ON COOPERATIVE ELLIPTIC SYSTEMS: PRINCIPAL EIGENVALUE AND CHARACTERIZATION OF THE MAXIMUM PRINCIPLE

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The purpose of this set of notes is to present the connection between the classical maximum principle with the principal eigenvalue of the elliptic operator. We will start with the maximum principle for single equations and proceed to the case of cooperative (or weakly-coupled) systems. By adopting an idea due to G. Sweers, we give a characterization of the principal eigenvalue for a cooperative system in terms of the spectral radius of a related positive compact operator, which leads to an eigenvalue comparison criterion. As an application, we present a recent result concerning the vanishing viscosity limit of the principal eigenvalue. For simplicity we only treat the Dirichlet case in this note, and we remark that analogous results in the Neumann and Robin cases follow with minor modifications of the proofs. This is joint work with Y. Lou.

1. INTRODUCTION

Suppose Ω is a smooth and bounded domain in $\mathbb{R}^n$ and $u \in C^1(\bar{\Omega}) \cap C^2(\Omega)$ and $f \in C^1(\bar{\Omega})$ satisfies

$$-\Delta u + \mu u = f \geq 0 \quad \text{in } \Omega \quad \text{and} \quad u \geq 0 \quad \text{on } \partial\Omega. \tag{1}$$

It is well-known that if $\mu = 0$, then $\inf_\Omega u = \inf_{\partial\Omega} u$ and u cannot achieve an interior minimum unless it is a constant. On the contrary, if $\mu > 0$, then one can easily see that sometimes u can attain a positive minimum in the interior. More generally, we have

Theorem 1. *Suppose $\mu \geq 0$ and $Lu \geq 0$.*

(i) $\inf_\Omega u \geq \inf_{\partial\Omega} \min\{u, 0\}$. ($\Leftrightarrow$ $\sup_\Omega -u \leq \sup_{\partial\Omega} \sup\{-u, 0\}$)

(ii) *u cannot achieve an interior non-positive minimum unless it is a constant.*

In particular, if $u|_{\partial\Omega} \geq 0$, then either $u \equiv 0$ or $u > 0$ in Ω with $\partial_\nu u(x_0) < 0$ for all $x_0 \in \partial\Omega$ such that $u(x_0) = 0$.

Note: (i) and (ii) are called the weak and strong maximum prinicples respectively.

What if $\mu > 0$? Does the maximum principle still hold? In general, consider the following single equation.

$$\begin{cases} Lu := -a_{ij} D_{ij} u + B_j D_j u + cu = f \geq 0 & \text{in } \Omega, \\ u \geq 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where Ω is a smooth and bounded domain in $\mathbb{R}^n$, $a_{ij}, b_j, c \in C(\bar{\Omega})$, and for some positive constants σ_1, σ_2 we have

$$\sigma_1 |\xi|^2 \leq a_{ij}(x) \xi_i \xi_j \leq \sigma_2 |\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^n.$$

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Definition 1. *We say that the maximum principle (MP) holds for* (2) *when (i) for each* $f \in C(\bar{\Omega})$, $f \geq 0$, *then any solution* u *of* (2) *satisfies* $u \geq 0$*; (ii) in addition, if either* $u|_{\partial\Omega} \not\equiv 0$ *or* $f \not\equiv 0$*, then* $u > 0$ *in* Ω *and* $\partial_\nu u < 0$ *on* $\{x_0 \in \partial\Omega : u = 0\}$.

Lemma 1.1 (Weak maximum principle)**.** *If* $c \geq 0$ *and* $Lu \geq 0$*, then* $\inf_\Omega u \geq \inf_{\partial\Omega} \min\{u, 0\}$.

Proof. It is readily seen that if $Lu > 0$, then a strong maximum principle holds: u cannot assume an interior minimum. In general, $L(-e^{\gamma x_1}) > 0$ in Ω if γ is chosen large. Hence for all $\epsilon > 0$, $L(u - \epsilon e^{\gamma x_1}) > 0$ and $\inf_\Omega u - \epsilon e^{\gamma x_1} = \inf_{\partial\Omega} u - \epsilon e^{\gamma x_1}$ and the lemma follows by taking $\epsilon \to 0$. □

Lemma 1.2 (Hopf's Lemma)**.** *Assume* Ω *satisfies a uniform interior sphere condition. Suppose* $u \geq 0$ *and* $Lu \geq 0$*, then either* $u \equiv 0$ *or* $u > \epsilon d(x)$ *in* Ω *for some* $\epsilon > 0$*, where* $d(x) = \mathrm{dist}(x, \partial\Omega)$.

