first equation of (1.2), we can derive that

$$\begin{cases}\tilde{u}_t=d\Delta\tilde{u}-\chi\nabla w\cdot\nabla\tilde{u}+\tilde{u}\left[1+\frac{\chi\varepsilon}{d}\Delta w-\frac{\chi\mu+d\sigma}{d}w-\left(1+\frac{\chi\lambda}{d}w\right)e^{-\frac{\chi}{d}w}\tilde{u}\right], & x\in\Omega,\\ \frac{\partial\tilde{u}}{\partial\nu}=0, & x\in\partial\Omega.\end{cases} \tag{2.70}$$

From Theorem 1.1, we know there exists a constant $c_1>0$ such that

$$\|u(\cdot,t)\|_{L^\infty}+\|w(\cdot,t)\|_{L^\infty}\leq c_1. \tag{2.71}$$

Moreover, from Lemma 2.13, we know that

$$\|\Delta w(\cdot,t)\|_{L^\infty}\leq c_2,\quad \text{for all}\quad t\geq 1. \tag{2.72}$$

Hence using (2.71) and (2.72), we can find a constant $c_3>0$ such that

$$\left\|1+\frac{\chi\varepsilon}{d}\Delta w-\frac{\chi\mu+d\sigma}{d}w-\left(1+\frac{\chi\lambda}{d}w\right)e^{-\frac{\chi}{d}w}\tilde{u}\right\|_{L^\infty}\leq c_3,\qquad \text{for all}\quad t\geq 1. \tag{2.73}$$

With (2.73) in hand, and applying Harnack's inequality (see [20, Theorem 2.5] and [23, Page 12]), from (2.70) we can find a constant $c_4>0$ such that

$$\sup_{x\in\Omega}\tilde{u}(x,t)\leq c_4\inf_{x\in\Omega}\tilde{u}(x,t),\quad \text{for all}\quad t\geq 1. \tag{2.74}$$

Noting that $\tilde{u}=ue^{\frac{\chi}{d}w}$ and using (2.71), one derives from (2.74) that

$$\sup_{x\in\Omega}u(x,t)\leq c_5\inf_{x\in\Omega}u(x,t),\quad \text{for all } t\geq 1. \tag{2.75}$$

On the other hand, since $\|w(\cdot,t)\|_{L^\infty}\to 0$ as $t\to\infty$, there exists $t_1>0$ such that for all $t\geq t_1>1$, we have

$$\sigma\|w(\cdot,t)\|_{L^\infty}\leq\frac{1}{2}. \tag{2.76}$$

Integrating the first equation of (1.2) and using (2.76), then for all $t\geq t_1>1$ one has

$$\begin{aligned}\frac{d}{dt}\int_\Omega u&=\int_\Omega u-\int_\Omega u^2-\sigma\int_\Omega uw\\ &\geq(1-\sigma\|w\|_{L^\infty})\int_\Omega u-\int_\Omega u^2\\ &\geq\frac{1}{2}\int_\Omega u-\int_\Omega u^2.\end{aligned} \tag{2.77}$$

Furthermore using (2.75), we can derive that

$$\int_\Omega u^2\leq\sup_{x\in\Omega}u(x,t)\cdot\int_\Omega u\leq c_5\inf_{x\in\Omega}u(x,t)\cdot\int_\Omega u\leq\frac{c_5}{|\Omega|}\left(\int_\Omega u\right)^2,\quad \text{for all } t\geq t_1>1. \tag{2.78}$$

Substituting (2.78) into (2.77) yields

$$\frac{d}{dt}\int_\Omega u\geq\frac{1}{2}\int_\Omega u-\frac{c_5}{|\Omega|}\left(\int_\Omega u\right)^2,$$

which allows us to obtain

$$\liminf_{t\to\infty}\int_\Omega u\geq\frac{|\Omega|}{2c_5}>0. \tag{2.79}$$

Using (2.75) again, we can derive that for all $t \geq t_1$ that

$$\frac{1}{|\Omega|}\int_\Omega u(x,t) \leq \sup_{x\in\Omega} u(x,t) \leq c_5 \inf_{x\in\Omega} u(x,t), \tag{2.80}$$

which together with (2.79) gives

$$\liminf_{t\to\infty} \inf_{x\in\Omega} u(x,t) \geq \frac{1}{2c_5^2} > 0$$

and hence (2.69) holds. □

Lemma 2.16. *Assume the assumption* (1.3) *holds. Then the solution* (u,w) *of* (1.2) *satisfies*

$$\lim_{t\to\infty} \|u(\cdot,t)-1\|_{L^2} = 0 \tag{2.81}$$

Proof. Multiplying the first equation of (1.2) by $\frac{u-1}{u}$, and integrating the results by parts, we have

$$\begin{aligned}\frac{d}{dt}\int_\Omega (u-1-\ln u) + d\int_\Omega \frac{|\nabla u|^2}{u^2} + \int_\Omega (u-1)^2 &= -\chi\int_\Omega \frac{\nabla u\cdot\nabla w}{u} - \sigma\int_\Omega uw + \sigma\int_\Omega w \\ &\leq d\int_\Omega \frac{|\nabla u|^2}{u^2} + \frac{\chi^2}{4d}\int_\Omega |\nabla w|^2 + \sigma\int_\Omega w,\end{aligned}$$

which gives

$$\frac{d}{dt}\int_\Omega (u-1-\ln u) + \int_\Omega (u-1)^2 \leq \frac{\chi^2}{4d}\int_\Omega |\nabla w|^2 + \sigma\int_\Omega w. \tag{2.82}$$

Using Taylor's expansion, we have

$$u-1-\ln u = \frac{1}{2\zeta^2}(u-1)^2 \geq 0, \tag{2.83}$$

where ζ between 1 and u. Using the boundedness of $\|u(\cdot,t)\|_{L^\infty}$ and Lemma 2.15, we know that for some $T_1 \geq 1$, there exist two positive constants c_1 and c_2 such that

$$c_1 \leq u(\cdot,t) \leq c_2 \quad \text{for some} \quad t \geq T_1,$$

which combined with (2.83) gives

$$\frac{1}{c_2^2}\int_\Omega (u-1)^2 \leq \int_\Omega (u-1-\ln u) \leq \frac{1}{c_1^2}\int_\Omega (u-1)^2, \quad \text{for some} \quad t \geq T_1. \tag{2.84}$$

Substituting (2.84) into (2.82), we can derive for all $t \geq T_1$ that

$$\frac{d}{dt}\int_\Omega (u-1-\ln u) + c_1^2\int_\Omega (u-1-\ln u) \leq \frac{\chi^2}{4d}\int_\Omega |\nabla w|^2 + \sigma\int_\Omega w. \tag{2.85}$$

On the other hand, using the Gagliardo-Nirenberg inequality and (2.62), we can derive that

$$\|\nabla w\|_{L^2} \leq c_3(\|D^2 w\|_{L^2}^{\frac{2}{3}}\|w\|_{L^1}^{\frac{1}{3}} + \|w\|_{L^1}) \leq c_4(\|w\|_{L^1}^{\frac{1}{3}} + \|w\|_{L^1}) \quad \text{for} \quad t \geq T_1,$$

which together with (2.68) gives

$$\frac{\chi^2}{4d}\int_t^{t+1}\int_\Omega |\nabla w|^2 + \sigma\int_t^{t+1}\int_\Omega w \to 0, \text{ as } t\to\infty. \tag{2.86}$$

With (2.86), we apply Lemma 2.4 to (2.85) and get

$$\int_\Omega (u-1-\ln u) \to 0, \text{ as } t\to\infty. \tag{2.87}$$

Then the combination of (2.84) and (2.87) gives

$$\int_\Omega (u-1)^2 \leq c_2^2\int_\Omega (u-1-\ln u) \to 0 \text{ as } t\to\infty,$$

which gives (2.81). Then we complete the proof of Lemma 2.16. □

Next, we show that $\|u(\cdot,t)-1\|_{L^\infty}\to 0$ as $t\to\infty$.

Lemma 2.17. *Assume the assumption* (1.3) *holds. Then the solution* (u,w) *of* (1.2) *satisfies*

$$\lim_{t\to\infty}\|u(\cdot,t)-1\|_{L^\infty}=0. \tag{2.88}$$

Proof. By Lemma 2.13, we have

$$\sup_{t\geq 1}\|u(\cdot,t)\|_{C^\theta(\bar\Omega)}\leq c_1.$$

Thanks to the Arzelà-Ascoli Theorem, we may pass to a sequence $t=t_k\to\infty$ and assume that

$$u(\cdot,t_k)\to v \quad \text{uniformly in } C^0(\bar\Omega).$$

But then $v\equiv 1$ due to Lemma 2.16. Hence, by the uniqueness of subsequential limit, it follows that $u(\cdot,t)\to 1$ as $t\to\infty$ uniformly in $\bar\Omega$. Then we complete the proof of Lemma 2.17. □

Now we are in a position to prove global stability result asserted in Theorem 1.1.

