EXISTENCE OF POSITIVE STEADY-STATE SOLUTIONS TO THE SKT COMPETITION SYSTEM WITH CROSS-DIFFUSION

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Abstract. This paper is concerned with the following stationary Shigesada-Kawasaki-Teramoto competition system with cross-diffusion

$$\begin{cases} d\Delta u + u(r - u - bv) = 0, & \text{in } \Omega, \\ \mu\Delta[(1 + ku)v] + v(r - v - cu) = 0, & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & \text{on } \partial\Omega, \end{cases}$$

where u and v represent the densities of two competing species, Ω is a bounded domain in $\mathbb{R}^n (n \geq 1)$ and ν denotes the outer unit normal to $\partial\Omega$. All coefficients d, μ, b, c, r, k are assumed to be positive constants. The existence and stability/instability of non-constant positive solutions of the above system has been widely studied in the literature but confined to large $k > 0$ and small $d > 0$ (or $d > 0$ close to some particular number) with $\mu \in (0, \infty]$. In this paper, we establish the existence/nonexistence of non-constant positive solutions for any $k, d > 0$ and large $\mu > 0$, which fills some gaps left out in the existing results. First, we show there are no positive solutions in the case of $b < 1 < c$ for large $\mu > 0$. Then by studying the shadow system of the above system as $\mu \to \infty$, we establish the existence of positive solutions for large $\mu > 0$ in other various ranges of $b, c > 0$ including all possible competitions: weak, strong-weak and strong. In particular, we find some conditions under which multiple positive solutions exist. Then we show the existence of positive solutions for some $\mu > 0$ in the case of weak competition $0 < b, c < 1$.

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1. Introduction

A variety of mathematical models have been employed to investigate the effect of dispersal on population dynamics [39, 51], as well as how the species interaction affects the selection and evolution of dispersal strategies [9, 18, 30]. However, much of the theoretical studies are devoted to the case of random (unconditional) dispersal where the movement of species is modeled as a random diffusion process [33, 36]. In comparison, the mathematical studies of models incorporating conditional dispersal strategies, which take into account factors such as avoidance effect, population pressure, crowding effect, and competition of species, to name a few, have received relatively less attention and there are many open questions related to conditional dispersal strategies [2]. Among them, cross-diffusion (the process by which the density gradient of one species induces an advective flux of another species) has often been used to interpret many observed patterns and evolutionary processes in living organisms, such as chemotaxis [14], preytaxis [13], pattern formation [23], biofilm [38], Turing pattern [7], spatial segregation [43] and so on. These reaction-cross-diffusion systems have received enormous attention due to their rich mathematical structures that enable the modeling of many important physical phenomena.

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In this paper, we are concerned with the following simplified STK cross-diffusion model proposed by Shigesada-Kawasaki-Teramoto [43]

$$\begin{cases} u_t = d\Delta u + u(r - u - bv), & x \in \Omega, t > 0, \\ v_t = \mu\Delta[(1 + ku)v] + v(r - v - cu), & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \end{cases} \tag{1.1}$$

where $u(x, t)$ and $v(x, t)$ represent the densities of two competing species at the location x and time t, Ω is a bounded domain in $\mathbb{R}^n (n \geq 1)$ and ν denotes the outer unit normal to $\partial\Omega$. All coefficients d, μ, b, c, r, k are assumed to be positive constants throughout this paper, where in particular the parameter $r > 0$ is referred to as the (spatially homogeneous) resource available in the environment. The term $\Delta[(1 + ku)v]$ says that the rate of departure of species v from location x is proportional to $1 + ku(x, t)$, which is an increasing function of the density $u(x, t)$ of the first species. The coefficient k is called the cross-diffusion coefficient measuring the biased movement of the species v in response to the population pressure from the species u.

If $k = 0$, then the dispersal strategy of v is unconditional upon the density of u, and the competition model (1.1) becomes the classical diffusive Lotka-Volterra competition model under zero Neumann boundary conditions. In this case, the system admits a comparison principle and the theory of monotone dynamical systems can be applied to classify the long-time dynamics of the system [10, 44]. A result by Kishimoto and Weinberger [15] asserts that (1.1) has no stable nontrivial positive steady state on a convex domain. It is well known that in the case of weak-strong competition (i.e. $b < 1 < c$ or $b > 1 > c$), (1.1) has no positive steady state (cf. [25, 32]), i.e. coexistence is impossible. However, if the resource is spatially heterogeneous, namely $r = r(x)$ is not constant, then the global dynamics are much more complicated and the species may coexist in the case of weak-strong competition, depending on the size of dispersal rates d and μ (see [10]). Therefore, an interesting question is whether two competing species can coexist in the case of weak-strong competition if one adopts density-dependent dispersal [2] given that the resource is spatially homogeneous. The quasilinear cross-diffusion system (1.1) with $k > 0$ is a prominent mathematical model highly pertinent to this question and has attracted tremendous attention in the past few decades. The existence of global-in-time solutions has been established in [1, 19]. For the steady states, the first analytical work was due to [31] which showed that (1.1) admits positive transition-layer steady states when μ and k are sufficiently large but $d > 0$ is sufficiently small in some strong competition case $b, c > 1$. Later the stability/instability of such steady states was investigated in [12]. The existence/nonexistence of positive steady states in some larger parameter regimes were obtained in [25]. In a celebrated work [26], Lou and Ni established the uniform boundedness of nontrivial steady states, and derived three types of limiting shadow systems determining all the possible asymptotic behavior of steady states as the cross-diffusion parameter k in (1.1) tends to infinity (see [26, Theorem 1.4 and Theorem 4.1]).

Theorem 1.1 ([26, Theorem 4.1]). *Let $\Omega \subset \mathbb{R}^n (1 \leq n \leq 3)$ be a bounded domain with smooth boundary. Suppose $b \neq 1$, $c \neq 1$ and $r/\mu \neq \lambda_j$ for all $j \in \mathbb{N}$, where λ_j denotes the eigenvalues of $-\Delta$ subject to homogeneous Neumann boundary condition. Let (u_i, v_i) be positive non-constant steady states of (1.1) with $(\mu, k) = (\mu_i, k_i)$ and $\mu_i k_i \to \infty$. Then the following conclusions hold.*

(a) *If $k_i \to \infty$ and $\mu_i \to \mu \in (0, \infty)$, then either* (i) *or* (ii) *occurs;*
(b) *If $k_i \to \infty$ and $\mu_i \to \infty$, then either* (i*) *or* (ii) *occurs;*
(c) *If $k_i \to k \in [0, \infty)$, then $k > 0$ and* (iii) *occurs; where*

(i) *$(k_i u_i, v_i) \to (w, v)$ uniformly, where (w, v) is a positive solution of*

$$\begin{cases} d\Delta w + w(r - bv) = 0, & \text{in } \Omega, \\ \mu\Delta[(1 + w)v] + v(r - v) = 0, & \text{in } \Omega, \\ \frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

(i*) $(k_i u_i, v_i) \to (w, \xi/(1+w))$ *uniformly, where* $\xi > 0$ *and* w *is a positive solution of*

$$\begin{cases} d\Delta w + w\left(r - b\xi/(1+w)\right) = 0, & \text{in } \Omega, \\ \int_\Omega \frac{r}{1+w}\, dx = \xi \int_\Omega \frac{1}{(1+w)^2}\, dx, & \\ \frac{\partial w}{\partial \nu} = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

(ii) $(u_i, v_i) \to (u, \xi/u)$ *uniformly, where* $\xi > 0$ *and* u *is a positive solution of*

$$\begin{cases} d\Delta u + u\left(r - u\right) = b\xi, & \text{in } \Omega, \\ \int_\Omega \frac{1}{u}\left(r - \xi/u - cu\right)\, dx = 0, & \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

(iii) $(u_i, v_i) \to (u, \xi/(1+ku))$ *uniformly, where* $\xi > 0$ *and* u *is a positive solution of*

$$\begin{cases} d\Delta u + u\big(r - u - \frac{b\xi}{1+ku}\big) = 0 & \text{in } \Omega, \\ \int_\Omega \frac{1}{1+ku}\left(r - cu\right)\, dx = \xi \int_\Omega \frac{1}{(1+ku)^2}\, dx, & \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.5}$$

The classification given in Theorem 1.1 provides a framework to study the steady-state solutions of the quasilinear system (1.1) and has stimulated lots of studies on the existence and/or stability/instability of limiting systems (1.2), (1.4) and (1.4) in various ranges of parameters (cf. [17, 20, 21, 25, 27, 28, 33, 34, 46, 49, 50] in one dimension and [16, 29] in multi-dimensions). But all these works have essentially assumed that $d > 0$ is either small or close to some particular number (or lies in certain range), and $k > 0$ is sufficiently large. We refer to a recent work [21] and references therein for a brief review of the above-mentioned works. See also [22] for more recent developments.

As we know, the limiting shadow system (1.5) has never been investigated in the literature. Indeed the limiting system (1.5) results from the case $\mu \to \infty$ and $0 < k < \infty$, which clearly varies from limiting systems (1.2), (1.3) and (1.4), all of which require $k \to \infty$. Hence the study of the existence/nonexistence of solutions to (1.5) is of interest in its own right. The main goal of this paper is to study the existence or nonexistence of non-constant steady state solution of (1.1) satisfying

$$\begin{cases} d\Delta u + u(r - u - bv) = 0, & \text{in } \Omega, \\ \mu\Delta[(1+ku)v] + v(r - v - cu) = 0, & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & \text{on } \partial\Omega \end{cases} \tag{1.6}$$

in some parameter regimes not covered by the existing studies mentioned above. Our first result is concerned with the nonexistence of positive solutions to (1.5) and (1.6). Specifically, we will show that system (1.5) does not admit non-constant positive solutions for $b \le 1 \le c$, while if $b < 1 < c$, then system (1.6) have no positive non-constant solutions for large μ.

Theorem 1.2. *Let* $d, k, b, c > 0$. *Then the following results hold.*

(1) *If* $b \le 1 \le c$, *then system* (1.5) *does not admit non-constant positive solutions for any* $\mu > 0$;
(2) *If* $b < 1 < c$, *then there exists* $\underline{\mu} > 0$ *such that* (1.6) *has no non-constant positive solutions for any* $\mu > \underline{\mu}$.

It is well known [32] that in the absence of cross-diffusion (i.e., $k = 0$), the weak ($b < 1 < c$) competitor v does not persist for any $d, \mu > 0$. Theorem 1.2 (2) implies that it will not persist either even if it adopts the dispersal strategy to avoid the stronger competition u (i.e. $k > 0$) when its diffusion rate μ is sufficiently large for given $b, c, d, k > 0$.

Then a natural question is whether (1.6) has positive solutions outside the parameter regime given in Theorem 1.2. It turns out this is a very challenging question. In this paper, we can address this question in one dimension due to the crucial observation (see Proposition 1.1 below) that the existence of solutions of the limiting shadow system (1.5) implies the existence of solutions to the SKT system (1.6) for $\mu \gg 1$ under some non-degeneracy conditions. Therefore we shall restrict our attention in one dimensional case in what follows. Without loss of generality, we assume $\Omega = (0, L)$ with $L > 0$ and rewrite the system (1.5) as

$$\begin{cases} u_{xx} + u\left(r - u - \frac{b\xi}{1+ku}\right) = 0 & \text{in } \ (0,\mathrm{L}), \\ \int_0^L \frac{1}{1+ku}\,(r - cu)\,dx = \xi \int_0^L \frac{1}{(1+ku)^2}\,dx, & \\ u_x = 0, & x = 0, L. \end{cases} \tag{1.7}$$

and (1.6) as

$$\begin{cases} u_{xx} + u(r - u - bv) = 0, & \text{in } (0, L), \\ \mu[(1 + ku)v]_{xx} + v(r - v - cu) = 0, & \text{in } (0, L), \\ u_x = v_x = 0, & x = 0, L, \end{cases} \tag{1.8}$$

where we set $d = 1$ for the simplicity of notation and $k > 0$ is fixed for the rest of this paper.

To study the existence of nonconstant solutions of the limiting (1.7) with that of the original system (1.8), we define a weak form of non-degeneracy as follows.

Definition 1.1. *We say that a nonconstant solution (u^*, ξ^*) of (1.7) is nondegenerate if the linear operator $T : \{\phi \in W^{2,2}([0, L]) :\ \phi_x(0) = \phi_x(L) = 0\} \to L^2([0, L])$ given by*

$$T(\phi) = \phi_{xx} + \phi\left(r - 2u^* - \frac{b\xi^*}{(1+ku^*)^2}\right),$$

is invertible.

Proposition 1.1. *Suppose (1.7) has a non-constant solution (u^*, ξ^*). If it is nondegenerate and satisfies*

$$\int_0^L \left(\frac{T^{-1}\big(\frac{bu^*}{1+ku^*}\big)(3\xi^* k - (kr + c)(1 + ku^*))}{(1 + ku^*)^3} - \frac{1}{(1 + ku^*)^2} \right) dx \neq 0, \tag{1.9}$$

where T^{-1} is the inverse of operator T given in Definition 1.1, then system (1.8) admits a non-constant positive solution (u^μ, v^μ) for $\mu \gg 1$. Moreover,

$$(u^\mu, v^\mu) \to (u^*, \tfrac{\xi^*}{1+u^*}) \qquad \text{as } \mu \to +\infty.$$

Remark 1.1. *Condition 1.9 is to ensure that system (1.7) when linearized at the solution (u^*, ξ^*) does not admit a zero eigenvalue. We believe it is a generic condition that is satisfied except for a small subset of parameter values. However, it is not easy to check its validity analytically.*

Thanks to Proposition 1.1, it remains to explore the existence and structure of non-constant solutions of the limiting shadow system (1.7). To this end, we classify the monotone increasing solution of (1.7) since every non-constant solution of the shadow system (1.7) can be constructed from monotone solutions by reflection (see Lemma 3.1). For the shadow system (1.7), we have the following conclusions on the existence and nonexistence of the monotone increasing solutions.

Theorem 1.3. *Suppose that*

$$rk > 1 \qquad \text{and} \qquad L > L_* := \frac{\sqrt{k}\pi}{\sqrt{rk + 3 - 2\sqrt{2(1 + kr)}}}. \tag{1.10}$$

Then there exists $b_* \in \left(\frac{1}{1+rk}, 1\right)$ and $c^* \in (1, \infty)$ such that the following results hold.

(i) *If $b < b_*$ (resp. $c > c^*$) , then system (1.7) has no strictly increasing positive solutions for any $c > 0$ (resp. for any $b > 0$).*

(ii) *If $b > b_*$, then there is a single bounded interval I_b such that system* (1.7) *has a strictly increasing positive solution u^* for some $\xi = \xi^*$ if and only if $c \in I_b$. Consequently for any $b > b_*$ and $c \in I_b$, system* (1.8) *with $\mu \gg 1$ admits a non-constant positive solution if (u^*, ξ^*) satisfies* (1.9).

(iii) *If $c < c^*$, then there is a single interval I_c such that system* (1.7) *has a strictly increasing positive solution u^* for some $\xi = \xi^*$ if and only if $b \in I_c$. Therefore for any $c > c^*$ and $b \in I_c$, system* (1.8) *with $\mu \gg 1$ admits a non-constant positive solution if (u^*, ξ^*) satisfies* (1.9).

Theorem 1.3 gives the existence of increasing positive solutions of (1.7) and hence (1.8). However, given $b \in (b_*, \infty)$ (resp. $c \in (0, c^*)$), the size of I_b (resp. I_c) is obscure and can not be explicitly identified. Below we present a more decisive result.

Theorem 1.4. *Let the conditions in* (1.10) *hold. Define*

$$z^{\pm} = (rk - 1 - k\pi^2/L^2 \pm \sqrt{(k\pi^2/L^2 - rk + 1)^2 - 8k\pi^2/L^2})/4k.$$

Then there exists a small constant $\epsilon = \epsilon(k, r, L) > 0$ such that the following results hold.

