A HAMILTON-JACOBI APPROACH TO ROAD-FIELD REACTION-DIFFUSION MODELS

CHRISTOPHER HENDERSON AND KING-YEUNG LAM

1. Comparison principle

In this section, we develop a comparison principle, and, thus, a uniqueness theorem, for strong solutions to the slightly more general equation

$$\min\{w - \ell t, w_t + H_\mathrm{f}(w_x, w_y)\} = 0 \qquad \text{in } (0,\infty)\times\mathbb{H} \tag{1.1}$$

and

$$\min\{w - \ell t, w_t + F(w_x, w_y)\} = 0 \qquad \text{on } (0,\infty)\times\mathbb{R}\times\{0\}, \tag{1.2}$$

where $\ell \in \mathbb{R}$. Here $H_f(q,p)$ and $H_r(q)$ are both convex and coercive, and

$$F(q,p) = \max\{H_\mathrm{f}^-(q,p), H_r(q)\},$$

where H_f^- is the decreasing part of H_f given, for each fixed q, by

$$H_\mathrm{f}^-(q,p) = \begin{cases} H_\mathrm{f}(q,p) & \text{for } p < p_*, \\ H_\mathrm{f}(q,p_*) & \text{for } p \geq p_*, \end{cases}$$

where $p_* = p_*(q)$ is the minimum point of $p \mapsto H_\mathrm{f}(q,p)$. For simplicity, we will assume in the following $H_\mathrm{f}(q,p) = q^2 + p^2 = 1$, so that $H_\mathrm{f}^- = q^2 + (p\wedge 0)^2 + 1$. This addition of the ℓ-term allows us to, in the sequel, reduce to an equation with a sub-homogeneity property (cf. (1.14)).

The main idea of the proof is due to [3] (see also [1] for general junction conditions). Our proof is new for unbounded viscosity solutions. We later apply it to uniqueness for unbounded viscosity solutions with discontinuous initial data, although in the simplified setting of a convex Hamiltonian. Our main contributions are the localization of the Lions-Souganidis argument as well as the simplification of certain steps by leveraging the Lipschitz continuity of subsolutions.

Finally, our argument is mainly showing that the comparison theorem holds for the Kirchhoff (Neumann) problem. Here we use only that B is convex (in our case $B(q) = p_q$). Solutions of the problem with Kirchhoff junction conditions are connected to the strong boundary conditions via Lemma 1.4.

Our first main result is the following:

Theorem 1.1 (Comparison principle)**.** *Fix any $T > 0$. Let $\underline{w}$ and $\overline{w}$ be, respectively, strong sub- and supersolutions to (1.1)-(1.2) on $(0,T)\times\overline{\mathbb{H}}$. If $\underline{w}(0,x,y) \leq \overline{w}(0,x,y)$ for $(x,y)\in\overline{\mathbb{H}}$, then*

$$\underline{w} \leq \overline{w}. \qquad \text{on } [0,T)\times\overline{\mathbb{H}}.$$

Before we prove Theorem 1.1, we show how to deduce uniqueness of (possibly infinite) solutions to (1.1) from it. This is our second main result.

Corollary 1.2 (Uniqueness)**.** *Any two functions $w : [0,\infty)\times\overline{\mathbb{H}} \to \mathbb{R}\cup\{+\infty\}$ satisfying*

(i) On $[0,\infty)\times\overline{\mathbb{H}}$, w is lower semicontinuous and a strong supersolution to (1.1),

(ii) On $(0,\infty)\times\overline{\mathbb{H}}$, w is finite-valued, continuous, and is a strong subsolution to (1.1), and

(iii) For $t > 0$, $w(t,0,0) \leq \ell t$, while at $t = 0$, we have $w(0,0,0) \geq 0$ and $w(0,x,y) = +\infty$ for $(x,y) \neq 0$.

Proof. We argue by contradiction. Fix any two functions w and $\tilde{w}$ that satisfy (i), (ii) and (iii). By the arbitrariness of w and $\tilde{w}$, we need only show that $w \leq \tilde{w}$.

Any supersolution must necessarily satisfy

$$\tilde{w}(t,x,y) \geq \ell t \qquad \text{for all } (t,x,y) \in (0,\infty)\times\bar{\mathbb{H}}.$$

Using (iii), we see that

$$-|\ell|\tau + w(\tau,0,0) \leq 0 \leq \tilde{w}(0,0,0).$$

(Christopher Henderson) Department of Mathematics, University of Arizona, Tucson, AZ, 85719, USA
(King-Yeung Lam) Department of Mathematics, The Ohio State University, Columbus, OH, 43210, USA
E-mail addresses: ckhenderson@math.arizona.edu, lam.184@osu.edu.

Moreover, again using (iii), we have, for all $(x, y) \neq (0,0)$,

$$-|\ell|\tau + w(\tau, x, y) < +\infty = \tilde{w}(0, x, y).$$

Hence

$$-|\ell|\tau + w(\tau, x, y) \le \tilde{w}(0, x, y) \qquad \text{for all } (x, y) \in \overline{\mathbb{H}}.$$

Finally, we immediately see that

$$-|\ell|\tau + w(t+\tau, x, y)$$

is a subsolution to (1.1).

