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LIMITING PROFILES OF SEMILINEAR ELLIPTIC EQUATIONS WITH LARGE ADVECTION IN POPULATION DYNAMICS

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Dedicated to Louis Nirenberg on the occasion of his 85th birthday

Abstract. Limiting profiles of solutions to a 2×2 Lotka-Volterra competition-diffusion-advection system, when the strength of the advection tends to infinity, are determined. The two species, competing in a heterogeneous environment, are identical except for their dispersal strategies: One is just random diffusion while the other is "smarter" - a combination of random diffusion and a directed movement up the environmental gradient. With important progress made, it has been conjectured in [2] and [3] that for large advection the "smarter" species will concentrate near a selected subset of positive local maximum points of the environment function. In this paper, we establish this conjecture in one space dimension, with the peaks located and the limiting profiles determined, under mild hypotheses on the environment function.

1. **Introduction.** An interesting but perhaps curious phenomenon in the competition of two species with different (random) dispersal rates but otherwise being identical is that *the slower diffuser always prevails*! More precisely, consider the following Lotka-Volterra competition-diffusion system

$$\begin{cases} U_t = d_1\Delta U + U(m(x) - U - V) & \text{in } \Omega \times (0,T), \\ V_t = d_2\Delta V + V(m(x) - U - V) & \text{in } \Omega \times (0,T), \\ \frac{\partial U}{\partial \nu} = \frac{\partial V}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,T), \end{cases} \tag{1}$$

where $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator; U and V representing the densities of two competing species with the dispersal rates d_1 and d_2, respectively, are therefore nonnegative; the habitat Ω is a bounded smooth domain in $\mathbf{R}^n$ with the unit outward normal ν on the boundary $\partial\Omega$, and $m(x)$ represents the *local* intrinsic growth rate at $x \in \Omega$ which may change sign in Ω, and will always be assumed to be *nonconstant*.

It is well known that if $\int_\Omega m > 0$, the single equation

$$\begin{cases} \theta_t = d\Delta\theta + \theta(m(x) - \theta) & \text{in } \Omega \times (0,T), \\ \frac{\partial \theta}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,T), \end{cases} \tag{2}$$

has a *unique positive steady state* θ_d, for every $d > 0$. In [4], it was established that *if* $d_1 < d_2$*, then the solution* (U,V) *of* (1) *always converges to* $(\theta_{d_1}, 0)$ *as* $t \to \infty$,

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regardless of initial values; in other words, $(\theta_{d_1}, 0)$ is globally asymptotically stable, and the slower diffuser U always wipes out its faster-moving competitor V.

However, individuals do not always move around just randomly. Incorporating "directed movements" into (1) seems to be a reasonable step further in understanding population dynamics. In a series of very interesting papers [1],[2],[3], the following 2×2 system is considered

$$\begin{cases} U_t = \nabla \cdot (d_1 \nabla U - \alpha U \nabla m) + U(m(x) - U - V) & \text{in } \Omega \times (0, T), \\ V_t = d_2 \Delta V + V(m(x) - U - V) & \text{in } \Omega \times (0, T), \\ d_1 \frac{\partial U}{\partial \nu} - \alpha U \frac{\partial m}{\partial \nu} = \frac{\partial V}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0, T), \end{cases} \tag{3}$$

where the parameter α is assumed to be positive; i.e. it is further assumed that the species U is "smarter", as it also moves up the gradient direction of the resources $m(x)$. (The species V is still assumed to disperse just randomly for the comparison purposes.) Then it is proved in [2] and [3] that *for α large, U and V could co-exist! Furthermore, system (3) seems to exhibit interesting concentration phenomena for sufficiently large α. More precisely, it is proved that if $\int_\Omega m > 0$ and m has exactly one critical point $x_0 \in \Omega$ which is a nondegenerate positive global maximum of $m(x)$ and $\frac{\partial m}{\partial \nu} \leq 0$ on $\partial\Omega$, then, for any positive steady state (U_α, V_α) of (3), we have $V_\alpha \to \theta_{d_2}$ uniformly in Ω and*

$$U_\alpha(x) \to \begin{cases} 0, & \text{if } x \neq x_0, \\ 2^{n/2}[m(x_0) - \theta_{d_2}(x_0)], & \text{at } x = x_0, \end{cases}$$

as $\alpha \to \infty$. (Here, x_0 being *nondegenerate* means that $\det(D^2 m(x_0)) \neq 0$.) Note that by maximum principle, we have

$$\max_{\bar{\Omega}} m > \max_{\bar{\Omega}} \theta_{d_2}$$

for all $d_2 > 0$. (See [2].)

This result has led the authors of [2] and [3] to conjecture that *for general $m(x)$ (which may have multiple local maximum points in Ω), any positive steady states (U_α, V_α) of (3) must concentrate at all local maximum points of m in $\bar{\Omega}$.* (See e.g. P.631 of [3].)

The purpose of this paper is to establish the above conjecture in the case $n = 1$, under additional mild hypotheses on m. It turns out that the above conjecture has to be modified slightly. To describe our result, we first set $n = 1$, $\Omega = (-1, 1)$ and *let $\mathfrak{M}$ be the set of all positive local maximum points of m in $\bar{\Omega} = [-1, 1]$.* The steady state equations for (3) now reduce to

$$\begin{cases} (d_1 U' - \alpha U m')' + U(m - U - V) = 0 & \text{in } (-1, 1), \\ d_2 V'' + V(m - U - V) = 0 & \text{in } (-1, 1), \\ d_1 U' - \alpha U m' = 0 = V' & \text{at } x = \pm 1. \end{cases} \tag{4}$$

Our main result for (4) now reads as follows.

Theorem 1.1. *Suppose that $\int_\Omega m > 0$, $\mathfrak{M} \subseteq (-1, 1)$ with $x m'(x) \leq 0$ at $x = \pm 1$, and that all critical points of m are nondegenerate. Let (U_α, V_α) be a positive solution of (4). Then, as $\alpha \to \infty$, it holds that*

(i) $V_\alpha \to \theta_{d_2}$ *in* $C^{1,\beta}$;

(ii) for any $x_0 \in \mathfrak{M}$ and any $r > 0$ small,

$$\|U_\alpha(x) - \max\{\sqrt{2}\,[m(x_0) - \theta_{d_2}(x_0)]\,, 0\} e^{\alpha[m(x) - m(x_0)]}\|_{L^\infty(x_0 - r, x_0 + r)} \to 0;$$

(iii) for any neighborhood $\mathfrak{N}$ *of* $\mathfrak{M}$, $U_\alpha \to 0$ *in* $(-1,1)\backslash\mathfrak{N}$ *uniformly and exponentially.*

From Theorem 1.1 we see that not only the peaks of U are located, the profiles of U for large α near its concentrations are also determined. In particular, we have proved that $\|U_\alpha\|_{L^\infty}$ remains uniformly bounded in α. Theorem 1.1 also seems interesting from a biological point of view. It says that although the species U has the ability of moving upward the gradient of $m(x)$, it will "survive" only at those local maximum points of m where m is strictly larger than θ_{d_2}; in other words, a local maximum point of m could be a "trap" for the species U if m is less than or equal to θ_{d_2} there!