(i) *If $b \in (1, 1+\epsilon)$ and $c \in (1-\epsilon, 1)$, then* (1.7) *admits at least two increasing positive solutions.*

(ii) *Assume $b \in (1-\epsilon, 1)$ and $c \in (1-\epsilon, 1)$. If $\frac{1/b-1}{1-c} < \frac{z^-}{r-z^-}$, then* (1.7) *at least admits two increasing positive solutions, which are non-degenerate; if $\frac{1/b-1}{1-c} \in \left(\frac{z^-}{r-z^-}, \frac{z^+}{r-z^+}\right)$, then* (1.7) *admits at least one increasing positive solution.*

(iii) *Assume $b \in (1, 1+\epsilon)$ and $c \in (1, 1+\epsilon)$. If $\frac{1-1/b}{c-1} > \frac{z^+}{r-z^+}$, then* (1.7) *at least admits two increasing positive solutions, which are non-degenerate; if $\frac{1/b-1}{1-c} \in \left(\frac{z^-}{r-z^-}, \frac{z^+}{r-z^+}\right)$, then* (1.7) *admits at least one increasing positive solution.*

Under the same conditions, if the solution of (1.7) *satisfies* (1.9), *then system* (1.8) *admits the same number of non-constant positive solutions as* (1.7).

While Theorem 1.2 asserts that system (1.8) does not have any non-constant positive solution if $b < 1 < c$, Theorem 1.4 (i) says that there are some $b, c > 0$ with $c < 1 < b$ such that (1.8) with $\mu \gg 1$ admits some non-constant positive solutions. Theorem 1.4 (ii) implies that there are some $b, c > 0$ with $b, c < 1$ or $b, c > 1$ such that (1.8) with $\mu \gg 1$ admits some non-constant positive solutions. In the following theorem proved by the global bifurcation theorem, we show (1.8) may admit at least one non-constant positive solutions for any $0 < b, c < 1$ (weak competition) and some $\mu > 0$.

Theorem 1.5. *Let $0 < b, c < 1$ and $\lambda_i = \frac{\pi^2 i^2}{L^2}, i = 0, 1, 2, \cdots$. Define $\mu_{\lambda_i} = \frac{(1-bc)u^+v^+}{(bku^+v^+ - (1+ku^+)(\lambda_i + u^+))\lambda_i}$, where $u^+ = \frac{(1-b)r}{1-bc} > 0$ and $v^+ = \frac{(1-c)r}{1-bc} > 0$. Suppose that j is a positive integer such that $\frac{(bkv^+ - (1+ku^+))u^+}{1+ku^+} \in (\lambda_j, \lambda_{j+1}]$. If there exists $i \in \{1, 2, \cdots, j\}$ such that*

$$\mu_{\lambda_i} \neq \mu_{\lambda_m} \text{ for any } m \in \{1, 2, \cdots, j\} \text{ and } m \neq i,$$

then (1.8) *admits at least one non-constant positive solution whenever*

$$\mu > \mu_{\lambda_i} \text{ and } \mu \notin \{\mu_{\lambda_1}, \mu_{\lambda_2}, \cdots, \mu_{\lambda_j}\}.$$

This paper is organized as follows. In Section 2, we prove the nonexistence of positive solutions to (1.7) and (1.8). In Section 3, we classify the monotone increasing solutions of (1.7), which form the building blocks of all nonconstant solutions (see Propositions 3.2 and 3.3). These enables us to conclude the existence of non-constant solutions of system (1.8) with large $\mu > 0$ as claimed in Theorem 1.3. In Section 4, the existence results in the case of weak competition for some $\mu > 0$ (i.e. Theorem 1.5) are proved via the global bifurcation theory [3, 42].

2. Nonexistence of positive solution of system (1.6)

This section is devoted to proving Theorem 1.2. We first prove the following result for system (1.5).

Proposition 2.1. *If $b \le 1 \le c$, then system* (1.5) *has no non-constant positive solutions.*

Proof. If not, assume that (1.5) has a non-constant positive solution u. Multiplying the first equation of (1.5) by $\frac{1}{u(1+ku)}$ and integrating the resulting equation over Ω, one obtains

$$\int_\Omega \frac{1}{1+ku}\left(r-u-\frac{b\xi}{1+ku}\right)\mathrm{d}x = -d\int_\Omega \frac{\Delta u}{u(1+ku)}\mathrm{d}x = -\int_\Omega \frac{|\nabla u|^2(1+2ku)}{u^2(1+ku)^2}\mathrm{d}x < 0,$$

which along with the condition $b \le 1 \le c$ implies that

$$\int_\Omega \frac{1}{1+ku}\left(r-cu-\frac{b\xi}{b(1+ku)}\right)\mathrm{d}x < 0.$$

This contradicts the second identity of (1.5), and hence completes the proof. □

To prove that system (1.6) does not admit any positive solution for large μ, we first establish several preparatory lemmas.

Lemma 2.1. *Let (u,v) be a positive solution of* (1.6)*. Suppose there is a constant $A>0$ such that*

$$\sup_{\bar\Omega}(1+ku)v \le A\inf_{\bar\Omega}(1+ku)v, \quad \|u\|_{L^\infty(\Omega)} \le A,$$

then there exists a constant C_A such that

$$\sup_{\bar\Omega} v \le C_A \inf_{\bar\Omega} v.$$

Here $C_A = A(1+kA)$ depends on A only.

Proof. Indeed, it is obvious that

$$\sup_{\bar\Omega} v \le \sup_{\bar\Omega}\frac{1}{1+ku}\sup_{\bar\Omega}(1+ku)v \le A\inf_{\bar\Omega}(1+ku)v \le A\sup_{\bar\Omega}(1+ku)\inf_{\bar\Omega} v \le C_A\inf_{\bar\Omega} v.$$

This completes the proof. □

Lemma 2.2. *Let $b,c,r,k>0$ be given. For any μ_0, there is $\delta_0=\delta_0(\mu_0)>0$ such that any positive solution (u,v) of* (1.8) *with $\mu\in[\mu_0,\infty)$ satisfies that $\inf_{\bar\Omega} u + \inf_{\bar\Omega} v \ge \delta_0$.*

Proof. Suppose to the contrary that there is a sequence of steady states (μ_j,u_j,v_j) of (1.6) such that

$$\inf_{\bar\Omega} u_j + \inf_{\bar\Omega} v_j \to 0, \quad \text{as} \quad j\to\infty. \tag{2.1}$$

First, by the weak maximum principle, we observe that

$$\sup_{\bar\Omega}|u_j| \le r.$$

For any j, let $x_0\in\overline{\Omega}$ be such that $(1+ku_j(x_0))v_j(x_0) = \max_{x\in\overline{\Omega}}(1+ku_j(x))v_j(x)$. Then

$$\Delta(1+ku(x_0)v(x_0)) \le 0 \quad \text{and} \quad r-cu(x_0)-v(x_0)\ge 0,$$

which suggests that $v(x_0)\le r$ and hence

$$\sup_{\bar\Omega}|(1+ku_j)v_j| \le (1+ku_j(x_0))v_j(x_0) \le r(1+kr),$$

namely, u_j and $(1+ku_j)v_j$ are bounded in $C([0,L])$ uniformly in j. Thus there is a constant C_0 independent of j so that

$$\sup_{\bar\Omega}(|u_j|+|(1+ku_j)v_j|) \le C_0. \tag{2.2}$$

It follows that both u_j and $w_j := (1+ku_j)v_j$ satisfy the Harnack inequality uniformly in j, as both satisfy homogeneous linear elliptic equations with L^∞ coefficients under (2.2) alongside the condition $\mu > \mu_0$

$$\Delta u_j + \left(r - u_j - b\frac{w_j}{1+ku_j}\right) u_j = 0 \quad \text{and} \quad \Delta w_j + \frac{r - \frac{w_j}{1+ku_j} - cu_j}{\mu_j(1+ku_j)} w_j = 0, \tag{2.3}$$

and the homogeneous Neumann boundary condition. By Lemma 2.1 and (2.1), we deduce that $u_j \to 0$ and $v_j \to 0$ uniformly in $\bar{\Omega}$. Now, if we divide the first equation of (1.8) by u_j, and integrate the result by parts, we get

$$0 = \int_\Omega \frac{|\nabla u_j|^2}{(u_j)^2} \mathrm{d}x + \int_\Omega (r - u_j - bv_j)\, \mathrm{d}x.$$

Sending $j \to \infty$, we deduce $\int_\Omega r \mathrm{d}x \le 0$, which is a contradiction. Therefore (2.1) is false and there is a constant $\delta_0 > 0$ such that $\inf_{\bar{\Omega}} u + \inf_{\bar{\Omega}} v \ge \delta_0$ for all $\mu \in [\mu_0, \infty)$. Particularly this $\delta_0 > 0$ can be chosen to depend only on μ_0 but independent of $\mu \in [\mu_0, \infty)$, since the L^∞ bound of the coefficients of the elliptic equations (2.3) only depends on μ_0. The proof is thus completed. □

Lemma 2.3. *Consider the problem*

$$-\mu_j \Delta w_j = F_j \quad \text{in } \Omega, \qquad \text{and} \qquad \frac{\partial w_j}{\partial \nu} = 0 \quad \text{on } \partial\Omega$$

where $\mu_j \to \infty$ as $j \to \infty$. If $\{F_j\}$ is uniformly bounded in L^2, then $\mu_j \int_\Omega |\nabla w_j|^2 \mathrm{d}x \to 0$ as $j \to \infty$.

Proof. First, one observes that $\int_\Omega F_j \mathrm{d}x = 0$ by integrating the equation along with the boundary condition. Multiplying the equation of w_j and integrating the result with Hölder inequality, we get

$$\mu_j \int_\Omega |\nabla w_j|^2 \mathrm{d}x = \int_\Omega w_j F_j \mathrm{d}x = \int_\Omega (w_j - \overline{w}_j) F_j \mathrm{d}x \le \|w_j - \bar{w}_j\|_{L^2} \|F_j\|_{L^2} \tag{2.4}$$

where $\bar{w}_j = \frac{1}{L}\int_\Omega w_j \mathrm{d}x$. Then applying the Poincaré inequality: $\|w_j - \bar{w}_j\|_{L^2} \le c\|\nabla w_j\|_{L^2}$ for some some constant $c > 0$ into (2.4), one finds a constant $C > 0$ depending on c and the L^2-norm of F_j such that

$$\|w_j - \bar{w}_j\|_{L^2} \le \frac{C}{\mu_j}.$$

Now sending $j \to \infty$ in (2.4), we obtain the desired conclusion. □

With the help of the above lemmas, we now prove Theorem 1.2.

Proof of Theorem 1.2. The result of Theorem 1.2-(1) directly follows from Proposition 2.1. We proceed to prove Theorem 1.2-(2). Suppose to the contrary that there is a sequence of positive solutions (μ_j, u_j, v_j) of (1.6) with $\mu_j \to \infty$ as $j \to \infty$ and $b < 1 < c$. Now, we claim that $\inf_{\bar{\Omega}} v_j \not\to 0$. Indeed, u_j and $w_j := (1+ku_j)v_j$ satisfy the Harnack inequality with constant independent of j (as explained in Lemma 2.2). It then follows that v_j also has Harnack inequality (Lemma 2.1). If $\inf_{\bar{\Omega}} v_j \to 0$, then $v_j \to 0$ uniformly by the Harnack inequality. It then follows from the equation of u_j that $u_j \to r$ or 0 uniformly and hence $r - v_j - cu_j \to (1-c)r$ or r uniformly. Since $c > 1$, this means $r - v_j - cu_j$ does not change sign for j sufficiently large. This is impossible since $\int_\Omega v_j(r - v_j - cu_j)\mathrm{d}x = 0$. Hence, $\inf_{\bar{\Omega}} v_j \ge \delta_1$ for some $\delta_1 > 0$ independent of j.

Next, we divide the equation of u_j by $u_j(1+ku_j)$ and integrate the result by parts to obtain

$$\int_\Omega \frac{1}{1+ku_j}(r - u_j - bv_j) = -\int_\Omega \frac{|\nabla u_j|^2(1+2ku_j)}{(u_j)^2(1+ku_j)^2} \mathrm{d}x < 0, \tag{2.5}$$

where the strict inequality results from that (1.6) has no constant positive solution for $b < 1 < c$. Dividing the equation of v_j by $w_j := (1 + ku_j)v_j$, we have

$$\mu_j \int_\Omega \frac{|\nabla w_j|^2}{(w_j)^2}\mathrm{d}x + \int_\Omega \frac{1}{1+ku_j}(r - v_j - cu_j)\mathrm{d}x = 0. \tag{2.6}$$

Combining (2.5) and (2.6), we get

$$\mu_j \int_\Omega \frac{|\nabla w_j|^2}{(w_j)^2}\mathrm{d}x = \int_\Omega \frac{cu_j + v_j - r}{1+ku_j}\mathrm{d}x > \int_\Omega \frac{(c-1)u_j + (1-b)v_j}{1+ku_j}\mathrm{d}x.$$

Using Lemma 2.2, we obtain

$$\mu_j \int_\Omega \frac{|\nabla w_j|^2}{(w_j)^2}\mathrm{d}x > (1-b)\int_\Omega \frac{\delta_0}{1+kC_0}\mathrm{d}x, \tag{2.7}$$

where C_0 is the uniform bound for u_j obtained in (2.2). However, Lemma 2.3 and the fact that $\inf_{\bar\Omega} v_j \geq \delta_1$ implies that $w_j \geq \delta_1$, and hence

$$0 \leq \mu_j \int_\Omega \frac{|\nabla w_j|^2}{(w_j)^2}\mathrm{d}x \leq \frac{1}{\delta_1^2}\cdot \mu_j \int |\nabla w_j|^2 \mathrm{d}x \to 0.$$

Then sending $j \to \infty$ in (2.7), we obtain

$$0 \geq (1-b)\int_\Omega \frac{\delta_0}{1+kC_0}\mathrm{d}x.$$

This is a contradiction and hence the proof of Theorem 1.2 is completed.

3. Existence and multiplicity of positive solutions of system (1.7) and (1.8)

In this section, we establish the existence and multiplicity of positive solutions of system (1.8) when μ is large. First, inspired by Proposition 1.1, we consider the shadow system (1.7).

The following lemma says that every nonconstant solution of the shadow system can be constructed from monotone solutions by reflection.

Lemma 3.1. *Let $(u,\xi) \in C^2([0,L]) \times [0,\infty)$ be a nonnegative solution of the shadow system (1.7). If u is nonconstant, then there exists $m \in \mathbb{N}$ such that $(u\big|_{[0,L/m]}, \xi)$ is a strictly monotone solution of (1.7) with the domain $(0,L)$ replaced by $(0,L/m)$. Furthermore,*

$$u(x) = \begin{cases} u(x - jL/m) & \text{when } x \in [jL/m, (j+1)L/m],\ j \text{ even}, \\ u((j+1)L/m - x) & \text{when } x \in [jL/m, (j+1)L/m],\ j \text{ odd}. \end{cases} \tag{3.1}$$

In particular, if u is nonconstant and increasing in $[0,L]$, then $u' > 0$ in $(0,L)$.

Proof. Since $u'(L) = 0$, the following is well defined:

$$x^* = \inf\{x \in (0,L]:\ u'(x) = 0\}.$$

First, we claim that $x^* > 0$. Suppose, not, then there exists a sequence $x_j \searrow 0$ such that $u'(x_j) = 0$ for all j. By Rolle's theorem, there exists $y_j \in (x_{j+1}, x_j)$ such that $y_j \to 0$ and $u''(y_j) = 0$ for all j. Sending $j \to \infty$, we deduce that $u''(0) = 0$. By uniqueness of ODE, it follows that $u(x) \equiv u(0)$ for all x, which is impossible as u is nonconstant. Hence, $x^* > 0$.