Applying Theorem 1.1, it follows that

$$-|\ell|\tau + w(t+\tau, x, y) \le \tilde{w}(t, x, y) \quad \text{for all } (t, x, y) \in (0,\infty) \times \overline{\mathbb{H}}.$$

By the continuity of w on $(0,\infty) \times \overline{H}$, We can then let $\tau \to 0$ to obtain $w \le \tilde{w}$ on $(0,\infty) \times \overline{\mathbb{H}}$, as desired. □

1.1. **Strong solution implies Kirchhoff solution.** We follow the idea of [3], as presented in [1]. For this purpose, we associate a Kirchhoff condition to strong sub- and supersolutions.

Definition 1.3. Let $\overline{w} : (0,\infty) \times \overline{\mathbb{H}} \to \mathbb{R}$, and let $(t_0, x_0, y_0) \in (0,\infty) \times \overline{\mathbb{H}}$ be given. We say that the constant vector $(-\lambda, q, p)$ is an element of the subdifferential at (t_0, x_0, y_0), a set denoted by $D^-\overline{w}(t_0, x_0, y_0)$, if there exists $r_0 > 0$ such that

$$\begin{aligned} w(t,x,y) \ge w(t_0,x_0,y_0) + (-\lambda, q, p)\cdot(t-t_0, x-x_0, y-y_0) + o(|t-t_0| + |x-x_0| + |y-y_0|) \\ \text{for } (t,x,y) \in \{(t',x',y') \in (0,\infty)\times\overline{\mathbb{H}} : \ |(t'-t_0, x'-x_0, y'-y_0)| < r_0\}. \end{aligned} \tag{1.3}$$

The superdifferential at (t_0, x_0, y_0), denoted $D^+w(t_0, x_0, y_0)$ is defined similarly up to reversing the inequality in (1.3).

For a given function $\phi(t, x)$ (that is, not depending on y), we denote by the sub and superdifferentials, denoted $D^-_{t,x}\phi(t_0, x_0)$ and $D^+_{t,x}\phi(t_0, x_0)$, analogously.

We now state the Kirchhoff condition:

$$-w_y + B(w_x) = 0. \tag{1.4}$$

We choose the notation B to match that of Lions and Souganidis [3]. Then a (weak) solution of (1.1)-(1.4) is one such that

$$\begin{cases} \min\{w - \ell t, \max\{-w_y + B(w_x), w_t + H_{\rm f}(w_x, w_y)\}\} \ge 0 & \text{on } (0,\infty)\times\mathbb{R}\times\{0\}, \\ \min\{w - \ell t, -w_y + B(w_x), w_t + H_{\rm f}(w_x, w_y)\} \le 0 & \text{on } (0,\infty)\times\mathbb{R}\times\{0\}. \end{cases} \tag{1.5}$$

The first inequality in (1.5) corresponds to supersolutions (along with the condition that w is lower semicontinuous), while the second inequality in (1.5) corresponds to subsolutions (along with the condition that w is upper semicontinuous). Let us point out that if $\bar{w}$ is a supersolution to (1.1)-(1.1), it must be that

$$w \ge \ell t \qquad \text{on } (0,\infty)\times\bar{\mathbb{H}}. \tag{1.6}$$

We now show that strong solutions satisfy the Kirchhoff condition. This was originally observed by Lions and Souganidis in [3] in a slightly different context, and we follow their proof.

Lemma 1.4. *Let w be a strong subsolution (resp. strong supersolution) of (1.1)-(1.2), then it satisfies the Kirchhoff condition (1.1)-(1.4) with coefficient*

$$B(q_0) = p_{q_0},$$

where we recall that, for each q_0, the number $p_{q_0} \in [0,\infty)$ is given by the larger root of $p \mapsto H_{\rm f}(q_0, p) - H_{\rm r}(q_0)$ if $H_{\rm r}(q_0) \ge \min H_{\rm f}(q_0, \cdot)$, or $p_{q_0} = \mathrm{argmin} H_{\rm f}(q_0,)$ otherwise. The expression $H_{\rm r}(q_0)$ is called the flux-limiter [2].

Proof. For simplicity, we assume the case $H_f(q, p) = q^2 + p^2 + 1$, and $H_{\rm r}(q) = q^2 + (p_q)^2 + 1$, where

$$p_q = \begin{cases} 0 & \text{if } q^2 \le \frac{1}{D-1}, \\ g^{-1}(q) & \text{if } q^2 > \frac{1}{D-1}. \end{cases} \tag{1.7}$$

Here $g : [0,\infty) \to [1/\sqrt{D-1}, \infty)$ is the increasing function

$$g(p) = \sqrt{\frac{1}{D-1}\left[p^2 + 1 + \frac{\mu p}{\kappa\nu + p}\right]}.$$

First, we assume w is a strong subsolution. It suffices to check the condition on the boundary $\{y = 0\}$. Fix $(t_0, x_0, 0)$ and $(-\lambda, p_0, q_0) \in D^+w(t_0, x_0, 0)$. Then

$$\max\{H_{\rm f}^-(q_0, p_0), H_{\rm r}(q_0)\} = F(q_0, p_0) \le \lambda.$$

If $H_{\rm f}(q_0, p_0) \le \lambda$ or $w(t_0, x_0, 0) \le \ell$, then we are finished by (1.5)

It remains only to argue in the case where $H_{\rm f}(q_0, p_0) > \lambda$ and $w(t_0, x_0, 0) > \ell$. In this case, we must show that

$$p_0 \ge p_{q_0}. \tag{1.8}$$

Since

$$H_{\rm f}^-(q_0, p_0) \le \lambda < H_{\rm f}(q_0, p_0), \tag{1.9}$$

it follows that $p_0 > 0$ (recall (??)).