In particular, if $c \geq 0$ and $u \geq 0$ on $\partial\Omega$, then $u \geq 0$ by weak maximum principle and the conclusion of Lemma 1.2 holds. i.e. The weak MP and strong MP are equivalent.

Proof. $Lu - \min\{c, 0\}u \geq Lu \geq 0$. Hence we may assume without loss that $c \geq 0$. It remains to show that if $u > 0$ in some $B_R(0) \subset \Omega$, then $u > \epsilon(R - |x|)$ in $B_R(0)$ for some $\epsilon > 0$. (Since then there cannot be any interior point x_0 where $u(x_0) = 0$, and that $\partial_\nu u < 0$ on the bondary.) Now $\inf_{B_{R/2}(0)} u = \epsilon_1 > 0$ and one can apply the weak maximum principle to $v = u - \frac{\epsilon_1 R^\sigma}{2^\sigma - 1}(|x|^{-\sigma} - R^{-\sigma})$ ($Lv > 0$ for σ large) to yield the desired lower estimate. □

More generally, the following is proven in [Walter1989].

Theorem 2. *Suppose* $h \in C(\bar{\Omega}) \cap C^2(\Omega)$ *satisfies*

$$Lh \geq 0 \quad \text{in } \Omega \quad \text{and } h(x) > 0 \quad \text{in } \Omega. \tag{3}$$

Then, for any $u \in C^1(\bar{\Omega}) \cap C^2(\Omega)$ *such that*

$$Lu \geq 0 \quad \text{in } \Omega, \quad u \geq 0 \quad \text{on } \partial\Omega,$$

one of the following holds

(i) $u = \beta h$ *for some* $\beta < 0$*;*
(ii) $u \equiv 0$ *in* Ω*;*
(iii) $u > 0$ *in* Ω *and* $\partial_\nu u(x_0) < 0$ *for all* $x_0 \in \partial\Omega$ *such that* $u(x_0) = 0$.

In particular, if either $Lh \not\equiv 0$ *or* $h \not\equiv 0$ *on* $\partial\Omega$*, then (i) is impossible and the MP holds.*

Remark 1.1. *When* $c \geq 0$*, then any positive constants* $h \equiv h_0 > 0$ *is a strict supersolution satisfying the assumption of Theorem 2.*

Proof. If $u \geq 0$, then by Lemma 1.2, either $u \equiv 0$ or $u > 0$ in Ω.

If $u < 0$ somewhere, then ($\partial_\nu h|_{\partial\Omega} < 0$ Lemma 1.2) there exists a minimal $\mu > 0$ such that $v := u + \mu h \geq 0$ in Ω. Now by Lemma 1.2 again either $v \equiv 0$ (i.e. $u = \beta h$ for some $\beta < 0$) or $v > \epsilon d(x)$ in Ω. Suppose the latter holds, then by minimality of μ, for all m large, there exists $x_m \in \Omega$ such that

$$u(x_m) + \left(\mu - \frac{1}{m}\right) h(x_m) = v(x_m) - \frac{1}{m} h(x_m) < 0 \quad \Leftrightarrow \quad \left(\mu - \frac{1}{m}\right) h(x_m) < -u(x_m) \leq M d(x_m).$$

But this contradicts
$$v(x_m) - \frac{1}{m}h(x_m) > \epsilon d(x_m) - \frac{M}{\mu m - 1}d(x_m).$$
□

Example 1. *Let $L = -\Delta - \mu$, then the prinicpal eigenfunction $-\Delta\phi_1 - \mu\phi_1 = (\mu_1 - \mu)\phi_1 > 0$ if $\mu < \mu_1$ serves as a strict super solution and hence the MP holds up to $\mu < \mu_1$.*

Hence it is important to consider the eigenvalue problem:
$$\begin{cases} L\phi = -a_{ij}D_{ij}\phi + b_jD_j\phi + c\phi = \mu\phi & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega, \end{cases} \tag{4}$$
The main result for the single equation in this note is the following.