Proof of Theorem 1.1 (global stabilization). The global stability of $(1,0)$ stated in Theorem 1.1 is a consequence of Lemma 2.14 and Lemma 2.17. □

3. Non-constant steady state: Proof of Theorem 1.2

3.1. **Existence.** In this section, we first use the Schauder fixed point theorem to prove the existence of non-constant positive solutions to the system (1.4) under some conditions on parameters. Then we show that the solution is unique if $h_0=\sup_{x\in\partial\Omega}h(x)$ is small. Before proving our main results, we first use the transformation

$$V=Ue^{\frac{\chi}{d}W} \tag{3.1}$$

to rewrite the elliptic system (1.4) as follows:

$$\begin{cases}0=d\nabla\cdot(e^{-\frac{\chi}{d}W}\nabla V)+e^{-\frac{\chi}{d}W}V(1-e^{-\frac{\chi}{d}W}V)-\sigma e^{-\frac{\chi}{d}W}WV, & x\in\Omega,\\ 0=\varepsilon\Delta W-\mu W-\lambda e^{-\frac{\chi}{d}W}WV, & x\in\Omega,\\ \nabla V\cdot\nu=0,\quad \nabla W\cdot\nu=\xi(h(x)-W), & x\in\partial\Omega.\end{cases} \tag{3.2}$$

Our idea of using the Schauder fixed point theorem to prove the existence of positive solutions to the system (3.2) can be roughly described below. Given any non-negative function $V\in C^0(\bar\Omega)$, we first consider the following elliptic problem

$$\begin{cases}0=\varepsilon\Delta W-\mu W-\lambda e^{-\frac{\chi}{d}W}WV, & x\in\Omega,\\ \nabla W\cdot\nu=\xi(h(x)-W), & x\in\partial\Omega\end{cases} \tag{3.3}$$

and show that (3.3) admits a non-negative solution $W\in C^{1+\alpha}(\bar\Omega)$. This generates a solution map $\mathcal{T}_1: C^0(\bar\Omega)\to C^{1+\alpha}(\bar\Omega)$ such that $\mathcal{T}_1(V)=W$. With this generated solution $W(x)$, we further consider the problem

$$\begin{cases}0=d\nabla\cdot(e^{-\frac{\chi}{d}W}\nabla V)+e^{-\frac{\chi}{d}W}V(1-e^{-\frac{\chi}{d}W}V)-\sigma e^{-\frac{\chi}{d}W}WV, & x\in\Omega,\\ \nabla V\cdot\nu=0, & x\in\partial\Omega,\end{cases} \tag{3.4}$$

and show that (3.4) admits a non-negative solution $\widehat{V}$. This generates another solution map $\mathcal{T}_2: C^{1+\alpha}(\bar\Omega)\to C^2(\bar\Omega)$ such that $\mathcal{T}_2(W)=\widehat{V}$. Now we define a composite map $\mathcal{T}=\mathcal{T}_2\circ\mathcal{T}_1: C^0(\bar\Omega)\to C^0(\bar\Omega)$ and show that $\mathcal{T}$ has a fixed point, namely there is a $V\in C^0(\bar\Omega)$ such that $\mathcal{T}(V)=V$ and $W=\mathcal{T}_1(V)$. Then this pair (V,W) yields a classical solution to (3.2).

In the following, we shall denote $\mathcal{T}_1(V)=:W[V]$ and $\mathcal{T}_2(W)=:V[W]$. We first show the existence of solutions to (3.3) for given $V\in C^0(\bar\Omega)$.

Lemma 3.1. *For any given non-negative function $V \in C^0(\bar{\Omega})$, system (3.3) has a solution $W[V] \in C^{1+\alpha}(\bar{\Omega})$ with $0 < \alpha < 1 - \frac{n}{p}$ satisfying*

$$0 \le W[V] \le h_0, \quad \text{for all} \quad x \in \bar{\Omega}, \tag{3.5}$$

where $h_0 = \sup_{x\in\partial\Omega} h(x)$.

Proof. Take a sequence $V_j \in C^{\theta_2}(\bar{\Omega})$ with $\theta_2 \in (0,1)$ such that $V_j \to V$ uniformly in Ω. Observe that 0 and h_0 form a pair of strict sub/supersolutions for (3.3). It follows by monotone iteration scheme that there exists a minimal solution $W_j = W[V_j] \in W^{2,p}(\Omega)$ such that $0 < W_j < h_0$; see, e.g. [23, Corollary 1.2.4]. Moreover, by L^p estimate, we see that W_j is bounded uniformly in $W^{2,p}(\Omega)$. Passing to a subsequence, we may assume that W_j converges weakly in $W^{2,p}(\Omega)$ and strongly in $C^{1+\alpha}(\bar{\Omega})$ to a strong solution $W \in W^{2,p}(\Omega)$ to (3.3). This proves the existence of $W = W[V] \in C^{1+\alpha}(\bar{\Omega})$ satisfying (3.5), where $0 < \alpha < 1 - \frac{n}{p}$ due to the Sobolev imbedding (cf. [21]) for $p > n$

$$\|W\|_{C^{1+\beta}(\bar{\Omega})} \le c_1 \|W\|_{W^{2,p}(\Omega)} \le c_2 \qquad \text{with} \ \ \beta = 1 - \frac{n}{p}. \tag{3.6}$$

□

Lemma 3.2. *If $h_0\chi < d$, then the solution obtained in Lemma 3.1 is unique, and the mapping $V \to W[V]$ from $C^0(\bar{\Omega})$ to $C^{1+\alpha}(\bar{\Omega})$ is continuous.*

Proof. Our proof is divided into the following two steps:

Step 1. We first prove the uniqueness of solutions under the condition $h_0\chi < d$. Assume that $W_1 \in H^1(\Omega)$ and $W_2 \in H^1(\Omega)$ are two different solutions of (3.3). Then it holds that

$$\begin{cases} \varepsilon\Delta(W_1 - W_2) = \mu(W_1 - W_2) + \lambda V(e^{-\frac{\chi}{d}W_1}W_1 - e^{-\frac{\chi}{d}W_2}W_2), & x \in \Omega, \\ \nabla(W_1 - W_2)\cdot\nu + \xi(W_1 - W_2) = 0, & x \in \partial\Omega. \end{cases} \tag{3.7}$$

Multiplying (3.7) by $W_1 - W_2$, and integrating the resulting equation by parts, we have

$$\begin{aligned} &-\varepsilon\int_\Omega |\nabla(W_1 - W_2)|^2 - \varepsilon\xi\int_{\partial\Omega}(W_1 - W_2)^2 \\ &= \mu\int_\Omega (W_1 - W_2)^2 + \lambda\int_\Omega V(e^{-\frac{\chi}{d}W_1}W_1 - e^{-\frac{\chi}{d}W_2}W_2)(W_1 - W_2) \\ &= \int_\Omega [\mu + \lambda V f'(\overline{W})](W_1 - W_2)^2, \end{aligned} \tag{3.8}$$

where $f'(z) = \left(1 - \frac{\chi z}{d}\right)e^{-\frac{\chi}{d}z}$ and $\overline{W} = \theta_1 W_1 + (1-\theta_1)W_2$ with $\theta_1 \in (0,1)$. Since $0 < W_i \le h_0$ for $i = 1, 2$, we have $0 < \overline{W} \le h_0$ and then $f'(\overline{W}) > 0$ in the case of $h_0\chi < d$. On the other hand, since $V \in C^0(\bar{\Omega})$ is a nonnegative function, we know there exists a constant $K > 0$ such that $0 \le V \le K$ and then

$$\mu + \lambda V f'(\overline{W}) \ge \mu > 0, \tag{3.9}$$

if $h_0\chi < d$. Then combining (3.8) and (3.9), we have $\mu\int_\Omega (W_1 - W_2)^2 \le 0$, which gives $W_1 = W_2$.

Step 2. We show the mapping $V \to W[V]$ is a continuous function from $C^0(\bar{\Omega})$ to $C^{1+\alpha}(\bar{\Omega})$. In fact, if we assume that $\{V_i\}_{i\in\mathbb{N}}$ is a sequence in $C^0(\bar{\Omega})$ satisfying $\lim\limits_{i\to\infty} V_i = V$, but

$$\lim_{i\to\infty} W[V_i] \ne W[V], \quad \text{in} \ \ C^{1+\alpha}(\bar{\Omega}). \tag{3.10}$$

Then there exists a subsequence $\{V_{i_j}\}_{j\in\mathbb{N}}$ and a constant $\delta > 0$ such that

$$\|W[V_{i_j}] - W[V]\|_{C^{1+\alpha}(\bar{\Omega})} \ge \delta \quad \text{for all} \ \ j \in \mathbb{N}. \tag{3.11}$$

From (3.6), we know that there exists a $\beta^* \in (\alpha, 1)$ with $0 < \alpha < \beta$ such that the sequence $\{W[V_{i_j}]\}_{j\in\mathbb{N}} \in C^{1+\beta^*}(\bar{\Omega})$. Then by the Arzelá-Ascoli theorem there exists a sub-sequence

$\{W[V_{i_{j_k}}]\}_{k\in\mathbb{N}}$ converge to $\widehat{W}$ in $C^{1+\alpha}(\bar{\Omega})$. Since for any $k \in \mathbb{N}$, the function $W[V_{i_{j_k}}]$ is a weak solution of

$$\begin{cases}\varepsilon\Delta W[V_{i_{j_k}}] = \mu W[V_{i_{j_k}}] + \lambda e^{-\frac{\chi}{d}W[V_{i_{j_k}}]}W[V_{i_{j_k}}]V_{i_{j_k}}, & x\in\Omega,\\ \nabla W[V_{i_{j_k}}]\cdot\nu = \xi(h(x) - W[V_{i_{j_k}}]), & x\in\partial\Omega.\end{cases} \tag{3.12}$$

Then sending $k\to\infty$ in (3.12), we obtain that $\widehat{W}$ is a weak solution of

$$\begin{cases}\varepsilon\Delta\widehat{W} = \mu\widehat{W} + \lambda e^{-\frac{\chi}{d}\widehat{W}}\widehat{W}V, & x\in\Omega,\\ \nabla\widehat{W}\cdot\nu = \xi(h(x)-\widehat{W}), & x\in\partial\Omega.\end{cases} \tag{3.13}$$

Noticing Lemma 3.1 with Step 1 implies that (3.13) admits a unique solution $W[V]$ in $C^{1+\alpha}(\bar{\Omega})$ for any given non-negative function $V\in C^0(\bar{\Omega})$, it hence follows that $\widehat{W} = W[V]$, which contradicts (3.11). This asserts that the mapping $V\to W[V]$ is continuous from $C^0(\bar{\Omega})$ to $C^{1+\alpha}(\bar{\Omega})$. The proof of Lemma 3.2 is finished. □

Remark 3.1. *To prove the mapping $V\to W[V]$ is a continuous function, we used the uniqueness of solutions to* (3.4)*, which is ensured by* (3.9)*. Here $\chi h_0 < d$ is a sufficient condition to prove* (3.9)*. However, for all $0\le\overline{W}\le h_0$ we can easily derive that $f'(\overline{W})\ge -e^{-2}$, which together with the fact $0\le V\le K$ gives*

$$\mu + \lambda V f'(\overline{W}) \ge \mu - \lambda K e^{-2}. \tag{3.14}$$

Then from (3.14)*, we know that if $\mu > \lambda K e^{-2}$, the results in Lemma 3.2 are still hold. Hence the condition $\chi h_0 < d$ in Lemma 3.2 can be replaced for large μ or small λ.*

Next, we shall study the existence of solutions for the system (3.4). First, we construct the sub- and super-solution of the problem (3.4) as follows.