By construction, we have that

$$H_{\rm f}(q_0, p_{q_0}) \le H_{\rm r}(q_0) \le \lambda. \tag{1.10}$$

Since $p_{q_0} \ge 0$ and $H_{\rm f}(q_0, \cdot)$ is increasing on $[0, \infty)$, we deduce from (1.9)-(1.10) that (1.8), as desired.

Next, assume that w is a strong supersolution and $(-\lambda, p_0, q_0) \in D^- w(t_0, x_0, 0)$. Then

$$\max\{H_{\rm f}^-(q_0, p_0), H_{\rm r}(q_0)\} \ge \lambda. \tag{1.11}$$

We are finished if

$$-\lambda + H_{\rm f}(q_0, p_0) \ge 0.$$

Hence, we consider when

$$H_{\rm f}(q_0, p_0) < \lambda,$$

in which case we need to show that $p_0 \le p_{q_0}$. By (1.11), we divide into two cases: (i) $H_{\rm f}^-(q_0, p_0) \ge \lambda$; (ii) $H_{\rm r}(q_0) \ge \lambda$.

In case (i), $H_{\rm f}^-(q_0, p_0) > H_{\rm f}(q_0, p_0)$, which implies that $p_0 < 0$. By definition, $0 \le p_{q_0}$. Thus, the proof is complete in this case.

In case (ii), $H_{\rm f}(q_0, p_0) < \lambda \le H_{\rm r}(q_0)$. Whence we conclude that $H_{\rm r}(q_0) = H_{\rm f}(q_0, p_{q_0})$. It follows that

$$H_{\rm f}(q_0, |p_0|) = H_{\rm f}(q_0, p_0) < H_{\rm f}(q_0, p_{q_0}).$$

Since $H_{\rm f}$ is increasing on $[0, \infty)$ and $p_{q_0} \ge 0$, we deduce that

$$p_0 \le |p_0| < p_{q_0}.$$

This completes the proof. ☐

The key of the comparison principle is the following lemma due to Lions and Souganidis [3, Lemma 3.1].

Lemma 1.5. *Assume that there exists $H_0 \in C(\mathbb{R})$, $p_0, q_0 \in \mathbb{R}$, and $a, b \in \mathbb{R}$ such that*

(i) $p_0 \ge q_0$, $a + H_0(p_0) \le 0 \le b + H_0(q_0)$;
(ii) $\min(p', a + H_0(p')) \le 0$ *for each* $p' \in (-\infty, p_0]$,
(iii) $\max(q', b + H_0(q')) \ge 0$ *for each* $q' \in [q_0, \infty)$.

Then $a \le b$.

We state one final technical result that is necessary in the proof of Theorem 1.1.

Lemma 1.6. *Given an open set Q, suppose that there is a constant C such that*

$$w(t, x, y) + C_0(t^2 + x^2) \quad \text{is convex in } (t, x) \text{ for each } y, \text{ with } (t, x, y) \in Q. \tag{1.12}$$

Then for any (t_0, x_0, y_0), the set of subdifferential $D^-_{t,x} w(t_0, x_0, y_0)$ in the (t, x) variable is nonempty, and

$$w(t_0, x_0, y_0) \le w(0, 0, y_0) + (-\lambda, q) \cdot (t_0, x_0) + C_0(t_0^2 + x_0^2), \tag{1.13}$$

for any $(t_0, x_0, y_0) \in Q$ and any element $(-\lambda, q) \in D^-_{t,x} w(t_0, x_0, y_0)$.

Proof. By (1.12), $w(t, x, y) + C_0(|t - t_0|^2 + |x - x_0|^2)$ is convex and, thus, has at least one supporting hyperplane. This implies that $D^-_{t,x} w(t_0, x_0, y_0)$ is nonempty.

Let $(-\lambda, q) \in D^-_{t,x} w(t_0, x_0, y_0)$. Combining the definition of the subdifferential and the (t, x)-convexity in (1.12), it follows that, for all (t, x),

$$w(t_0, x_0, y_0) + (-\lambda, q) \cdot (t - t_0, x - x_0) \le w(t, x, y_0) + C_0(|t - t_0|^2 + |x - x_0|^2).$$

The conclusion (1.13) follows by setting $(t, x) = (0, 0)$. This completes the proof. ☐

1.2. **Proof of the comparison principle.** We provide a self-contained proof here adapting the ideas of [3] (see also [1, Chap. 17]). Before beginning, let us point out that the main differences between our setting and that of [3] are: (1) B is a function, not a constant, (2) the initial data need not be bounded and uniformly continuous, (3) our Hamilton-Jacobi equation (1.5) involves an obstacle.

Proof of Theorem 1.1. Let $\ell_0 = \max\{H(0,0), H_{\rm r}(0)\}$. By replacing ℓ by $\ell+\ell_0$ and $\overline{w}$ and $\underline{w}$ with $\overline{w}+\ell_0 t$ and $\underline{w}+\ell_0 t$, we may assume that

$$H(0,0) \le 0 \quad \text{and} \quad H_{\rm r}(0) \le 0.$$

Then, by convexity,

$$H(\lambda q, \lambda p) \le \lambda H_{\rm f}(q,p) \quad \text{and} \quad H_{\rm r}(\lambda q) \le \lambda H_{\rm r}(q) \quad \text{for all } p, q \in \mathbb{R},\ 0 < \lambda < 1. \tag{1.14}$$

Next, assume to the contrary that $\inf_{(0,T]\times\mathbb{H}}(\overline{w} - \underline{w}) < 0$.