By construction, u is strictly monotone in $[0, x^*]$, as u' does not change sign in $(0, x^*)$. By uniqueness of ODE again, we easily see that $u(x) = u(x^* - x)$ for $x \in [x^*, 2x^*]$. Repeating the argument, we have

$$u(x) = \begin{cases} u(x - jx^*) & \text{when } x \in [jx^*, (j+1)x^*],\ j \text{ even}, \\ u((j+1)x^* - x) & \text{when } x \in [jx^*, (j+1)x^*],\ j \text{ odd}. \end{cases}$$

It follows that $L = mx^*$ for some $m \in \mathbb{N}$.

Finally, if $m = 1$, then $x^* = L$, so the definition of x^* implies $u' > 0$ in $(0,L)$. This completes the proof. □

To find a positive solution of (1.7), we first investigate the regular boundary value problem

$$\begin{cases} u_{xx} + u\left(r - u - \frac{b\xi}{1+ku}\right) = 0, & \text{in } (0, L), \\ u_x(0) = u_x(L) = 0, \end{cases} \tag{3.2}$$

for given positive parameters ξ, b, r, k. Motivated by Lemma 3.1 (see also [11, 47]), it is equivalent to consider the existence/non-existence of increasing solutions of (3.2) for arbitrary $L > 0$. To simplify notations, we define for $u \geq 0$,

$$h(u) = (r - u)(1 + ku), \qquad g(u) = \frac{h(u) - b\xi}{1 + ku} = r - u - \frac{b\xi}{1 + ku},$$
$$f(u) = ug(u), \quad \text{and} \quad F(u) = \int_0^u f(\tau)\, d\tau. \tag{3.3}$$

With the maximum principle, we can obtain the following result.

Proposition 3.1. *Let u be a positive solution of* (3.2)*. Then $0 < u < r$ on $[0, L]$.*

Next, we establish the necessary condition for (3.2) admitting a strictly increasing positive solution.

Lemma 3.2. *For any $L > 0$, let u be a nonnegative, nonconstant and increasing solution of* (3.2)*. Then $rk > 1$ and $b\xi \in \left(h(0), h\left(\frac{rk-1}{2k}\right)\right)$, where $h(0) = r$ and $h\left(\frac{rk-1}{2k}\right) = \frac{(rk+1)^2}{4k}$.*

Proof. Denote the boundary values of $\alpha := u(0)$ and $\beta := u(L)$, and let $E(x) = \frac{u_x^2}{2} + F(u(x))$. Then $E_x(x) \equiv 0$. Since $u_x(0) = u_x(L) = 0$, we have

$$\frac{1}{2}|u_x(x)|^2 + F(u(x)) \equiv F(\alpha) \quad \text{and} \quad F(\alpha) = F(\beta) = B_0.$$

Combining $u_x > 0$ in $(0, L)$ (see Lemma 3.1) and $E(x) \equiv F(\alpha) = F(\beta)$, one observes that

$$F(u(x)) < F(\alpha) \quad \text{for any } x \in (0, L).$$

So, there exists some $z \in (\alpha, \beta) \subset (0, r)$ at which $F(\cdot)$ takes a minimum value. This along with (3.3) and the fact that zeros of $f(u)$ are isolated implies that there exists some small $\epsilon > 0$ such that $f(z) = 0$ and

$$f(s) < 0 \text{ in } (z - \epsilon, z) \ \text{ and } \ f(s) > 0 \text{ in } (z, z + \epsilon).$$

Since $f(\cdot)$ and $g(\cdot)$ have the same sign in $(0, r)$ by (3.3), we have $g(z) = 0$ and

$$g(s) < 0 \text{ in } (z - \epsilon, z) \ \text{ and } \ g(s) > 0 \text{ in } (z, z + \epsilon). \tag{3.4}$$

Obviously, $g(z) = 0$ if and only if $b\xi = h(z)$.

Claim: $rk > 1$.

Indeed, if $rk \leq 1$, then $h(\cdot)$ is strictly decreasing in $(0, r)$, which combined with $g(s) = \frac{h(s)-b\xi}{1+ks}$ indicates that (3.4) cannot hold. Thus, we have $rk > 1$.

It is trivial to show that

$$h_x(x) \begin{cases} > 0, & x \in \left(0, \frac{rk-1}{2k}\right), \\ = 0, & x = \frac{rk-1}{2k}, \\ < 0, & x \in \left(\frac{rk-1}{2k}, r\right) \end{cases} \quad \text{and} \quad \max_{u \in (0,r)} h(u) = h\left(\frac{rk-1}{2k}\right) = \frac{(rk+1)^2}{4k}. \tag{3.5}$$

From (3.4), (3.5), and $g(s) = \frac{h(s)-b\xi}{1+ks}$, it follows that $b\xi = h(z) \in \left(h(0), h\left(\frac{rk-1}{2k}\right)\right)$. ☐

From now on, we assume that $rk > 1$ and $b\xi \in \left(h(0), h\left(\frac{rk-1}{2k}\right)\right)$. Given $b\xi \in \left(h(0), h\left(\frac{rk-1}{2k}\right)\right)$, by (3.5), one has that there exits $0 < z_- < \frac{rk-1}{2k} < z_+ < r$ such that

$$b\xi = h(z_-) = h(z_+) \quad \text{and} \quad f(u) \begin{cases} < 0, & u \in (0, z_-) \cup (z_+, r), \\ > 0, & u \in (z_-, z_+), \\ = 0, & u = 0, z_-, z_+, \end{cases} \tag{3.6}$$

where

$$z_\pm = \frac{kr - 1 \pm \sqrt{(1-kr)^2 - 4k(b\xi - r)}}{2k}. \tag{3.7}$$

Definition 3.1. *Define $\alpha_0 \in [0, z_-)$ as follows.*

- *If $F(0) \le F(z_+)$, then take $\alpha_0 = 0$.*
- *If $F(0) > F(z_+)$, then we define α_0 to be the unique number in $(0, z_-)$ such that $F(\alpha_0) = F(z_+)$.*

Lemma 3.3. *Given $b\xi \in \left(h(0), h\left(\frac{rk-1}{2k}\right)\right)$, for any $\alpha \in (\alpha_0, z_-)$, (3.2) admits a strictly increasing solution for some $L_\alpha > 0$ with $u(0) = \alpha$ and $u(L_\alpha) = \beta$, where $\beta \in (z_-, z_+)$ and $F(\alpha) = F(\beta)$.*

Proof. Based on the definition of α_0, (3.3), and (3.6), we have $F(\alpha) < F(z_+)$, which combined with (3.6) implies that there exits unique $\beta \in (z_-, z_+)$ such that $F(\alpha) = F(\beta)$.

Since the proofs are similar, we only consider the case $F(0) \le F(z_+)$. Then $\alpha_0 = 0$. Denote the unique solution to the initial value problem

$$u_{xx} + f(u) = 0, \quad u(0) = \alpha \in (\alpha_0, z_-), \quad u_x(0) = 0 \tag{3.8}$$

by $u(x;\alpha)$. By (3.6), one has $u_{xx}(0;\alpha) = -f(\alpha) > 0$ and therefore $u(x;\alpha)$ is initially increasing. Let $E(x) = \frac{(u_x(x;\alpha))^2}{2} + F(u(x;\alpha))$. Then we have that

$$E_x(x) \equiv 0 \quad \text{and} \quad E(x) \equiv F(\alpha). \tag{3.9}$$

Claim: there exists some finite $L_\alpha > 0$ such that $u_x(x;\alpha) > 0$ in $(0, L_\alpha)$ and $u_x(L_\alpha;\alpha) = 0$. If not, we assume that

$$u_x(x;\alpha) > 0 \text{ in } (0,\infty). \tag{3.10}$$

This together with (3.9) gives that

$$F(u(x;\alpha)) < F(\alpha) \quad \text{for any } \; x > 0,$$

which together with (3.3), (3.6), and the definition of β yields that

$$u(x;\alpha) < \beta \quad \text{for any } \; x > 0. \tag{3.11}$$

Let $u_\infty = \lim_{x\to\infty} u(x;\alpha)$. Then $\alpha < u_\infty \le \beta$. Moreover, from (3.10) and (3.11), it follows that $\lim_{x\to\infty} u_{xx}(x;\alpha) = 0$, which combined with (3.8) yields that $f(u_\infty) = 0$. Recall that z_- is the only zero of f in $(\alpha, \beta]$ and one obtains that $u_\infty = z_-$. This further implies that $u_{xx}(x;\alpha) > 0$ in $(0,\infty)$, which contradicts (3.11). Therefore, the claim holds. Moreover, by (3.9) and the definition of β, one has that $u(L_\alpha;\alpha) = \beta$. Thus, (3.2) admits a strictly increasing solution $u(x;\alpha)$ with $L = L_\alpha$, $u(0;\alpha) = \alpha$ and $u(L_\alpha;\alpha) = \beta$. These facts complete the proof. □

To obtain more precise information for the existence, we shall study the function L_α, $\alpha \in (\alpha_0, z_-)$. Multiplying (3.8) by $u_x(x;\alpha)$ and integrating the resulting equation over $(0, x)$, we have

$$u_x(x;\alpha) = \sqrt{2(F(\alpha) - F(u(x;\alpha)))}, \;\; x \in (0, L_\alpha).$$

Dividing both sides by $\sqrt{2(F(\alpha) - F(u(x;\alpha)))}$ and integrating the resulting equation over $(0, L_\alpha)$, it follows that

$$L_\alpha = \int_\alpha^\beta \frac{\mathrm{d}u}{\sqrt{2(F(\alpha) - F(u))}}, \tag{3.12}$$

which is a singular integral. Next, inspired by the approaches in [11, 40, 41], we shall apply several change of variables to transform the singular integral into a regular one.

Next, define $p_0 = \sqrt{2(F(\alpha_0) - F(z_-)}$, where α_0 is as in Definition 3.1 and z_- satisfies (3.7). Define the mapping $\gamma : [-p_0, p_0] \to [\alpha_0, z_+]$ by

$$F(\gamma(s)) - F(z_-) = \frac{s^2}{2}, \quad \text{sign } s = \text{sign}(\gamma(s) - z_-) = \text{sign}(f(\gamma(s))), \tag{3.13}$$

Since $\gamma' > 0$ and it is clear that $[\alpha, \beta]$ iscontained in the image of γ, then $s = \gamma^{-1}(u)$ is well defined and is strictly increasing in (α, β) due to the facts that $0 < \alpha < z_- < \beta < r$, (3.6) and the definition of $F(u)$.

Similarly, for each $\alpha \in (\alpha_0, z_-)$, associate $p > 0$ by

$$\frac{p^2}{2} = F(\alpha) - F(z_-) > 0. \tag{3.14}$$

Note that

$$\frac{dp}{d\alpha} < 0 \tag{3.15}$$

and $\alpha \in (\alpha_0, z_-)$ iff $p \in (0, p_0)$. Then, one obtains $L_\alpha = \int_{-p}^{p} \frac{\gamma'(s)\mathrm{d}s}{\sqrt{p^2 - s^2}}$. Let $s = -p\cos t$, $0 \le t \le \pi$, and we have

$$L_\alpha = \int_0^\pi \gamma'(-p\cos t)\mathrm{d}t. \tag{3.16}$$

For later use, we first express $\gamma'(s)$, $\gamma''(s)$ and $\gamma'''(s)$ as functions of $u \in (\alpha, \beta)$, following the calculation similar to that in [40, pp. 4-6]. Differentiating the identity (3.13) with respect to s, one obtains

$$f(u)\gamma'(s) = s.$$

Let

$$\tilde{F}(u) = F(u) - F(z_-).$$

This together with (3.13) yields that $\gamma'(s) = \frac{\sqrt{2\tilde{F}(u)}}{|f(u)|} > 0$, as long as $s \neq 0$ or $u \neq z_-$. For $s = 0$, by the L'Hopital's rule, one arrives at

$$\gamma'(0) = \lim_{u \to z_-} \frac{\sqrt{2\tilde{F}(u)}}{|f(u)|} = \frac{1}{\sqrt{f'(z_-)}},$$

which further implies that

$$\lim_{\alpha \to z_-} L_\alpha = \frac{\pi}{\sqrt{f'(z_-)}} := L_0. \tag{3.17}$$

Differentiating the identity $f(u)\gamma'(s) = s$ with respect to s further gives

$$f'(u)[\gamma'(s)]^2 + f(u)\gamma''(s) = 1$$

and

$$f''(u)[\gamma'(s)]^3 + 3f'(u)\gamma'(s)\gamma''(s) + f(u)\gamma'''(s) = 0.$$

This further suggests that

$$\gamma''(s) = \frac{f^2 - 2f'\tilde{F}}{f^3}(u), \quad \gamma''(0) = -\frac{f''}{3(f')^2}(z_-),$$

and

$$\gamma'''(s) = -\frac{\gamma'(s)}{f^4(u)}H(u), \quad \gamma'''(0) = \frac{[5(f'')^2 - 3f'f'''](z_-)}{12(f'(z_-))^{7/2}}, \tag{3.18}$$

where $f'(z_-) = z_- g'(z_-) > 0$ and

$$H(u) = 2f(u)f''(u)\tilde{F}(u) + 3f'(u)[f^2(u) - 2f'(u)\tilde{F}(u)]. \tag{3.19}$$

The following calculus lemma will be useful later.

Lemma 3.4. *$H(z_-) = 0$ and $H(u) < 0$ for $u \in (0, z_-) \cup (z_-, z_+)$. In particular,*

$$\gamma'''(s) > 0 \quad \textit{for } s \in \mathrm{Dom}\,(\gamma), \quad \textit{and } s \neq 0. \tag{3.20}$$

Proof. Clearly, $H(z_-) = 0$ derived from the facts that $f(z_-) = 0$ and $\tilde{F}(z_-) = 0$. Direct computations show that

$$f'(u) = r - 2u - \frac{b\xi}{(1+ku)^2}, \; f''(u) = -2 + \frac{2kb\xi}{(1+ku)^3}, \; \text{ and } \; f'''(u) = \frac{-6k^2 b\xi}{(1+ku)^4} < 0. \tag{3.21}$$

In particular, observe that $u \mapsto f''(u)$ changes sign exactly once, and that f changes sign exactly three times at $u = 0$, z_-, z_+ (see (3.6)). It follows that f' changes sign exactly twice at some c_- and c_+ such that $0 < c_- < z_- < c_+ < z_+$ and moreover

$$f'' > 0 \text{ in } [0, c_-], \quad f'' < 0 \text{ in } [c_+, z_+], \quad \text{and} \quad f'(u) \begin{cases} < 0, & \text{for } u \in (0, c_-) \cup (c_+, z_+), \\ = 0, & \text{for } u = c_-, c_+, \\ > 0, & \text{for } u \in (c_-, c_+). \end{cases} \tag{3.22}$$

If $u \in (0, c_-]$, by (3.6), and (3.22), one sees that

$$f(u) < 0, \; f''(u) > 0, \; \tilde{F}(u) > 0, \; \text{ and } \; f'(u) \le 0,$$

which suggests that $H < 0$ in $(0, c_-]$.

If $u \in [c_+, z_+)$, from (3.6), and (3.22), it follows that

$$f(u) > 0, \; f''(u) < 0, \; \tilde{F}(u) > 0, \; \text{ and } \; f'(u) \le 0.$$

This further gives that $H < 0$ in $[c_+, z_+)$.

We now consider the case $u \in (c_-, z_-) \cup (z_-, c_+)$. First differentiating (3.19) with respect to u yields

$$H'(u) = 2f(u)f'''(u)\tilde{F}(u) + 5f''(u)[f^2(u) - 2f'(u)\tilde{F}(u)]. \tag{3.23}$$

Multiplying (3.19) and (3.23) by $5f''$ and $3f'$, respectively, subtracting the resulting identities, one gets

$$5f''(u)H(u) - 3f'(u)H'(u) = 2f\tilde{F}G(u), \tag{3.24}$$

where

$$G(u) = 5[f''(u)]^2 - 3f'(u)f'''(u) > 0.$$

Next we make a claim.