Theorem 3. *Problem (4) has a principal eigenvalue $\mu_1 = \mu_1(L)$, with the property that*

(i) *μ_1 is real and simple,*
(ii) *μ_1 is the unique eigenvalue corresponding to a non-negative eigenfunction,*
(iii) *the corresponding eigenfunction ϕ_1 can be chosen so that $\phi_1 > 0$ in Ω and $\partial_\nu\phi_1 < 0$ on $\partial\Omega$,*
(iv) *$\mu_1 < \operatorname{Re}\mu$ for all eigenvalue $\mu \neq \mu_1$.*

Moreover, MP holds for (2) if and only if $\mu_1 > 0$.

2. The Krein-Rutman Theorem

Let X be a Banach space.

Definition 2. *$K \subset X$ is a cone if K is closed, convex such that $sK \subset K$ for all $s \geq 0$ and $K \cap (-K) = \{0\}$.*

A given cone K induces a partial ordering:
$$u, v \in X, \quad u \leq v \quad \text{iff} \quad v - u \in K.$$
In this case X is called a partially ordered Banach space with positive cone K.

Definition 3.
(i) *If $\overline{K - K} = X$, then K is a total cone.*
(ii) *If $K^o \neq \emptyset$, then K is a solid cone.*

Note that *any solid cone is total*: Suppose $B_{2r}(z) \subset K$, then for all $y \in X$ we have $z, z + ry \in K$, which gives $y \in r^{-1}K - r^{-1}K = K - K$.

Definition 4. *We write $v > u$ if $v - u \in K \setminus \{0\}$ and $v \gg u$ if $v - u \in K^o$.*

We say that $T : X \to X$ (i) is positive if $T(K) \subset K$ and (ii) is strongly positive if $T(K \setminus \{0\}) \subset K^o$.

Example 2. *$X = L^p(\Omega)$, $K = \{\text{non-negative functions}\}$, then $K - K = X$ but $K^0 = \emptyset$.*

Example 3. *$X = C(\bar{\Omega})$, then $K = \{\text{non-negative functions}\}$ is a solid cone.*

Example 4. *$X = C^1_0(\bar{\Omega}) = \{u \in C^1(\bar{\Omega}) : u|_{\partial\Omega} = 0.\}$, then $K = \{\text{non-negative functions}\}$ also forms a solid cone with $K^o = \{u \in K : u > 0 \text{ in } \Omega \text{ and } \partial_\nu u < 0 \text{ on } \partial\Omega\}$.*

Theorem 4. *Let X be a Banach space with positive cone $K \subset X$ and $T : X \to X$ is a compact linear operator.*

(a) *Suppose K is a total cone and $T : X \to X$ is positive with positive spectral radius: $r(T) := \limsup_{m\to\infty} \sqrt[m]{\|T^m\|} > 0$. Then $r(T)$ is an eigenvalue of T with an eigenvector $u \in K \setminus \{0\}$.*

(b) *Suppose K is a solid cone and T is strongly positive, then*
   (a) *$r(T) > 0$ is a simple eigenvalue with eigenvector $v \in K^o$. There is no other eigenvalue with a positive eigenvector.*
   (b) *$|\lambda| < r(T)$ for all eigenvalues $\lambda \neq r(T)$.*

See Theorem 19.2 and Ex. 12 in [Deimling1985] for the proof of part (a) and Theorem 1.2 in [Du2006] for the derivation of (b) from (a).

3. Principal Maximum for Single Equation

Let $L = -a_{ij}D_{ij} + b_j D_j + c$ be as in the introduction section. First we demonstrate the existence of principal eigenvalue $\mu(L)$. Then by Lemma 1.2, $(L + \beta)$ satisfies a strong MP if $\beta + c \geq 0$. Moreover, one can observe that the positivity readily implies uniqueness of the following problem

$$Lu + \beta u = f \quad \text{in } \Omega \quad \text{and} \quad u = 0 \quad \text{on } \partial\Omega.$$

i.e. (by Fredholm's alternative) $T := (L + \beta)^{-1} : X \to X$ exists and is strongly positive, where $X = C^1_0(\bar{\Omega})$ and K be the set of non-negative functions, as in Example 3. By Theorem 4(b), $r(T) > 0$ is a simple eigenvalue with an eigenfunction $u \in K^o$. Hence

$$\frac{1}{r(T)} u = Lu + \beta u \Leftrightarrow Lu = \left(\frac{1}{r(T)} - \beta\right) u$$

and $\mu_1 := \frac{1}{r(T)} - \beta$ is a simple eigenvalue of L with eigenfunction $u \in K^o$. Furthermore, define $B : X \to X$ by

$$Bu := e^{tL}u|_{t=1},$$

where e^{-tL} is the semigroup generated by L, i.e. for each $u \in X$, $v(x,t) = e^{-tL}u(x)$ is the unique solution to

$$\begin{cases} v_t + Lv = 0 & \text{in } \Omega \times (0, \infty), \\ v(x, 0) = u(x) & \text{in } \Omega. \end{cases}$$