In the next three steps, we perform a reduction to the case of a bounded domain, $\underline{w}$ being locally Lipschitz in $(0,\infty)\times\overline{\mathbb{H}}$ and semi-convex in (t,x), and $\overline{w}$ being semi-concave in (t,x) and continuous on the hyperplane $\{y = 0\}$.

 # Step one: Without loss of generality, we may assume that $\underline{w}$ is bounded from above.

We claim that $\underline{w}_K = \min\{\underline{w}, K\}$ is a strong subsolution to (1.1)-(1.2) for each $K > 0$. Indeed, take a sequence $\{g_j\}$ of smooth functions satisfying

$$0 \le g_j'(r) \le 1 \qquad \text{and} \qquad g_j(r) \nearrow \min\{r, K\} \quad \text{for } r \in \mathbb{R}.$$

Notice that

$$|\nabla g_j(\underline{w})| \le |\nabla w|. \tag{1.15}$$

Then $\hat{w} = g_j(\underline{w})$ is a viscosity subsolution to (1.1)-(1.2). Indeed, we clearly need only check the set $\{(t,x,y) :\ \hat{w} - \ell t > 0\}$, in which, thanks to (1.14) and (1.15),

$$\partial_t \hat{w} + G(\nabla \hat{w}) \le g_j'(\underline{w})[\partial_t \underline{w} + G(\nabla \underline{w})] \quad \text{for } G = H_{\rm f},\ H_{\rm f}^-, \text{ or } H_{\rm r}.$$

Using the stability of strong subsolutions (see, e.g., [1, Theorem 14.2.1]), we take $j \to \infty$ and deduce that $\underline{w}_K = \min\{\underline{w}, K\}$ is a strong subsolution to (1.1)-(1.2).

Notice that, if we prove that $\underline{w}_K \le \overline{w}$ for all K, then we deduce that $\underline{W} \le \overline{w}$ in the limit $K \to \infty$. We may, thus, assume that $\underline{w}$ is bounded from above.

 # Step two: reduction to a strict subsolution. Without loss of generality, we may assume that there is $\eta > 0$ such that

$$\limsup_{t\to T-} \underline{w} = \limsup_{|x|+|y|\to\infty} \underline{w} = -\infty \quad \text{and} \quad \min_{\{0\}\times\overline{\mathbb{H}}} (\overline{w} - \underline{w}) > 0, \tag{1.16}$$

while

$$\begin{cases} \min\{\underline{w} - \ell t, \partial_t \underline{w} + H_{\rm f}(\nabla \underline{w}) + 2\eta\} \le 0 & \text{on } (0,T)\times\mathbb{H}, \\ \min\{\underline{w} - \ell t, \partial_t \underline{w} + F(\underline{w}_x, \underline{w}_y) + 2\eta\} \le 0 & \text{on } (0,T)\times\partial\mathbb{H}. \end{cases} \tag{1.17}$$

It is easy to see that

$$\tilde{w}_1(t,x,y) = -\frac{K}{T-t} - \log(1 + |x|^2 + |y|^2) + \ell t$$

is a strong subsolution to (1.1)-(1.2) for K sufficiently large. Thanks to the convexity of H, $H_{\rm f}^-$ and $H_{\rm r}$, the function

$$\underline{w}_\mu = (1-\mu)\underline{w} + \mu \tilde{w}_1$$

satisfies (1.16)-(1.17) for any $0 < \mu < 1$ (recall that $\underline{w}$ is bounded from above by the previous step). Again, it suffices to show that $\underline{w}_\mu \le \overline{w}$ for all sufficiently small $\mu > 0$.

 # Step three: reduction to a compact portion of the boundary. Let us note that, due to (1.6) and the work in Step two, there is $R > 0$ such that the

$$\inf_{[0,T]\times\overline{\mathbb{H}}} (\overline{w} - \underline{w}) = \min_{Q_R} (\overline{w} - \underline{w}) < 0,$$

where

$$Q_R = \{(t,x,y) \in [0,T]\times\overline{\mathbb{H}} : {}^1\!/\!_R < t < T - {}^1\!/\!_R, |x| + |y| \le R\}.$$

We now show we need only consider $((0,T)\times\partial\mathbb{H}) \cap Q_R$. Let

$$(\bar{t}, \bar{x}, \bar{y}) = \operatorname*{argmin}_{Q_R} (\underline{w} - \overline{w}) = \operatorname*{argmin}_{[0,T]\times\mathbb{H}} (\underline{w} - \overline{w}).$$

If $\bar{y} > 0$, we may apply a standard doubling of variables argument to deduce a contradiction. Hence, we proceed with the rest of the proof assuming that $\bar{y} = 0$. Here the doubling of variables method has inherent difficulty when the point of contradiction occurs at the boundary. We proceed following the ideas of [3], with some minor modification taking advantage of the Lipschitz continuity of the subsolution.

Step four: reduction to semiconvex/concave functions. Without loss of generality, we may assume that $\underline{w}$ (resp. $\overline{w}$) is semi-convex (resp. semi-concave) in the (t,x) variable for each y with a uniform constant C. We can also assume that $\underline{w}$ is locally Lipschitz, and $\overline{w}$ is continuous on $\{(t,x,y) \in \overline{\mathbb{H}} : \ y = 0\}$.