Lemma 3.3. *Let $W\in C^0(\bar{\Omega})$ be a given function with $0\le W\le h_0$ for all $x\in\bar{\Omega}$ and suppose that $\sigma h_0 < 1$. Define two constants*

$$V_* = \min_{x\in\bar{\Omega}}(e^{\frac{\chi}{d}W} - \sigma e^{\frac{\chi}{d}W}W) = \min_{x\in\bar{\Omega}} e^{\frac{\chi}{d}W}(1-\sigma W), \tag{3.15}$$

and

$$V^* = \max_{x\in\bar{\Omega}}(e^{\frac{\chi}{d}W} - \sigma e^{\frac{\chi}{d}W}W) = \max_{x\in\bar{\Omega}} e^{\frac{\chi}{d}W}(1-\sigma W). \tag{3.16}$$

Then V_* and V^* are sub-solution and super-solution of (3.4)*, respectively*

Proof. The proof follows by a straightforward computation and is omitted. □

Next, we shall show that system (3.4) admits a unique solution between V_* and V^*. Precisely, we prove the following results.

Lemma 3.4. *Let $W\in C^{1+\alpha}(\bar{\Omega})$ be a given function. Then system* (3.4) *admits a unique positive solution $V[W]\in C^{2+\alpha}(\bar{\Omega})$ satisfying*

$$V_* \le V[W](x) \le V^* \qquad \text{for all } x\in\bar{\Omega}, \tag{3.17}$$

where V_* and V^* are defined by (3.15) *and* (3.16)*, respectively.*

Proof. First, we introduce a solution space as follows:

$$\mathcal{X} = \{V\in C^0(\bar{\Omega}) : V_*\le V\le V^*\ \},$$

which is closed and convex. For given $\widetilde{V}\in\mathcal{X}$, we assume that $V\in H^1(\Omega)$ is a weak solution of the following problem

$$\begin{cases}-d\nabla\cdot(e^{-\frac{\chi}{d}W}\nabla V) + \kappa V = \kappa\widetilde{V} + e^{-\frac{\chi}{d}W}\widetilde{V}(1-e^{-\frac{\chi}{d}W}\widetilde{V}) - \sigma e^{-\frac{\chi}{d}W}W\widetilde{V}, & x\in\Omega,\\ \nabla V\cdot\nu = 0, & x\in\partial\Omega,\end{cases} \tag{3.18}$$

where $\kappa > 0$ is a constant chosen later. This defines a solution operator $\Phi_1[\widetilde{V}] = V$. Next, we show the operator Φ_1 is continuous and $\Phi_1[\mathcal{X}]\subset\mathcal{X}$ is relatively compact in $\mathcal{X}$.

We first show the operator Φ_1 is continuous. Since $W \in C^{1+\alpha}(\bar{\Omega})$, we can rewrite the first equation of (3.18) as follows

$$-de^{-\frac{\chi}{d}W}\Delta V + \chi e^{-\frac{\chi}{d}W}\nabla V \cdot \nabla W + \kappa V = \left(\kappa + e^{-\frac{\chi}{d}W} - \sigma e^{-\frac{\chi}{d}W} W\right)\widetilde{V} - e^{-\frac{2\chi}{d}W}\widetilde{V}^2. \tag{3.19}$$

Then using the L^p-estimate together with the Sobolev imbedding theorem and Agmon-Douglas-Nirenberg theorem, it follows from (3.19) with the boundary condition $\nabla V \cdot \nu = 0$ on $\partial\Omega$ that

$$\begin{aligned}
\|\Phi_1[\widetilde{V}_1] - \Phi_1[\widetilde{V}_2]\|_{L^\infty} \le & \|\Phi_1[\widetilde{V}_1] - \Phi_1[\widetilde{V}_2]\|_{C^{1+\beta}(\bar{\Omega})} \\
\le & c_1\|(\kappa + e^{-\frac{\chi}{d}W} - \sigma e^{-\frac{\chi}{d}W}W)(\widetilde{V}_1 - \widetilde{V}_2) - e^{-\frac{2\chi}{d}W}(\widetilde{V}_1^2 - \widetilde{V}_2^2)\|_{L^\infty} \\
\le & c_1(\kappa + 1 + \sigma\|W\|_{L^\infty})\|\widetilde{V}_1 - \widetilde{V}_2\|_{L^\infty} + c_1(\|\widetilde{V}_1\|_{L^\infty} + \|\widetilde{V}_2\|_{L^\infty})\|\widetilde{V}_1 - \widetilde{V}_2\|_{L^\infty} \\
\le & c_2\|\widetilde{V}_1 - \widetilde{V}_2\|_{L^\infty},
\end{aligned}$$

where

$$c_2 := c_1\left(\kappa + 1 + \sigma\|W\|_{L^\infty} + \|\widetilde{V}_1\|_{L^\infty} + \|\widetilde{V}_2\|_{L^\infty}\right) < \infty$$

due to $W \in C^{1+\alpha}(\bar{\Omega})$ and $\widetilde{V}_i \in C^0(\bar{\Omega})$ for $i = 1, 2$. Hence the continuity of the operator Φ_1 is proved.

Second, we show that $\Phi_1[\mathcal{X}]$ is relatively compact in $\mathcal{X}$. In fact, due to $W \in C^{1+\alpha}(\bar{\Omega})$ and $\widetilde{V} \in C^0(\bar{\Omega})$, we have

$$\|(\kappa + e^{-\frac{\chi}{d}W} - \sigma e^{-\frac{\chi}{d}W}W)\widetilde{V} - e^{-\frac{2\chi}{d}W}\widetilde{V}^2\|_{L^\infty} \le (\kappa + 1 + \sigma\|W\|_{L^\infty} + \|\widetilde{V}\|_{L^\infty})\|\widetilde{V}\|_{L^\infty} \le c_3$$

and

$$\|\chi e^{-\frac{\chi}{d}W}\nabla W\|_{L^\infty} \le \chi\|\nabla W\|_{L^\infty} \le c_4.$$

Then by the elliptic regularity applied to (3.19), we have $V \in C^{1+\beta}(\bar{\Omega})$ for all $\beta \in (0,1)$, which implies that $\Phi_1[\mathcal{X}]$ is relatively compact in $\mathcal{X}$.

At last, we show that $\Phi_1[\mathcal{X}] \subset \mathcal{X}$ for large κ. If $\widetilde{V}(x) \ge V_*$, we let $\widetilde{V}(x) = V_* + g(x)$, where $g(x)$ is a continuous non-negative function satisfying $0 \le g(x) \le V^* - V_*$ for all $x \in \bar{\Omega}$. Then from (3.18), we have

$$\begin{aligned}
& - d\nabla \cdot [e^{-\frac{\chi}{d}W}\nabla(V_* - V)] + \kappa(V_* - V) \\
& = -\kappa g - e^{-\frac{2\chi}{d}W}\widetilde{V}(e^{\frac{\chi}{d}W} - \widetilde{V} - \sigma e^{\frac{\chi}{d}W}W) \\
& = -\kappa g - (V_* + g)e^{-\frac{2\chi}{d}W}(e^{\frac{\chi}{d}W} - V_* - \sigma e^{\frac{\chi}{d}W}W) + g(V_* + g)e^{-\frac{2\chi}{d}W} \\
& \le [(V_* + g)e^{-\frac{2\chi}{d}W} - \kappa]g \\
& \le (V_* + V^* - \kappa)g,
\end{aligned} \tag{3.20}$$

where we have used the facts $0 < V_* \le e^{\frac{\chi}{d}W}(1 - \sigma W)$ and $0 \le g(x) \le V^*$. Then multiplying (3.20) by $[(V_* - V)_+]^2$ and integrating the result by parts, we have

$$\begin{aligned}
& 2d\int_\Omega e^{-\frac{\chi}{d}W}|\nabla(V_* - V)|^2(V_* - V)_+ + \kappa\int_\Omega[(V_* - V)_+]^3 \\
& \le \int_\Omega(V_* + V^* - \kappa)g[(V_* - V)_+]^2.
\end{aligned} \tag{3.21}$$

By choosing κ large enough such that $\kappa \ge \kappa_1 := V_* + V^*$, from (3.21), we have $\int_\Omega[(V_* - V)_+]^3 = 0$, and hence $V \ge V_*$.

Using similar arguments as above, we can find a constant $\kappa_2 > 0$ such that $V \le V^*$ if $\kappa \ge \kappa_2$. Hence, choosing $\kappa \ge \kappa_* := \max\{\kappa_1, \kappa_2\}$, we have $V_* \le V \le V^*$, which implies $\Phi_1[\mathcal{X}] \subset \mathcal{X}$ for $\kappa \ge \kappa_*$.

In summary, we have proved the operator $\Phi_1 : \mathcal{X} \to \mathcal{X}$ is continuous and $\Phi_1[\mathcal{X}] \subset \mathcal{X}$ is relatively compact in $\mathcal{X}$. Then by the Schauder fixed point theorem, there exist a fixed point

$V \in \mathcal{X}$ such that $\Phi_1[V] = V$. That is, for a given $W \in C^{1+\alpha}(\bar\Omega)$, there exist a solution $V \in \mathcal{X}$ to the system (3.4) and hence

$$\begin{cases} -de^{-\frac{\chi}{d}W}\Delta V + \chi e^{-\frac{\chi}{d}W}\nabla V \cdot \nabla W + V = \mathcal{H}(V,W), & x \in \Omega, \\ \nabla V \cdot \nu = 0, & x \in \partial\Omega, \end{cases} \tag{3.22}$$

where

$$\mathcal{H}(V,W) = V\left(1 + e^{-\frac{\chi}{d}W} - e^{-\frac{2\chi}{d}W}V - \sigma e^{-\frac{\chi}{d}W}W\right).$$

Since $V \in \mathcal{X}$ and $W \in C^{1+\alpha}(\bar\Omega)$, we have that $\|\mathcal{H}(V,W)\|_{L^\infty} \le c_5$. Then applying the elliptic regularity estimate to (3.22), we can derive $V[W] \in C^{1+\alpha}(\bar\Omega)$ with some $\alpha \in (0,1)$. Then using again the fact $W \in C^{1+\alpha}(\bar\Omega)$, one has $\|\mathcal{H}(V,W)\|_{C^\alpha(\bar\Omega)} \le c_6$, and hence $V[W] \in C^{2+\alpha}(\bar\Omega)$ follows by the elliptic regularity again. Moreover the positivity of $V[W]$ follows from the fact $V \ge V_* > 0$.