Claim: There exists $\delta > 0$ such that $H(u) < 0$ for $u \in (z_- - \delta, z_-) \cup (z_-, z_- + \delta)$. Indeed, by (3.18), one has

$$\gamma'''(0) = \frac{G(z_-)}{12(f'(z_-))^{7/2}} > 0,$$

and so $\gamma'''(s) > 0$ for s close to zero. This together with (3.18) and (3.22) implies that $H(u) < 0$ for all u close to z_- but not equal to z_-. This proves the claim.

We proceed by the argument of contradiction. Assume that there exists some $\chi \in (z_-, c_+)$ such that

$$H < 0 \text{ in } (z_-, \chi) \quad \text{and} \quad H(\chi) = 0.$$

Then, using also (3.22), we have

$$H'(\chi) \ge 0, \; f'(\chi) > 0, \; f(\chi) > 0, \; \tilde{F}(\chi) > 0, \quad \text{and} \quad G(\chi) > 0,$$

which contradicts (3.24). Hence, $H < 0$ in (z_-, c_+). Similarly, if there exists some $\chi_1 \in (c_-, z_-)$ such that

$$H(\chi_1) = 0 \quad \text{and} \quad H < 0 \text{ in } (\chi_1, z_-).$$

Then, by (3.22), we have

$$H'(\chi_1) \le 0, \; f'(\chi_1) > 0, \; f(\chi_1) < 0, \; \tilde{F}(\chi_1) > 0, \quad \text{and} \quad G(\chi_1) > 0,$$

which also contradicts (3.24). Thus, $H < 0$ in (c_-, z_-).

Finally, (3.20) follows by combining the above with $\gamma'(s) > 0$ and (3.18). This completes the proof. □

Lemma 3.5. *Let $\alpha \in (\alpha_0, z_-)$. Then $\frac{dL_\alpha}{d\alpha} < 0$.*

Proof. Since $\frac{dp}{d\alpha} < 0$ (thanks to (3.15)), it is enough to prove the an equivalent inequality $\frac{dL_\alpha}{dp} > 0$. By (3.16), we have

$$\frac{dL_\alpha}{dp} = -\int_0^\pi \cos t\gamma''(-p\cos t)dt \text{ and } \frac{d^2L_\alpha}{dp^2} = \int_0^\pi \cos^2 t\gamma'''(-p\cos t)dt.$$

Using (3.20), we obtain

$$\frac{d^2L_\alpha}{dp^2} > 0.$$

Combining with $\frac{dL_\alpha}{dp}(0) = -\gamma''(0)\int_0^\pi \cos t dt = 0$, implies that $\frac{dL_\alpha}{dp} > 0$ for all $p > 0$. This completes the proof. □

Now we provide the necessary and sufficient condition for (3.2) admitting a strictly increasing positive solution.

Lemma 3.6. *The scalar equation* (3.2) *admits a strictly increasing positive solution if and only if*

$$rk > 1, \;\; b\xi \in \left(r, \frac{(rk+1)^2}{4k}\right) \quad \text{and} \quad L > L_0 = \frac{\pi}{\sqrt{f'(z_-)}}. \tag{3.25}$$

Furthermore, if b, ξ, r, k, *and* L *are given numbers such that* (3.25) *holds, then the scalar equation* (3.2) *has exactly one strictly increasing solution* u, *and* u *must be non-degenerate. Let* $u(0) = \alpha$ *and* $u(L) = \beta$. *Then* $\alpha \in (\alpha_0, z_-)$, $\beta \in (z_-, z_+)$, *and* $\frac{\partial\alpha}{\partial L} < 0$, *and* $\frac{\partial\beta}{\partial L} > 0$.

Proof. Given $b\xi \in \left(r, \frac{(rk+1)^2}{4k}\right)$, the mapping $\alpha \mapsto L_\alpha$ is decreasing by Lemma 3.5, with the domain (α_0, z_-) and the range (L_0, ∞).

We claim that $\lim_{\alpha\to\alpha_0} L_\alpha = \infty$. Actually, if $F(0) > F(z_+)$, then $\alpha_0 > 0$ and $F(\alpha_0) = F(z_+)$, in which case $u(x;\alpha_0)$ is defined for $x \in [0,\infty)$, $u_x(x;\alpha_0) > 0$ for all $x > 0$ and $\lim_{x\to\infty} u(x;\alpha_0) = z_+$. If $F(0) \le F(z_+)$, then $\alpha_0 = 0$ and $u(x;0) \equiv 0$, for which one still has $L_\alpha \to \infty$ as $\alpha \to \alpha_0$ by the continuous dependence on initial conditions.

Hence, $\alpha \mapsto L_\alpha$ is a strictly decreasing homeomorphism with domain (α_0, z_-) and range (L_0, ∞). This combined with Lemma 3.3 implies that the existence and uniqueness results as stated. Furthermore, in view of Lemmas 3.3 and 3.5, we obtain the properties for α and β.

It remains to show that u must be non-degenerate. Differentiating the relation $u_x(L_\alpha;\alpha) = 0$ with respect to α, one obtains

$$u_{xx}(L_\alpha;\alpha)\frac{\mathrm{d}L_\alpha}{\mathrm{d}\alpha} + \omega_x(L_\alpha,\alpha) = 0, \;\; \omega(L_\alpha;\alpha) = \frac{\partial u(x;\alpha)}{\partial\alpha}\Big|_{x=L_\alpha},$$

which gives that $\omega_x(L_\alpha;\alpha) = f(\beta)\frac{\mathrm{d}L_\alpha}{\mathrm{d}\alpha} < 0$. Differentiating (3.8) with respect to α, one obtains

$$\begin{cases} \omega_{xx} + f'(u)\omega = 0, & x \in (0, L_\alpha) \\ \omega(0;\alpha) = 1, \; \omega_x(0;\alpha) = 0, \; \omega_x(L_\alpha;\alpha) < 0. \end{cases} \tag{3.26}$$

Claim: The only solution to the linear problem

$$\begin{cases} \phi_{xx} + f'(u)\phi = 0, & x \in (0, L_\alpha), \\ \phi_x(x) = \phi_x(L_\alpha) = 0 \end{cases} \tag{3.27}$$

is the trivial solution. Indeed, multiplying equations (3.26) and (3.27) by ϕ and ω, respectively, subtracting the resulting equations and integrating it over $(0, L_\alpha)$, we have

$$\omega_x(L_\alpha;\alpha)\phi(L_\alpha) = 0,$$

which further implies that $\phi(L_\alpha) = 0$. By the uniqueness of the solution of ODEs, one has that $\phi \equiv 0$, which shows that the claim holds. Thus, u is non-degenerate, which completes the proof. □

In the following, we denote the quantity $b\xi$ by τ for simplicity. In the following, we will treat the first zero z_- of $f(u)$ on $(0,r)$ as a function of τ. Recalling (3.7), we see that

$$z_- = z_-(\tau) = \frac{kr - 1 - \sqrt{(1-kr)^2 - 4k(\tau - r)}}{2k} \tag{3.28}$$

is a well defined function for $\tau \in \left(r, \frac{(rk+1)^2}{4k}\right)$. Also, $z_-(\tau)$ is a strictly increasing function, with

$$\lim_{\tau\to r} z_-(\tau) = r \quad \text{and} \quad \lim_{\tau\to\frac{(rk+1)^2}{4k}} z_-(\tau) = \frac{rk-1}{2k}.$$

With the one-to-one correspondence between τ and z_-, we can define $\frac{d}{dz_-} = \frac{1}{\frac{dz_-}{d\tau}}\frac{d}{d\tau}$.

Next, we try to understand the existence of solutions to (3.2) when the interval length L is fixed. Given $\tau \in \left(r, \frac{(rk+1)^2}{4k}\right)$, we have $f'(z_-) = r - 2z_- + \frac{z_- - r}{1+kz_-}$, where we have used $f(z_-) = 0$. Then, one has

$$\frac{\partial f'(z_-)}{\partial z_-} = -2 + \frac{1+kr}{(1+kz_-)^2}. \tag{3.29}$$

Let

$$z^* = \frac{\sqrt{2(1+kr)} - 2}{2k} \tag{3.30}$$

which is characterized by $\left.\frac{\partial f'(z_-)}{\partial z_-}\right|_{z_-=z^*} = 0$. It is trivial to show that

$$\max_{z_-\in\left(0,\frac{rk-1}{2k}\right)} f'(z_-) = f'(z_-)|_{z_-=z^*} = r - 2z^* + \frac{z^*-r}{1+kz^*} = \frac{rk+3-2\sqrt{2(1+kr)}}{k}. \tag{3.31}$$

Let $L_* = \frac{\pi}{\sqrt{f'(z_-)|_{z_-=z^*}}} = \frac{\sqrt{k}\pi}{\sqrt{rk+3-2\sqrt{2(1+kr)}}}$. We have the following result.

Lemma 3.7. *The following results on* (3.2) *hold.*

(i) *If $L \le L_*$, then* (3.2) *does not admit strictly increasing solution for any $\tau > 0$.*

(ii) *If $L > L_*$ and $rk > 1$, then there exist two numbers $\tau_- < \tau_+$ such that $(\tau_-,\tau_+) \subset \left(r, \frac{(rk+1)^2}{4k}\right)$, and* (3.2) *admits a strictly increasing solution if and only if $\tau \in (\tau_-,\tau_+)$, Furthermore, for each $\tau \in (\tau_-,\tau_+)$,* (3.2) *admits exactly one strictly increasing solution u, and u is non-degenerate. Here $\tau_{L^\pm}$ are defined in* (3.32) *and* (3.33).

Proof. For assertion (i). Suppose (3.2) has a strictly increasing solution for some $\tau > 0$, then

$$L > \pi[f'(z_-)]^{-1/2} = \pi\left(r - 2z_- + \frac{z_- - r}{1+kz_-}\right)^{-1/2} \quad \text{for some } z_-.$$

It then follows from (3.31) and the definition of L_* that $L > L_*$.

Next, we prove assertion (ii). By (3.29) and (3.31), one concludes that $f'(z_-)$ is increasing in $(0, z^*)$ and it is decreasing in $\left(z^*, \frac{rk-1}{2k}\right)$, and in particular that $f'(z_-)$ has a unique maximum value at $z_- = z^*$. Given $L > L_*$, there exists two numbers $z^- < z^+$, such that $z^* \in (z^-, z^+) \subset \left(0, \frac{rk-1}{2k}\right)$, and $f'(z_-)|_{z_-=z^-} = f'(z_-)|_{z_-=z^+} = \frac{\pi^2}{L^2}$. One can verify that

$$z^\pm = \left(rk - 1 - k\pi^2/L^2 \pm \sqrt{(k\pi^2/L^2 - rk + 1)^2 - 8k\pi^2/L^2}\right)/4k. \tag{3.32}$$

Let

$$\tau_- = h(z^-) = (r - z^-)(1+kz^-) \text{ and } \tau_+ = h(z^+) = (r - z^+)(1+kz^+). \tag{3.33}$$

So, for $\tau \in (\tau_-,\tau_+)$, it holds that $f'(z_-) = f'(z_-(\tau)) > \frac{\pi^2}{L^2}$, i.e., $L > L_0$. By Lemma 3.6, there must be a unique, increasing, and non-degenerate solution to (3.2). The proof is completed. □

Clearly, as $L \to L_*$, the set (τ_-, τ_+) shrinks to an empty set; whereas as $L \to \infty$, (τ_-, τ_+) expands to the interval $\left(r, \frac{(rk+1)^2}{4k}\right)$.

Next, we consider the existence of positive solutions to system (1.7). Given $L > L_*$, Lemma 3.7 says that (3.2) admits exactly one strictly increasing solution u_τ for $\tau \in (\tau_-, \tau_+)$, and admits no strictly increasing solutions for $\tau \notin (\tau_-, \tau_+)$. Define

$$\zeta(\tau) = \int_0^L \frac{1}{1+ku_\tau}\left(r - cu_\tau - \frac{\tau}{b(1+ku_\tau)}\right)\mathrm{d}x, \quad \tau \in (\tau_-, \tau_+).$$

Let

$$c_\tau = \frac{\int_0^L \frac{r}{1+ku_\tau}\mathrm{d}x}{\int_0^L \frac{u_\tau}{1+ku_\tau}\mathrm{d}x} \quad \text{and} \quad b_{\tau,c} = \begin{cases} \frac{\int_0^L \frac{\tau}{(1+ku_\tau)^2}\mathrm{d}x}{\int_0^L \frac{r-cu_\tau}{1+ku_\tau}\mathrm{d}x} & \text{for } c \in (0, c_\tau), \\ +\infty & \text{for } c \in [c_\tau, \infty). \end{cases} \tag{3.34}$$

One sees that $c_\tau > 1$ due to the fact that $u_\tau < r$.

Lemma 3.8. *The following results on (1.7) hold.*

(i) *If $L \le L_*$ or $rk \le 1$, then (1.7) does not admit any increasing solution.*

(ii) *If $L > L_*$ and $rk > 1$, the following statements hold.*

(ii.1) *Fix all the parameters except ξ $\left(\xi = \frac{\tau}{b}\right)$. If $0 \in \left(\min_{\tau\in(\tau_-,\tau_+)} \zeta(\tau), \max_{\tau\in(\tau_-,\tau_+)} \zeta(\tau)\right)$, then (1.7) admits a strictly increasing positive solution. If $0 \notin \left(\min_{\tau\in(\tau_-,\tau_+)} \zeta(\tau), \max_{\tau\in(\tau_-,\tau_+)} \zeta(\tau)\right)$, then (1.7) does not have any strictly increasing positive solution.*

(ii.2) *Given $\tau \in (\tau_-, \tau_+)$, if $c \ge c_\tau$, then (1.7) does not have any strictly increasing positive solution. If $c < c_\tau$, (1.7) admits a strictly increasing positive solution if and only if $b = b_{\tau,c}$ and $\xi = \frac{\tau}{b_{\tau,c}}$. Specially, we have*

$$c_\tau \to \frac{r}{z^\pm} \text{ and } b_{\tau,c} \to \frac{r-z^\pm}{r-cz^\pm} \text{ as } \tau \to \tau_\pm. \tag{3.35}$$

Here $z^\pm$, $\tau_\pm$, c_τ, and $b_{\tau,c}$ are defined in (3.32), (3.33), and (3.34).

Proof. Assertion (i) follows directly from assertion (i) of Lemma 3.7.

By the non-degeneracy of u_τ, one sees that $\zeta(\tau)$ is a smooth function of $\tau \in (\tau_-, \tau_+)$. From statement (ii) of Lemma 3.7, it follows that statement (ii.1) and the first part of statement (ii.2) hold.

Finally, (3.35) follows from the above analysis. □

Fix $rk > 1$ and $L > L_*$. Statement (ii.2) of Lemma 3.8 indicates that (1.7) has a strictly increasing positive solution if and only if

$$(\tau, c, b) \in \Gamma_{r,k,L} := \{(\tau, c, b) | \tau \in (\tau_-, \tau_+), c \in (0, c_\tau), b = b_{\tau,c}\}.$$

Next, we study the shape of $\Gamma_{r,k,L}$.

