Critial Slope Lemmas

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This note is the proof of the critical slope lemmas taken from Imbert and Monneau. The proof for the one-dimensional case is detailed in [1, Lemmas 2.9 & 2.10]. The multi-dimensional case, stated below, is contained in [2], but the proof is omitted. Here we supply the proof for the multi-dimensional case, which is essentially the same as the one-dimensional case.

Define X^+ as the half space

$$X^+ = \{(t,x,y) : \; y \geq 0\}.$$

Fix a point $(\bar{t},\bar{x},0) \in X^+$, and define the 3-dimensional half ball

$$B_r^+ = B_r^+(\bar{t},\bar{x},0) = \{(t,x,y) \in X^+ : \; |(t-\bar{t}, x-\bar{x}, y)| < r.$$

Lemma 0.1 ([2, Lemma A.9]). *Let $u : B_1^+ \to \mathbb{R}$ be a lower semicontinuous function and suppose $\phi(t,x,y)$ is a test function that touches $u(t,x,y)$ from below at some $(\bar{t},\bar{x})$. Define the critical slope at $(\bar{t},\bar{x},0)$*

$$\underline{p} = \sup\{p : \; \exists r > 0, \; u(t,x,y) \geq \phi(t,x,0) + py \;\; \textit{for all } (t,x,y) \in B_\delta^+(\bar{t},\bar{x},0)\}. \tag{0.1}$$

If $\underline{p} < +\infty$, and u and is a viscosity supersolution of

$$u_t + H(u_x, u_y) = 0 \tag{0.2}$$

then

$$\phi_t(\bar{t},\bar{x},0) + H(\phi_x(\bar{t},\bar{x},0), \underline{p}) \geq 0. \tag{0.3}$$

Remark 0.2. Note that $\underline{p}$ is well-defined as the existence of test function implies the set of subdifferential is nonempty.

Proof. By the definition of $\underline{p}$, there exists $\delta > 0$ and $(t_\varepsilon, x_\varepsilon, y_\varepsilon) \in B_{\delta/2}^+(\bar{t},\bar{x},0)$ such that

$$u(t,x,y) \geq \phi(t,x,0) + (\underline{p} - \varepsilon)y \quad \text{for all } (t,x,y) \in B_\delta^+(\bar{t},\bar{x},0), \tag{0.4}$$

$$u(t_\varepsilon, x_\varepsilon, y_\varepsilon) \leq \phi(t_\varepsilon, x_\varepsilon, 0) + (\underline{p} + \varepsilon)y_\varepsilon. \tag{0.5}$$

Now consider a smooth function $\Psi : \mathbb{R}^3 \to [-1, 0]$ such that

$$\Psi(t,x,y) = 0 \quad \text{in } B_{1/2}(0), \quad \Psi(t,x,y) = -1 \quad \text{in } \mathbb{R}^3 B_1(0)$$

and define

$$\Phi(t,x,y) = \phi(t,x) + 2\varepsilon\Psi_\delta(t,x,y) + (\underline{p} + \varepsilon)y,$$

where $\Psi_\delta(t,x,y) = \delta\Psi\left(\frac{t-\bar t}{\delta}, \frac{x-\bar x}{\delta}, \frac{y}{\delta}\right)$ is bounded in C^1 uniformly in δ. Then we have

$$\Phi(t,x,y) = \phi(t,x,0) - 2\varepsilon\delta + (\underline{p}+\varepsilon)y \le u(t,x,y) \quad \text{on } \partial B_\delta^+(\bar t,\bar x,0)\cap\{y>0\}, \tag{0.6}$$

which is satisfied on the curved part of the boundary of $B_\delta^+(\bar t,\bar x,0)$.

$$\Phi(t,x,0) \le \phi(t,x,0) \le u(t,x,0) \text{ on } \{(t,x,y)\in \partial B_\delta^+ \bar t,\bar x,0)\cap\{y>0\}, \tag{0.7}$$

which is satisfied on the hyperplane part of the boundary of $B_\delta^+(\bar t,\bar x,0)$.

$$\Phi(t_\varepsilon,x_\varepsilon,y_\varepsilon) = \phi(t_\varepsilon,x_\varepsilon) + (\underline{p}+\varepsilon)y_\varepsilon > u(t_\varepsilon,x_\varepsilon,y_\varepsilon). \tag{0.8}$$