Finally, we shall show the uniqueness of solutions for the system (3.4) with given $W \in C^{1+\alpha}(\bar\Omega)$ based on some ideas from [8]. We assume that the problem (3.4) has two different positive solutions V_1 and V_2. Then they solve the following equations weakly

$$-\frac{d\nabla \cdot (e^{-\frac{\chi}{d}W}\nabla V_i)}{V_i} = (1 - \sigma W)e^{-\frac{\chi}{d}W} - e^{-\frac{2\chi}{d}W}V_i \quad \text{for} \quad i = 1, 2,$$

and hence

$$-\frac{d\nabla \cdot (e^{-\frac{\chi}{d}W}\nabla V_1)}{V_1} + \frac{d\nabla \cdot (e^{-\frac{\chi}{d}W}\nabla V_2)}{V_2} = -e^{-\frac{2\chi}{d}W}(V_1 - V_2). \tag{3.23}$$

Multiplying (3.23) by $V_1^2 - V_2^2$, and integrating the result by parts, we have

$$\begin{aligned} &- \int_\Omega e^{-\frac{2\chi}{d}W}(V_1 - V_2)^2(V_1 + V_2) \\ &= d\int_\Omega e^{-\frac{\chi}{d}W}\nabla V_1 \cdot \nabla\left(V_1 - \frac{V_2^2}{V_1}\right) + d\int_\Omega e^{-\frac{\chi}{d}W}\nabla V_2 \cdot \nabla\left(V_2 - \frac{V_1^2}{V_2}\right) \\ &= d\int_\Omega e^{-\frac{\chi}{d}W}\left(|\nabla V_1|^2 - 2\frac{V_2}{V_1}\nabla V_1 \cdot \nabla V_2 + \frac{V_2^2}{V_1^2}|\nabla V_1|^2\right) \\ &\quad + d\int_\Omega e^{-\frac{\chi}{d}W}\left(|\nabla V_2|^2 - 2\frac{V_1}{V_2}\nabla V_1 \cdot \nabla V_2 + \frac{V_1^2}{V_2^2}|\nabla V_2|^2\right) \\ &= d\int_\Omega e^{-\frac{\chi}{d}W}\left(\left|\nabla V_1 - \frac{V_1}{V_2}\nabla V_2\right|^2 + \left|\nabla V_2 - \frac{V_2}{V_1}\nabla V_1\right|^2\right) \ge 0 \end{aligned}$$

which implies that $V_1 = V_2$. This completes the proof. □

Lemma 3.5. *The mapping $W \to V[W]$ is a continuous function from $C^{1+\alpha}(\bar\Omega)$ into $C^0(\bar\Omega)$.*

Proof. We shall show the conclusion by using the uniqueness of solutions for (3.4). Let $\{W_i\}_{i\in\mathbb{N}}$ be a sequence in $C^{1+\alpha}(\bar\Omega)$ satisfying

$$\lim_{i\to\infty} W_i = W.$$

Arguing by contradiction, we assume

$$\lim_{i\to\infty} V[W_i] \ne V[W]. \tag{3.24}$$

Then there exist a constant $\delta_1 > 0$ and a subsequence $\{W_{i_j}\}_{j\in\mathbb{N}}$ such that for all $j \in \mathbb{N}$

$$\|V[W_{i_j}] - V[W]\|_{L^\infty} \ge \delta_1. \tag{3.25}$$

Since $\{V[W_{i_j}]\}_{j\in\mathbb{N}}$ is uniformly bounded in $C^{2+\alpha}(\bar\Omega)$ and hence equi-continuous in $C^2(\bar\Omega)$, we can use the Arzelá-Ascoli theorem to find a sub-sequence $\{V[W_{i_{j_k}}]\}_{k\in\mathbb{N}}$ and $\widehat{V}$ such that

$$\lim_{k\to\infty} \|V[W_{i_{j_k}}] - \widehat{V}\|_{C^2(\bar\Omega)} = 0. \tag{3.26}$$

Since for any $k \in \mathbb{N}$, the function $V[W_{i_{j_k}}] > 0$ satisfies

$$\begin{cases} 0 = d\nabla \cdot (e^{-\frac{\chi}{d}W_{i_{j_k}}} \nabla V[W_{i_{j_k}}]) + e^{-\frac{\chi}{d}W_{i_{j_k}}} V[W_{i_{j_k}}](1 - e^{-\frac{\chi}{d}W_{i_{j_k}}} V[W_{i_{j_k}}] - \sigma W_{i_{j_k}}), & x \in \Omega, \\ \nabla V[W_{i_{j_k}}] \cdot \nu = 0, & x \in \partial\Omega, \end{cases}$$

which together with (3.26) gives

$$\begin{cases} 0 = d\nabla \cdot (e^{-\frac{\chi}{d}W} \nabla \widehat{V}) + e^{-\frac{\chi}{d}W} \widehat{V}(1 - e^{-\frac{\chi}{d}W} \widehat{V}) - \sigma e^{-\frac{\chi}{d}W} W \widehat{V}, & x \in \Omega, \\ \nabla \widehat{V} \cdot \nu = 0, & x \in \partial\Omega, \end{cases} \tag{3.27}$$

by taking $k \to \infty$. Since the problem (3.27) has a unique positive solution $V[W]$ for given $W \in C^{1+\alpha}(\bar{\Omega})$, we derive that $\widehat{V} = V[W]$, which contradicts (3.25). The proof is complete. □

Now we are ready to prove the existence and uniqueness of solutions to (1.4) asserted in Theorem 1.2.

Proof of Theorem 1.2 (existence and uniqueness). We divide the proof into two steps.

Step 1: Existence. Consider the existence of solutions in the following closed and convex solution space

$$\mathcal{X} := \{V \in C^0(\bar{\Omega}) : V_* \le V \le V^*\},$$

with V_* and V^* are defined by (3.15) and (3.16), respectively. From Lemma 3.1 and Lemma 3.2, we know that for any $V \in \mathcal{X}$, there exists a continuous operator $\mathcal{T}_1$: $C^0(\bar{\Omega}) \to C^{1+\alpha}(\bar{\Omega})$ such that $W = \mathcal{T}_1[V] \in C^{1+\alpha}(\bar{\Omega})$ which solves (3.3). For this $W = \mathcal{T}_1[V]$, from Lemma 3.4 and Lemma 3.5, we can define a continuous operator $\mathcal{T}_2$: $C^{1+\alpha}(\bar{\Omega}) \to C^0(\bar{\Omega})$ such that $V = \mathcal{T}_2[W]$ solving (3.4). Define a composition operator $\mathcal{T}$: $C^0(\bar{\Omega}) \to C^0(\bar{\Omega})$ as follows:

$$\mathcal{T}[V] := \mathcal{T}_2 \circ \mathcal{T}_1[V] = \mathcal{T}_2[\mathcal{T}_1[V]].$$

Clearly, $\mathcal{T}$ is continuous since both operators $\mathcal{T}_1$ and $\mathcal{T}_2$ are continuous.

From Lemma 3.1, we know that $0 \le W \le h_0$ and $W \in C^{1+\alpha}(\bar{\Omega})$, which together with Lemma 3.4 gives $V = \mathcal{T}_2[W] \in C^{2+\alpha}(\bar{\Omega})$ and

$$V_* \le V \le V^*,$$

and hence $\mathcal{T}[\mathcal{X}] \subset \mathcal{X}$ is relatively compact in $\mathcal{X}$. Hence, by the Schauder fixed point theorem, the operator $\mathcal{T}$ has a fixed point $V \in \mathcal{X}$, which in fact belongs to $C^{2+\alpha}(\bar{\Omega})$. Moreover, by the Schauder estimates, we have $W = \mathcal{T}_1[V] \in C^{2+\alpha}(\bar{\Omega})$. Hence the pair

$$(\mathcal{T}[V], \mathcal{T}_1[V]) = (V, W)$$

yields a classical solution to the problem (3.2).

Next, we shall prove the solution pair (V, W) obtained above is positive. Since $0 < V_* \le V \le V^*$, we only need to prove $W(x) > 0$ for all $x \in \bar{\Omega}$. To this end, we first rewrite (3.3) as

$$\begin{cases} 0 = \varepsilon \Delta W - c(x)W, & x \in \Omega, \\ \nabla W \cdot \nu = \xi(h(x) - W), & x \in \partial\Omega, \end{cases} \tag{3.28}$$

with $c(x) = \mu + \lambda e^{-\frac{\chi}{d}W} V > 0$. Using the maximum principle [22, Lemma 3.5], we know that W can not achieve non-positive minimum in Ω and hence for all $x \in \Omega$ it holds that $W(x) > 0$. Then to show $W(x) > 0$ for $x \in \bar{\Omega}$, it remains to show $W(x) > 0$ if $x \in \partial\Omega$. Assume that there exist some $x_0 \in \partial\Omega$ such that $W(x_0) < W(x)$ for all $x \in \partial\Omega$. If $W(x_0) \le 0$, we can use the Hopf's boundary point lemma [22, Lemma 3.4] to derive $\xi(h(x_0) - W(x_0)) = \nabla W(x_0) \cdot \nu < 0$, which contradict the fact $W(x_0) \le 0$, and thus $W(x) > 0$ for $x \in \bar{\Omega}$. Hence we prove that (3.2) admits a positive classical solution $(V(x), W(x)) \in C^{2+\alpha}(\bar{\Omega}) \times C^{2+\alpha}(\bar{\Omega})$ satisfying

$$0 < V_* \le V(x) \le V^* \text{ and } 0 < W(x) \le h_0, \text{ for } x \in \bar{\Omega},$$

which yields a positive classical solution $(U(x), W(x))$ to (1.4) satisfying (1.7) by (3.1).