By way of proving Theorem 1.1, we first consider the following closely related single equation which was proposed in [1] to model the population dynamics of a single species

$$\begin{cases} u_t = \nabla\cdot(d\nabla u - \alpha u\nabla m) + u(m(x)-u) & \text{in } \Omega\times(0,T), \\ d\frac{\partial u}{\partial\nu} - \alpha u\frac{\partial m}{\partial \nu} = 0 & \text{on } \partial\Omega\times(0,T). \end{cases} \tag{5}$$

Similarly, it was established in [1] that *if m is positive somewhere in Ω, then for any $d>0$ and α sufficiently large,* (5) *has a unique positive steady state u_α which is globally asymptotically stable* (among all nonnegative nonzero solutions.) Again, similarly, from the fact that $\|u_\alpha\|_{L^\infty} \geq \|m\|_{L^\infty}$ for α large (proved in [2]), it was conjectured *that the concentration set of u_α, as $\alpha\to\infty$, is precisely the set of all local maximum points of m.* In a recent paper [5], this conjecture was verified under additional mild hypotheses on m. Here we are able to further determine the limiting profile of the positive steady state u_α for α large in the 1-dimensional case $n=1$.

For the positive steady state u_α of (5) for $n=1$

$$\begin{cases} (u'-\alpha u m')' + u(m-u) = 0 & \text{in } (-1,1), \\ u' - \alpha u m' = 0 & \text{at } x=\pm 1, \end{cases} \tag{6}$$

(here we have set $d=1$, for simplicity) we have the following limiting profile of u_α. Again, as a consequence $\|u_\alpha\|_{L^\infty}$ remains uniformly bounded in α.

Theorem 1.2. *Suppose that $\mathfrak{M}\subseteq(-1,1)$ with $xm'(x)\leq 0$ at $x=\pm1$ and that all critical points of m are non-degenerate. Then, for any $r>0$ small and any $x_0\in\mathfrak{M}$, we have*

(i) $u_\alpha \to 0$ *uniformly and exponentially in* $(-1,1)\backslash\cup_{x_0\in\mathfrak{M}}(x_0-r,x_0+r)$,

(ii) $\|u_\alpha - \sqrt{2}m(x_0)e^{\alpha[m(x)-m(x_0)]}\|_{L^\infty(x_0-r,x_0+r)} \to 0$,

as $\alpha\to\infty$.

Comparing Theorems 1.1 and 1.2, we remark that the extra condition $\int_\Omega m > 0$ in Theorem 1.1 is needed only to guarantee the existence of θ_{d_2}. Our proofs, lengthy but elementary, are presented in Sections 2 and 3, respectively. Further remarks on extending Theorems 1.1 and 1.2 are included in Section 4.

2. **Proof of Theorem 1.2.**

In this section, we will prove Theorem 1.2. For convenience, we will sometimes suppress the sub-index α in u_α when there is no confusion. We will assume throughout the rest of this paper that $\Omega=(-1,1)$ and that $m(x)$ satisfies the following conditions:

(M1): $m(x)\in C^3([-1,1])$ and $xm'(x)\leq 0$ at $x=\pm1$.

(M2): $\mathfrak{M}\subseteq(-1,1)$ and all critical points of m are nondegenerate.

(M3): $\max_{\overline{\Omega}} m > 0$.

Note that **(M2)** implies $m(x)$ has only a finite number of local maximum points. First, we recall the following facts about (6).

Theorem 2.1. *Suppose that m satisfies* **(M1),(M2) and (M3)**. *Then the following statements hold.*

(i) For every α large, (6) has a unique positive solution u_α, which is globally stable among all nonnegative nontrivial solutions of (5).
(ii) $u_\alpha \to 0$ in $L^2(-1,1)$ as $\alpha \to \infty$.
(iii) $\|u_\alpha\|_{L^\infty} \le \|m\|_{L^\infty} + \alpha\|\Delta m\|_{L^\infty}$.
(iv) For each $x_0 \in \mathfrak{M}$ and any $r > 0$,

$$\liminf_{\alpha\to\infty}\left(\max_{B_r(x_0)} u_\alpha\right) \ge m(x_0).$$

(v) For each neighborhood $\mathfrak{N}$ of $\mathfrak{M}$, there exists $b > 0$ such that $0 \le u_\alpha \le e^{-b\alpha}$ in $(-1,1)\backslash\mathfrak{N}$.

Proof. Part (i) follows from Propositions 2.1,2.3 of [1], since $\int_{-1}^1 e^{\alpha m} m > 0$ for large α (by **(M3)**). Parts (ii) and (iii) are proved in Theorem 3.5 and Lemma 3.3 of [2]. (Note that although the extra condition $\int_\Omega m > 0$ is assumed in [2], it is not required in the proofs of Theorem 3.5 and Lemma 3.3 there.) Parts (iv) and (v) are established in Theorems 1.4 and 1.5 of [5]. □

To analyze (6), we first integrate (6) from -1 to x,

$$u'(x) - \alpha u(x)m'(x) + \int_{-1}^x u(m-u) = 0,$$

i.e.

$$(\ln u)' = \frac{u'}{u} = \alpha m' - \frac{1}{u}\int_{-1}^x u(m-u). \tag{7}$$

Hence, for any x, $x_\alpha \in (-1,1)$ we have

$$\ln u(x) - \ln u(x_\alpha) = \alpha(m(x) - m(x_\alpha)) - \int_{x_\alpha}^x \frac{1}{u(z)}\left(\int_{-1}^z u(m-u)\right)dz,$$

and we have derived the following basic formula which we will use repeatedly in this section:

$$\frac{u(x)}{u(x_\alpha)} = \exp\left[\alpha\left(m(x) - m(x_\alpha)\right) - \int_{x_\alpha}^x \frac{1}{u(z)}\left(\int_{-1}^z u(m-u)\right)dz\right]. \tag{8}$$

We first estimate the integral in (8).

Lemma 2.2. *There exists a constant $C > 0$ independent of α, such that*

$$\left|\int_{-1}^z u_\alpha(m-u_\alpha)\right| \le C\|u_\alpha\|_{L^2(-1,1)}$$

for all $z \in (-1,1)$ whenever u exists.

Proof. Fix α large so that the positive solution u for (6) exists. By integrating (6), we have $\int_{-1}^1 u(m-u)dx = 0$ and hence $\|u\|_{L^2} \le \|m\|_{L^2}$. Now,

$$\left|\int_{-1}^z u(m-u)\right| \le \|m\|_{L^2}\|u\|_{L^2} + \|u\|_{L^2}^2 \le (2\|m\|_{L^2})\|u\|_{L^2}.$$

□

As m has only a finite number of nondegenerate interior local maximum points, there exist a small positive constant ϵ_0 and a positive constant C_0 such that $m'' < -C_0$ and $m > 0$ on

$$\mathfrak{N} \equiv \cup_{x_0 \in \mathfrak{M}} (x_0 - \epsilon_0, x_0 + \epsilon_0).$$

From Part (v) of Theorem 2.1, we have $u \leq e^{-b\alpha}$ on $\Omega \backslash \mathfrak{N}$ for some constant $b > 0$. Hence, if we set $\delta_1 = \alpha^{-\frac{17}{32}}$ and $\delta_2 = \alpha^{-\frac{1}{4}}$, we have, for $i = 1, 2$,

$$I_{\delta_i} \equiv \{x \in \Omega \mid u_\alpha(x) > \delta_i\} \subseteq \mathfrak{N}$$

for all large α. Note that $I_{\delta_1} \supseteq I_{\delta_2}$.