Then it is well-known that B is strongly positive. Moreover, if μ is an eigenvalue of L, then $e^{-\mu}$ is an eigenvalue of B with the same eigenspace. Hence the principal eigenvalue of B guaranteed by Theorem 4(b) is $e^{-\mu_1}$ and we have for all eigenvalue $\mu \neq \mu_1$ of L,

$$|e^{-\mu}| < e^{-\mu_1} \Leftrightarrow \mu_1 < \operatorname{Re} \mu.$$

This proves the first part of Theorem 3. Now the second part of Theorem 3 follows readily from Theorem 2, as any positive principal eigenfunction ϕ_1 is a strict supersolution:

$$L\phi_1 = \mu_1 \phi_1 > 0 \quad \text{in } \Omega, \quad \phi_1 = 0 \quad \text{on } \partial\Omega.$$

Here we give an alternative functional analytic proof of the second part of Theorem 2, which can be generalized to study the maximum principle in cooperative systems. Given any $\beta > 0$ such that

$\beta + c > 0$, and $\mu \in (-\beta, \infty)$, define $A_{\mu,\beta} : C_0^1(\bar{\Omega}) \to C_0^1(\bar{\Omega})$ by $A_{\mu,\beta}u := (L+\beta)^{-1}[(\mu+\beta)u]$, where $(L+\beta)^{-1}f$ is the unique solution to

$$Lu + \beta u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega.$$

Lemma 3.1. *Suppose $\mu + \beta > 0$, i.e. $A_{\mu,\beta} : X \to X$ is strongly positive, then*

(i) $r(A_{\mu,\beta}) < 1 \iff \mu < \mu_1(L)$;
(ii) $r(A_{\mu,\beta}) = 1 \iff \mu = \mu_1(L)$;
(iii) $r(A_{\mu,\beta}) > 1 \iff \mu > \mu_1(L)$.

Proof. Again let $\phi_1 \in K^o$ be a principal eigenfunction of L corresponding to $\mu_1(L)$, then

$$A_{\mu,\beta}\phi_1 = \frac{\mu+\beta}{\mu_1(L)+\beta}\phi_1.$$

Hence by the characterization that $r(A_{\mu,\beta})$ being the only eigenvalue of $A_{\mu,\beta}$ corresponding to an eigenfunction in $K \setminus \{0\}$, we have $r(A_{\mu,\beta}) = \frac{\mu+\beta}{\mu_1(L)+\beta}$. This proves the lemma. □

Proof of second part of Theorem 3. It is clear that if $\mu_1 \leq 0$, then the principal eigenfunction provides a counter example to the MP. Suppose $\mu_1 > 0$, then by Lemma 3.1(i), for any β large (so that $(L+\beta)^{-1}$ is strongly positive), $r(A_{0,\beta}) < 1$. Hence, $\sum_{j=0}^{\infty} A_{0,\beta}^j : X \to X$ is well-defined and strongly positive. Suppose we have

$$Lu = f \geq 0,$$

then it is equivalent to

$$Lu + \beta u = \beta u + f.$$

By definition of $A_{0,\beta}$, we have

$$u = A_{0,\beta}u + (L+\beta)^{-1}f.$$

Finally, taking $\sum_{j=0}^{\infty} A_{0,\beta}^j$ on both sides, we have $u = Tf$, where

$$T := \sum_{j=0}^{\infty} A_{0,\beta}^j (L+\beta)^{-1} : X \to X$$

is a strongly positive operator. This completes the proof. □

We end this section on single equation with the following observation:

Corollary 3.2. *Assume $c \leq 0$, and that $\mu_1(L) > 0$. Suppose (2) holds, then $u \geq 0$ and*

$$-a_{ij}D_{ij}u + b_jD_ju \geq -cu \geq 0,$$

which implies that

$$\inf_{\Omega} u = \inf_{\partial\Omega} u \geq 0. \tag{5}$$

For example, let $L = -\Delta - \mu$, for some $\mu \in [0, \mu_1)$. If

$$-\Delta u - \mu u \geq 0 \quad \text{in } \Omega, \quad u \geq 0 \quad \text{on } \partial\Omega,$$

then (5) holds.