Step 2: Uniqueness. To show the uniqueness of the positive classical solution $(U(x), W(x))$ under the assumptions that h_0 is small, we first show that

$$\|W\|_{W^{1,\infty}} \to 0 \quad \text{as} \quad h_0 \to 0. \tag{3.29}$$

In fact, using [24, Theorem 9.4 in Chap. 1], we can find a bounded function $g \in C^\infty(\bar{\Omega})$ satisfying

$$g(x) = h(x) \quad \text{and} \quad \nabla g \cdot \nu = 0 \quad \text{on } \partial\Omega.$$

Letting $Z(x) = g(x) - W(x)$ for $x \in \Omega$, we can derive from (1.4) that

$$\begin{cases} \varepsilon\Delta Z - \mu Z = \varepsilon\Delta g - \mu g - \lambda UW, & x \in \Omega, \\ \nabla Z \cdot \nu + \xi Z = 0, & x \in \partial\Omega. \end{cases} \tag{3.30}$$

Then applying Agmon–Douglis–Nirenberg L^p estimates and using the facts $g \in C^\infty(\bar{\Omega}), U \in C^{2+\alpha}(\bar{\Omega})$ and $0 < W(x) \le h_0$, from (3.30) we have

$$\begin{aligned} \|Z\|_{W^{2,p}} \le c_1\|\varepsilon\Delta g - \mu g - \lambda UW\|_{L^p} &\le c_2(\|\Delta g\|_{L^\infty} + \|g\|_{L^\infty} + \|U\|_{L^\infty}\|W\|_{L^\infty}) \\ &\le c_3(1 + h_0), \end{aligned}$$

for all $p > 1$ and hence

$$\|W\|_{W^{2,p}} = \|g - Z\|_{W^{2,p}} \le c_4(1 + h_0). \tag{3.31}$$

Choosing $p = 2n$ in (3.31), and using the Gagliardo-Nirenberg inequality and the fact $0 < W(x) \le h_0$, one has

$$\begin{aligned} \|W\|_{W^{1,\infty}} = \|\nabla W\|_{L^\infty} + \|W\|_{L^\infty} &\le c_5\|W\|_{W^{2,2n}}^{\frac{2}{3}}\|W\|_{L^\infty}^{\frac{1}{3}} + \|W\|_{L^\infty} \\ &\le c_6(1 + h_0)^{\frac{2}{3}}h_0^{\frac{1}{3}} + h_0, \end{aligned}$$

which gives (3.29).

Assume (U_1, W_1) and (U_2, W_2) are two different solutions of (1.4). Then from (1.7), we can derive that

$$\ell_* e^{-\frac{\chi h_0}{d}} \le U_i \le \ell^* \quad \text{for} \quad i = 1, 2, \tag{3.32}$$

where $\ell_* = \min\limits_{0\le z\le h_0}\{e^{\frac{\chi}{d}z}(1 - \sigma z)\}$ and $\ell^* = \max\limits_{0\le z\le h_0}\{e^{\frac{\chi}{d}z}(1 - \sigma z)\}$.

Let $\widetilde{U} = U_1 - U_2$ and $\widetilde{W} = W_1 - W_2$. Then from (1.4) we have

$$\begin{cases} 0 = \nabla\cdot[d\nabla\widetilde{U} + \chi(\widetilde{U}\nabla W_1 + U_2\nabla\widetilde{W})] + \widetilde{U} - \widetilde{U}(U_1 + U_2) - \sigma\widetilde{U}W_1 - \sigma U_2\widetilde{W}, & x \in \Omega, \\ 0 = \varepsilon\Delta\widetilde{W} - \mu\widetilde{W} - \lambda\widetilde{U}W_1 - \lambda U_2\widetilde{W}, & x \in \Omega, \\ [d\nabla\widetilde{U} + \chi(\widetilde{U}\nabla W_1 + U_2\nabla\widetilde{W})]\cdot\nu = 0, & x \in \partial\Omega, \\ \nabla\widetilde{W}\cdot\nu + \xi\widetilde{W} = 0, & x \in \partial\Omega. \end{cases} \tag{3.33}$$

Multiplying (3.33) by $\widetilde{U}$, and integrating the result by parts, we have

$$\begin{aligned} &d\int_\Omega |\nabla\widetilde{U}|^2 + \int_\Omega (U_1 + U_2)\widetilde{U}^2 + \sigma\int_\Omega W_1\widetilde{U}^2 \\ &= -\chi\int_\Omega (\widetilde{U}\nabla W_1 + U_2\nabla\widetilde{W})\cdot\nabla\widetilde{U} + \int_\Omega \widetilde{U}^2 - \sigma\int_\Omega U_2\widetilde{W}\widetilde{U}. \end{aligned} \tag{3.34}$$

On the other hand, we can use the Hölder inequality and Young's inequality to derive

$$\begin{aligned} &-\chi\int_\Omega (\widetilde{U}\nabla W_1 + U_2\nabla\widetilde{W})\cdot\nabla\widetilde{U} \\ &\le \frac{d}{2}\int_\Omega |\nabla\widetilde{U}|^2 + \frac{\chi^2}{2d}\int_\Omega |\widetilde{U}\nabla W_1 + U_2\nabla\widetilde{W}|^2 \\ &\le \frac{d}{2}\int_\Omega |\nabla\widetilde{U}|^2 + \frac{\chi^2}{d}\|\nabla W_1\|_{L^\infty}^2\int_\Omega \widetilde{U}^2 + \frac{\chi^2}{d}\|U_2\|_{L^\infty}^2\int_\Omega |\nabla\widetilde{W}|^2, \end{aligned} \tag{3.35}$$

and

$$-\sigma\int_\Omega U_2\widetilde{W}\widetilde{U}\le\sigma^2\int_\Omega U_2\widetilde{W}^2+\frac{1}{4}\int_\Omega U_2\widetilde{U}^2. \tag{3.36}$$

Substituting (3.35) and (3.36) into (3.34), we have

$$\frac{d}{2}\int_\Omega|\nabla\widetilde{U}|^2+\int_\Omega\left(U_1+\frac{3}{4}U_2-1-\frac{\chi^2}{d}\|\nabla W_1\|^2_{L^\infty}\right)\widetilde{U}^2\le\frac{\chi^2\|U_2\|^2_{L^\infty}}{d}\int_\Omega|\nabla\widetilde{W}|^2+\sigma^2\int_\Omega U_2\widetilde{W}^2,$$

which, together with (3.32), gives

$$\begin{aligned}&\frac{d}{2}\int_\Omega|\nabla\widetilde{U}|^2+\left(\frac{7\ell_*e^{-\frac{\chi h_0}{d}}}{4}-1-\frac{\chi^2}{d}\|\nabla W_1\|^2_{L^\infty}\right)\int_\Omega\widetilde{U}^2\\&\quad\le\frac{\chi^2|\ell^*|^2}{d}\int_\Omega|\nabla\widetilde{W}|^2+\sigma^2\int_\Omega U_2\widetilde{W}^2.\end{aligned} \tag{3.37}$$

We multiply the second equation of (3.33) by $\widetilde{W}$ to obtain

$$\begin{aligned}\varepsilon\int_\Omega|\nabla\widetilde{W}|^2+\mu\int_\Omega\widetilde{W}^2+\varepsilon\xi\int_{\partial\Omega}\widetilde{W}^2+\lambda\int_\Omega U_2\widetilde{W}^2&=-\lambda\int_\Omega W_1\widetilde{U}\widetilde{W}\\&\le\mu\int_\Omega\widetilde{W}^2+\frac{\lambda^2}{4\mu}\|W_1\|^2_{L^\infty}\int_\Omega\widetilde{U}^2,\end{aligned}$$

which yields

$$\varepsilon\int_\Omega|\nabla\widetilde{W}|^2+\lambda\int_\Omega U_2\widetilde{W}^2\le\frac{\lambda^2}{4\mu}\|W_1\|^2_{L^\infty}\int_\Omega\widetilde{U}^2. \tag{3.38}$$

Let $\gamma_*:=\frac{\chi^2|\ell^*|^2}{\varepsilon d}+\frac{\sigma^2}{\lambda}$. Then multiplying (3.38) by γ_*, and adding the result to (3.37), we have

$$\frac{d}{2}\int_\Omega|\nabla\widetilde{U}|^2+\left(\frac{7\ell_*e^{-\frac{\chi h_0}{d}}}{4}-1-\frac{\chi^2}{d}\|\nabla W_1\|^2_{L^\infty}-\frac{\lambda^2\gamma_*}{4\mu}\|W_1\|^2_{L^\infty}\right)\int_\Omega\widetilde{U}^2\le 0. \tag{3.39}$$

Noting that $\ell_*=\min_{0\le z\le h_0}\{e^{\frac{\chi}{d}z}(1-\sigma z)\}$, we have

$$\ell_*\to 1\quad\text{as}\quad h_0\to 0. \tag{3.40}$$

On the other hand, from (3.29) we have $\|W_1\|_{W^{1,\infty}}\to 0$ as $h_0\to 0$, which together with (3.40) gives

$$\lim_{h_0\to 0}\left(\frac{7\ell_*e^{-\frac{\chi h_0}{d}}}{4}-1-\frac{\chi^2}{d}\|\nabla W_1\|^2_{L^\infty}-\frac{\lambda^2\gamma_*}{4\mu}\|W_1\|^2_{L^\infty}\right)=\frac{3}{4}.$$

This yields a small constant $h_*<h^*:=\min\left\{\frac{1}{\sigma},\frac{d}{\chi}\right\}$ such that the following holds if $h_0<h_*$

$$\frac{d}{2}\int_\Omega|\nabla\widetilde{U}|^2+c_7\int_\Omega\widetilde{U}^2\le 0, \tag{3.41}$$

for some positive constant $c_7\in(0,\frac{3}{4})$. Then (3.41) implies $\widetilde{U}=0$, that is $U_1=U_2$.

Finally with $\widetilde{U}=0$ and the fact $U_2\ge\ell_*e^{-\frac{\chi h_0}{d}}>0$, from (3.38) we obtain

$$\varepsilon\int_\Omega|\nabla\widetilde{W}|^2+\lambda\ell_*e^{-\frac{\chi h_0}{d}}\int_\Omega\widetilde{W}^2\le 0,$$

which gives $\widetilde{W}=0$ and hence $W_1=W_2$. Then the uniqueness of solutions to (1.4) is proved. □

3.2. **Global stability.** In this subsection, we further impose that Ω is two-dimensional and show that the positive non-constant steady state (U, W) is globally asymptotically stable if $h_0 > 0$ is sufficiently small. We first use the Harnack inequality again and the fact $\sigma h_0 < 1$ to derive that $\inf_{x\in\Omega} u(x,t)$ has a positive lower bound as time is large.

Lemma 3.6. *Let (u, w) be the solution of the system (1.2) with $h(x,t) = h(x)$ obtained in Theorem 1.1 and $h_0 = \sup_{x\in\partial\Omega} h(x)$. If $\sigma h_0 < 1$, then*

$$\liminf_{t\to\infty} \inf_{x\in\Omega} u(x,t) \geq \zeta_2, \tag{3.42}$$

where $\zeta_2 > 0$ is a constant independent of t.