Proposition 2.3. *For each $x_0 \in \mathfrak{M}$, and $i = 1, 2$, $I_{\delta_i} \cap (x_0 - \epsilon_0, x_0 + \epsilon_0)$ is nonempty and connected, for α large. In other words, I_{δ_i} consists of exactly $\#\mathfrak{M}$ disjoint intervals for α large.*

Proof. To prove the connectedness by contradiction, suppose that there are at least two connected components of $I_{\delta_i} \cap (x_0 - \epsilon_0, x_0 + \epsilon_0)$. Then, there exists a local minimum point $\bar{x} \in (x_0 - \epsilon_0, x_0 + \epsilon_0)$ such that $u(\bar{x}) \leq \delta_i$, $u'(\bar{x}) = 0$ and $u''(\bar{x}) \geq 0$. Writing (6) as

$$u'' - \alpha m' u' + u(m - u - \alpha m'') = 0,$$

we see that,

$$u(\bar{x}) \geq m(\bar{x}) - \alpha m''(\bar{x}) \geq \inf_\Omega m + \alpha C_0 \geq 1 > \delta_i,$$

for α sufficiently large, a contradiction. $I_{\delta_i} \cap (x_0 - \epsilon_0, x_0 + \epsilon_0)$ being nonempty is a consequence of Theorem 2.1 (iv). □

Now, let $x_\alpha \in I_{\delta_i} \cap (x_0 - \epsilon_0, x_0 + \epsilon_0)$ be a maximum point of u_α in $I_{\delta_i}(x_0) \equiv I_{\delta_i} \cap (x_0 - \epsilon_0, x_0 + \epsilon_0)$; i.e.

$$u_\alpha(x_\alpha) = \max\{u_\alpha(x) \mid x \in I_{\delta_i} \cap (x_0 - \epsilon_0, x_0 + \epsilon_0)\} \tag{9}$$

Observe that x_α does not depend on $i = 1, 2$, by Part (iv) of Theorem 2.1. From (7) we deduce that for α large,

$$\begin{aligned} |m'(x_\alpha)| &= \frac{1}{\alpha u_\alpha(x_\alpha)} \left| \int_{-1}^{x_\alpha} u_\alpha(m - u_\alpha) \right| \\ &\leq \frac{C}{\alpha} \|u\|_{L^2} \end{aligned}$$

by Lemma 2.2 and Theorem 2.1 (iv). Since

$$x_\alpha \in (x_0 - \epsilon_0, x_0 + \epsilon_0),\ m''(x_0) \neq 0,\ m'(x_0) = 0,$$

Mean Value Theorem implies that

$$|x_\alpha - x_0| \leq \frac{C}{\alpha} \|u_\alpha\|_{L^2} = o(\frac{1}{\alpha}) \tag{10}$$

by Theorem 2.1 (ii).

Next, we turn to estimating $|I_{\delta_i}|,\ i = 1, 2$.

Lemma 2.4. *For any $M > 0$ and any $x_0 \in \mathfrak{M}$, both $(x_0 - \frac{M}{\sqrt{\alpha}}, x_0 + \frac{M}{\sqrt{\alpha}})$ and $(x_\alpha - \frac{M}{\sqrt{\alpha}}, x_\alpha + \frac{M}{\sqrt{\alpha}})$ are contained in I_{δ_2} for α large.*

Proof. To prove this by contradiction, suppose that there exist $M_0 > 0$ and a sequence $\alpha_j \to \infty$ with $z_{\alpha_j} \in \partial I_{\delta_2}$, such that

$$|z_{\alpha_j} - x_{\alpha_j}| \leq \frac{M_0}{\sqrt{\alpha_j}}$$

(10) then implies that

$$|z_{\alpha_j} - x_0| \leq \frac{(1+M_0)}{\sqrt{\alpha_j}}$$

From (8) and (10) it follows that

$$\begin{aligned}
\frac{u(z_{\alpha_j})}{u(x_{\alpha_j})} &= \exp\left[\alpha_j(m(z_{\alpha_j}) - m(x_{\alpha_j})) - \int_{x_{\alpha_j}}^{z_{\alpha_j}} \frac{1}{u(z)}\left(\int_{-1}^{z} u(m-u)\right) dz\right] \\
&\geq \exp\left[\alpha_j\left(m(z_{\alpha_j}) - m(x_0) + m(x_0) - m(x_{\alpha_j})\right) - \int_{x_{\alpha_j}}^{z_{\alpha_j}} \frac{C}{u(z)} \|u\|_{L^2}\right] \\
&\geq \exp\left[\alpha_j\{m(z_{\alpha_j}) - m(x_0) + O(|x_{\alpha_j} - x_0|^2)\} - \frac{C}{\delta_2}|z_{\alpha_j} - x_{\alpha_j}| \|u\|_{L^2}\right] \\
&= \exp\left[\alpha_j\{\frac{1}{2}m''(x_0)|z_{\alpha_j} - x_0|^2 + o(\frac{1}{\alpha_j^2})\} - C\alpha_j^{\frac{1}{4}}\frac{M_0}{\sqrt{\alpha_j}}\|u\|_{L^2}\right] \\
&\geq \exp\left[\alpha_j\left\{\frac{1}{2}m''(x_0)\frac{(M_0+1)}{\alpha_j}\right\} - o(\alpha_j^{-\frac{1}{4}})\right] \\
&\to \exp[\frac{1}{2}m''(x_0)(M_0+1)^2] > 0
\end{aligned}$$

as $\alpha_j \to \infty$.

On the other hand, $\frac{u(z_{\alpha_j})}{u(x_{\alpha_j})} \to 0$ as $\alpha_j \to \infty$ since $u(x_{\alpha_j}) \geq \frac{m(x_0)}{2} > 0$ for α large and $u(z_{\alpha_j}) = \delta_2 \to 0$, a contradiction. Thus $(x_\alpha - \frac{M}{\sqrt{\alpha}}, x_\alpha + \frac{M}{\sqrt{\alpha}}) \subseteq I_{\delta_2}$. The fact that $(x_0 - \frac{M}{\sqrt{\alpha}},\ x_0 + \frac{M}{\sqrt{\alpha}}) \subseteq I_{\delta_2}$ for α large now follows from (10). ☐

Now we come to the upper estimate of $|I_{\delta_2}|$.

Proposition 2.5. *For α large $|I_{\delta_2}| = o(\frac{1}{\alpha^c})$ for any $0 < c < \frac{1}{2}$. In particular, $|I_{\delta_2}| = o(\frac{1}{\alpha^{1/3}})$.*

Proof. Fix $\frac{1}{4} < c < \frac{1}{2}$. Suppose the assertion is false. Then for some $x_0 \in \mathfrak{M}$ there is a sequence $\alpha_j \to \infty$ such that for each j, there exists $z_{\alpha_j} \in I_{\delta_2}(x_0)$ with $|z_{\alpha_j} - x_0| = \frac{k_1}{\alpha_j^c}$, for some constant $k_1 > 0$. From (8) and (10) it follows that (for simplicity we suppress the subindex j).

$$\begin{aligned}
\frac{u(z_\alpha)}{u(x_\alpha)} &\leq \exp[\alpha(m(z_\alpha) - m(x_0) + m(x_0) - m(x_\alpha)) \\
&\qquad + \frac{C}{\delta_2}|z_\alpha - x_\alpha| \|u\|_{L^2}] \\
&\leq \exp[-\alpha k_2|z_\alpha - x_0|^2 + o(1) + Ck_1\alpha^{\frac{1}{4}-c}\|u\|_{L^2}] \\
&\leq \exp(-k_3\alpha^{1-2c})
\end{aligned}$$

for α large, where k_2, k_3 are two positive constants.