4. Maximum Principle for Cooperative System

Now we consider the following linear cooperative problem:

$$(6)\qquad \begin{cases} L_k u_k := -a^k_{ij} D_{ij} u_k + b^k_j D_j u_k + c^k u_k = h_{kl} u_l + f_k & \text{in } \Omega,\ 1 \leq k \leq N \\ u_k = 0 & \text{on } \partial\Omega,\ 1 \leq k \leq N, \end{cases}$$

where $a^k_{ij}, b^k_j, c^k \in C^(\bar{\Omega})$; for some $\sigma_1, \sigma_2 > 0$,

$$\sigma_1 |\xi|^2 \leq a^k_{ij}(x)\xi_i\xi_j \leq \sigma_2 |\xi|^2 \quad \text{for all } 1 \leq k \leq N,\ x \in \Omega,\ \xi \in \mathbb{R}^n;$$

and that h_{ij} is cooperative and irreducible, i.e.

$$h_{ij} \geq 0 \quad \text{when } i \neq j, \text{ and}$$

$$\text{there does not exist a partition } \alpha \cup \beta = \{1, 2, ..., N\} \text{ such that } h_{ij} = 0 \text{ for } i \in \alpha \text{ and } j \in \beta.$$

The requirement that H is irreducible is equivalent to saying that the system cannot "decoupled", i.e. it cannot be broken down into smaller systems. Alternatively, we can write (6) in vector notation, by letting $L = \operatorname{diag} L_k$, $H = \{h_{ij}\}$, $u = \{u_k\}$ and $f = \{f_k\}$,

$$(7)\qquad \begin{cases} Lu = Hu + f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

In this section, we are going to show the existence of principal eigenvalue for the cooperative problem

$$(8)\qquad \begin{cases} L_k \phi_k = h_{kl}\phi_l + \lambda\phi_k & \text{in } \Omega,\ 1 \leq k \leq N \\ \phi_k = 0 & \text{on } \partial\Omega,\ 1 \leq k \leq N, \end{cases}$$

or in vector notation,

$$(9)\qquad \begin{cases} L\phi = H\phi + \lambda\phi & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega, \end{cases}$$

In this section, set $X = [C^1_0(\bar{\Omega})]^N$, $K = \{u \in X : u_k \geq 0 \text{ in } \Omega \text{ for all } k\}^N$. One can deduce that

$$K^o = \{u \in X : u_k > 0 \text{ in } \Omega \text{ and } \partial_\nu u_k < 0 \text{ on } \partial\Omega \text{ for all } k\}.$$

We are going to show the counterpart of Theorem 3.

Theorem 5. *System* (4) *has a principal eigenvalue $\lambda_1 = \lambda_1(L - H)$, with the following properties:*

(i) *μ_1 is real and simple;*
(ii) *μ_1 is the unique eigenvalue corresponding to an eigenfunction in K (non-negative eigenfunction);*
(iii) *the corresponding eigenfunction ϕ_1 lies in K^o;*
(iv) *$\lambda_1 < \operatorname{Re}\lambda$ for all eigenvalue $\lambda \neq \lambda_1$.*

Moreover, MP holds for (2) *if and only if $\lambda_1 > 0$, i.e. $(L - H)^{-1} : X \to X$ exists and is strongly positive.*

First we demonstrate the existence of principal eigenvalue $\lambda_1(L - H)$. WLOG assume $h_{kl} \geq 0$ for all k, l. Let $\beta > 0$ be large so that for all k, $(L_k + \beta)^{-1} : C^1_0(\bar{\Omega}) \to C^1_0(\bar{\Omega})$ (under Dirichlet b.c.) exists and is strongly positive. Define for such β and $\lambda \in (-\beta, \infty)$, $A_{\lambda,\beta} : X \to X$ by

$$(A_{\lambda,\beta} u)_k := (L_k + \beta)^{-1}[h_{kl} u_l + (\lambda + \beta) u_k] \quad \text{for } 1 \leq k \leq N,$$

or in vector notation

$$A_{\lambda,\beta}u := (L+\beta I)^{-1}[Hu + (\lambda+\beta)u].$$

Then $A_{\lambda,\beta}$ is compact, linear and strongly positive. By Theorem 4(b), $r(A_{\lambda,\beta}) > 0$ is a simple eigenvalue with eigenvector in K^o. Moreover, it is the only eigenvalue corresponding to an eigenvector in K. We establish the counterpart of Lemma 3.1. Note that Lemma 4.1 will become useful when we estimate the vanishing viscosity limit of $\lambda_1(L-H)$ in Section 5.