It follows that $u-\Phi$ has an interior maximum point $\bar P_\varepsilon = (\bar t_\varepsilon,\bar x_\varepsilon,\bar y_\varepsilon)\in B_\delta^+(\bar t,\bar x,0)$, which implies

$$\phi_t(\bar P_\varepsilon) + 2\varepsilon(\Psi_\delta)_t(\bar P_\varepsilon) + H(\partial_x\phi(\bar P_\varepsilon) + 2\varepsilon(\Psi_\delta)_x(\bar P_\varepsilon), 2\varepsilon(\Psi_\delta)_y(\bar P_\varepsilon) + \underline{p}+\varepsilon) \ge 0.$$

Since $|(\Psi_\delta)_t| + |\nabla\Psi_\delta|$ are uniformly bounded in δ, we may take $\varepsilon\to 0$ to deduce (0.3). □

Lemma 0.3 ([2, Lemma A.9]). *Let $u: B_1^+\to\mathbb{R}$ be a lower semicontinuous function and suppose $\phi(t,x,y)$ is a test function that touches $u(t,x,y)$ from below at some $(\bar t,\bar x)$. Define the critical slope at $(\bar t,\bar x,0)$*

$$\underline{p} = \sup\{p:\ \exists r>0,\ u(t,x,y)\ge\phi(t,x,0)+py \ \textit{ for all } (t,x,y)\in B_\delta^+(\bar t,\bar x,0)\}. \tag{0.9}$$

If $\underline{p}<+\infty$, and u and is a viscosity supersolution of

$$u_t + H(u_x,u_y) = 0 \tag{0.10}$$

then

$$\phi_t(\bar t,\bar x,0) + H(\phi_x(\bar t,\bar x,0),\underline{p}) \ge 0. \tag{0.11}$$

Lemma 0.4 ([2, Lemma A.10]). *Let $u: B_1^+\to\mathbb{R}$ be a upper semicontinuous function and suppose $\phi(t,x,y)$ is a test function that touches $u(t,x,y)$ from above at some $(\bar t,\bar x)$. Define the critical slope at $(\bar t,\bar x,0)$*

$$\overline{p} = \inf\{p:\ \exists r>0,\ u(t,x,y)\le\phi(t,x,0)+py \ \textit{ for all } (t,x,y)\in B_\delta^+(\bar t,\bar x,0)\}. \tag{0.12}$$

If $\overline{p}>-\infty$, and u and is a viscosity subsolution of (0.10), then

$$\phi_t(\bar t,\bar x,0) + H(\phi_x(\bar t,\bar x,0),\overline{p}) \le 0. \tag{0.13}$$

Furthermore, if H is coercive and u satisfies the weak continuity assumption, namely,

$$\limsup_{(t,x,y)\to(\bar t,\bar x,0)} u(t,x,y) = u(\bar t,\bar x,0) \tag{0.14}$$

then $\overline{p}>-\infty$.

Proof. We only prove that $\underline{p}>-\infty$ since this is the main difference with the proof of the previous lemma.

Assume that $\underline{p}=-\infty$, then there exists $p_n\to-\infty$ and $r_n\searrow 0$ such that

$$\phi(t,x,0)+p_n y \ge u(t,x,y) \quad \text{in } B_n = B_{r_n}^+.$$

By replacing ϕ by $\phi + (t-\bar{t})^2 + (x-\bar{x})^2 + y^2$ if necessary, we may assume that

$$u(t,x,y) < \phi(t,x,0) + p_n y \quad \text{in } B_n = B^+_{r_n} \setminus \{(\bar{t},\bar{x},0)\}. \tag{0.15}$$

In particular, there exists $\delta_n > 0$ such that $\phi(t,x,0) + p_n y \geq u + \delta_n$ on the curved part of $\partial B^+_{r_n}$. Since u satisfies (0.14), there exists $P_\varepsilon = (t_\varepsilon, x_\varepsilon, y_\varepsilon) \to \bar{P} = (\bar{t},\bar{x},0)$ such that $y_\varepsilon > 0$ and $u(P_\varepsilon) \to u(\bar{P})$.