On the other hand, from Theorem 2.1 (iii) we have

$$\frac{u(z_\alpha)}{u(x_\alpha)} \geq \frac{\delta_2}{\|u\|_{L^\infty}} \geq \frac{\delta_2}{\|m\|_{L^\infty} + \alpha\|\Delta m\|_{L^\infty}} \geq \frac{k_4}{\alpha^{5/4}}$$

for some constant $k_4 > 0$, a contradiction. □

Theorem 2.6.

$$\frac{u_\alpha(x)}{u_\alpha(x_\alpha)} \exp\left[-\frac{\alpha}{2}m''(x_0)(x-x_0)^2\right] \to 1$$

uniformly in $I_{\delta_2}(x_0)$ *for each* $x_0 \in \mathfrak{M}$ *as* $\alpha \to \infty$. *In particular,*

$$\frac{1}{2}u_\alpha(x_\alpha)e^{\frac{\alpha}{2}m''(x_0)(x-x_0)^2} \leq u_\alpha(x) \leq 2u_\alpha(x_\alpha)e^{\frac{\alpha}{2}m''(x_0)(x-x_0)^2}$$

in $I_{\delta_2}(x_0)$ *for all* α *large.*

Proof. By (8) again we have, for $x \in I_{\delta_2}(x_0)$,

$$\begin{aligned}
&\left|\frac{u(x)}{u(x_\alpha)} \exp\left[-\frac{\alpha}{2}m''(x_0)(x-x_0)^2\right] - 1\right| \\
&= \left|\exp\left[\alpha(m(x) - m(x_\alpha)) - \int_{x_\alpha}^x \frac{\int_1^z u(m-u)}{u(z)}dz - \frac{\alpha}{2}m''(x_0)(x-x_0)^2\right] - 1\right| \\
&= |g_1(x) - g_2(x)| \exp \xi(x)
\end{aligned}$$

where

$$g_1(x) = \alpha(m(x) - m(x_\alpha)) - \frac{\alpha}{2}m''(x_0)(x-x_0)^2 \tag{11}$$

$$g_2(x) = \int_{x_\alpha}^x \frac{1}{u(z)}\left(\int_{-1}^z u(m-u)\right)dz \tag{12}$$

and $\xi(x)$ lies in between 0 and $g_1(x) - g_2(x)$. Now, our assertion follows from the following observations:

$$\begin{aligned}
|g_1(x)| &\leq \alpha\left|m(x) - m(x_0) - \frac{1}{2}m''(x_0)(x-x_0)^2\right| + \alpha\,|m(x_0) - m(x_\alpha)| \\
&\leq \alpha \cdot O(|x-x_0|^3) + \alpha \cdot O(|x_0 - x_\alpha|^2) \to 0.
\end{aligned}$$

by (10) and Proposition 2.5, and

$$\begin{aligned}
|g_2(x)| &\leq \frac{C}{\delta_2}|x - x_\alpha|\|u\|_{L^2} \\
&\leq C\alpha^{\frac{1}{4}}|I_{\delta_2}| \cdot \|u\|_{L^2} \to 0.
\end{aligned}$$

by Proposition 2.5 and Theorem 2.1 (ii). □

Eventually we will show that $\|u_\alpha\|_{L^\infty}$ is uniformly bounded for all α large. The following is the first step.

Lemma 2.7. $\|u_\alpha\|_{L^\infty}^2 \leq C\sqrt{\alpha}\int_\Omega u_\alpha^2$ *for* α *large. In particular,* $\|u_\alpha\|_{L^\infty} = o(\alpha^{\frac{1}{4}})$ *for* α *large.*

Proof.

$$\begin{aligned}\int_\Omega u^2 &\geq \int_{I_{\delta_2}(x_0)} u^2 \\ &\geq \frac{1}{4}u^2(x_\alpha)\int_{I_{\delta_2}(x_0)} \exp[\alpha m''(x_0)(x-x_0)^2]dx \\ &\geq \frac{1}{4}u^2(x_\alpha)\int_{-M}^{M} \exp(m''(x_0)y^2)dy \cdot \frac{1}{\sqrt{\alpha}} \\ &\geq \frac{C}{\sqrt{\alpha}}u^2(x_\alpha).\end{aligned}$$

for any $M > 0$, by Theorem 2.6 and Lemma 2.4, where $y = \sqrt{\alpha}(x - x_0)$. Our assertion now follows from Theorem 2.1 (ii). □

Lemma 2.8. *$\int_\Omega u_\alpha^2 = O(\alpha^{-\frac{1}{4}})$ for α large.*

Proof. From (6) and Theorem 2.6 we have

$$\begin{aligned}\int_\Omega u^2 &= \int_\Omega mu \leq C\int_\Omega u \\ &= C\left(\int_{[u\leq\delta_2]} u + \int_{[u>\delta_2]} u\right) \\ &= C\left(\int_{[u\leq\delta_2]} u + \sum_{x_0\in\mathfrak{M}}\int_{I_{\delta_2(x_0)}} u\right) \\ &\leq C|\Omega|\delta_2 + \sum_{x_0\in\mathfrak{M}} 2u(x_\alpha)\int_{I_{\delta_2}(x_0)} \exp\left[\frac{\alpha}{2}m''(x_0)(x-x_0)^2\right] \\ &\leq C\alpha^{-\frac{1}{4}} + o(\alpha^{\frac{1}{4}})\sum_{x_0\in\mathfrak{M}} \frac{1}{\sqrt{\alpha}}\int_{\mathbf{R}} \exp\left[\frac{1}{2}m''(x_0)y^2\right]dy \\ &= O(\alpha^{-\frac{1}{4}}).\end{aligned}$$

□

To estimate I_{δ_1}, we begin with the following counterpart of Proposition 2.5.

Proposition 2.9. *For α large, $|I_{\delta_1}| = o(\frac{1}{\alpha^c})$ for any $0 < c < \frac{1}{2}$. In particular, $|I_{\delta_1}| = o(\frac{1}{\alpha^{13/32}})$.*

Proof. Fix $\frac{7}{16} < c < \frac{1}{2}$. Suppose that the assertion is false. Then for some $x_0 \in \mathfrak{M}$ there is a sequence $\alpha_j \to \infty$ such that for each j, there exists $z_{\alpha_j} \in I_{\delta_1}(x_0)$, with $|z_{\alpha_j} - x_0| = \frac{k_1}{\alpha^c}$, for some constant $k_1 > 0$. From (8), (10) and Lemma 2.8, it follows that (again we suppress the subindex j, for simplicity)

$$\begin{aligned}\frac{u(z_\alpha)}{u(x_\alpha)} &\leq \exp\left[\alpha(m(z_\alpha) - m(x_0)) + \alpha(m(x_0) - m(x_\alpha)) + \frac{1}{\delta_1}C|z_\alpha - x_\alpha|\|u\|_{L^2}\right] \\ &\leq \exp\left[-\alpha k_2|z_\alpha - x_0|^2 + o(1) + C\alpha^{\frac{17}{32}-c}\|u\|_{L^2}\right] \\ &\leq \exp\left[-k_3\alpha^{1-2c} + o(1) + C\alpha^{\frac{17}{32}-c-\frac{1}{8}}\right] \\ &\leq \exp\left[-k_4\alpha^{1-2c}\right]\end{aligned}$$

for α large, where k_2, k_3, k_4 are positive constants. On the other hand, from Theorem 2.1 (iii) we have

$$\frac{u(z_\alpha)}{u(x_\alpha)} \geq \frac{\delta_1}{\|u\|_{L^\infty} + \alpha\|\Delta m\|_{L^\infty}} \geq \frac{k_5}{\alpha^{\frac{49}{32}}},$$

a contradiction. □

Now we have the counterpart of Theorem 2.6 for I_{δ_1}.