Proof. Let $\widehat{w}(x,t)$ be the solution of the following problem

$$\begin{cases} \widehat{w}_t = \varepsilon\Delta\widehat{w} - \mu\widehat{w}, & x \in \Omega, t > 0, \\ \nabla\widehat{w}\cdot\nu = \xi(h(x) - \widehat{w}), & x \in \partial\Omega, t > 0, \\ \widehat{w}(x,0) = w_0(x), & x \in \Omega. \end{cases} \tag{3.43}$$

Then using the comparison principle, one has

$$0 < w(x,t) \leq \widehat{w}(x,t). \tag{3.44}$$

Using the method of energy estimates, we can derive from the system (3.43) that

$$\lim_{t\to\infty} \|\widehat{w}(x,t) - w_*(x)\|_{L^\infty} = 0, \tag{3.45}$$

where $w_*(x) \in C^2(\bar{\Omega})$ is the solution of (1.6). In fact, letting $\widehat{v}(x,t) = \widehat{w}(x,t) - w_*(x)$, then from (3.43) and (1.6) one has

$$\begin{cases} \widehat{v}_t = \varepsilon\Delta\widehat{v} - \mu\widehat{v}, & x \in \Omega, t > 0, \\ \nabla\widehat{v}\cdot\nu + \xi\widehat{v} = 0, & x \in \partial\Omega, t > 0, \\ \widehat{v}(x,0) = \widehat{v}_0(x) = w_0(x) - w_*(x), & x \in \Omega. \end{cases} \tag{3.46}$$

Then we multiply the first equation of (3.46) by $\widehat{v}$ and integrate the result by parts to obtain

$$\frac{d}{dt}\int_\Omega \widehat{v}^2 + \varepsilon\int_\Omega |\nabla\widehat{v}|^2 + \mu\int_\Omega \widehat{v}^2 + \xi\varepsilon\int_{\partial\Omega} \widehat{v}^2 = 0,$$

which gives

$$\|\widehat{v}(\cdot,t)\|_{L^2} \leq \|w_0(x) - w_*\|_{L^2} e^{-\frac{\mu}{2}t} \leq c_1 e^{-\frac{\mu}{2}t}. \tag{3.47}$$

Applying the parabolic regularity, we can derive from (3.46) that $\|\widehat{v}(\cdot,t)\|_{W^{1,\infty}} \leq c_2$. Then we use the Gagliardo-Nirenberg inequality and the fact (3.47) to derive

$$\|\widehat{w}(\cdot,t) - w_*\|_{L^\infty} = \|\widehat{v}(\cdot,t)\|_{L^\infty} \leq c_3\|\widehat{v}(\cdot,t)\|^{\frac{1}{2}}_{W^{1,\infty}}\|\widehat{v}(\cdot,t)\|^{\frac{1}{2}}_{L^2} \leq c_4 e^{-\frac{\mu}{4}t},$$

which proves (3.45).

Noting that $0 < w_*(x) \leq h_0$ and using (3.44)-(3.45), we can find $t^* > 1$ such that for all $t \geq t^* > 1$ that

$$0 < w(x,t) \leq h_0 + \frac{1-\sigma h_0}{2\sigma} = \frac{1+\sigma h_0}{2},$$

and hence

$$\sigma\|w(\cdot,t)\|_{L^\infty} \leq \frac{1+\sigma h_0}{2} < 1, \text{ for all } t \geq t^* > 1. \tag{3.48}$$

Integrating the first equation of (1.2) and using (3.48), one has

$$\frac{d}{dt}\int_\Omega u = \int_\Omega u - \int_\Omega u^2 - \sigma\int_\Omega uw \geq (1 - \sigma\|w\|_{L^\infty})\int_\Omega u - \int_\Omega u^2 \geq c_1\int_\Omega u - \int_\Omega u^2$$

with $c_1 = \frac{1-\sigma h_0}{2} > 0$.

Then the remaining proofs are the same as those for Lemma 2.15 and will be omitted for brevity. This completes the proof. □

Next, we shall study the global stability of the positive (coexistence) steady state under the smallness assumptions on h_0. More precisely, we have the following results.

Lemma 3.7. *Let (u, w) be the global solution of (1.2) with $h(x,t) = h(x) \gneqq 0$ and $h_0 = \sup_{x\in\partial\Omega} h(x)$. Assume (U, W) is the positive non-constant steady state of (1.4) obtained in Theorem 1.2. Then there exists a $\hbar > 0$ and $t_* > 1$ such that the following estimate holds for all $0 < h_0 \le \hbar$*

$$\|u(\cdot,t) - U\|_{L^\infty} + \|w(\cdot,t) - W\|_{L^\infty} \le c_1 e^{-c_2 t}, \text{ for all } t \ge t_*, \tag{3.49}$$

where c_1 and c_2 are two positive constants independent of t.

Proof. Let $\widetilde{u} = u - U$ and $\widetilde{w} = w - W$. From (1.2) and (1.4) we can derive that

$$\begin{cases} \widetilde{u}_t = \nabla\cdot(d\nabla\widetilde{u} + \chi u\nabla\widetilde{w} + \chi\widetilde{u}\nabla W) + \widetilde{u}(1-u-U) - \sigma\widetilde{u}w - \sigma U\widetilde{w}, \\ \widetilde{w}_t = \varepsilon\Delta\widetilde{w} - \mu\widetilde{w} - \lambda u\widetilde{w} - \lambda W\widetilde{u}, \\ (d\nabla\widetilde{u} + \chi u\nabla\widetilde{w} + \chi\widetilde{u}\nabla W)\cdot\nu = 0, \quad \nabla\widetilde{w}\cdot\nu + \xi\widetilde{w} = 0, \\ \widetilde{u}(x,0) = u_0(x) - U, \quad \widetilde{w}(x,0) = w_0(x) - W. \end{cases} \tag{3.50}$$

Then integrating the first equation of (3.50) multiplied by $\widetilde{u}$, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega \widetilde{u}^2 + d\int_\Omega|\nabla\widetilde{u}|^2 + \int_\Omega \widetilde{u}^2(u+U-1) + \sigma\int_\Omega w\widetilde{u}^2 \\ &= -\chi\int_\Omega u\nabla\widetilde{w}\cdot\nabla\widetilde{u} - \chi\int_\Omega \widetilde{u}\nabla W\cdot\nabla\widetilde{u} - \sigma\int_\Omega U\widetilde{w}\widetilde{u}. \end{aligned} \tag{3.51}$$

Noting the facts $\|u(\cdot,t)\|_{L^\infty} \le M$ and $\|U\|_{L^\infty} \le c_1$ due to $U \in C^2(\bar\Omega)$, we can use Young's inequality to derive that

$$-\chi\int_\Omega u\nabla\widetilde{w}\cdot\nabla\widetilde{u} \le \chi\|u\|_{L^\infty}\int_\Omega|\nabla\widetilde{w}||\nabla\widetilde{u}| \le \frac{d}{4}\|\nabla\widetilde{u}\|_{L^2}^2 + \frac{\chi^2M^2}{d}\|\nabla\widetilde{w}\|_{L^2}^2, \tag{3.52}$$

and

$$-\chi\int_\Omega \widetilde{u}\nabla W\cdot\nabla\widetilde{u} \le \chi\|\nabla W\|_{L^\infty}\|\widetilde{u}\|_{L^2}\|\nabla\widetilde{u}\|_{L^2} \le \frac{d}{4}\|\nabla\widetilde{u}\|_{L^2}^2 + \frac{\chi^2}{d}\|\nabla W\|_{L^\infty}^2\|\widetilde{u}\|_{L^2}^2, \tag{3.53}$$

as well as

$$-\sigma\int_\Omega U\widetilde{w}\widetilde{u} \le \sigma\|U\|_{L^\infty}\|\widetilde{w}\|_{L^2}\|\widetilde{u}\|_{L^2} \le \frac{\zeta_2}{4}\|\widetilde{u}\|_{L^2}^2 + \frac{\sigma^2c_1^2}{\zeta_2}\|\widetilde{w}\|_{L^2}^2. \tag{3.54}$$

Then substituting (3.53), (3.52) and (3.54) into (3.51), we obtain

$$\begin{aligned} &\frac{d}{dt}\int_\Omega \widetilde{u}^2 + d\int_\Omega|\nabla\widetilde{u}|^2 + 2\int_\Omega \widetilde{u}^2\left(u+U-1-\frac{\zeta_2}{4} - \frac{\chi^2}{d}\|\nabla W\|_{L^\infty}^2\right) + 2\sigma\int_\Omega w\widetilde{u}^2 \\ &\le \frac{2\chi^2M^2}{d}\|\nabla\widetilde{w}\|_{L^2}^2 + \frac{2\sigma^2c_1^2}{\zeta_2}\|\widetilde{w}\|_{L^2}^2 \\ &\le c_2(\|\nabla\widetilde{w}\|_{L^2}^2 + \|\widetilde{w}\|_{L^2}^2), \end{aligned} \tag{3.55}$$

where $c_2 = \frac{2\chi^2M^2}{d} + \frac{2\sigma^2c_1^2}{\zeta_2}$. We multiply the second equation of (3.50) by $\widetilde{w}$ and integrate the result by part to obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega \widetilde{w}^2 + \varepsilon\int_\Omega|\nabla\widetilde{w}|^2 + \mu\int_\Omega\widetilde{w}^2 + \lambda\int_\Omega u\widetilde{w}^2 + \varepsilon\xi\int_{\partial\Omega}\widetilde{w}^2 &= -\lambda\int_\Omega W\widetilde{u}\widetilde{w} \\ &\le \frac{\mu}{2}\int_\Omega\widetilde{w}^2 + \frac{\lambda^2\|W\|_{L^\infty}^2}{2\mu}\int_\Omega\widetilde{u}^2, \end{aligned}$$

which gives

$$\frac{d}{dt}\int_\Omega \widetilde{w}^2 + 2\varepsilon\int_\Omega|\nabla\widetilde{w}|^2 + \mu\int_\Omega\widetilde{w}^2 + 2\lambda\int_\Omega u\widetilde{w}^2 + 2\varepsilon\xi\int_{\partial\Omega}\widetilde{w}^2 \le \frac{\lambda^2\|W\|_{L^\infty}^2}{\mu}\int_\Omega\widetilde{u}^2. \tag{3.56}$$