Consider the sup-convolution of $\underline{w}$ and inf-convolution of $\overline{w}$: for $\varepsilon, \alpha > 0$,

$$\underline{w}^\varepsilon(t,x,y) = \max_{t',x'} \left\{ \underline{w}(t',x',y) - \frac{(|(t',x') - (t,x)|^2 + \varepsilon^4)^\alpha}{\varepsilon^\alpha} \right\} \quad \text{and}$$
$$\overline{w}^\varepsilon(t,x,y) = \min_{t',x'} \left\{ \overline{w}(t',x',y) + \frac{(|(t',x') - (t,x)|^2 + \varepsilon^4)^\alpha}{\varepsilon^\alpha} \right\}.$$

By [1, Propositions 2.4.4 and 2.4.9] and the boundedness of $\overline{w}$ and $\underline{w}$, there exists some small constant $\alpha > 0$ such that the following statements hold.

(i) $\underline{w}^\varepsilon$ and $\overline{w}^\varepsilon$ are strong sub and strong supersolutions in $\overline{Q_R}$, repectively;
(ii) $\underline{w}^\varepsilon$ and $-\overline{w}^\varepsilon$ are semi-convex in the (t,x) variables;
(iii) $\limsup \underline{w}^\varepsilon = \underline{w}$ and $\liminf \overline{w}^\varepsilon = \overline{w}$, where the limits are taken as $\varepsilon \to 0$.

Moreover, the subsolution $\underline{w}^\varepsilon$ is Lipschitz continuous in all variables, since $\partial_t \underline{w}^\varepsilon + H_{\mathrm{f}}(\partial_x \underline{w}^\varepsilon, \partial_y \underline{w}^\varepsilon) \leq 0$ and coercivity of H_{f} implies

$$|\partial_y \underline{w}^\varepsilon| \leq C(|\partial_t \underline{w}^\varepsilon|, |\partial_x \underline{w}^\varepsilon|), \text{ this is true for } \underline{w}, \text{ but not } \overline{w}, \text{ and C is independent of } \varepsilon$$

for some constant C that depends on $\partial_t \underline{w}^\varepsilon$ and $\partial_x \underline{w}^\varepsilon$ but independent of ε. Here the right hand side is bounded due to regularization by sup-convolution.

This proves that $\underline{w}^\varepsilon$ is Lipschitz continuous in $(0,\infty) \times \mathbb{H}$, and the Lipschitz constant can be chosen uniformly for all interiori points. On the boundary, we have $\partial_t \underline{w}^\varepsilon + H_f^-(\partial_x \underline{w}^\varepsilon, \partial_y \underline{w}^\varepsilon) \leq 0$, so the same reasoning yields that $\partial_y \underline{w}^\varepsilon$ is bounded uniformly from below by the (bounded) constant $C(|\partial_t \underline{w}^\varepsilon|, |\partial_x \underline{w}^\varepsilon|)$. This, together with the upper semicontinuity property, implies that $\underline{w}^\varepsilon$ is uniformly Lipschitz in $(0,\infty) \times \overline{\mathbb{H}}$.

In the following, we replace $\underline{w}$ (resp. $\overline{w}$) by its sup-convolution (resp. inf-convolution) for some fixed small ε. It remains to modify $\overline{w}$ so it is continuous on the hyperplane $\{y = 0\}$. [1]

To this end, we will replace $\overline{w}$ by $\tilde{w}_1(t,x,y) = \min\{\overline{w}(t,x,y), \overline{w}(t,x,0) + K'y\}$. We claim that $\tilde{w}_1$ is a strong supersolution to (1.1)-(1.2) in $Q_{\tau,R}$. Indeed, at each point $P_0 = (t_0,x_0,y_0)$, if $y_0 = 0$, then the set of subdifferential of $\tilde{w}_1$ is a subset of that of $\overline{w}$. Moreover, when $y_0 > 0$ it is easy to see that $(t,x,y) \mapsto \overline{w}(t,x,0) + K'y$ is a supersolution to (**??**) in the interior points of $\mathbb{H}$ provided K' is large enough, since $\overline{w}$ is Lipschitz continuous in the (t,x) variable, with a uniform in y constant (recall that ε is fixed). Now, $\overline{w}$ is already lower semi-continuous, so that

$$\liminf_{(t,x,y) \to (t_0,x_0,0)} \overline{w}(t,x,y) \geq \overline{w}(t_0,x_0,0).$$

Noting that $(t,x,y) \mapsto \overline{w}(t,x,0) + Ky$ is upper semi-continuous on $\partial\mathbb{H} = \{(t,x,0) : \ t > 0, \, x \in \mathbb{R}\}$, it follows that $\tilde{w}_1$ is continuous on the hyperplane $(0,\infty) \times \partial\mathbb{H}$, and remains semi-concave in (t,x) (since minimum of semi-concave functions are semi-concave).

A key consequence of the semiconvexity of $\underline{w}$ and semiconcavity of $\overline{w}$ in the variables (t,x) (they are now replaced by the associated sup/inf-convolutions with small enough ε) is that both of $\underline{w}$, $\overline{w}$ are differentiable in (t,x) at the maximum point $\bar{P} = (\bar{t}, \bar{x}, 0)$, and we denote the derivatives as follows

$$\bar{q} := \underline{w}_x(\bar{P}) = \overline{w}_x(\bar{P}), \quad -\bar{\lambda} := \underline{w}_t(\bar{P}) = \overline{w}_t(\bar{P}). \tag{1.18}$$

Step five: convergence of sub/superdifferential.