Lemma 3.9. *Fixing $rk > 1$ and $L > L_*$, for any $\tau \in (\tau_-, \tau_+)$, $b_{\tau,c}$ is strictly increasing in $c \in (0, c_\tau)$ and $c_\tau > 1$. Moreover, we have*

$$\lim_{c\to 0} b_{\tau,c} = \frac{\int_0^L \frac{\tau}{(1+ku_\tau)^2}\mathrm{d}x}{\int_0^L \frac{r}{1+ku_\tau}\mathrm{d}x} > \frac{1}{1+kr} \quad \text{and} \quad \lim_{c\to c_\tau} b_{\tau,c} = \infty. \tag{3.36}$$

Proof. The proposition follows directly from $\tau_- > r$, $u_\tau < r$, and (3.34). □

Remark 3.1. *Given $\tau \in (\tau_-, \tau_+)$, let*

$$\Gamma_{\tau,r,k,L} = \{(c, b) | c \in (0, c_\tau),\ b = b_{\tau,c}\}.$$

Lemma 3.9 shows that $\Gamma_{\tau,r,k,L}$ is an increasing curve. Moreover, if there exist distinct $\tau_1, \tau_2, \cdots, \tau_n \in (\tau_-, \tau_+)$ such that $\bigcap_{i=1}^n \Gamma_{\tau_i,r,k,L} \neq \emptyset$, then for any $(b, c) \in \bigcap_{i=1}^n \Gamma_{\tau_i,r,k,L}$, system (1.7) admits at least n increasing positive solutions.

Proposition 3.2. *Given r, k, L satisfying $rk > 1$ and $L > L_*$, we have the following results.*

(i) *There exists $b_* \in \left(\frac{1}{1+rk}, 1\right)$ such that the following results hold.*

(i.1) *If $b < b_*$, then (1.7) does not admit any strictly increasing positive solution for any c and τ.*

(i.2) *If $b > b_*$, then there is a single bounded interval I_b such that the system (1.7) has a strictly increasing positive solution (for one or more values of τ) if and only if $c \in I_b$. Moreover, if $b \in (b_*, 1)$, then for any $c \in I_b$, we have $c < 1$.*

(ii) *There exists $c^* \in (1, \infty)$ such that the following results hold.*

(ii.1) *If $c > c^*$, then (1.7) does not admit any strictly increasing positive solution for any b and τ.*

(ii.2) *If $c < c^*$, then there is a single interval I_c such that the system (1.7) has a strictly increasing positive solution (for one or more values of τ) if and only if $b \in I_c$. Moreover, if $c \in [1, c^*)$, then for any $b \in I_c$, we have $b > 1$.*

Proof. For statement (i), define

$$b_* = \inf_{\tau \in (\tau_-, \tau_+)} b_{\tau,0}.$$

From (3.35), (3.36), and $u_\tau < r$, it follows that $b_* \in (\frac{1}{1+rk}, 1)$. If $b < b_*$, from statement (ii.2) of Lemma 3.8, it follows that (1.7) does not admit any strictly increasing positive solution for any c and τ. Given $b > b_*$, based on the definition of b_*, there exists $\tau_0 \in (\tau_-, \tau_+)$ such that

$$b_{\tau_0,0} < b,$$

which combined with statement (i) and Lemma 3.8 yields that (1.7) admits a strictly increasing positive solution with $\tau = \tau_0$ and appropriate $c \in (0, c_{\tau_0})$.

Next, fix $b > b_*$ and define I_b to be the set of c such that the shadow system has a solution, i.e.

$$I_b = \{c > 0 : \ (\tau, c, b) \in \Gamma_{r,k,L} \ \text{ for some } \tau\}.$$

The boundedness of I_b follows from statement (ii.2) of Lemma 3.8 and (3.34). Furthermore, if $b \in (b_*, 1)$, then for any $c \in I_b$, by Proposition 2.1, we have $c < 1$.

It remains to show the connectedness of I_b. We first claim: If (1.7) admits a strictly increasing positive solution with some (τ_1, c_1) and (τ_2, c_2) (without loss of generality, assume $c_1 < c_2$), then for any $c \in [c_1, c_2]$, (1.7) admits a strictly increasing positive solution with some appropriate τ.

If not, assume that there exists $c_0 \in (c_1, c_2)$ such that $b \neq b_{\tau,c_0}$ for any $\tau \in (\tau_-, \tau_+)$. Define

$$A_{b,c_0} = \{\tau \in (\tau_-, \tau_+) : \ b_{\tau,c_0} < b\} \quad \text{and} \quad B_{b,c_0} = \{\tau \in (\tau_-, \tau_+) : \ b_{\tau,c_0} > b\},$$

where $b_{\tau,c}$ is given in (3.34). Then, we have

$$A_{b,c_0} \text{ and } B_{b,c_0} \text{ are open subsets of } (\tau_-, \tau_+) \text{ and } A_{b,c_0} \cup B_{b,c_0} = (\tau_-, \tau_+). \tag{3.37}$$

Combining the facts that $b = b_{\tau_1,c_1} = b_{\tau_2,c_2}$, $c_1 < c < c_2$, and Lemma 3.9, one obtains from the monotone increasing property of $c \mapsto b_{\tau,c}$ that

$$b_{\tau_1,c_0} > b \quad \text{and} \quad b_{\tau_2,c_0} < b,$$

which means that both A_{b,c_0} and B_{b,c_0} are nonempty. This contradicts the connectedness of (τ_-, τ_+). So, the claim holds, which suggests that there is a *single* interval I_b such that the system (1.7) has a strictly increasing positive solution (for one or more values of τ) if and only if $c \in I_b$. This proves assertion (i).

Finally, similar to the arguments proving the statement (i), letting

$$c^* = \sup_{\tau \in (\tau_-, \tau_+)} c_\tau,$$

one can show that the statement (ii) holds. □

Proposition 3.2 gives the existence/non-existence of the increasing positive solution of (1.7). However, given $b \in (b_*, \infty)$ (resp. $c \in (0, c^*)$), the size of I_b (resp. I_c) can not be explicitly characterized. Below we give some more decisive information for I_b and I_c.

Proposition 3.3. *Given $L > L_*$ and $rk > 1$, let $z^{\pm}$ be defined in (3.32) which indicates $z^- < z^+$ and $\frac{z^+}{r-z^+} > \frac{z^-}{r-z^-}$. Then there exists some $\epsilon = \epsilon(k, r, L) > 0$ such that the following results hold.*

(i) *If $b \in (1, 1+\epsilon)$ and $c \in (1-\epsilon, 1)$, then (1.7) at least admits two increasing positive solutions.*

(ii) *Assume $b \in (1-\epsilon, 1)$ and $c \in (1-\epsilon, 1)$. If $\frac{1/b-1}{1-c} < \frac{z^-}{r-z^-}$, then (1.7) at least admits two increasing positive solutions, which are non-degenerate; if $\frac{1/b-1}{1-c} \in \left(\frac{z^-}{r-z^-}, \frac{z^+}{r-z^+}\right)$, then (1.7) admits at least one increasing positive solution.*

(iii) *Assume $b \in (1, 1+\epsilon)$ and $c \in (1, 1+\epsilon)$. If $\frac{1-1/b}{c-1} > \frac{z^+}{r-z^+}$, then (1.7) at least admits two increasing positive solutions, which are non-degenerate; if $\frac{1/b-1}{1-c} \in \left(\frac{z^-}{r-z^-}, \frac{z^+}{r-z^+}\right)$, then (1.7) admits at least one increasing positive solution.*

Proof. Given $L > L_*$, by Lemma 3.7, there exist two numbers $\tau_- < \tau_+$ such that $(\tau_-, \tau_+) \subset (h(0), h(u_h))$, and (3.2) admits increasing solution if and only if $\tau \in (\tau_-, \tau_+)$. Given $\tau \in (\tau_-, \tau_+)$, then (3.2) admits an increasing solution denoted by u_τ.

Claim: $\int_0^L \frac{1}{1+ku_\tau}\left(r - u_\tau - \frac{\tau}{1+ku_\tau}\right) \mathrm{d}x < 0$. Indeed, recall that u_τ satisfies

$$\begin{cases} u_{\tau_{xx}} + u_\tau \left(r - u_\tau - \frac{\tau}{1+ku_\tau}\right) = 0, & \text{in } (0, L), \\ u_{\tau_x}(0) = u_{\tau_x}(L) = 0. \end{cases} \tag{3.38}$$

Dividing the first equation of (3.38) by u_τ and integrating it over $(0, L)$, one finds that

$$\int_0^L \left(r - u_\tau - \frac{\tau}{1+ku_\tau}\right) \mathrm{d}x = -\int_0^L \frac{(u_{\tau_x})^2}{u_\tau^2} \mathrm{d}x < 0, \tag{3.39}$$

due to the fact that u_τ is a strictly increasing solution of (3.38).

Recall that $f(y)$ satisfies

$$f(y) \begin{cases} < 0, & y \in (u_\tau(0), z_-(\tau)), \\ = 0, & y = z_-(\tau), \\ > 0, & y \in (z_-(\tau), u_\tau(L)), \end{cases}$$

which combined with the fact that u_τ is a strictly increasing solution on $(0, L)$ yields a unique $x_0 \in (0, L)$ such that $u_\tau(x_0) = z_-(\tau)$ and

$$f(u_\tau(x)) \begin{cases} < 0, & x \in (0, x_0), \\ = 0, & x = x_0, \\ > 0, & x \in (x_0, L). \end{cases} \tag{3.40}$$

From (3.39) and (3.40), it follows that

$$\begin{aligned} &\int_0^L \frac{1}{1+ku_\tau}\left(r - u - \frac{\tau}{1+ku_\tau}\right) \mathrm{d}x \\ &= \int_0^{x_0} \frac{1}{1+ku_\tau}\left(r - u - \frac{\tau}{1+ku_\tau}\right) \mathrm{d}x + \int_{x_0}^L \frac{1}{1+ku_\tau}\left(r - u - \frac{\tau}{1+ku_\tau}\right) \mathrm{d}x \\ &< \int_0^{x_0} \frac{1}{1+kz(\tau)}\left(r - u - \frac{\tau}{1+ku_\tau}\right) \mathrm{d}x + \int_{x_0}^L \frac{1}{1+kz(\tau)}\left(r - u - \frac{\tau}{1+ku_\tau}\right) \mathrm{d}x \\ &= \frac{1}{1+kz(\tau)} \int_0^L \left(r - u - \frac{\tau}{1+ku_\tau}\right) \mathrm{d}x < 0. \end{aligned}$$

Therefore, the claim holds. Recall that $h(z^*) \in (\tau_-, \tau_+)$ ($z^* = \frac{\sqrt{2(1+kr)}-2}{2k}$ is defined in (3.30)), then it is trivial to show that there exists $\epsilon = \epsilon(r, k, L) > 0$ such that

$$\zeta(h(z^*)) < 0 \text{ for } b \in (1-\epsilon, 1+\epsilon) \text{ and } c \in (1-\epsilon, 1+\epsilon). \tag{3.41}$$

On the other hand, as $\tau \to \tau_-$ (resp. τ_+), then $z_-(\tau) \to z^-$ (resp. z^+) and $u_\tau \to z^-$ (resp. z^+) in $(0, L)$. By Lemma 3.7, one sees that

$$0 < z^- < z^+ < u_h = \frac{rk-1}{2k} < \frac{r}{2}.$$

For statement (i), that is $b > 1 > c$, then we have

$$\lim_{\tau\to\tau_-} \zeta(\tau) = \frac{L}{1+kz^-}\left(r - cz^- - \frac{r-z^-}{b}\right) > \frac{L}{1+kz^-}\left(r - z^- - (r-z^-)\right) = 0$$

and

$$\lim_{\tau\to\tau_+} \zeta(\tau) = \frac{L}{1+kz^+}\left(r - cz^+ - \frac{r-z^+}{b}\right) > \frac{L}{1+kz^+}\left(r - z^+ - (r-z^+)\right) = 0.$$

Since $\zeta(\tau)$ is a continuous function of $\tau \in (\tau_-, \tau_+)$, one can conclude that there exists two numbers $\tau_1 < \tau_2$ such that $\tau_- < \tau_1 < \tau_2 < \tau_+$ and $\zeta(\tau_1) = \zeta(\tau_2)$=0. So, by Lemma 3.8, we have that (1.7) at least admits two increasing positive solutions.

For statement (ii), if $\frac{1/b-1}{1-c} < \frac{z^-}{r-z^-}$, one can show that

$$\begin{aligned}\lim_{\tau\to\tau_-} \zeta(\tau) &= \frac{L}{1+kz^-}\left(r - cz^- - \frac{r-z^-}{b}\right)\\ &> \frac{L}{1+kz^-}\left(r - cz^- + (c-1)z^- + z^- - r\right) = 0.\end{aligned}$$

If $\frac{1/b-1}{1-c} < \frac{z^+}{r-z^+}$, then

$$\begin{aligned}\lim_{\tau\to\tau_+} \zeta(\tau) &= \frac{L}{1+kz^+}\left(r - cz^+ - \frac{r-z^+}{b}\right)\\ &> \frac{L}{1+kz^+}\left(r - cz^+ + (c-1)z^+ + z^+ - r\right) = 0.\end{aligned}$$

Similar to the arguments as that in proving statement (i), one can prove statement (ii).

Finally, following the approach same as that in proving statements (i) and (ii), one can prove the statement. This completes the proof. □

Remark 3.2. *We have some comments related to Proposition 3.3.*

- *For statements (ii) and (iii), we will show that all the case may occur. For example, let*

$$b = 1 - \varrho_1\epsilon \quad \textit{and} \quad c = 1 - \varrho_2\epsilon,$$

where $\varrho_1,\ \varrho_2 \in (0,1)$. *One can choose appropriate* ϱ_1 *and* ϱ_2 *such that* $\frac{1/b-1}{1-c} < \frac{z^-}{r-z^-}$ *or* $\frac{1/b-1}{1-c} \in \left(\frac{z^-}{r-z^-}, \frac{z^+}{r-z^+}\right)$ *holds.*

- *Given* $b \in (1, 1+\epsilon)$, *statement (i) of Proposition 3.3 and statement (i) of Proposition 3.2 yield that* $(1-\epsilon, 1) \subset \overline{I_b}$. *Moreover, statement (iii) of Proposition 3.3 and statement (i) of Proposition 3.2 suggest that* $\left(1, \min\left\{1 + \frac{(b-1)(r-z^+)}{bz^+}, 1+\epsilon\right\}\right) \subset \overline{I_b}$. *These facts combined with statement (ii) of Proposition 3.2 further imply that*

$$\left(1-\epsilon, \min\left\{1 + \frac{(b-1)(r-z^+)}{bz^+}, 1+\epsilon\right\}\right) \subset \overline{I_b}$$

- *Symmetrically, given* $c \in (1-\epsilon, 1)$, *statement (i) of Proposition 3.3 and statement (ii) of Proposition 3.2 yield that* $(1, 1+\epsilon) \subset \overline{I_c}$. *Moreover, statement (ii) of Proposition 3.3 and statement (ii) of Proposition 3.2 suggest that* $\left(\max\left\{\frac{r-z^-}{r-cz^-}, 1-\epsilon\right\}, 1\right) \subset \overline{I_c}$. *These facts combined with statement (ii) of Proposition 3.2 further imply that*

$$\left(\max\left\{\frac{r-z^-}{r-cz^-}, 1-\epsilon\right\}, 1+\epsilon\right) \subset \overline{I_c}.$$

Now we are in a position to prove Theorem 1.3 and Theorem 1.4. Before embarking on this, we prove Proposition 1.1.