Theorem 2.10.

$$\frac{u_\alpha(x)}{u_\alpha(x_\alpha)} \exp\left[-\frac{\alpha}{2}m''(x_0)(x-x_0)^2\right] \to 1 \tag{13}$$

uniformly in $I_{\delta_1(x_0)}$ *for each* $x_0 \in \mathfrak{M}$ *as* $\alpha \to \infty$*. In particular, we have, for each* $\epsilon > 0$*,*

$$(1-\epsilon)u_\alpha(x_\alpha)e^{\frac{\alpha}{2}m''(x_0)(x-x_0)^2} \leq u_\alpha(x) \leq (1+\epsilon)u_\alpha(x_\alpha)e^{\frac{\alpha}{2}m''(x_0)(x-x_0)^2}, \tag{14}$$

and

$$(1-\epsilon)u_\alpha(x_\alpha)e^{\alpha(m(x)-m(x_0))} \leq u_\alpha(x) \leq (1+\epsilon)u_\alpha(x_\alpha)e^{\alpha(m(x)-m(x_0))} \tag{15}$$

uniformly in I_{δ_1}*, for* α *large.*

Proof. As in the proof of Theorem 2.6, we have, for $x \in I_{\delta_1}(x_0)$,

$$\left|\frac{u(x)}{u(x_\alpha)} \exp\left[-\frac{\alpha}{2}m''(x_0)(x-x_0)^2\right] - 1\right| = |g_1(x) - g_2(x)| \exp \xi(x)$$

where g_1 and g_2 are given in (11) and (12) respectively, and $\xi(x)$ lies in between 0 and $g_1(x) - g_2(x)$. $g_1(x)$ and $g_2(x)$ can be estimated in a similar fashion as in the proof of Theorem 2.6:

$$\begin{aligned}|g_1(x)| &\leq \left|\alpha[m(x) - m(x_0) - \frac{1}{2}m''(x_0)(x-x_0)^2]\right| + \alpha|m(x_0) - m(x_\alpha)| \\ &\leq \alpha\left[O(|x-x_0|^3) + O(|x_0 - x_\alpha|^2)\right] \to 0\end{aligned}$$

in view of Proposition 2.9 and (10). Similarly,

$$\begin{aligned}|g_2(x)| &\leq C\frac{1}{\delta_1}|x - x_\alpha|\|u\|_{L^2} \\ &\leq o\left(\alpha^{\frac{17}{32}-\frac{13}{32}-\frac{1}{8}}\right) \to 0\end{aligned}$$

by Lemma 2.8 and Proposition 2.9. Thus (13) and (14) hold. (15) follows from the fact that

$$\begin{aligned}&\exp\left[-\frac{\alpha}{2}m''(x_0)(x-x_0)^2\right]\exp\left[\alpha(m(x) - m(x_0))\right] \\ &= \exp\left[\alpha O(|x-x_0|^3)\right] \to 1\end{aligned}$$

for $x \in I_{\delta_1}$, by Proposition 2.9. □

Next we show that $\|u_\alpha\|_{L^\infty}$ is uniformly bounded in α large.

Theorem 2.11. $\|u_\alpha\|_{L^\infty}$ *is uniformly bounded for all* α *large.*

Proof. Let $u(x_\alpha) = \|u\|_{L^\infty}$, Lemma 2.7 and (14) imply that

$$
\begin{aligned}
u^2(x_\alpha) &\le C\sqrt{\alpha}\int_\Omega u^2 = C\sqrt{\alpha}\int_\Omega mu \le C\sqrt{\alpha}\int_\Omega u \\
&= C\sqrt{\alpha}\left(\int_{[u\le\delta_1]} u + \int_{[u>\delta_1]} u\right) \\
&\le C\sqrt{\alpha}\left(|\Omega|\delta_1 + \sum_{x_0\in\mathfrak{M}} Cu(x_\alpha)\int_{I_{\delta_1(x_0)}} \exp\left[\frac{\alpha}{2}m''(x_0)(x-x_0)^2\right]dx\right) \\
&\le C|\Omega|\alpha^{-\frac{17}{32}+\frac{1}{2}} + \sum_{x_0\in\mathfrak{M}} Cu(x_\alpha)\sqrt{\alpha}\int_{\mathbf{R}} \exp\left[\frac{1}{2}m''(x_0)y^2\right]\frac{dy}{\sqrt{\alpha}}
\end{aligned}
$$

Therefore,

$$\|u\|^2_{L^\infty} \le C(1+\|u\|_{L^\infty}).$$

Since C is independent of α, $\|u\|_{L^\infty}$ must be uniformly bounded for all α large. □

Theorem 2.12. *For each $x_0 \in \mathfrak{M}$, we have*

$$\lim_{\alpha\to\infty} u_\alpha(x_\alpha) = \sqrt{2}m(x_0)$$

where x_α is given by (9).

Proof. Integrating the equation (6) from $x_0-\epsilon_0$ to $x_0+\epsilon_0$ gives

$$(u'-\alpha um')\Big|_{x_0-\epsilon_0}^{x_0+\epsilon_0} + \int_{x_0-\epsilon_0}^{x_0+\epsilon_0} mu = \int_{x_0-\epsilon_0}^{x_0+\epsilon_0} u^2 \tag{16}$$

First, we claim that for some $\tilde{b} > 0$,

$$(u'-\alpha um')\Big|_{x_0-\epsilon_0}^{x_0+\epsilon_0} \le e^{-\tilde{b}\alpha}, \tag{17}$$

for all α large. By Theorem 2.1 (v), at $x = x_0 \pm \epsilon_0$, $|\alpha um'| \le C\alpha e^{-b\alpha} \le Ce^{-\tilde{b}\alpha}$ for some $\tilde{b} > 0$. Thus, to show (17) it suffices to prove

$$u'(x_0 \pm \epsilon_0) \le Ce^{-\tilde{b}\alpha}. \tag{18}$$

We will prove (18) only for the case $x_0 - \epsilon_0$, as the other case can be handled in a similar fashion.

Case 1. $x_0 = \min \mathfrak{M}$.

Then, integrating the equation (6) from -1 to $x_0 - \epsilon_0$, we obtain

$$u'(x_0-\epsilon_0) = \alpha(um')(x_0-\epsilon_0) - \int_{-1}^{x_0-\epsilon_0} u(m-u) \tag{19}$$

by the no-flux boundary condition at -1. Now every term on the right-hand side of (19) is bounded by $C\alpha e^{-b\alpha}$ or $Ce^{-b\alpha}$, therefore our assertion follows.

Case 2. $x_0 > \min \mathfrak{M}$.

Without loss of generality we may assume that there exists $x_1 \in \mathfrak{M}$ such that $\mathfrak{M}$ has no other points in the interval (x_1, x_0). Then, by Theorem 2.1 (iv),(v), there exists $\tilde{x} \in (x_1, x_0)$ such that $u'(\tilde{x}) = 0$ and $u < e^{-b\alpha}$ in between $\tilde{x}$ and $x_0 - \epsilon_0$.

Now, we integrate (6) from $\tilde{x}$ to $x_0 - \epsilon_0$,

$$\begin{aligned}
(u' - \alpha u m')\Big|_{\tilde{x}}^{x_0-\epsilon_0} &= -\int_{\tilde{x}}^{x_0-\epsilon_0} u(m-u) \\
u'(x_0 - \epsilon_0) &= \alpha u m'\Big|_{\tilde{x}}^{x_0-\epsilon_0} - \int_{\tilde{x}}^{x_0-\epsilon_0} u(m-u) \\
&< C\alpha e^{-b\alpha} \le e^{-\tilde{b}\alpha}
\end{aligned}$$

for α large, where $\tilde{b}$ is a positive constant and (18) is established.

From (16) and (17) we obtain

$$\sqrt{\alpha}\int_{x_0-\epsilon_0}^{x_0+\epsilon_0} mu = \sqrt{\alpha}\int_{x_0-\epsilon_0}^{x_0+\epsilon_0} u^2 + O(\sqrt{\alpha}e^{-\tilde{b}\alpha}). \tag{20}$$

Next, we need the following technical lemma.