Claim. *Let $\bar{P} = (\bar{t}, \bar{x}, 0)$ and $\bar{q}$, $\bar{\lambda}$ be specified above. Let $P_j = (t_j, x_j, y_j)$ such that $P_j \to \bar{P}$ be given. If $(-\lambda_j, q_j)$ is an element in the subdifferential of $(t,x) \mapsto \underline{w}(t,x,y_j)$ (resp. superdifferential of $\overline{w}$) at (t_j,x_j), then $(-\lambda_j, q_j) \to (-\bar{\lambda}, \bar{q})$.*

The claim follows from [1, Proposition 5.1.1(v)]. (By construction, there exists $C > 0$ independent of j such that $f_j(t,x) = \underline{w}(t,x,y_j) + C(|x - x_j|^2 + |t - t_j|^2)$ is convex for all j, so that $(-\lambda_j, q_j)$ defines a supporting hyperplane for f_j. Moreover, the continuity of $\underline{w}$ implies that $f_j \to \underline{w}(t,x,0) + C(|x - \bar{x}|^2 + |t - \bar{t}|^2)$ uniformly, so that the limit $\lim(-\lambda_j, q_j)$ again defines a supporting hyperplane, and is uniquely identified as the derivative $D_{(t,x)}\underline{w}(\bar{t}, \bar{x}, 0) = (-\bar{\lambda}, \bar{q})$ as specified in (1.18). Recall that $\underline{w}$ is differentiable in (t,x) at $\bar{P}$.)

Define

$$N(r_1) = (\bar{t} - r_1, \bar{t} + r_1) \times (\bar{x} - r_1, \bar{x} + r_1) \times (0, r_1).$$

[1] Q: Why is there no such issue with $\underline{w}$? A: Because $\underline{w}$ is Lipschitz continuous in y, but $\overline{w}$ might be discontinuous in y.

Recalling the fact that $\underline{w}$ is Lipschitz continuous in $\overline{Q_{\tau,R}}$ (so that the p-component of the superdifferential is uniformly bounded), we can take $r_1 > 0$ small enough so that (i) $\underline{w} - \ell t > 0$ in $\overline{N(r_1)}$ and (ii) for any element $(-\lambda, q, p)$ belonging to the subdifferential of $\underline{w}$ at some point $P \in N(r_1)$, we have

$$|\lambda - \bar{\lambda}| + |q - \bar{q}| \to 0 \quad \text{as } P \to \bar{P} = (\bar{t}, \bar{x}, 0), \tag{1.19}$$

and

$$|\lambda - \bar{\lambda}| + |q - \bar{q}| + \big|(-\lambda + H_{\rm f}(q,p)) - (-\bar{\lambda} + H_{\rm f}(\bar{q},p))\big| < \eta. \tag{1.20}$$

 # Step six: defining $u(y)$, $v(y)$, $\underline{p}_u, \overline{p}_u$ and $\underline{p}_v$

Next, we would like to reduce to the one-dimensional problem. To this end define

$$u(y) = \underline{w}(\bar{t}, \bar{x}, y) \quad \text{and} \quad v(y) = \overline{w}(\bar{t}, \bar{x}, y)$$

and let

$$\underline{p}_u = \liminf_{y \to 0+} \frac{u(y) - u(0)}{y}, \quad \overline{p}_u = \limsup_{y \to 0+} \frac{u(y) - u(0)}{y}, \quad \underline{p}_v = \liminf_{y \to 0+} \frac{v(y) - v(0)}{y}.$$

Claim. $\underline{p}_u, \overline{p}_u, \underline{p}_v$ *are finite, and satisfies*

$$\underline{p}_v \le \underline{p}_u \le \overline{p}_u. \tag{1.21}$$

Indeed, $\underline{p}_u$, $\overline{p}_u$ are finite since $\underline{w}$ is Lipschitz continuous in $\overline{Q_{\tau,R}}$. Also, $u - v$ attains a local maximum at $y = 0$, it follows that $v(y) - v(0) \ge u(y) - u(0) \ge -C|y|$. Taking liminf in $y \to 0+$, we see that $\underline{p}_v$ is also finite. By construction, it immediately holds that $\underline{p}_v \le \underline{p}_u \le \overline{p}_u$.

 # Step seven: reduction to one-dimensional problem.

We claim that the set $D^+\underline{w}(\bar{P})$ of superdifferential of $\underline{w}$ and the set $D^-\overline{w}(\bar{P})$ of subdifferential of $\overline{w}$ satisfy

$$D^+\underline{w}(\bar{P}) \supseteq \{(-\bar{\lambda}, \bar{q}, p) : \ p \ge \overline{p}_u\} \ \text{ and } \ D^-\overline{w}(\bar{P}) \supseteq \{(-\bar{\lambda}, \bar{q}, p) : \ p \le \underline{p}_v\}.$$

This is proved in [1, Lemma 15.3.4] by regularization argument. Below, we give an alternative proof here for the convenience of the reader. We only prove the first assertion as the latter is analogous. Since $\bar{P}$ lies on the boundary, it suffices to show that $(-\bar{\lambda}, \bar{q}, \overline{p}_u) \in D^+\underline{w}(\bar{P})$, i.e.

$$\underline{w}(t,x,y) \le \underline{w}(\bar{t}, \bar{x}, 0) + (-\bar{\lambda}, \bar{q}, \overline{p}_u) \cdot (t - \bar{t}, x - \bar{x}, y) + o(y + \sqrt{(t-\bar{t})^2 + (x - \bar{x})^2}). \tag{1.22}$$