Lemma 4.1. *The principal eigenvalue $\lambda_1 = \lambda_1(L-H)$ exists. Moreover, the following statements hold.*

(i) $r(A_{\lambda,\beta}) < 1 \quad \Longleftrightarrow \quad \lambda < \lambda_1(L-H)$;

(ii) $r(A_{\lambda,\beta}) = 1 \quad \Longleftrightarrow \quad \lambda = \lambda_1(L-H)$;

(iii) $r(A_{\lambda,\beta}) > 1 \quad \Longleftrightarrow \quad \lambda > \lambda_1(L-H)$.

Note that different from the single equation case, the existence of λ_1 is part of the conclusion here.

Proof.

Claim 1. *If $r(A_{\lambda,\beta}) = 1$, then λ is a simple eigenvalue of $L-H$ equipped with a positive eigenfunction.*

Claim 1 follows from Theorem 4(b).

Claim 2. *There exists at most one eigenvalue λ of $L-H$ corresponding to a non-negative eigenfunction. Moreover, $\lambda < \operatorname{Re}\lambda'$ for all eigenvalue $\lambda' \neq \lambda$ of $L-H$.*

Claim 2 follows from the application of Theorem 4(b) to the semigroup operator $e^{-t(L-H)}$. Since it is similar to the single equation case, we omit the details here. Now by Claims 1 and 2, it remains to show

(I) If $r(A_{\lambda_0,\beta}) < 1$, then there exists $\lambda > \lambda_0$ such that $r(A_{\lambda,\beta}) = 1$.

(II) If $r(A_{\lambda_0,\beta}) > 1$, then $r(A_{\lambda,\beta}) > 1$ for all $\lambda > \lambda_0$. Moreover, there exists some $\beta' > 0$ and $\lambda' \in (-\beta', \infty)$ such that $r(A_{\lambda',\beta'}) = 1$.

Note that (I) and (II) gives the "$\Longrightarrow$" direction of (i) and (iii) and the opposite directions follows automatically. For (I), suppose $r(\lambda_0, \beta) < 1$ for some $\lambda_0 > -\beta$. Observe that for $\lambda > \lambda_0$, and any fixed $u \in K^o$, there exists $\epsilon > 0$ such that

$$(A_{\lambda,\beta}u)_k > (L_k+\beta)^{-1}[(\lambda+\beta)u_k] > (\lambda+\beta)\epsilon u_k.$$

Hence we see that $r(A_{\lambda,\beta}) > 1$ for all large λ. By continuous dependence of $r(A_{\lambda,\beta})$ on λ, we have that $r(A_{\lambda',\beta}) = 1$ for some $\lambda' > \lambda_0$. Hence $\lambda_1 = \lambda' > \lambda_0$. This proves (I).

For (II), suppose $r(A_{\lambda_0,\beta}) > 1$ with eigenvector $\tilde{u} \in K^o$, then for all $\lambda > \lambda_0$,

$$A_{\lambda,\beta}\tilde{u} = A_{\lambda_0,\beta}\tilde{u} + (\lambda-\lambda_0)(L+\beta I)^{-1}\tilde{u} \geq r(A_{\lambda_0,\beta})\tilde{u}$$

and yields $r(A_{\lambda,\beta}) > 1$ for all $\lambda > \lambda_0$. Hence λ_1, if exists, has to be strictly smaller than λ_0. We now turn the exsitence of λ_1 in case (II). First, we claim that $\|(L_k+\beta)^{-1}\| \to 0$ as $\beta \to \infty$. To see the claim, observe that if for some k,

$$L_k u_k + \beta u_k = f_k, \quad u_k|_{\partial\Omega} = 0$$

then by maximum principle,

$$(\sup_\Omega u_k)(-|c_-^k|_\infty + \beta) \le \sup_\Omega f_k \quad \text{and} \quad -(\inf_\Omega u_k)(-|c_-^k|_\infty + \beta) \le -\inf_\Omega f_k.$$

Combining, we have

$$|u_k|_\infty \le C\beta^{-1}|f_k|_\infty.$$

Hence by elliptic regularity, $\|u_k\|_{C_0^1(\bar\Omega)} = o(|f_k|_\infty)$ as $\beta \to \infty$. i.e. $\|(L_k+\beta)^{-1}\| \to 0$. This implies that $r(A_{-\beta',\beta'}) < 1$ for some $\beta' > 0$ large. Then again by similar arguments as in (I), we derive the existence of $\lambda_1 = \lambda_1(L-H)$. This proves (II) and completes the proof of the lemma. □

Now we are in a position to prove Theorem 5.