Lemma 2.13.

$$\begin{aligned}
\limsup_{\alpha\to\infty} \sqrt{\alpha}\int_{x_0-\epsilon_0}^{x_0+\epsilon_0} mu_\alpha &= m(x_0)\left(\int_{\mathbf{R}} e^{\frac{1}{2}m''(x_0)y^2}\,dy\right)\limsup_{\alpha\to\infty} u_\alpha(x_\alpha), \\
\liminf_{\alpha\to\infty} \sqrt{\alpha}\int_{x_0-\epsilon_0}^{x_0+\epsilon_0} mu_\alpha &= m(x_0)\left(\int_{\mathbf{R}} e^{\frac{1}{2}m''(x_0)y^2}\,dy\right)\liminf_{\alpha\to\infty} u_\alpha(x_\alpha), \\
\limsup_{\alpha\to\infty} \sqrt{\alpha}\int_{x_0-\epsilon_0}^{x_0+\epsilon_0} u_\alpha^2 &= \left(\int_{\mathbf{R}} e^{m''(x_0)y^2}\,dy\right)\limsup_{\alpha\to\infty} u_\alpha^2(x_\alpha), \\
\liminf_{\alpha\to\infty} \sqrt{\alpha}\int_{x_0-\epsilon_0}^{x_0+\epsilon_0} u_\alpha^2 &= \left(\int_{\mathbf{R}} e^{m''(x_0)y^2}\,dy\right)\liminf_{\alpha\to\infty} u_\alpha^2(x_\alpha).
\end{aligned}$$

We postpone the proof of Lemma 2.13 and continue to prove Theorem 2.12. Taking limsup and liminf respectively as $\alpha \to \infty$ on both sides of (20) we have

$$\left(\limsup_{\alpha\to\infty} u(x_\alpha)\right) m(x_0)\int_{\mathbf{R}} e^{\frac{1}{2}m''(x_0)y^2}\,dy = \left(\limsup_{\alpha\to\infty} u(x_\alpha)\right)^2 \int_{\mathbf{R}} e^{m''(x_0)y^2}\,dy,$$

and

$$\left(\liminf_{\alpha\to\infty} u(x_\alpha)\right) m(x_0)\int_{\mathbf{R}} e^{\frac{1}{2}m''(x_0)y^2}\,dy = \left(\liminf_{\alpha\to\infty} u(x_\alpha)\right)^2 \int_{\mathbf{R}} e^{m''(x_0)y^2}\,dy.$$

Since $0 < m(x_0) \le \liminf_{\alpha\to\infty} u(x_\alpha) \le \limsup_{\alpha\to\infty} u(x_\alpha) < \infty$, we obtain

$$\limsup_{\alpha\to\infty} u(x_\alpha) = \liminf_{\alpha\to\infty} u(x_\alpha) = \sqrt{2}m(x_0)$$

and our conclusion follows. □

It remains to prove Lemma 2.13. We will only show the first equality as the rest are similar.

Proof of Lemma 2.13. First, observe that

$$\left|\sqrt{\alpha}\int_{(x_0-\epsilon_0, x_0+\epsilon_0)\setminus I_{\delta_1}} m(x)u(x)\;dx\right| \le \sqrt{\alpha}\delta_1\|m\|_{L^\infty}\cdot 2\epsilon_0 \to 0 \tag{21}$$

as $\alpha \to \infty$. Now for any $\epsilon > 0$, (14) implies that

$$\sqrt{\alpha}(1-\epsilon)u(x_\alpha)\int_{I_{\delta_1}(x_0)} m(x)e^{\frac{\alpha}{2}m''(x_0)(x-x_0)^2}dx \le \sqrt{\alpha}\int_{I_{\delta_1}(x_0)} mu\; dx \le \sqrt{\alpha}(1+\epsilon)u(x_\alpha)\int_{I_{\delta_1}(x_0)} m(x)e^{\frac{\alpha}{2}m''(x_0)(x-x_0)^2}dx. \tag{22}$$

We compute, for any constant $M > 0$, by Lemma 2.4,

$$\int_{-M}^{M} m_\alpha(y)e^{\frac{1}{2}m''(x_0)y^2}dy \le \sqrt{\alpha}\int_{I_{\delta_1}(x_0)} m(x)e^{\frac{\alpha}{2}m''(x_0)(x-x_0)^2}dx \le \int_{\mathbf{R}} m_\alpha(y)e^{\frac{1}{2}m''(x_0)y^2}dy. \tag{23}$$

where $y = \sqrt{\alpha}(x - x_0)$ and $m_\alpha(y) = m(x)$. For $-M \le y \le M$, we have

$$x_0 - \frac{M}{\sqrt{\alpha}} \le x \le x_0 + \frac{M}{\sqrt{\alpha}}$$

and thus for α large, $|m_\alpha(y) - m(x_0)| \to 0$ as $\alpha \to \infty$. This implies that

$$\left|\int_{-M}^{M} m_\alpha(y)e^{\frac{1}{2}m''(x_0)y^2}dy - m(x_0)\int_{-M}^{M} e^{\frac{1}{2}m''(x_0)y^2}dy\right| \to 0$$

as $\alpha \to \infty$. On the other hand,

$$\int_{\mathbf{R}\setminus(-M,M)} (|m_\alpha(y)| + m(x_0))e^{\frac{1}{2}m''(x_0)y^2}dy \to 0$$

as $M \to \infty$, since $m''(x_0) < 0$. Hence,

$$\left|\int_{\mathbf{R}} m_\alpha(y)e^{\frac{1}{2}m''(x_0)y^2}dy - \int_{\mathbf{R}} m(x_0)e^{\frac{1}{2}m''(x_0)y^2}dy\right| \to 0$$

as $\alpha \to \infty$, and (23) becomes, for any $M > 0$,

$$m(x_0)\int_{-M}^{M} e^{\frac{1}{2}m''(x_0)y^2}dy + o(1) \le \sqrt{\alpha}\int_{I_{\delta_1}(x_0)} m(x)e^{\frac{\alpha}{2}m''(x_0)(x-x_0)^2}dx \le m(x_0)\int_{\mathbf{R}} e^{\frac{1}{2}m''(x_0)y^2}dy$$

holds for α large. Thus

$$\lim_{\alpha\to\infty}\sqrt{\alpha}\int_{I_{\delta_1}(x_0)} m(x)e^{\frac{\alpha}{2}m''(x_0)(x-x_0)^2}dx = m(x_0)\int_{\mathbf{R}} e^{\frac{1}{2}m''(x_0)y^2}dy$$

since M can be arbitrarily large. Now from (22) we conclude that

$$(1-\epsilon)\left[\limsup_{\alpha\to\infty} u(x_\alpha)\right] m(x_0)\int_{\mathbf{R}} e^{\frac{1}{2}m''(x_0)y^2}dy \le \limsup_{\alpha\to\infty}\sqrt{\alpha}\int_{I_{\delta_1}(x_0)} mu \le (1+\epsilon)\left[\limsup_{\alpha\to\infty} u(x_\alpha)\right] m(x_0)\int_{\mathbf{R}} e^{\frac{1}{2}m''(x_0)y^2}dy$$

Combining (21) and the inequality above we have

$$\limsup_{\alpha\to\infty}\sqrt{\alpha}\int_{x_0-\epsilon_0}^{x_0+\epsilon_0} mu = m(x_0)\left(\int_{\mathbf{R}} e^{\frac{1}{2}m''(x_0)y^2}dy\right)\limsup_{\alpha\to\infty} u(x_\alpha)$$

This finishes the proof. □

Theorem 1.2 now follows from Proposition 2.3, Theorems 2.10 and 2.12.

3. **Proof of Theorem 1.1.** As before, in this section we assume $\Omega = (-1, 1)$ and $m(x) \in C^3([-1, 1])$ satisfies **(M1),(M2),(M3)**. In addition, here we assume $\int_{-1}^{1} m > 0$. Let (U_α, V_α) be a coexistence steady-state of (4). The existence of (U_α, V_α) for large α is established in [2] and generalized in [3]. Again, in this section, the sub-indices d_2 and α will be suppressed when there is no confusion.