Proof of Proposition 1.1. Define $\mathcal{L}: H_0^2(0,L)\times\mathbb{R}\times\bar{H}_0^2(0,L)\times[0,+\infty)\to L^2(0,L)\times\mathbb{R}\times\bar{L}^2(0,L)$ by

$$\mathcal{L}(u,\xi,\zeta,\nu)=\begin{pmatrix} u_{xx}+u\left[r-u-\frac{b(\xi+\zeta)}{1+ku}\right] \\ \int_0^L \frac{\xi+\zeta}{1+ku}\left(r-cu-\frac{\xi+\zeta}{1+ku}\right)\mathrm{d}x \\ \zeta_{xx}+\nu\left[\frac{\xi+\zeta}{1+ku}\left(r-cu-\frac{\xi+\zeta}{1+ku}\right)-\frac{1}{L}\int_0^L \frac{\xi+\zeta}{1+ku}\left(r-cu-\frac{\xi+\zeta}{1+ku}\right)\mathrm{d}x\right]\end{pmatrix},$$

where $H_0^2(0,L)=\{u\in H^2(0,L)|u_x(0)=u_x(L)=0\}$, $\bar{H}_0^2(0,L)=\{u\in H_0^2(0,L)|\int_0^L u\mathrm{d}x=0\}$, and $\bar{L}^2(0,L)=\{u\in L^2(0,L)|\int_0^L u\mathrm{d}x=0\}$. Then, we have

$$\begin{aligned} &D_{(u,\xi,\zeta)}\mathcal{L}|_{(u,\xi,\zeta,\nu)=(u^*,\xi^*,0,0)}(\phi,\psi,\eta) \\ &=\begin{pmatrix} \phi_{xx}+\phi\left[r-2u^*-\frac{b\xi^*}{(1+ku^*)^2}\right]-\frac{bu^*(\psi+\eta)}{1+ku^*} \\ \int_0^L\left[\frac{\xi^*\phi(3\xi^*k-(kr+c)(1+ku^*))}{(1+ku^*)^3}+\frac{\eta}{1+ku^*}\left(r-cu^*-\frac{\xi^*}{1+ku^*}\right)-\frac{\xi^*(\psi+\eta)}{(1+ku^*)^2}\right]\mathrm{d}x \\ \eta_{xx}\end{pmatrix}.\end{aligned}$$

Next we claim: $D_{(u,\xi,\zeta)}\mathcal{L}|_{(u,\xi,\zeta,\nu)=(u^*,\xi^*,0,0)}$ *is non-degenerate.* To prove this claim, it suffices to show that the following problem

$$\begin{cases} \phi_{xx}+\phi\left[r-2u^*-\frac{b\xi^*}{(1+ku^*)^2}\right]-\frac{bu^*(\psi+\eta)}{1+ku^*}=0, & \text{in } (0,L), \\ \int_0^L\left[\frac{\xi^*\phi(3\xi^*k-(kr+c)(1+ku^*))}{(1+ku^*)^3}+\frac{\eta}{1+ku^*}\left(r-cu^*-\frac{\xi^*}{1+ku^*}\right)-\frac{\xi^*(\psi+\eta)}{(1+ku^*)^2}\right]\mathrm{d}x=0 & \\ \eta_{xx}=0, & \text{in } (0,L), \end{cases} \tag{3.42}$$

only admits the trivial solution in $H_0^2(0,L)\times\mathbb{R}\times\bar{H}_0^2(0,L)$. The third equation of (3.42) and the definition of $\bar{H}_0^2(\Omega)$ suggest that $\eta\equiv 0$. Hence, we have

$$\begin{cases} \phi_{xx}+\phi\Big[r-2u^*-\frac{b\xi^*}{(1+ku^*)^2}\Big)-\frac{bu^*\psi}{1+ku^*}=0, & \text{in } (0,L), \\ \int_0^L\left[\frac{\xi^*\phi(3\xi^*k-(kr+c)(1+ku^*))}{(1+ku^*)^3}-\frac{\xi^*\psi}{(1+ku^*)^2}\right]\mathrm{d}x=0. & \end{cases} \tag{3.43}$$

From Lemma 3.1, it follows that there exists some $m\in\mathbb{N}$ such that u^* satisfies (3.1).

If $m=1$, combining Lemmas 3.1 and 3.6, one has that operator T is invertible. By the first equation of (3.43), one obtains

$$\phi=\psi T^{-1}\left(\frac{bu^*}{1+ku^*}\right).$$

Hence, (1.9) suggests that (3.43) only admits the trivial solution $(0,0)$.

If $m\geq 2$, for any $\psi\in\mathbb{R}$, consider the following truncated problem

$$\begin{cases} \phi^*_{xx}+\phi^*\left[r-2u^*-\frac{b\xi^*}{(1+ku^*)^2}\right]=\frac{bu^*\psi}{1+ku^*}, & \text{in } (0,L/m), \\ \phi^*_x(0)=\phi^*_x(L/m)=0. & \end{cases}$$

By Lemma 3.6, we have $\phi^*=\psi T^{-1}_{(0,L/m)}\left(\frac{bu^*}{1+ku^*}\right)$, where

$$T_{(0,L/m)}(\phi)=\phi_{xx}+\phi\Big(r-2u^*-\frac{b\xi^*}{(1+ku^*)^2}\Big),$$

for $\phi\in C^2(0,L/m)\cap C^1([0,L/m])$ satisfying $\phi_x(0)=\phi_x(L/m)=0$. Then by the symmetry, one has

$$\phi(x)=\begin{cases} \phi^*(x-jL/m) & \text{when } x\in[jL/m,(j+1)L/m],\ j \text{ is even}, \\ \phi^*((j+1)L/m-x) & \text{when } x\in[jL/m,(j+1)L/m],\ j \text{ is odd}. \end{cases}$$

Thus, $T^{-1}\left(\frac{bu^*}{1+ku^*}\right)$ is well-defined. Moreover, (1.9) also yields that (3.43) only admits the trivial solution $(0,0)$.

Finally, based on the implicit function theorem, there exists small $\delta^* > 0$ such that for any $\nu \in (0,\delta^*)$, there exists $(u_\nu, \xi_\nu, \zeta_\nu)$ near $(u^*, \xi^*, 0)$ such that $L(u_\nu, \xi_\nu, \zeta_\nu, \nu) = 0$, which implies that for any $\nu \in (0,\delta^*)$, the steady-state problem (1.8) admits a non-constant positive solution $(u_\mu, \xi_\nu + \zeta_\nu)$. This completes the proof.

□

Proof of Theorem 1.3 and Theorem 1.4. Combining the results in Proposition 3.2 with Proposition 1.1, one obtains Theorem 1.3 directly. Theorem 1.4 is a consequence of Proposition 3.3 alongside Proposition 1.1.

4. Weak competition $0 < b, c < 1$

Though we have proved the existence of non-constant positive solutions to system (1.8) in Theorem 1.3 by studying the shadow system (1.7), the admissible parameter regime is somewhat narrow (see also Theorem 1.4, where μ is particulary required to be large. In this section, we shall employ the global bifurcation theory to show that system (1.8) may admit non-constant positive solutions for any $0 < b, c < 1$ and $\mu > 0$ which largely expands the admissible parameter regimes given in Theorem 1.3 (see also Theorem 1.4) for the case of weak competition.

A nonlinear problem can be formulated as an abstract equation $F(\rho, u) = 0$, where $F : \mathbb{R} \times X :\to Y$ is a nonlinear differentiable mapping, and X, Y are Banach spaces. We introduce a celebrated global bifurcation Theorem [42, Theorems 4.3]. For more results about the bifurcation theory, we refrer to references [3, 4, 6, 24, 37]. Recall that a Fredholm operator is a bounded linear mapping F from a Banach space B_1 to another Banach space B_2 such that the mapping has a finite-dimensional null space $\text{Ker}(F)$, has a closed range $\text{Ran}(F)$ with a finite co-dimension. We say the index of F is zero if the dimension of $\text{Ker}(F)$ is equal to the co-dimension of $\text{Ran}(F)$.

Theorem 4.1. *[42, Theorems 4.3] Let V be an open connected subset of $\mathbb{R} \times X$ and $(\rho_0, u_0) \in V$, and let F be a continuously differentiable mapping from V into Y. Suppose that*

(i) *$F(\rho, u_0) = 0$ for $(\rho, u_0) \in V$;*
(ii) *The partial derivative $D_{\rho u}F(\rho, u)$ exists and is continuous in (ρ, u) near (ρ_0, u_0);*
(iii) *$D_u F(\rho_0, u_0)$ is a Fredholm operator and $\text{dimKer}(F_u(\rho_0, u_0)) = \text{codimRan}(F_u(\rho_0, u_0)) = 1$;*
(iv) *$D_{\rho,u}F(\rho_0, u_0)\phi_0 \notin \text{Ran}(F_u(\rho_0, u_0))$ where $\phi_0 \in X =$ span $\{\text{Ker}(F_u(\rho_0, u_0))\}$.*

Let Z be any complement of $\text{span}\{\phi_0\}$ in X. Then there exist an open interval $I_1 = (-\epsilon, \epsilon)$ and continuous functions $\rho : I_1 \to \mathbb{R}$, $\psi : I_1 \to Z$, such that $\rho(0) = \rho_0$, $\psi(0) = 0$, and, if $u(s) = u_0 + s\phi_0 + s\psi(s)$ for $s \in I_1$, then $F(\rho(s), u(s)) = 0$. Moreover, $F^{-1}(\{0\})$ near (ρ_0, u_0) consists precisely of $u = u_0$ and the curves $\Gamma = \{(\rho(s), u(s)) : s \in I_1\}$. If in addition, $D_u F(\rho, u)$ is a Fredholm operator for all $(\rho, u) \in V$, then the curve Γ is contained in $\mathcal{C}$, which is a connected component of closure of S where $S = \{(\rho, u) \in V : F(\rho, u) = 0, u \neq u_0\}$; and either $\mathcal{C}$ is not compact in V, or $\mathcal{C}$ contains a point (ρ^*, u_0) with $\rho^* \neq \rho_0$.

4.1. **Applying abstract bifurcation theory to** (1.8)**.**

In this subsection, we will apply abstract bifurcation Theorem 4.1 to obtain the existence of non-constant positive solutions of (1.8), where $d(u) = 1 + ku$. We shall fix all the parameters except μ and treat μ as a bifurcation parameter. The positive solutions will be the ones bifurcating from the constant steady states (u^+, v^+), where

$$u^+ = \frac{(1-b)r}{1-bc} > 0 \ \text{ and } \ v^+ = \frac{(1-c)r}{1-bc} > 0,$$

due to $0 < b, c < 1$.

We recall a well-known result. The eigenvalue problem

$$\begin{cases} -\phi_{xx} = \lambda\phi, & x \in (0, L), \\ \phi_x(0) = \phi_x(L) = 0, \end{cases}$$

has a sequence of simple eigenvalues $\lambda_0 < \lambda_1 < \cdots < \lambda_n \cdots$, where

$$\lambda_i = \frac{\pi^2 i^2}{L^2}, \quad i = 0, 1, 2, \cdots,$$

with normalized eigenfunctions given by

$$\phi_i(x) = \begin{cases} \frac{1}{\sqrt{L}}, & i = 0, \\ \sqrt{\frac{2}{L}} \cos(\pi i x/L), & i > 0. \end{cases}$$

The set of eigenfunctions forms an orthonormal basis in $L^2(0, L)$.

Let $Y = L^2(0, L) \times L^2(0, L)$ be the Hilbert space with the inner product

$$(U_1, U_2)_Y = (u_1, u_2)_{L^2(0,L)} + (v_1, v_2)_{L^2(0,L)}$$

for $U_1 = (u_1, v_1), U_2 = (u_2, v_2)$, and $X = \{(u, v) |\ u, v \in H^2_N(0, L)\}$. Here

$$H^2_N(0, L) = \{u \in H^2(0, L) |\ u_x(0) = u_x(L) = 0\}.$$

We regard X as a Banach space with usual H^2 norm. Define the map $F : (0, \infty) \times X \to Y$ by

$$F(\mu, u, v) = \begin{pmatrix} u_{xx} + u(r - u - bv) \\ \mu((1 + ku)v)_{xx} + v(r - v - cu) \end{pmatrix}.$$

Then the solutions of the boundary value problem (1.8) are exactly zeros of this map. For any $\mu > 0$, we have that

$$F(\mu, u^+, v^+) = 0.$$

For any fixed $(u, v) \in X$, the Frechet derivative is given by

$$D_{(u,v)}F(\mu, u, v)(\phi, \psi) = \begin{pmatrix} \phi_{xx} + \phi(r - 2u - bv) - bu\psi \\ \mu k(v\phi)_{xx} + \mu((1 + ku)\psi)_{xx} + \psi(r - 2v - cu) - cv\phi \end{pmatrix}. \tag{4.1}$$

By Remark 2.5 of case 3 in [42], $D_{(u,v)}F(\mu, u, v)(\phi, \psi)$ is elliptic and satisfies Agmon's condition. Therefore by [42, Theorem 3.3 and Remark 3.4], one obtains that

$$D_{(u,v)}F(\mu, u, v) : X \to Y \text{ is a Fredholm operator with zero index.} \tag{4.2}$$

The necessary condition for bifurcation to occur at the ,constant steady state (μ, u^+, v^+) is that the null space

$$\ker(D_{(u,v)}F(\mu, u^+, v^+)) \neq \{0\}.$$

We study the eigenvalues of the operator $D_{(u,v)}F(\mu, u^+, v^+)$. The eigenvalue τ with corresponding eigenfunction (ϕ, ψ) of operator $D_{(u,v)}F(\mu, u^+, v^+)$ satisfy

$$\begin{cases} \phi_{xx} - u^+\phi - bu^+\psi = \tau\phi, & x \in (0, L), \\ \mu(1 + ku^+)\psi_{xx} + \mu kv^+\phi_{xx} - v^+\psi - cv^+\phi = \tau\psi, & x \in (0, L), \\ \phi_x(0) = \phi_x(L) = \psi_x(0) = \psi_x(L) = 0. \end{cases} \tag{4.3}$$

Let

$$\phi = \sum_{j=0}^{\infty} t_j \cos\frac{j\pi x}{L}, \text{ and } \psi = \sum_{j=0}^{\infty} s_j \cos\frac{j\pi x}{L}. \tag{4.4}$$

Substituting (4.4) into (4.3), we get

$$\begin{cases} (-\lambda_j - u^+ - \tau)\, t_j - bu^+ s_j = 0, \\ (-\mu kv^+\lambda_j - cv^+)\, t_j + (-\mu(1 + ku^+)\lambda_j - v^+ - \tau)\, s_j = 0. \end{cases} \tag{4.5}$$

Then (4.3) has nonzero solutions if and only if

$$(\lambda_j + u^+ + \tau)\,(\mu(1 + ku^+)\lambda_j + v^+ + \tau) - bu^+v^+\,(\mu k\lambda_j + c) = 0 \text{ for some } j \geq 0.$$

Define

$$A_j(\tau) = \tau^2 + (\lambda_j + u^+ + \mu(1 + ku^+)\lambda_j + v^+)\tau + B_j(\tau),$$

where

$$B_j = (\lambda_j + u^+)(\mu(1+ku^+)\lambda_j + v^+) - bu^+v^+(\mu k\lambda_j + c)$$
$$= \lambda_j\mu[(\lambda_j + u^+)(1+ku^+) - bku^+v^+] + (1-bc)u^+v^+.$$

For each $j \in \mathbb{N}$, $A_j(\tau) = 0$ admits two roots $\tau_{j,1}$ and $\tau_{j,2}$, which satisfy

$$\tau_{j,1} + \tau_{j,2} = -(\lambda_j + u^+ + \mu(1+ku^+)\lambda_j + v^+) < 0 \ \text{ and } \ \tau_{j,1}\tau_{j,2} = B_j.$$

It is well-known that if any eigenvalue τ of the operator $D_{(u,v)}F(\mu, u^+, v^+)$ satisfies $\mathrm{Re}\tau < 0$, then (u^+, v^+) is linearly stable. If the operator $D_{(u,v)}F(\mu, u^+, v^+)$ has an eigenvalue τ with $\mathrm{Re}\tau > 0$, then (u^+, v^+) is linearly unstable. Therefore, if there exits $j \in \mathbb{N}$ such that $B_j < 0$, then (u^+, v^+) is linearly unstable; while (u^+, v^+) is linearly stable if $B_j > 0$ for all $j \in \mathbb{N}$. Hence, if $\frac{(bkv^+-(1+ku^+))u^+}{1+ku^+} \le \lambda_1$, then for each $j \in \mathbb{N}$, we have $B_j > 0$ and (u^+, v^+) is linearly stable for any $\mu > 0$; while if $\frac{(bkv^+-(1+ku^+))u^+}{1+ku^+} > \lambda_1$, then we have

$$B_1 = \begin{cases} > 0, & \text{for } \ \mu < \frac{(1-bc)u^+v^+}{(bku^+v^+-(1+ku^+)(\lambda_1+u^+))\lambda_1}, \\ = 0, & \text{for } \ \mu = \frac{(1-bc)u^+v^+}{(bku^+v^+-(1+ku^+)(\lambda_1+u^+))\lambda_1}, \\ < 0, & \text{for } \ \mu > \frac{(1-bc)u^+v^+}{(bku^+v^+-(1+ku^+)(\lambda_1+u^+))\lambda_1}. \end{cases}$$

Moreover, if $\frac{(bkv^+-(1+ku^+))u^+}{1+ku^+} \in (\lambda_j, \lambda_{j+1}]$ for some $j \ge 1$, then operator $D_{(u,v)}F(\mu, u^+, v^+)$ has zero eigenvalue only when

$$\mu = \mu_{\lambda_i} > 0, \ \ \mu_{\lambda_i} = \frac{(1-bc)u^+v^+}{(bku^+v^+ - (1+ku^+)(\lambda_i + u^+))\lambda_i}, \ \ i = 1, 2, \cdots, j, \tag{4.6}$$

and

$$(u^+, v^+) \text{ is linearly} \begin{cases} \text{stable}, & \text{for } \ \mu < \min\{\mu_{\lambda_1}, \mu_{\lambda_2}, \cdots, \mu_{\lambda_j}\}, \\ \text{unstable}, & \text{for } \ \mu > \min\{\mu_{\lambda_1}, \mu_{\lambda_2}, \cdots, \mu_{\lambda_j}\}. \end{cases}$$

To obtain the existence of non-constant positive solutions of (1.8), we first establish several lemmas.