Proof of Theorem 5. We have already proved the first part of the theorem concerning the existence and properties of $\lambda_1 = \lambda_1(L-H)$. As before, if $\lambda_1 \le 0$, then the principal eigenfunction provides a counter example to show that the maximum principle does not hold. Suppose $\lambda_1 > 0$, then by Lemma 4.1(i), $r(A_{0,\beta}) < 1$ for some $\beta > 0$ and we can show as before that for each $f \in X$, (7) has a unique solution u, given by Tf, where $T : X \to X$ is a positive compact linear operator given by

$$T = \sum_{j=0}^\infty A_{0,\beta}^j (L+\beta I)^{-1}.$$

And hence the strong maximum principle holds for (7). □

5. Asymptotic Behavior of λ_1 when Diffusion Coefficient Tends to Zero

Consider

$$\begin{cases} DL\Phi = H\Phi + \lambda\Phi & \text{in } \Omega, \\ \Phi = 0 & \text{on } \partial\Omega. \end{cases} \tag{10}$$

Here $D = \mathrm{diag}\{d_k\}$, $L = \mathrm{diag}\{L_k\}$ is as in the previous section and $\Phi = \{\phi_k\}$. Suppose $H = \{h_{kl}\}$ is cooperative and irreducible. WLOG, we may assume that $h_{ij}(x) \ge 0$ for all i, j.

We have seen that (10) has a principal eigenvalue λ_1. In this section, we are interested in the asymptotic behavior of λ_1 as $\max\{d_k\} \to 0$.

To state our result, we recall the counterpart of Krein-Rutman's Theorem for non-negative matrices:

Theorem 6 (Perron-Frobenius)**.** *Suppose $H \in \mathbb{R}^{N\times N}$ is non-negative, then $\sigma(H) := r(H) > 0$ is an eigenvalue of H with a nonnegative eigenvector $\alpha \in (\mathbb{R}_+ \cup \{0\})^N$. Moreover if H is irreducible, then $\sigma(H)$ is simple with $\alpha \in (\mathbb{R}_+)^N$ and $\sigma(H)$ is the only eigenvalue of H possessing a non-negative eigenvector.*

The main result of this section is the following theorem contained in [LamLou2013]:

Theorem 7 (Lam-Lou(2013))**.** *Let λ_1 be the principal eigenvalue of* (10)*, then*

$$\lim_{\max\{d_k\}\to 0} \lambda_1 = -\max_{\bar\Omega} \sigma(H(x)).$$

Remark 5.1. *In case $d_i \sim d_j$ for all i, j, Theorem 7 is proved earlier in* [Dancer2009]*, by considering the limiting system in either R^N or half space with constant coefficient. In general, there is no limiting system and we have to rely on a different method.*

To show the Theorem, we will need the following boundary Lipschitz estimate:

Proposition 1. *Suppose $L = -a_{ij}D_{ij} + b_j D_j + c$ is as defined in the introduction section, then for all $f \in C_0^1(\bar{\Omega})$, $u_d := (dL+1)^{-1}f$ exists for all d small and satisfies*

$$\sup_{\Omega} \frac{|u_d - f|}{\mathrm{dist}(x, \partial\Omega)} \to 0 \qquad \text{as } d \to 0.$$

Proof. The proof follows from a careful application of the barrier method from the proof of Hopf's lemma. Please refer [LamLou2013] for details. □

Proof of Theorem 7. For simplicity, we assume $h_{ij} > 0$ in $\bar{\Omega}$ for all i, j. The general case can be proven by approximation arguments. We shall treat the lim sup and lim inf separately. First we show

$$\liminf_{\max\{d_k\}\to 0} \lambda_1 \geq -\max_{\bar{\Omega}} \sigma(H(x)).$$

As in the previous section, define $A_{\lambda,\beta}u = (DL + \beta I)^{-1}[Hu + (\lambda+\beta)I]$. Then by Lemma 4.1, it suffices to show that for all fixed $\lambda < -\max_{\bar{\Omega}} \sigma(H(x))$, $r(A_{\lambda,|\lambda|+1}) < 1$ when $\max\{d_k\}$ is sufficiently small.