Lemma 3.1. *$0 \le U_\alpha \le u_\alpha$ and $0 \le V_\alpha \le \theta_{d_2}$ in $(-1, 1)$ where u_α is the unique positive solution of (6) and θ_{d_2} is the unique positive solution to*

$$\begin{cases} d_2\theta'' + \theta(m - \theta) = 0 & \text{in } (-1, 1) \\ \theta' = 0 & \text{at } x = -1, 1. \end{cases} \tag{24}$$

Proof. The existence and uniqueness of θ_{d_2} is standard. (See, e.g. Lemma 7.1 in [3]). By (4), U_α satisfies,

$$\begin{cases} (U' - \alpha U m')' + U(m - U) = UV \ge 0 & \text{in } (-1, 1) \\ U' - \alpha U m' = 0 & \text{at } \ x = -1, 1. \end{cases} \tag{25}$$

and V_α satisfies

$$\begin{cases} d_2V'' + V(m - V) = UV \ge 0 & \text{in } \ (-1, 1) \\ V' = 0 & \text{at } \ x = -1, 1. \end{cases} \tag{26}$$

(Here we have set $d_1 = 1$ for simplicity.) It follows that U_α and V_α are lower solutions of (6) and (24) respectively. Since u_α, θ_{d_2} are the unique positive steady-states of (6) and (24) respectively which are globally asymptotically stable, the inequalities follow from standard upper and lower solutions arguments. □

Lemma 3.2. *$V_\alpha \to \theta_{d_2}$ in $C^{1,\beta}([-1, 1])$ for any $0 < \beta < 1$.*

Proof. By Lemma 3.1, Theorem 2.11, $\{U_\alpha, V_\alpha\}_\alpha$ is bounded in $L^\infty(-1, 1)$ uniformly. Hence by (26), $\{V_\alpha\}$ is bounded in $C^2([-1, 1])$ uniformly and is therefore relatively compact in $C^{1,\beta}([-1, 1])$ for any $0 < \beta < 1$.

Next, take an arbitrary subsequence $\{V_{\alpha_i}\}_i$ such that $V_{\alpha_i} \to V$ in $C^{1,\beta}([-1, 1])$ for some $V \in C^{1,\beta}([-1, 1])$. Then V satisfies $d_2V'' + V(m - V) = 0$ weakly, i.e. for any $\psi \in H^1(-1, 1)$,

$$-d_2 \int_{-1}^{1} V'\psi' + \int_{-1}^{1} \psi V(m - V) = 0.$$

Take, for $x_0 \in [-1, 1)$

$$\psi_{\epsilon,x_0} = \begin{cases} 1 & x < x_0 \\ \frac{x_0+\epsilon-x}{\epsilon} & x_0 \le x < x_0 + \epsilon \\ 0 & x \ge x_0 + \epsilon \end{cases}$$

$$\psi_{\epsilon,1} = \begin{cases} 1 & x < 1 - \epsilon \\ \frac{1-x}{\epsilon} & 1 - \epsilon \le x \le 1. \end{cases}$$

Now, letting $\epsilon \to 0_+$, we have

$$d_2V'(x_0) + \int_{-1}^{x_0} V(m - V) = 0, \qquad \forall x_0 \in [-1, 1].$$

We then have $V' \in C^1([-1, 1])$, i.e. $V \in C^2([-1, 1])$ and so V satisfies (24) in the classical sense. Hence $V \equiv \theta_{d_2}$ by uniqueness. Thus, $V_\alpha \to \theta_{d_2}$ in $C^{1,\beta}([-1, 1])$ for any $0 < \beta < 1$. □

The following result is contained in Theorem 1.8 of [5].

Lemma 3.3. *For any $r > 0$ and $x_0 \in \mathfrak{M}$,*

$$\liminf_{\alpha\to\infty} \max_{B_r(x_0)} U_\alpha \geq m(x_0) - \theta(x_0).$$

Lemma 3.4. *Suppose that* $\liminf_\alpha \sup_{[x_0-\epsilon_0,x_0+\epsilon_0]} U_\alpha > 0$*, then*

$$\frac{U_\alpha(x)}{U_\alpha(x_\alpha)} \exp\{\alpha[m(x_0) - m(x)]\} \to 1 \tag{27}$$

uniformly in $[x_0 - \frac{M}{\sqrt{\alpha}}, x_0 + \frac{M}{\sqrt{\alpha}}]$*, for all $M > 0$ as $\alpha \to \infty$, where* $U_\alpha(x_\alpha) = \sup_{[x_0-\epsilon_0,x_0+\epsilon_0]} U_\alpha$ *and* $x_\alpha \in (x_0 - \frac{M}{\sqrt{\alpha}}, x_0 + \frac{M}{\sqrt{\alpha}})$ *for α large.*

Proof. The existence of x_α follows from Lemma 3.1, Theorems 2.10 and 2.12. Also, by (10) and its proof, $\alpha' m(x_\alpha) = o(1)$ and $\alpha[m(x_0) - m(x_\alpha)] = o(1)$. Now, let

$$w(x) = U_\alpha(x) \exp\{\alpha[m(x_0) - m(x)]\}.$$

By Lemma 3.1, Theorems 2.10 and 2.12, w is bounded in $L^\infty[x_0 - \frac{M}{\sqrt{\alpha}}, x_0 + \frac{M}{\sqrt{\alpha}}]$ uniformly in α. Moreover, it satisfies

$$\begin{cases} (\exp\{\alpha[m(x) - m(x_0)]\}w')' + F_\alpha = 0 \quad \text{in } [x_0 - \frac{M}{\sqrt{\alpha}}, x_0 + \frac{M}{\sqrt{\alpha}}] \\ w(x_\alpha) = U_\alpha(x_\alpha) \exp\{\alpha[m(x_0) - m(x_\alpha)]\} \\ w'(x_\alpha) = -w(x_\alpha)\alpha m'(x_\alpha) \end{cases} \tag{28}$$

where $F_\alpha = U_\alpha(m - U_\alpha - V_\alpha)$ is bounded in $L^\infty[x_0 - \frac{M}{\sqrt{\alpha}}, x_0 + \frac{M}{\sqrt{\alpha}}]$ uniformly in α. Thus,

$$\exp\{\alpha[m(x) - m(x_0)]\}w'(x) = -w(x_\alpha)\alpha m'(x_\alpha) \exp\{\alpha[m(x_\alpha) - m(x_0)]\} - \int_{x_\alpha}^x F_\alpha,$$

and

$$w(x) - w(x_\alpha) = \int_{x_\alpha}^x \Bigg[\exp\{\alpha[m(x_0) - m(y)]\}\times \\ (-w(x_\alpha)\alpha m'(x_\alpha) \exp\{\alpha[m(x_\alpha) - m(x_0)]\} - \int_{x_\alpha}^y F_\alpha)\Bigg] dy.$$

It is not hard to see that the integrand on the right-hand side is bounded in $L^\infty[x_0 - \frac{M}{\sqrt{\alpha}}, x_) + \frac{M}{\sqrt{\alpha}}]$ for each M. Therefore, since $|x - x_\alpha| \leq \frac{2M}{\sqrt{\alpha}} \to 0$, we see that $|w(x) - w(x_\alpha)| \to 0$ uniformly in $[x_0 - \frac{M}{\sqrt{\alpha}}, x_0 + \frac{M}{\sqrt{\alpha}}]$. Now $U_\alpha(x_\alpha)$ is bounded away from 0 as $\alpha \to \infty$, and $\alpha[m(x_0) - m(x_\alpha)] = o(1)$. The Lemma is proved. ☐