Lemma 4.1. *Let (u, v) be a positive solution of (1.8). Then we have*

$$0 < u < r \ \ \text{and} \ \ 0 < v < r(1+kr) \ \ \text{on } [0, L].$$

Proof. The result follows directly from the strong maximum principle and we omit the details. □

Lemma 4.2. *Given all the parameters except μ, if $bc < 1$, then there exits some small $\mu^* > 0$ such that (1.8) only admits the constant positive solution (u^+, v^+) for any $\mu \in (0, \mu^*)$, where*

$$\mu^* = \tfrac{4\eta u^+}{v^+} \cdot \tfrac{1+kr}{k^2r^2} \ \text{ and } \ \eta = \tfrac{4-2bc}{2b^2}.$$

Proof. We will show that (u^+, v^+) is globally asymptotically stable for (1.1) when $\mu < \mu^*$.

Let

$$\mathcal{F}(t) = \eta \int_0^L \left(u - u^+ - u^+ \ln \frac{u}{u^+}\right) \mathrm{d}x + \int_0^L \left(v - v^+ - v^+ \ln \frac{v}{v^+}\right) \mathrm{d}x$$

and

$$G(t) = \int_0^L \left[(u-u^+)^2 + (v-v^+)^2 + \frac{u_x^2}{u^2} + \frac{v_x^2}{v^2}\right] \mathrm{d}x,$$

where (u, v) is the unique positive solution of (1.1).

Claim 1: $\forall \epsilon > 0$, there exits some $T_\epsilon > 0$ (depending on u_0) such that $u \le (1+\epsilon)r$ for $t \ge T_\epsilon$. Consider the ODE:

$$\begin{cases} z_t = z(r - z), \\ z(0) = \|u_0\|_{L^\infty}. \end{cases}$$

It is trivial to show that $z(t) \to r$ exponentially as $t \to \infty$. Moreover, by comparison principle, one has

$$u(x, t) \le z(t) \quad \text{for } x \in (0, L), \ t \ge 0.$$

Therefore, claim 1 holds. Given $\mu < \mu^*$, one can choose small ϵ such that

$$\mu < \frac{4\eta u^+}{v^+} \cdot \frac{1 + k(1+\epsilon)r}{k^2(1+\epsilon)^2 r^2}.$$

Claim 2: there exists $\delta > 0$, such that $\frac{dF(t)}{dt} \le -\delta G(t)$ for $t > T_\epsilon$. Indeed, one can compute that

$$\begin{aligned}
\frac{dF(t)}{dt} &= \int_0^L \left[\frac{\eta(u-u^+)u_{xx}}{u} + \frac{\mu(v-v^+)((1+ku)v)_{xx}}{v}\right] \mathrm{d}x \\
&\quad + \int_0^L (\eta(u-u^+)(r-u-bv) + (v-v^+)(r-v-cu))\mathrm{d}x \\
&= -\int_0^L \begin{pmatrix} \frac{u_x}{u} & \frac{v_x}{v} \end{pmatrix} \begin{pmatrix} \eta u^+ & \frac{\mu uk v^+}{2} \\ \frac{\mu uk v^+}{2} & \mu(1+ku)v^+ \end{pmatrix} \begin{pmatrix} \frac{u_x}{u} \\ \frac{v_x}{v} \end{pmatrix} \mathrm{d}x \\
&\quad - \int_0^L \begin{pmatrix} u-u^+ & v-v^+ \end{pmatrix} \begin{pmatrix} \eta & \frac{\eta b + c}{2} \\ \frac{\eta b+c}{2} & 1 \end{pmatrix} \begin{pmatrix} u-u^+ \\ v-v^+ \end{pmatrix} \mathrm{d}x
\end{aligned}$$

From $bc < 1$, $\eta = \frac{4-2bc}{2b^2}$, and $\mu < \frac{4\eta u^+}{v^+} \cdot \frac{1+k(1+\epsilon)r}{k^2(1+\epsilon)^2r^2}$, it follows that

$$\eta > \frac{(\eta b+c)^2}{4} \text{ and } \mu\eta(1+ku)u^+v^+ > \frac{(\mu uk v^+)^2}{4} \quad \text{for } t > T_\epsilon.$$

Then, one obtains that the claim 2. Next, following the approach as that in proving [48, Lemma 3.2], one can show that (u^+, v^+) is globally asymptotically stable for (1.1). Therefore, (1.8) only admits the constant positive solution (u^+, v^+) for any $\mu \in (0, \mu^*)$. □

Now we are in a position to prove Theorem 1.5.

Proof of Theorem 1.5. We will prove the Theorem in two steps.

Step 1: local bifurcation. Recall that $X = \{(u,v)|\ u, v \in H_N^2(0,L)\}$, $Y = L^2(0,L) \times L^2(0,L)$, and $F(\mu, u^+, v^+) = 0$ for any $\mu > 0$. Let $V = (0,\infty) \times X$. By (4.6), one finds that

$$\mathrm{Ker} D_{(u,v)} F(\mu_{\lambda_i}, u^+, v^+) = s\left(bu^+, -(\lambda_i + u^+)\right) \cos\frac{ix\pi}{L}, \quad s \in \mathbb{R}$$

and

$$\dim(\mathrm{Ker} D_{(u,v)} F(\mu_{\lambda_i}, u^+, v^+)) = 1.$$

Direct computations show that

$$D_{\mu,(u,v)} F(\mu, u, v)(\phi, \psi) = \begin{pmatrix} 0 \\ k(v\phi)_{xx} + ((1+ku)\psi)_{xx} \end{pmatrix}.$$

By (4.2), to apply Theorem 4.1, it remains to check the transversality condition

$$D_{\mu,(u,v)} F(\mu_{\lambda_i}, u^+, v^+)(\phi_i, \psi_i) \notin \mathrm{Ran}(D_{(u,v)} F(\mu_{\lambda_i}, u^+, v^+)),$$

where $\phi_i = bu^+ \cos\frac{ix\pi}{L}$ and $\psi_i = -(\lambda_i + u^+)\cos\frac{ix\pi}{L}$. If this condition fails, then there exits ζ, η such that

$$\begin{cases} \zeta_{xx} - u^+\zeta - bu^+\eta = 0, & \text{in } (0,L), \\ \mu_{\lambda_i} k v^+ \zeta_{xx} + \mu_{\lambda_i}(1+ku^+)\eta_{xx} - v^+\eta - cv^+\zeta = \chi_i, & \text{in } (0,L), \\ \zeta_x(0) = \zeta_x(L) = \eta_x(0) = \eta_x(L) = 0, \end{cases} \tag{4.7}$$

where

$$\chi_i = kv^+(\phi_i)_{xx} + (1+ku^+)(\psi_i)_{xx} = \left[(\lambda_i + u^+)(1+ku^+) - bku^+v^+\right] \lambda_i \cos\frac{ix\pi}{L}.$$

Let

$$\zeta = \sum_{i=0}^{\infty} \hat{t}_i \cos\frac{ix\pi}{L}, \quad \eta = \sum_{i=0}^{\infty} \hat{s}_i \cos\frac{ix\pi}{L}. \tag{4.8}$$

Substituting (4.8) into (4.7), we have

$$\begin{cases} \lambda_i \hat{t}_i + u^+ \hat{t}_i + bu^+ \hat{s}_i = 0, \\ k\hat{t}_i \lambda_i \mu_{\lambda_i} v^+ + (1+ku^+)\hat{s}_i \lambda_i \mu_{\lambda_i} + v^+ \hat{s}_i + cv^+ \hat{t}_i = [bku^+ v^+ - (\lambda_i + u^+)(1+ku^+)]\, \lambda_i > 0. \end{cases}$$

From the definition of μ_{λ_i}, one obtains that this linear system has no solutions. Therefore, by Theorem 4.1, we have that there exist an open interval $I_1 = (-\epsilon, \epsilon)$ and continuous functions $\mu : I_1 \to \mathbb{R}$, $\sigma : I_1 \to Z$, such that $\mu(0) = \mu_{\lambda_i}$, $\sigma(0) = 0$, and, if $(u(s), v(s)) = (u^+, v^+) + s(\phi_i, \psi_i) + s\sigma(s)$ for $s \in I_1$, then $F(\mu(s), u(s), v(s)) = 0$. Here, Z be any complement of span$\{(\phi_i, \psi_i)\}$ in X. Moreover, $F^{-1}(\{0\})$ near the bifurcation point $(\mu_{\lambda_i}, u^+, v^+)$ consists precisely of $(u, v) = (u^+, v^+)$ and the curves $\Gamma = \{(\mu(s), u(s), v(s)) : s \in I_1\}$.

Step 2: global bifurcation. By (4.2) and Theorem 4.1, we obtain that the curve Γ is contained in $\mathcal{C}$, which is a connected component of closure of S with

$$S = \{(\mu, u, v) \in V : F(\mu, u, v) = 0,\ (u, v) \neq (u^+, v^+)\},$$

and either $\mathcal{C}$ is not compact in V, or $\mathcal{C}$ contains a point (μ^*, u^+, v^+) with $\mu^* \neq \mu_{\lambda_i}$. We now show that the first alternative must occur by using the approach in [11, 35, 45]. Indeed, if $\mathcal{C}$ is bounded, by Lemma 4.2, one obtains that it is compact, and $\mathcal{C}$ meets some other bifurcation points. Let $1 \leq i^* \leq j$ be such that $\mathcal{C}$ meets $(\mu_{\lambda_{i^*}}, u^+, v^+)$, but not $(\mu_{\lambda_m}, u^+, v^+)$ for any $\lambda_m > \lambda_{i^*}$, where $m \leq j$. Consider an auxiliary problem

$$\begin{cases} u_{xx} + u(r - u - bv) = 0, & \text{in } (0, \frac{L}{i^*}), \\ \mu[(1+ku)v]_{xx} + v(r - v - cu) = 0, & \text{in } (0, \frac{L}{i^*}), \\ u_x(0) = u_x(\frac{L}{i^*}) = v_x(0) = v_x(\frac{L}{i^*}) = 0. \end{cases} \tag{4.9}$$

We note here that if (4.9) admits a positive solution (u^*, v^*), then one can construct a solution (u, v) to (1.8) by a reflective and periodic extension. Let $x_n = \frac{nL}{i^*}$, $n = 0, 1, \cdots, i^*$, and define

$$(u, v)(x) = \begin{cases} (u^*, v^*)(x - x_{2n}), & \text{if } x_{2n} \leq x \leq x_{2n+1}, \\ (u^*, v^*)(x_{2n+2} - x), & \text{if } x_{2n+1} \leq x \leq x_{2n+2}. \end{cases}$$

It is easy to verify that $(\mu_{\lambda_{i^*}}, u^+, v^+)$ is also a bifurcation point of the problem (4.9). Let Λ_{i^*} denotes the bifurcation branch of this new problem that meets $(\mu_{\lambda_{i^*}}, u^+, v^+)$, then using the same argument above it is clear that it either meets infinity or meets $(\mu_{\lambda_{m^*}}, u^+, v^+)$ for some $\lambda_{m^*} > \lambda_{i^*}$. If the second case occurs, then by the above extension one sees that $\mathcal{C}$ meets $(\mu_{\lambda_{m^*}}, u^+, v^+)$, which violates the definition of $\mu_{\lambda_{i^*}}$; hence Λ_{i^*} meets infinity, and then by the extension again $\mathcal{C}$ meets infinity too. To show that the projection of $\mathcal{C}$ on the μ interval must be unbounded, we first establish some results.

Claim 1: $\forall\ (\mu, u, v) \in \mathcal{C}$, we have $u > 0$ and $v > 0$ on $[0, L]$. From step 1, it follows that $u, v > 0$ on $[0, L]$ for $(\mu, u, v) \in \mathcal{C}$ and (μ, u, v) close to $(\mu_{\lambda_i}, u^+, v^+)$. By Lemma 4.2, the projection of $\mathcal{C}$ on the μ has positive lower bound. Assume the claim is false. That is, there exits $(\mu_i, u_i, v_i) \in \mathcal{C}$ with $u_i, v_i > 0$ on $[0, L]$ and $(\mu_i, u_i, v_i) \to (\hat{\mu}, \hat{u}, \hat{v})$ as $i \to \infty$, where $(\hat{\mu}, \hat{u}, \hat{v}) \in \mathcal{C}$ with

$$\min\left\{\min_{x\in[0,L]} \hat{u},\ \min_{x\in[0,L]} \hat{v}\right\} = 0. \tag{4.10}$$

If $\min_{x\in[0,L]} \hat{u} = 0$, by maximum principle, one obtains that $\hat{u} \equiv 0$. Recall that $\hat{v}$ satisfies

$$\begin{cases} \hat{\mu} d(0) \hat{v}_{xx} + \hat{v}\,(r - \hat{v}) = 0, & \text{in } (0, L), \\ \hat{v}_x(0) = \hat{v}_x(L) = 0. \end{cases}$$

Hence, we have

$$\hat{v} \equiv 0 \text{ or } \hat{v} \equiv r. \tag{4.11}$$

Let $\hat{u}_i = \frac{u_i}{\|u_i\|_{L^\infty}}$. Applying the elliptic regularity (cf. [8]) and the Sobolev imbedding theorem, without loss of generality, we assume that $\hat{u}_i \to \hat{u}^\infty$ in $C^1([0,L])$ as $i \to \infty$ and $\hat{u}^\infty$ satisfies

$$\begin{cases} \hat{u}^\infty_{xx} + \hat{u}^\infty(r - b\hat{v}) = 0, & \text{in } (0,L), \\ \hat{u}^\infty_x(0) = \hat{u}^\infty_x(L) = 0. \end{cases}$$

This together with (4.11), and $\hat{u}^\infty \geq 0$ implies that $\hat{u}^\infty \equiv 0$, which contradicts $\|\hat{u}^\infty\|_{L^\infty} = 1$. Hence, $\hat{u} > 0$ on $[0,L]$. This combined with (4.10) suggests that $\min\limits_{x\in[0,L]} \hat{v} = 0$. Let $w_i = (1+ku_i)v_i$ and $\hat{w} = (1+k\hat{u})\hat{v}$. Then $\hat{w}$ satisfies

$$\begin{cases} \hat{\mu}\hat{w}_{xx} + \frac{\hat{w}}{1+k\hat{u}}\left(r - \hat{u} - \frac{\hat{w}}{1+k\hat{u}}\right) = 0, & \text{in } (0,L), \\ \hat{w}_x(0) = \hat{w}_x(L) = 0. \end{cases}$$

So, we have $\hat{w} \equiv 0$. Let $\hat{w}_i = \frac{w_i}{\|w_i\|_{L^\infty}}$. Similarly, one attains that $\hat{w}_i \to \hat{w}^\infty$ in $C^1([0,L])$ as $i \to \infty$ and $\hat{w}^\infty$ satisfies

$$\begin{cases} \hat{\mu}\hat{w}^\infty_{xx} + \frac{\hat{w}^\infty}{1+k\hat{u}}(r - \hat{u}) = 0, & \text{in } (0,L), \\ \hat{w}^\infty_x(0) = \hat{w}^\infty_x(L) = 0. \end{cases}$$

This further yields that $\hat{w}^\infty \equiv 0$ due to Lemma 4.1, which contradicts $\|\hat{w}^\infty\|_{L^\infty} = 1$. Therefore, Claim 1 holds.