To this end, denote

$$t_* = t - \bar{t}, \quad x_* = x - \bar{x}, \quad \text{and} \quad \delta = \sqrt{t_*^2 + x_*^2}.$$

By Lemma 1.6, there exists an element $(-\lambda, q) \in D^-_{t,x}\underline{w}(t,x,y)$ such that

$$\begin{aligned} \underline{w}(t,x,y) &\le u(y) + (-\lambda, q) \cdot (t_*, x_*) + C_0\delta^2 \\ &\le \underline{w}(\bar{t}, \bar{x}, 0) + \overline{p}_u y + o(y) + (-\lambda, q) \cdot (t_*, x_*) + C_0\delta^2 \\ &\le \underline{w}(\bar{t}, \bar{x}, 0) + (-\bar{\lambda}, \bar{q}, \overline{p}_u) \cdot (t_*, x_*, y) + o(y) + (|\lambda - \bar{\lambda}| + |q - \bar{q}|)\delta + C_0\delta^2 \end{aligned}$$

where we used $u(y) = \underline{w}(\bar{t}, \bar{x}, y) \le \underline{w}(\bar{t}, \bar{x}, 0) + \overline{p}_u y + o(y)$ (which follows from the definition of $\overline{p}_u$) in the second inequality. Finally, combining with (1.19), we obtain (1.22).

By Lemma 1.4, we have [[we might be better off using $B(\bar{q})$ in place of $p_{\bar{q}}$ below]]

$$\begin{cases} \min\{-p + p_{\bar{q}}, \mathcal{H}(p) + 2\eta\} \le 0 & \text{for all } p \ge \overline{p}_u, \\ \max\{-p + p_{\bar{q}}, \mathcal{H}(p)\} \ge 0 & \text{for all } p \ge \underline{p}_v, \end{cases} \tag{1.23}$$

where $\bar{\lambda}, \bar{q}$ are specified in (1.18), and $p_{\bar{q}}$ is as in (1.7), and

$$\mathcal{H}(p) = -\bar{\lambda} + H(\bar{q}, p).$$

Moreover, the critical slope lemmas (Lemmas **??** and **??**) asserts that

$$\mathcal{H}(\overline{p}_u) + 2\eta \le 0 \le \mathcal{H}(\underline{p}_v). \tag{1.24}$$

 # Step eight: $u(y)$ is subsolution to one-dimensional problem.

In this step, we show that $u(y) = \underline{w}(\bar{t}, \bar{x}, y)$ satisfies, in the viscosity sense,

$$\mathcal{H}(u') + \eta \le 0 \quad \text{in } (0, r_1).$$

Indeed, fix a point $y_1 \in (0, r_1)$ and a test function $\psi(y)$ such that $\psi(y_1) = u(y_1)$ and $u(y) - \psi(y)$ has a strict global maximum at y_1, we will show $\mathcal{H}(\psi'(y_1)) + \eta \leq 0$. Now, consider the test function $\tilde{\psi}(t,x,y) = (-\bar{\lambda}, \bar{q}) \cdot (t - \bar{t}, x - \bar{x}) + \phi(t,x) + \psi(y)$, where

$$\phi(t,x) = \eta\sqrt{\delta^2 + (t-\bar{t})^2 + (x-\bar{x})^2}.$$

Denote $Q = B_\delta \times (y_1 - \delta_1, y_1 + \delta_1)$. We claim that $G(t,x,y) = \underline{w}(t,x,y) - \tilde{\psi}(t,x,y)$ attains maximum over $\overline{Q}$ at an interior point. It is enough to check that for $\delta > 0$ small enough, we have

$$\text{(i)} \quad \max_{\overline{Q}} G \geq \phi(\bar{t},\bar{x}); \qquad \text{(ii)} \quad \max_{\partial Q} G < \phi(\bar{t},\bar{x}).$$

Assertion (i) follows from $u(y_1) = \psi(y_1)$, so that

$$\max_{\overline{Q}} G \geq G(\bar{t},\bar{x},y_1) = u(y_1) - \psi(y_1) + \phi(\bar{t},\bar{x}) = \phi(\bar{t},\bar{x}).$$

For assertion (ii), suppose $(t,x,y) \in \partial Q$. Then Lemma 1.6 gives

$$G(t,x,y) \leq u(y) - \psi(y) + (-\lambda + \bar{\lambda}, q - \bar{q}) \cdot (t - \bar{t}, x - \bar{x}) + C_0(|t-\bar{t}|^2 + |x-\bar{x}|^2) - \phi(t,x). \tag{1.25}$$

Suppose $y = y_1 \pm \delta_1$, then for δ sufficiently small,

$$G(t,x,y) \leq \sup_{y = y_1 \pm \delta_1} (u - \psi) + O(|t-\bar{t}| + |x - \bar{x}|) + \phi(\bar{t},\bar{x}) < \phi(\bar{t},\bar{x}),$$

as $\sup_{y=y_1\pm\delta_1}(u-\psi) < 0$ and is independent of δ. It remains to consider the case $|t-\bar{t}|^2 + |x-\bar{x}|^2 = \delta^2$. Then $\phi(t,x) = \eta\delta\sqrt{2}$, and (1.25) implies

$$\begin{aligned} G(t,x,y) &\leq u(y) - \psi(y) + (-\lambda + \bar{\lambda}, q - \bar{q}) \cdot (t-\bar{t}, x-\bar{x}) + C_0\delta^2 - \phi(t,x) \\ &\leq (|\lambda - \bar{\lambda}| + |q - \bar{q}|)\delta + C_0\delta^2 - \eta\delta\sqrt{2} \\ &\leq -(\sqrt{2}-1)\eta\delta + C_0\delta^2 < 0 \qquad \text{for } \delta \text{ small,} \end{aligned}$$

where we used (1.20) in the third inequality. This proves assertion (ii). We conclude that, for all δ sufficiently small, $\underline{w}(t,x,y) - \tilde{\psi}(t,x,y)$ attains its maximum in $\overline{Q} = \overline{B_\delta} \times [y_1 - \delta_1, y_1 + \delta_1]$ at an interior point $(t_\delta, x_\delta, y_\delta) \in Q$, i.e.