Lemma 3.5. *If $m(x_0) - \theta(x_0) > 0$, then (27) holds and*

$$\lim_\alpha U_\alpha(x_\alpha) = \sqrt{2}(m(x_0) - \theta(x_0)).$$

Proof. By Lemma 3.3, the assumption of Lemma 3.4 is satisfied. Therefore (27) holds. Now we proceed to evaluate $\lim_{\alpha\to\infty} U_\alpha(x_\alpha)$. We first claim that for some small constant $\epsilon_0 > 0$, and some $\tilde{b} > 0$,

$$\int_{x_0-\epsilon_0}^{x_0+\epsilon_0} U_\alpha(m - U_\alpha - V_\alpha)dx = O(e^{-\tilde{b}\alpha}). \tag{29}$$

By integrating (25) from $x_0 - \epsilon_0$ to $x_0 + \epsilon_0$, we obtain

$$\int_{x_0-\epsilon_0}^{x_0+\epsilon_0} U_\alpha(m - U_\alpha - V_\alpha)dx = -(U'_\alpha - \alpha U_\alpha m')\Big|_{x_0-\epsilon_0}^{x_0+\epsilon_0}.$$

By Lemma 3.1 and Theorem 2.1 (v), it suffices to show that $U'_\alpha(x_0 \pm \epsilon_0) = O(e^{-\tilde{b}\alpha})$. We shall only estimate $U'_\alpha(x_0+\epsilon_0)$, as the other can be handled in a similar fashion.

Case 1. $x_0 = \max \mathfrak{M}$.

Integrating the equation (25) from $x_0+\epsilon_0$ to 1, we obtain

$$U'_\alpha(x_0+\epsilon_0) = \alpha(U_\alpha m')(x_0+\epsilon_0) + \int_{x_0+\epsilon_0}^1 U_\alpha(m - U_\alpha - V_\alpha)dx$$

by the no-flux boundary condition at 1. Now every term on the right hand side is bounded by $C\alpha e^{-b\alpha}$, therefore our assertion follows.

Case 2. $x_0 < \max \mathfrak{M}$.

At least one of the following holds:

(i) $U_\alpha(x) \le O(e^{-b\alpha})$ in $[x_0+\epsilon_0, 1)$;

(ii) there exists $\tilde{x} \in (x_0, 1)$ such that $U'_\alpha(\tilde{x}) = 0$ and $U_\alpha(x) \le O(e^{-b\alpha})$ in the closed interval between $x_0+\epsilon_0$ and $\tilde{x}$.

The assertion follows as in *Case* 1 if (i) holds. If (ii) holds, integrate (25) from $\tilde{x}$ to $x_0+\epsilon_0$. Then

$$|U'_\alpha(x_0+\epsilon_0)| \le \left|\alpha(U_\alpha m')|_{\tilde{x}}^{x_0+\epsilon_0}\right| + \left|\int_{\tilde{x}}^{x_0+\epsilon_0} U_\alpha(m - U_\alpha - V_\alpha)dx\right|$$

and the assertion holds. Hence (29) is proved.

By changing coordinates $y = \sqrt{\alpha}(x - x_0)$ in (29),

$$\begin{aligned}\left|\int_{-M}^{M} U_\alpha(m - U_\alpha - V_\alpha)dy\right| &\le \alpha^{\frac{1}{2}}\left|\int_{(x_0-\epsilon_0, x_0+\epsilon_0)\setminus[x_0-\frac{M}{\sqrt{\alpha}}, x_0+\frac{M}{\sqrt{\alpha}}]} U_\alpha(m - U_\alpha - V_\alpha)\right| \\ &\quad + O(\alpha^{\frac{1}{2}}e^{-\tilde{b}\alpha}) \\ &\le C\int_{\mathbf{R}\setminus[-M,M]} e^{\frac{1}{2}m''(x_0)y^2}dy + O(\alpha^{\frac{-1}{32}}) + O(\alpha^{\frac{1}{2}}e^{-\tilde{b}\alpha})\end{aligned}$$

by Lemma 3.1, Theorem 2.10 and Theorem 2.12.

By taking $\alpha_i \to \infty$ such that $U_{\alpha_i}(x_{\alpha_i}) \to \limsup_\alpha U_\alpha(x_\alpha)$, making use of Lemmas 3.2 and 3.4, we have

$$\begin{aligned}&\Bigg|(\limsup U_\alpha(x_\alpha))(m(x_0) - \theta(x_0))\int_{-M}^{M} e^{\frac{1}{2}m''(x_0)y^2}dy \\ &\qquad -(\limsup U_\alpha(x_\alpha))^2\int_{-M}^{M} e^{m''(x_0)y^2}dy\Bigg| \\ &\le C\int_{\mathbf{R}\setminus[-M,M]} e^{\frac{1}{2}m''(x_0)y^2}dy.\end{aligned}$$

Take $M \to +\infty$, we have

$$P(\limsup U_\alpha(x_\alpha)) = 0 \text{ where } P(s) = \sqrt{2}(m(x_0) - \theta(x_0))s - s^2. \tag{30}$$

Similarly, we have

$$P(\liminf U_\alpha(x_\alpha)) = 0. \tag{31}$$

Now if $m(x_0) - \theta(x_0) > 0$, then by Lemmas 3.1 and 3.3,

$$+\infty > \limsup_{\alpha} U_\alpha(x_\alpha) \geq \liminf_{\alpha} U_\alpha(x_\alpha) \geq m(x_0) - \theta(x_0) > 0.$$

By (30) and (31), $\limsup_\alpha U_\alpha(x_\alpha) = \liminf_\alpha U_\alpha(x_\alpha) = \sqrt{2}(m(x_0) - \theta(x_0))$. □

Lemma 3.6. *If $m(x_0) - \theta(x_0) \leq 0$, then for each small $r > 0$, $U_\alpha \to 0$ uniformly in $(x_0 - r, x_0 + r)$.*

Proof. Suppose to the contrary that there exists a sequence $\alpha_i \to \infty$, such that $\lim_{\alpha_i}\left[\sup_{(x_0-\epsilon_0,x_0+\epsilon_0)} U_{\alpha_i}\right] > 0$. Then by the same arguments in the proof of (30),

$$P(\lim_i U_{\alpha_i}(x_{\alpha_i})) = 0 \text{ where } P(s) = \sqrt{2}(m(x_0) - \theta(x_0))s - s^2,$$

a contradiction, because P does not have any positive roots. □

We now prove Theorem 1.1.

Proof of Theorem 1.1. Part (i) follows from Lemma 3.2. Part (ii) is a consequence of Lemmas 3.4, 3.5 and 3.6. Finally, part (iii) follows from Lemma 3.1 and Theorem 2.1(v). □

4. Concluding remarks.

Although we have set for simplicity $d = d_1 = 1$ in Sections 2 and 3, the results in this paper hold true for any $d_1, d > 0$ with essentially the same proofs. Moreover, as stated in the concluding remarks in [5], the assumptions on critical points of $m(x)$ in $\{x \in \Omega : m(x) < 0\}$ can be substantially weakened. In fact, **(M2)** can be replaced by

(M2′): $\mathfrak{M} \subseteq (-1, 1)$ and all critical points of $m(x)$ in $\{x \in \Omega : m(x) > -\delta\}$ are nondegenerate for some $\delta > 0$.

Only **(M1)**,**(M2′)** and **(M3)** were needed to construct the upper solution in [5], which implies that $u_\alpha \to 0$ uniformly in $\{x \in \Omega : m(x) < 0\}$. As a consequence, we only need $\{x \in \Omega : |\nabla m(x)| = 0 \text{ and } m(x) \geq 0\}$ to be of measure zero (which is a consequence of **(M2′)**) to obtain $\|u_\alpha\|_{L^2} \to 0$, as $\alpha \to \infty$.

Finally, we remark that the multi-dimensional cases of (3) and (5) will be treated in a forthcoming paper.

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