We now introduce the following perturbed test function

$$\Psi(t,x,y) = \phi(t,x,0) + p_n y + \frac{|y_\varepsilon|^2}{y}.$$

Fix n and observe that $\Psi > u$ on both the curved and flat part of $\partial B^+_{r_n}$. Let $P'_\varepsilon = (t'_\varepsilon, x'_\varepsilon, y'_\varepsilon)$ be the minimum point of $\Psi - u$ in $\partial B^+_{r_n}$, then

$$(\phi + p_n \quad \cdot - u)(P'_\varepsilon) \leq (\Psi - u)(P'_\varepsilon) \leq (\Psi - u)(P_\varepsilon) \approx \phi(P_\varepsilon) - u(P_\varepsilon) + \frac{y_\varepsilon^2}{y_\varepsilon} + o(1) = o(1),$$

since ϕ touches u from above at $\bar{P}$. It follows that the minimum point P'_ε is achieved in the interior, so

$$\phi_t(P'_\varepsilon) + H(\phi_x(P'_\varepsilon), p_n - \frac{y_\varepsilon^2}{y^2}) \leq 0.$$

Denote $p_n^0 = \liminf_{\varepsilon\to 0}(p_n - \frac{y_\varepsilon^2}{y^2}) \in [-\infty, 0]$, then

$$\phi_t(\bar{P}) + H(\phi_x(\bar{P}), p_n^0) \leq 0,$$

which in particular implies $p_n^0 > -\infty$ and is bounded uniformly from below, independent of n. It follos that $\{p_n\}$ is also bounded from below, which is a contradiction. The proof is now complete. $\square$

1 Restating the lemmas using super/subdifferentials

This can be equivalently stated in terms of subdifferential.

Definition 1.1. Let $u : X^+ \to \mathbb{R}$, be given. We say that the constant vector $(-\lambda, q, p)$ is an element of the set $D^-_{X^+}u(P_0)$ (which is called the set of subdifferential of u at $P_0 = (t_0, x_0, y_0)$) provided that there exists $r_0 > 0$ such that

$$u(t,x,y) \geq u(t_0,x_0,y_0) + (-\lambda, q, p) \cdot (t-t_0, x-x_0, y-y_0) + o(|t-t_0| + |x-x_0| + |y-y_0|)$$

for $(t,x,y) \in B_r(t_0,x_0,y_0) \cap X^+$.

Similarly, we define the set $D^+_{X^+}u(P_0)$ of superdifferential of u at P_0 by reversing the inequality.

Lemma 1.2. *Let $u : B_1^+ \to \mathbb{R}$ be a lower semicontinuous function and suppose $D^-_{X^+}u(\bar{t},\bar{x},0)$ is nonempty. Fix an element $(-\lambda_0, q_0, p_0) \in D^-_{X^+}u(\bar{t},\bar{x},0)$ at $(\bar{t},\bar{x},0)$, and define the critical slope*

$$\underline{p} = \sup\{p : \ (-\lambda_0, q_0, p) \in D^-_{X^+}u(\bar{t},\bar{x},0)\}. \tag{1.1}$$

If $\underline{p} < +\infty$, and u and is a viscosity supersolution of (0.10), then

$$-\lambda_0 + H(q_0, \underline{p}) \geq 0. \tag{1.2}$$

Lemma 1.3. *Let* $u : B_1^+ \to \mathbb{R}$ *be a upper semicontinuous function and suppose* $D^+_{X^+}u(\bar{t}, \bar{x}, 0)$ *is nonempty. Fix an element* $(-\lambda_0, q_0, p_0) \in D^+_{X^+}u(\bar{t}, \bar{x}, 0)$ *at* $(\bar{t}, \bar{x}, 0)$*, and define the critical slope*

$$\overline{p} = \inf\{p : \; (-\lambda_0, q_0, p) \in D^+_{X^+}u(\bar{t}, \bar{x}, 0)\}. \tag{1.3}$$

If $\overline{p} > -\infty$*, and* u *and is a viscosity subsolution of* (0.10)*, then*

$$-\lambda_0 + H(q_0, \overline{p}) \leq 0. \tag{1.4}$$

Furthermore, $\overline{p} > -\infty$ *is verified if* u *satisfies the weak continuity assumtion.*

References

[1] Cyril Imbert and Régis Monneau. Flux-limited solutions for quasi-convex Hamilton-Jacobi equations on networks. Ann. Sci. Éc. Norm. Supér. (4), 50(2):357–448, 2017.

[2] Cyril Imbert and Régis Monneau. Quasi-convex Hamilton-Jacobi equations posed on junctions: the multi-dimensional case. Discrete Contin. Dyn. Syst., 37(12):6405–6435, 2017.