$$-\lambda + H(q, \psi'(y_\delta)) \leq -2\eta,$$

for some element $(-\lambda, q, \psi'(y_\delta))$ belonging to the superdifferential of $\underline{w}$ at the point $(t_\delta, x_\delta, y_\delta)$. Using (1.20), we deduce

$$\mathcal{H}(\psi'(y_\delta)) = -\bar{\lambda} + H(\bar{q}, \psi'(y_\delta)) \leq -\eta. \tag{1.26}$$

Claim. *$y_\delta \to y_1$ as $\delta \to 0$.*

Recall that $y \mapsto \underline{w}(\bar{t},\bar{x},y) - \psi(y)$ has a strict global maximum at y_1 in $[y_1 - \delta_1, y_1 + \delta_1]$. Since $(t_\delta, x_\delta) \to (\bar{t},\bar{x})$ as $\delta \to 0$, and that $\underline{w} - \psi$ is continuous in y, it follows that the maximum point y_δ of $y \mapsto \underline{w}(t_\delta, x_\delta, y) - \psi(y)$ converges to y_1. This proves the claim. Letting $\delta \to 0$ in (1.26), we obtain $\mathcal{H}(\psi'(y_1)) \leq \eta$. This completes Step eight.

 # Step nine: In this step, we show

$$\mathcal{H}(p) + \eta \leq 0 \qquad \text{for } p \in [\underline{p}_u, \bar{p}_u]. \tag{1.27}$$

First, recall that $u(y)$ is Lipschitz, so that the classical derivative u' exists in a set S such that $(0, r_1) \setminus S$ is of measure zero relative to $\mathbb{R}$, and $\mathcal{H}(u'(y)) \leq -\eta$ holds pointwise for each $y \in S$. It follows, by continuity of $\mathcal{H}$, that

$$\mathcal{H}(\inf_S u') + \eta \leq 0 \qquad \text{and} \qquad \mathcal{H}(\sup_S u') + \eta \leq 0. \tag{1.28}$$

The convexity of $\mathcal{H}$ then yields $\mathcal{H}(p) + \eta \leq 0$ for all $p \in [\inf_S u', \sup_S u']$.

Next, we claim that that

$$[\underline{p}_u, \bar{p}_u] \subset [\inf_S u', \sup_S u']. \tag{1.29}$$

Indeed, since u is Lipschitz,

$$\frac{u(y) - u(0)}{y} = \frac{1}{y}\int_{(0,y)\cap S} u'(s)\,ds \leq \sup_S u'.$$

Taking limsup as $y \to 0+$ in the above, we obtain $\bar{p}_u \leq \sup_S u'$. The proof for $\underline{p}_u \geq \inf_S u'$ is the same. This proves (1.29). Finally, (1.28) and (1.29) implies (1.27). This completes Step nine.

 # Step ten: applying the Lions-Souganidis lemma. Summarizing (1.21), (1.23), (1.24) and (1.27), we deduce

$$\begin{cases} \underline{p}_u \le \underline{p}_v, \quad \eta + \mathcal{H}(\underline{p}_u) \le 0 \le \mathcal{H}(\underline{p}_v), \\ \min\{-p' + p_{\bar{q}}, \eta + \mathcal{H}(p')\} \le 0 & \text{for all } p' \ge \underline{p}_u, \\ \max\{-p' + p_{\bar{q}}, \mathcal{H}(p')\} \ge 0 & \text{for all } p \le \underline{p}_v. \end{cases} \tag{1.30}$$

In view of (1.30), the assumptions of Lemma 1.5 are satisfied if we take

$$a = \eta, \quad b = 0, \quad p_0 = -\underline{p}_u - p_{\bar{q}}, \quad q_0 = -\underline{p}_v - p_{\bar{q}}, \quad H_0(p) = \mathcal{H}(-p - p_{\bar{q}}).$$

where $\mathcal{H}(p) = \eta - \bar{\lambda} + H(\bar{q}, p)$. Then Lemma 1.5 applies to deduce $a \le b$, which means $\eta \le 0$, which is a contradiction. □

References

[1] Guy Barles and Emmanuel Chasseigne. An Illustrated Guide of the Modern Approaches of Hamilton-Jacobi Equations and Control Problems with Discontinuities. working paper or preprint, April 2023.

[2] Cyril Imbert and Régis Monneau. Flux-limited solutions for quasi-convex Hamilton-Jacobi equations on networks. *Ann. Sci. Éc. Norm. Supér. (4)*, 50(2):357–448, 2017.

[3] Pierre-Louis Lions and Panagiotis E. Souganidis. Well-posedness for multi-dimensional junction problems with Kirchoff-type conditions. *Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl.*, 28(4):807–816, 2017.