Then multiplying (3.56) by $c_3 = \frac{c_2}{\varepsilon} + \frac{2c_2}{\mu}$ and adding the result to (3.55), we end up with

$$\begin{aligned}&\frac{d}{dt}\int_\Omega(\widetilde{u}^2+c_3\widetilde{w}^2)+d\int_\Omega|\nabla\widetilde{u}|^2+c_2\int_\Omega|\nabla\widetilde{w}|^2+c_2\int_\Omega\widetilde{w}^2\\&\quad+2\int_\Omega\widetilde{u}^2\left(u+U-1-\frac{\zeta_2}{4}-\frac{\chi^2}{d}\|\nabla W\|^2_{L^\infty}-\frac{c_3\lambda^2\|W\|^2_{L^\infty}}{\mu}\right)\le 0.\end{aligned}\tag{3.57}$$

Due to $U\to 1$ as $h_0\to 0$ and (3.29), there exists a constant $\hbar>0$ such that if $0<h_0\le\hbar$ one has

$$-\frac{\zeta_2}{8}\le U-1\le\frac{\zeta_2}{8}\tag{3.58}$$

and

$$\frac{\chi^2}{d}\|\nabla W\|^2_{L^\infty}+\frac{c_3\lambda^2\|W\|^2_{L^\infty}}{\mu}\le\frac{\zeta_2}{8}.\tag{3.59}$$

Noting the fact $\liminf\limits_{t\to\infty}\inf\limits_{x\in\Omega}u(x,t)\ge\zeta_2$ in (3.42), we know that there exists $T_*>1$ such that

$$\inf_{x\in\Omega}u(x,t)\ge\frac{3\zeta_2}{4}\qquad\text{for all}\quad t\ge T_*>1,$$

which combined with (3.58) and (3.59) gives

$$u+U-1-\frac{\zeta_2}{4}-\frac{\chi^2}{d}\|\nabla W\|^2_{L^\infty}-\frac{c_3\lambda^2\|W\|^2_{L^\infty}}{\mu}\ge\frac{\zeta_2}{4}>0,\tag{3.60}$$

for $0<h_0\le\hbar$ and $t\ge T_*>1$.

Substituting (3.60) into (3.57), we can find a $c_4>0$ such that

$$\frac{d}{dt}\int_\Omega(\widetilde{u}^2+c_3\widetilde{w}^2)+c_4\int_\Omega(\widetilde{u}^2+c_3\widetilde{w}^2)\le 0,\quad\text{for all } t\ge T_*>1,$$

which gives

$$\|u(\cdot,t)-U\|_{L^2}+\|w(\cdot,t)-W\|_{L^2}\le c_5e^{-\frac{c_4}{2}t},\quad\text{for all } t\ge T_*>1.\tag{3.61}$$

Next we recall the interpolation inequality

$$\|f\|_{L^\infty(\Omega)}\le c_6\|f\|^{\frac{n}{n+\theta}}_{C^\theta(\bar\Omega)}\|f\|^{\frac{\theta}{n+\theta}}_{L^1(\Omega)}\qquad\text{for } f\in L^1(\Omega)\cap C^\theta(\bar\Omega).\tag{3.62}$$

To see that, choose a point $\bar x\in\Omega$. Then

$$f(\bar x)\le\frac{c_7}{\epsilon^n}\int_{B_\epsilon(0)}\left\{f(x+\bar x)+\epsilon^\theta\frac{|f(\bar x)-f(x+\bar x)|}{|x|^\theta}\right\}dx\le c_8\left(\epsilon^{-n}\|f\|_{L^1}+\epsilon^\theta\|f\|_{C^\theta}\right).$$

By choosing $\epsilon=\left(\frac{\|f\|_{L^1(\Omega)}}{\|f\|_{C^\theta(\bar\Omega)}}\right)^{\frac{1}{\theta+n}}$, we deduce (3.62).

Along with the facts $\sup_{t\ge1}\|u(\cdot,t)\|_{C^\theta(\bar\Omega)}\le c_9$ from Lemma 2.13 and $U(x)\in C^2(\bar\Omega)$, and using (3.61) and (3.62) with $n=2$, for all $t\ge T_*>1$ one has

$$\begin{aligned}\|u(\cdot,t)-U\|_{L^\infty}&\le c_{10}\|u(\cdot,t)-U\|^{\frac{2}{2+\theta}}_{C^\theta}\|u(\cdot,t)-U\|^{\frac{\theta}{2+\theta}}_{L^1}\\&\le c_{11}\|u(\cdot,t)-U\|^{\frac{\theta}{2+\theta}}_{L^2}\le c_{12}e^{-\frac{\theta c_4}{2(2+\theta)}t}.\end{aligned}\tag{3.63}$$

On the other hand, with the facts $\|w(\cdot,t)\|_{W^{1,\infty}}\le c_{13}$ and $W(x)\in C^2(\bar\Omega)$, we can use Gagliadro-Nirenberg inequality to derive

$$\|w(\cdot,t)-W\|_{L^\infty}\le c_{14}\|w(\cdot,t)-W\|^{\frac12}_{W^{1,\infty}}\|w(\cdot,t)-W\|^{\frac12}_{L^2}\le c_{15}e^{-\frac{c_4}{4}t}.\tag{3.64}$$

Finally the combination of (3.63) and (3.64) gives (3.49). The proof of Lemma 3.7 is complete. □

Proof of Theorem 1.2 (global stability). The global stability result in Theorem 1.2 is a consequence of Lemma 3.7. □

4. Toxicant only steady state: Proof of Theorem 1.3

In this section, we again impose that Ω is two-dimensional and study the global dynamics of toxicant-only steady state $(0, w_*)$, where $0 < w_* \le \sup_{x\in\partial\Omega} h(x)$ is the unique non-constant positive solution of (1.6) (see the statement in the Introduction). We first show that the species u is uniformly persistent and hence the toxicant only steady state $(0, w_*)$ is uniformly strongly repelling if $\sigma \sup_{x\in\partial\Omega} h(x) < 1$, as described in the following lemma.

Lemma 4.1. *Let (u, w) be the solution of the time-dependent problem (1.2) with $u_0 \gneqq 0$. If $\sigma \sup_{\partial\Omega\times[0,\infty)} h(x, t) < 1$, then the species u is uniformly persistent, i.e. there exists $\delta_0 > 0$ independent of initial data such that*

$$\liminf_{t\to\infty} \inf_{x\in\Omega} u(\cdot, t) \ge \delta_0.$$

In particular, if $h(x, t) = h(x)$ satisfies $\sigma \sup_{x\in\partial\Omega} h(x) < 1$, then the toxicant only steady state $(0, w_*)$ is uniformly strongly repelling.

Proof. Fix a solution (u, w) of (1.2) with $u_0 \gneqq 0$ and denote

$$h_\infty := \sup_{\partial\Omega\times[0,\infty)} h(x, t).$$

Under the assumption $\sigma h_\infty < 1$, we may choose $\delta \in (0, 1)$ sufficiently small such that

$$\sigma h_\infty + 3\delta < 1.$$

Define

$$\mathcal{N}_\delta := \{u(x, t) \in C^0([0, \infty) \times \bar{\Omega}) : u < \delta \ \text{ for } \ (x, t) \in \bar{\Omega} \times [0, +\infty)\}.$$

Observe, by the maximum principle, that

$$w(x, t) \le h_\infty + \sup_{x\in\Omega} w_0 e^{-\mu t}.$$

So there exists $t_0 \ge 1$ such that

$$w(x, t) \le h_\infty + \frac{\delta}{\sigma} < \frac{1 - 2\delta}{\sigma} \quad \text{in } \Omega \times [t_0, \infty). \tag{4.1}$$

Next, suppose

$$u(\cdot, t) \in \mathcal{N}_\delta \quad \text{in } (t_1, t_2) \quad \text{for some } (t_1, t_2) \subset (t_0, \infty). \tag{4.2}$$

It is straightforward to see that $t_2 < \infty$, since

$$\frac{d}{dt}\int_\Omega u = \int_\Omega u(1 - u - \sigma w) \ge \delta \int_\Omega u \quad \text{in } (t_1, t_2),$$

which implies

$$\delta|\Omega| \ge \int_\Omega u(x, t) \ge e^{\delta(t-t_1)} \int_\Omega u(x, t_1) \quad \text{in } (t_1, t_2). \tag{4.3}$$

Therefore, it follows that

$$\limsup_{t\to\infty} \sup_{x\in\Omega} u(\cdot, t) \ge \delta,$$

which, combined with Harnack's inequality (2.75), gives

$$c_5 \limsup_{t\to\infty} \inf_{x\in\Omega} u(\cdot, t) \ge \limsup_{t\to\infty} \sup_{x\in\Omega} u(\cdot, t) \ge \delta, \tag{4.4}$$

where $c_5 > 0$ is the Harnack's constant in (2.75) and we may assume $c_5 > 1$ without loss of generality. Since δ and $c_5 > 0$ are independent of initial data so long as $u_0 \not\equiv 0$, the species u is said to be uniformly weakly persistent [32].

Next we will show that $\liminf_{t\to\infty} \inf_{x\in\Omega} u \ge \frac{\delta}{c_5^2}$, where $c_5 > 1$ is the Harnack's constant in (2.75).