To this end, let $\alpha(x)$ denote the positive eigenvector corresponding to $\sigma(H(x))$, normalized by $\sum \alpha_k^2(x) = 1$. Then by uniqueness it is easy to see that $\alpha(x)$ is continuous. Take any $\varphi \in C_0^1(\bar{\Omega})$ such that $\varphi > 0$ and $\partial_\nu \varphi < 0$ in $\partial\Omega$, and define $u = \varphi\alpha$. Then for all $\epsilon > 0$,

$$\begin{aligned}
A_{\lambda,|\lambda|+1}u &= (DL + (|\lambda|+1)I)^{-1}[Hu + (\lambda + |\lambda| + 1)u] \\
&= (DL + (|\lambda|+1)I)^{-1}[\varphi H\alpha + (\lambda + |\lambda| + 1)\varphi\alpha] \\
&= (DL + (|\lambda|+1)I)^{-1}[(\sigma(H(x)) + \lambda + |\lambda| + 1)u] \\
&\leq (DL + (|\lambda|+1)I)^{-1}[(\max_{\bar{\Omega}} \sigma(H(x)) + \lambda + |\lambda| + 1)u] \\
&< \frac{\max_{\bar{\Omega}} \sigma(H(x)) + \lambda + |\lambda| + 1}{|\lambda|+1}(1+\epsilon)u
\end{aligned}$$

whenever $\max\{d_k\}$ is sufficiently small, by Proposition 1. Hence if we take $\epsilon > 0$ small such that

$$\frac{\max_{\bar{\Omega}} \sigma(H(x)) + \lambda + |\lambda| + 1}{|\lambda|+1}(1+\epsilon) < 1,$$

then $\|A_{\lambda,|\lambda|+1}\| < 1$ and hence $r(A_{\lambda,|\lambda|+1}) < 1$. This proves the first part of the theorem.

Next, we recall the well-known domain and coefficient monotonicity of λ_1. (e.g. It can be proven by our characterization Lemma 4.1. See Proposition 3.4 of [LamLou2013].)

Lemma 5.1. *The principal eigenvalue subject to Dirichlet boundary condition is monotone decreasing in the domain and also in h_{kl} for all k, l.*

Now we proceed to show

$$\limsup_{\max\{d_k\}\to 0} \lambda_1 \le -\max_{\bar\Omega} \sigma(H(x)).$$

Again, it suffices to show that for $\lambda > -\max_{\bar\Omega}\sigma(H(x))$, $r(A_{\lambda,|\lambda|+1}) > 1$. Let $\delta > 0$ be a given small constant. Choose $B = B_r(x_0) \subset \Omega$ such that

$$\sigma(H') > \max_{\bar\Omega} \sigma(H(x)) - \delta$$

where $H' = \{h'_{ij}\} \in \mathbb{R}^{N\times N}$ and for each i,j, $h'_{ij} = \inf_B h_{ij}(x)$. Denote the principal eigenvalue of the following problem be λ'_1:

$$\begin{cases} DLu' = H'u' + \lambda' u' & \text{in } \tilde\Omega, \\ u' = 0 & \text{on } \partial\Omega, \end{cases}$$

Then by Lemma 5.1,

$$\lambda'_1 > \lambda_1. \tag{11}$$

Now take $\varphi \in C^1_0(\bar\Omega)$ as before, and define $u = \{u_k\}$ by $u = \varphi a$, where a is a non-negative eigenvector of the constant matrix H'. Then again,

$$A'_{\lambda,|\lambda+1}u \ge \frac{\sigma(H') + \lambda + |\lambda| + 1}{|\lambda| + 1}(DL + (|\lambda|+1)I)^{-1}[\varphi a] > cu$$

for some $c > 1$. Hence $r(A'_{\lambda,|\lambda|+1}) > 1$ and hence

$$\limsup_{\max\{d_k\}\to 0} \lambda'_1 \le -\sigma(H') < -\max_{\bar\Omega}\sigma(H(x)) + \delta.$$

The theorem now follows from (11) and letting $\delta \to 0$. □

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