Claim 2: $\forall\,(\mu,u,v) \in \mathcal{C}$, one has u and v are bounded in $H^2(0,L)$. Recall Lemma 4.2 and it is standard to show that the claim holds.

Combining Claim 1, Claim 2 and the fact that $\mathcal{C}$ meets infinity, one concludes that the projection of $\mathcal{C}$ on the μ interval must be unbounded. This completes the proof. □

Remark 4.1. *We have the following remarks.*

- *Let $\Gamma_+ = \{(\mu(s),u(s),v(s)) : s \in (0,\epsilon)\}$ and $\Gamma_- = \{(\mu(s),u(s),v(s)) : s \in (-\epsilon,0)\}$. Denote $\mathcal{C}^+$ (resp. $\mathcal{C}^-$) be the component of $\mathcal{C}\backslash\Gamma_-$ which contains Γ_+ (resp. the component of $\mathcal{C}\backslash\Gamma_+$ which contains Γ_-). Similarly, one can show that the $\mathcal{C}^+$ (resp. $\mathcal{C}^-$) meets infinity. We note here that we don't exclude the possibillity that $\mathcal{C}^+$ and $\mathcal{C}^-$ meet at some point.*
- *If $\frac{(bkv^+-(1+ku^+))u^+}{1+ku^+} \in (\lambda_1,\lambda_2]$, then we have*

$$(u^+,v^+) \text{ is linearly} \begin{cases} \text{stable}, & \text{for } \mu < \mu_{\lambda_1}, \\ \text{unstable}, & \text{for } \mu > \mu_{\lambda_1}. \end{cases}$$

  Moreover, applying the well-known index theory [5], one can show that (1.8) admits at least two non-constant positive solutions for $\mu > \mu_{\lambda_1}$ because the indices of $(0,0)$, $(r,0)$, $(0,r)$ are all equal to 0, the index of (u^+,v^+) is -1, and the sum of index of all the non-negative solutions of (1.8) is 1.
- *If $\frac{(bkv^+-(1+ku^+))u^+}{1+ku^+} \in (\lambda_2,\lambda_3]$, we assume $\mu_{\lambda_1} \neq \mu_{\lambda_2}$. Without loss of generality, we assume that $\mu_{\lambda_1} < \mu_{\lambda_2}$. For the case $\mu \in (\mu_{\lambda_1},\mu_{\lambda_2})$, by the index theory, one can show that (1.8) admits at least two non-constant positive solutions. For the case $\mu > \mu_{\lambda_2}$, if the positive solution bifurcating from μ_{λ_1} is non-degenerate, then (1.8) admits at least two non-constant positive solutions due to the facts that the indices of $(0,0)$, $(r,0)$, $(0,r)$ are equal to 0, the index of (u^+,v^+) is 1, the index of the non-constant positive solution is 1 or -1; and the sum of index of all the non-negative solution of (1.8) is 1.*

5. Summary and discussion

In this paper, we consider the existence and nonexistence of non-constant positive solutions to the one-dimensional stationary SKT system (1.8). Indeed the existence/nonexistence and stability of positive solutions to system (1.8) have been widely studied in the literature, but the results are confined to the case of strong cross-diffusion (i.e. $k \gg 1$). In this paper, we make a step forward by considering a fixed $k > 0$ and $\mu \gg 1$ at the first time. Our main results consist of two parts. The first part includes some non-existence and existence of positive solutions as $\mu \gg 1$. We first establish the non-existence of positive solutions for (1.8) with $\mu \gg 1$ in the case of $b < 1 < c$ (see Theorem 1.2). This implies that the cross-diffusion strategy of avoiding the strong competitor can not help the weak competitor to survive. Then by studying the existence of monotonic solutions to the shadow system of (1.8) as $\mu \to \infty$ for fixed $k > 0$, we obtain the existence of positive non-constant solutions of (1.8) under generic conditions (see Theorem 1.3) via the non-degeneracy condition (1.9) (see Proposition 1.1). More explicit existence conditions are further given in Theorem 1.4. The second part of our main results is the existence of non-constant positive solutions in the case of weak competition $0 < b, c < 1$ for any $\mu > 0$ given in Theorem 1.5 which is proved by the global bifurcation theory.

Various interesting open questions arise from our present study. For example, the stability (or instability) of non-constant positive solutions is yet to be studied. The existence result given in Theorem 1.4-(i) requires that b and c are sufficiently close to 1. Then one may ask whether system (1.1) admits non-constant positive solutions if $0 < c < 1 < b$ but b or c is not close to 1. The more interesting yet challenging question is to find threshold values of b and c so that the existence or non-existence of non-constant positive solutions can be determined.

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References

[1] Yung Sze Choi, Roger Lui, and Yoshio Yamada. Existence of global solutions for the Shigesada-Kawasaki-Teramoto model with strongly coupled cross-diffusion. *Discrete Contin. Dyn. Syst.*, 10(3):719–730, 2004.

[2] Chris Cosner. Reaction-diffusion-advection models for the effects and evolution of dispersal. *Discrete Contin. Dyn. Syst.*, 34(5):1701–1745, 2014.

[3] Michael G. Crandall and Paul H. Rabinowitz. Bifurcation from simple eigenvalues. *J. Func. Anal.*, 8:321–340, 1971.

[4] Michael G. Crandall and Paul H. Rabinowitz. Bifurcation, perturbation of simple eigenvalues and linearized stability. *Arch. Rational Mech. Anal.*, 52:161–180, 1973.

[5] Edward N. Dancer. On the indices of fixed points of mappings in cones and applications. *J. Math. Anal. Appl.*, 91(1):131–151, 1983.

[6] Edward N. Dancer. Bifurcation from simple eigenvalues and eigenvalues of geometric multiplicity one. *Bull. London Math. Soc.*, 34(5):533–538, 2002.

[7] Eamonn A. Gaffney, Andrew L. Krause, Philip K. Maini, and Chenyuan Wang. Spatial heterogeneity localizes Turing patterns in reaction-cross-diffusion systems. *Discrete Contin. Dyna. Syst.-B*, 28(12):6092–6125, 2023.

[8] David Gilbarg and Neil S. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin, 2001.

[9] Alan Hastings. Can spatial variation alone lead to selection for dispersal? *Theoretical Population Biology*, 24(3):244–251, 1983.

[10] Xiaoqing He and Wei-Ming Ni. Global dynamics of the Lotka-Volterra competition-diffusion system: diffusion and spatial heterogeneity I. *Comm. Pure Appl. Math.*, 69(5):981–1014, 2016.

[11] Jaeduck Jang, Wei-Ming Ni, and Moxun Tang. Global bifurcation and structure of Turing patterns in the 1-D Lengyel-Epstein model. *J. Dynam. Differential Equations*, 16(2):297–320, 2004.

[12] Yukio Kan-on. Stability of singularly perturbed solutions to nonlinear diffusion systems arising in population dynamics. *Hiroshima Math. J.*, 23(3):509–536, 1993.

[13] Peter Kareiva and Garrett Odell. Swarms of predators exhibit "preytaxis" if individual predators use area-restricted search. *Amer. Nat.*, 130(2):233–270, 1987.

[14] Evelyn F. Keller and Lee A. Segel. Initiation of slime mold aggregation viewed as an instability. *J. Theor. Biol.*, 26:399–415, 1970.

[15] Kazuo Kishimoto and Hans F. Weinberger. The spatial homogeneity of stable equilibria of some reaction-diffusion systems on convex domains. *J. Differential Equations*, 58(1):15–21, 1985.

[16] Kousuke Kuto. Limiting structure of shrinking solutions to the stationary Shigesada-Kawasaki-Teramoto model with large cross-diffusion. *SIAM J. Math. Anal.*, 47(5):3993–4024, 2015.

[17] Kousuke Kuto and Yoshio Yamada. Positive solutions for Lotka-Volterra competition systems with large cross-diffusion. *Appl. Anal.*, 89(7):1037–1066, 2010.

[18] King-Yeung Lam, Yuan Lou, and Frithjof Lutscher. Evolution of dispersal in closed advective environments. *J. Biol. Dyn.*, 9(sup1):188–212, 2015.

[19] Dung Le. Cross diffusion systems on n spatial dimensional domains. *Indiana Univ. Math. J.*, 51(3):625–643, 2002.

[20] Qing Li and Yaping Wu. Stability analysis on a type of steady state for the SKT competition model with large cross diffusion. *J. Math. Anal. Appl.*, 462(1):1048–1072, 2018.

[21] Qing Li and Yaping Wu. Existence and instability of some nontrivial steady states for the SKT competition model with large cross diffusion. *Discrete Contin. Dyn. Syst.*, 40(6):3657–3682, 2020.

[22] Qing Li and Qian Xu. On a limiting system of the triangular SKT competition model with large cross-diffusion and random-diffusion. *Nonlinear Anal. Real World Appl.*, 72:Paper No. 103865, 17, 2023.

[23] Chenli Liu et al. Sequential establishment of stripe patterns in an expanding cell population. *Science*, 334(6053):238–241, 2011.

[24] Julián López-Gómez. *Spectral theory and nonlinear functional analysis*, volume 426 of *Chapman & Hall/CRC Research Notes in Mathematics*. Chapman & Hall/CRC, Boca Raton, FL, 2001.

[25] Yuan Lou and Wei-Ming Ni. Diffusion, self-diffusion and cross-diffusion. *J. Differential Equations*, 131(1):79–131, 1996.

[26] Yuan Lou and Wei-Ming Ni. Diffusion vs cross-diffusion: an elliptic approach. *J. Differential Equations*, 154(1):157–190, 1999.

[27] Yuan Lou, Wei-Ming Ni, and Yaping Wu. On the global existence of a cross-diffusion system. *Discrete Contin. Dyn. Syst.*, 4(2):193–203, 1998.

[28] Yuan Lou, Wei-Ming Ni, and Shoji Yotsutani. On a limiting system in the Lotka-Volterra competition with cross-diffusion. *Discrete Contin. Dyn. Syst.*, 10(1-2):435–458, 2004.

[29] Yuan Lou, Wei-Ming Ni, and Shoji Yotsutani. Pattern formation in a cross-diffusion system. *Discrete Contin. Dyn. Syst.*, 35(4):1589–1607, 2015.

[30] Winsor H. Lowe and Mark A. McPeek. Is dispersal neutral? *Trends in Ecology & Evolution*, 29(8):444–450, 2014.

[31] Masayasu Mimura, Yasumasa Nishiura, Alberto Tesei, and Tohru Tsujikawa. Coexistence problem for two competing species models with density-dependent diffusion. *Hiroshima Math. J.*, 14(2):425–449, 1984.

[32] Piero De. Mottoni. Qualitative analysis for some quasilinear parabolic systems. *Inst. Math. Polish Acad. Sci. Zam.*, 190:11–79, 1979.

[33] Wei-Ming Ni. *The mathematics of diffusion*, volume 82 of *CBMS-NSF Regional Conference Series in Applied Mathematics*. SIAM, Philadelphia, PA, 2011.

[34] Wei-Ming Ni, Yaping Wu, and Qian Xu. The existence and stability of nontrivial steady states for S-K-T competition model with cross diffusion. *Discrete Contin. Dyn. Syst.*, 34(12):5271–5298, 2014.

[35] Yasumasa Nishiura. Global structure of bifurcating solutions of some reaction-diffusion systems. *SIAM J. Math. Anal.*, 13(4):555–593, 1982.

[36] Akira Okubo, Simon A. Levin, et al. *Diffusion and ecological problems: modern perspectives*, volume 14. Springer, 2001.

[37] Paul H. Rabinowitz. Some global results for nonlinear eigenvalue problems. *J. Functional Analysis*, 7:487–513, 1971.

[38] Kazi A. Rahman, A mixed-culture biofilm model with cross-diffusion Sudarsan, and Hermann J. Eberl. A mixed-culture biofilm model with cross-diffusion. *Bull. Math. Biol.*, 77:2086–2124, 2015.

[39] Ronen Ron, Ori Fragman-Sapir, and Ronen Kadmon. Dispersal increases ecological selection by increasing effective community size. *Proceedings of the National Academy of Sciences*, 115(44):11280–11285, 2018.

[40] Renate Schaaf. *Global solution branches of two-point boundary value problems*, volume 1458 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1990.

[41] Junping Shi. Semilinear Neumann boundary value problems on a rectangle. *Trans. Amer. Math. Soc.*, 354(8):3117–3154, 2002.

[42] Junping Shi and Xuefeng Wang. On global bifurcation for quasilinear elliptic systems on bounded domains. *J. Differential Equations*, 246(7):2788–2812, 2009.

[43] Nanako Shigesada, Kohkichi Kawasaki, and Ei Teramoto. Spatial segregation of interacting species. *J. Theoret. Biol.*, 79(1):83–99, 1979.

[44] Hal L. Smith. Monotone dynamical systems: reflections on new advances & applications. *Discrete Contin. Dyn. Syst.*, 37(1):485–504, 2017.

[45] Izumi Takagi. Point-condensation for a reaction-diffusion system. *J. Differential Equations*, 61(2):208–249, 1986.

[46] Li Wang, Yaping Wu, and Qian Xu. Instability of spiky steady states for S-K-T biological competing model with cross-diffusion. *Nonlinear Anal.*, 159:424–457, 2017.

[47] Xuefeng Wang and Qian Xu. Spiky and transition layer steady states of chemotaxis systems via global bifurcation and Helly's compactness theorem. *J. Math. Biol.*, 66(6):1241–1266, 2013.

[48] Zhi-An Wang and Jiao Xu. On the Lotka-Volterra competition system with dynamical resources and density-dependent diffusion. *J. Math. Biol.*, 82(1-2):Paper No. 7, 37, 2021.

[49] Yaping Wu. The instability of spiky steady states for a competing species model with cross diffusion. *J. Differential Equations*, 213(2):289–340, 2005.

[50] Yoshio Yamada. Global solutions for the Shigesada-Kawasaki-Teramoto model with cross-diffusion. In *Recent progress on reaction-diffusion systems and viscosity solutions*, pages 282–299. World Sci. Publ., Hackensack, NJ, 2009.

[51] Bo Zhang, Alex Kula, Keenan ML Mack, Lu Zhai, Arrix L Ryce, Wei-Ming Ni, Donald L DeAngelis, and J David Van Dyken. Carrying capacity in a heterogeneous environment with habitat connectivity. *Ecology letters*, 20(9):1118–1128, 2017.