If $\sup_{x\in\Omega} u \ge \delta$ for all $t \gg 1$, there is nothing to prove due to Harnack's inequality (2.75). Otherwise, by (4.4) we suppose that (4.2) holds for some t_1, t_2 such that

$$\sup_{x\in\Omega} u(\cdot, t_1) = \sup_{x\in\Omega} u(\cdot, t_2) = \delta, \quad \text{and} \quad \sup_{x\in\Omega} u(\cdot, t) < \delta \quad \text{for } t \in (t_1, t_2).$$

Then we proceed to derive a uniform lower bound for $\inf_{\Omega\times(t_1,t_2)} u$. Indeed, if $\sup_{x\in\Omega} u(\cdot,t_1) = \delta$, then $\int_\Omega u(x,t_1) \geq |\Omega|\frac{\delta}{c_5}$ thanks to Harnack's inequality (2.75) again. Then (4.3) implies that

$$\delta|\Omega| \geq e^{\delta(t_2-t_1)}|\Omega|\frac{\delta}{c_5},$$

which implies

$$|t_2 - t_1| \leq \frac{1}{\delta}\log c_5.$$

Applying (4.3) again, we obtain

$$\int_\Omega u(x,t) \geq e^{\delta(t-t_1)}|\Omega|\frac{\delta}{c_5} \geq |\Omega|\frac{\delta}{c_5} \quad \text{for all } t\in(t_1,t_2).$$

By Harnack's inequality, it follows that

$$c_5|\Omega|\inf_{x\in\Omega} u(\cdot,t) \geq \int_\Omega u(x,t) \geq |\Omega|\frac{\delta}{c_5} \quad \text{for all } t\in(t_1,t_2).$$

This completes the proof that u is uniformly bounded from below, in the sense that

$$\liminf_{t\to\infty}\inf_{x\in\Omega} u(\cdot,t) \geq \frac{\delta}{c_5^2}.$$

This asserts that the species u is uniformly persistent and the proof is complete. □

Next we consider whether $(0,w_*)$ is globally stable if $\sigma h_0 \geq 1$. In fact, we can show that $(0,w_*)$ is globally asymptotically stable provided $\sigma M_h > 1$, where $M_h := \min_{x\in\bar\Omega} \tilde{w}_*(x)$ and $\tilde{w}_*(x)$ is the solution of the system (1.8). More precisely, we have the following results:

Lemma 4.2. *Let (u,w) be the solution of the system (1.2) with $h(x,t) = h(x) \gneqq 0$ and $(0,w_*)$ is the corresponding toxicant-only steady state. Then if $\sigma M_h > 1$ with M_h is defined in Theorem 1.3, the toxicant only steady state $(0,w_*)$ is globally asymptotically stable with exponential decay rate.*

Proof. We divide our proof into two steps:

Step 1: We first show that $\|u(\cdot,t)\|_{L^\infty} \to 0$ exponentially as $t\to\infty$. From Theorem 1.1, there exists a constant $M > 0$ independent of t and σ such that $0 < u(x,t) \leq M$. Let $\tilde{w}(x,t)$ be the solution of the following system

$$\begin{cases} \tilde{w}_t = \varepsilon\Delta\tilde{w} - (\mu+\lambda M)\tilde{w}, & x\in\Omega, t>0,\\ \nabla\tilde{w}\cdot\nu = \xi(h(x)-\tilde{w}), & x\in\partial\Omega, t>0,\\ \tilde{w}(x,0) = w_0(x), & x\in\Omega. \end{cases} \tag{4.5}$$

Then using the comparison principle, we have

$$w(x,t) \geq \tilde{w}(x,t). \tag{4.6}$$

To proceed, we claim that

$$\lim_{t\to\infty}\|\tilde{w}(\cdot,t) - \tilde{w}_*(x)\|_{L^\infty} = 0, \tag{4.7}$$

where $0 < \tilde{w}_*(x) \leq h_0$ satisfies (1.8). The existence of unique non-constant positive solution for the system (1.8) can be proved by the method of super-lower solutions using the same arguments as for (1.6).

Let $M_h := \min_{x\in\bar\Omega}\tilde{w}_*(x)$. Then the combination of (4.6) and (4.7) implies there exists $t_* > 1$ such that

$$w(x,t) \geq \tilde{w}(x,t) \geq M_h - \delta_1, \quad \text{for all } \ t \geq t_*, \tag{4.8}$$

with $\delta_1 = \frac{\sigma M_h - 1}{2\sigma} > 0$. Integrating the first equation of (1.2) and using (4.8), for all $t > t_*$, we have

$$\frac{d}{dt}\int_\Omega u = \int_\Omega u(1-u-\sigma w) \leq (1-\sigma M_h + \sigma\delta_1)\int_\Omega u = \frac{1-\sigma M_h}{2}\int_\Omega u. \tag{4.9}$$

Since $\sigma M_h > 1$, we can derive from (4.9)

$$\|u(\cdot,t)\|_{L^1} \le e^{\frac{(1-\sigma M_h)(t-t_*)}{2}} \int_\Omega u(x,t_*) \le M_0 e^{\frac{(1-\sigma M_h)(t-t_*)}{2}}, \quad \text{for all} \quad t > t_*. \tag{4.10}$$

Using (4.10) and the interpolation inequality (3.62) with $n = 2$, we have

$$\|u(\cdot,t)\|_{L^\infty} \le c_1 \|u(\cdot,t)\|_{L^1}^{\frac{\theta}{\theta+2}} \|u(\cdot,t)\|_{C^\theta}^{\frac{n}{\theta+2}} \le c_2 e^{-\frac{\theta(\sigma M_h-1)}{2(2+\theta)}t} \quad \text{for } t > t_*. \tag{4.11}$$

Hence, it remains to show (4.7). In fact, letting $\tilde{v}(x,t) = \tilde{w}(x,t) - \tilde{w}_*(x)$, one has

$$\begin{cases} \tilde{v}_t = \varepsilon\Delta\tilde{v} - (\mu+\lambda M)\tilde{v}, & x \in \Omega, t > 0, \\ \nabla\tilde{v}\cdot\nu + \xi\tilde{v} = 0, & x \in \partial\Omega, t > 0, \\ \tilde{v}(x,0) = \tilde{v}_0(x) = w_0(x) - \tilde{w}_*(x), & x \in \Omega. \end{cases} \tag{4.12}$$

Then multiplying the first equation of (4.12) by $\tilde{v}$ and integrating the result by parts, we have

$$\frac{d}{dt}\int_\Omega \tilde{v}^2 + \varepsilon\int_\Omega |\nabla\tilde{v}|^2 + (\mu+\lambda M)\int_\Omega \tilde{v}^2 = \varepsilon\int_{\partial\Omega} \tilde{v}\nabla\tilde{v}\cdot\nu = -\xi\varepsilon\int_{\partial\Omega}\tilde{v}^2 \le 0,$$

and hence

$$\|\tilde{v}(\cdot,t)\|_{L^2}^2 \le \|\tilde{v}_0\|_{L^2}^2 e^{-(\mu+\lambda M)t} = \|w_0(x) - \tilde{w}_*(x)\|_{L^2}^2 e^{-(\mu+\lambda M)t}. \tag{4.13}$$

Since $\tilde{w}_*(x)$ is the steady state for the system (4.5), using the elliptic regularity estimates, one has $\tilde{w}_*(x) \in C^2(\bar{\Omega})$. Hence from (4.13), we have

$$\|\tilde{w}(\cdot,t) - \tilde{w}_*\|_{L^2} = \|\tilde{v}(\cdot,t)\|_{L^2} \le c_3 e^{-\frac{\mu+\lambda M}{2}t}. \tag{4.14}$$

On the other hand, using the parabolic regularity, from (4.5) we derive that $\|\tilde{w}(\cdot,t)\|_{W^{1,\infty}} \le c_4$, which together with $\tilde{w}_*(x) \in C^2(\bar{\Omega})$ gives

$$\|\tilde{w}(\cdot,t) - \tilde{w}_*\|_{W^{1,\infty}} \le c_5. \tag{4.15}$$

Then we can use the Gagliardo-Nirenberg inequality and (4.14)-(4.15) to derive

$$\|\tilde{w}(\cdot,t) - \tilde{w}_*\|_{L^\infty} \le c_6 \|\tilde{w}(\cdot,t) - \tilde{w}_*\|_{W^{1,\infty}}^{\frac{1}{2}} \|\tilde{w}(\cdot,t) - \tilde{w}_*\|_{L^2}^{\frac{1}{2}} \le c_7 e^{-\frac{\mu+\lambda M}{4}t},$$

which gives (4.7).

Step 2: Next, we shall show that $\|w(\cdot,t) - w_*\|_{L^\infty} \to 0$ with exponential decay rate as $t \to \infty$. To this end, we let $v(x,t) = w(x,t) - w_*(x)$. Then from (1.2) and (1.6), we see that v satisfies

$$\begin{cases} v_t = \varepsilon\Delta v - \mu v - \lambda uv - \lambda w_* u, & x \in \Omega, t > 0, \\ \nabla v\cdot\nu + \xi v = 0, & x \in \partial\Omega, t > 0, \\ v(x,0) = w_0(x) - w_*(x), & x \in \Omega. \end{cases} \tag{4.16}$$

Then multiplying the first equation of (4.16) by v, and integrating the result by parts and using the fact $0 < w_*(x) \le h_0$, we have

$$\begin{aligned} &\frac{d}{dt}\int_\Omega v^2 + \int_\Omega |\nabla v|^2 + \mu\int_\Omega v^2 + \lambda\int_\Omega uv^2 + \xi\int_{\partial\Omega} v^2 \\ &= -\lambda\int_\Omega w_* uv \le \lambda h_0 \int_\Omega |uv| \le \frac{\mu}{2}\int_\Omega v^2 + \frac{\lambda^2 h_0^2}{2\mu}\int_\Omega u^2, \end{aligned}$$

which, together with (4.11), gives

$$\frac{d}{dt}\int_\Omega v^2 + \frac{\mu}{2}\int_\Omega v^2 \le \frac{\lambda^2 h_0^2}{2\mu}\int_\Omega u^2 \le \frac{\lambda^2 h_0^2|\Omega|}{2\mu}\|u\|_{L^\infty}^2 \le c_8 e^{\frac{-(\sigma M_h-1)t}{3}}, \text{ for all } \ t > t_*. \tag{4.17}$$

Then solving (4.17), one can find $\alpha_1 = \frac{1}{2}\min\{\frac{\sigma M_h-1}{3}, \frac{\mu}{2}\} > 0$ such that

$$\|w(\cdot,t) - w_*\|_{L^2} = \|v(\cdot,t)\|_{L^2} \le c_9 e^{-\alpha_1 t}, \text{ for all } \ t > t_*,$$

which along with the Gagliardo-Nirenberg inequality as well as the facts $\|w(\cdot,t)\|_{W^{1,\infty}} \le c_1$ and $w_*(x) \in C^2(\bar{\Omega})$ gives

$$\|w(\cdot,t) - w_*\|_{L^\infty} \le \|w(\cdot,t) - w_*\|_{W^{1,\infty}}^{\frac{1}{2}} \|w(\cdot,t) - w_*\|_{L^2}^{\frac{1}{2}} \le c_{10} e^{\frac{-\alpha_1 t}{2}}, \text{ for all } \ t > t_*. \tag{4.18}$$

Then the combination of (4.11) and (4.18) completes the proof. □

Proof of Theorem 1.3. Theorem 1.3 is a consequence of Lemma 4.1 and Lemma 4.2. □

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