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Symmetry of Positive Solutions of Nonlinear Elliptic Equations in R^n

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We prove symmetry of positive solutions of two general types of nonlinear elliptic equations on R^n:

(a) $-\Delta u = g(u)$, $g(u) = O(u^\alpha)$ near $u = 0$ for suitable $\alpha > 1$, and
(b) $-\Delta u + u = g(u)$, $g(u) = O(u^\alpha)$ near $u = 0$, $\alpha > 1$.

Under certain additional assumptions on the nonlinearities, we show that positive solutions tending to zero at infinity are spherically symmetric about some point. In the case of equations of type (a), we also establish symmetry of positive solutions with isolated singularities.

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1. Introduction

In Gidas *et al.* [4], some special forms of the maximum principle and a device of Alexandroff were employed to study symmetry and related properties of positive solutions of nonlinear elliptic (and parabolic) equations in bounded domains and in the entire space. In bounded domains, the class of nonlinearities we could treat was quite general.[†] In contrast, in the entire space our results were rather special. The following is typical: Let $u(x)$ be positive, $C^{2+\mu}$, $0 < \mu < 1$, solution of

$$-\Delta u = g(u) \qquad \text{in } R^n, \quad n > 2,$$

such that $u(x) = O(|x|^{2-n})$ at infinity. Assume that for some $k \geq (n+2)/(n-2)$, $u^{-k}g(u)$ is Hölder continuous on $0 \leq u \leq u_0$, where u_0 is the maximum of $u(x)$. Then $u(x)$ is spherically symmetric about some point, and if that point is taken as the origin, then $u_r < 0$ for $r > 0$.

In this chapter we study spherical symmetry of positive solutions in the entire space and treat several classes of nonlinearities as well as some singular solutions. Our main results are

THEOREM 1. *Let $u(x)$ be a positive C^2 solution of*

$$-\Delta u = g(u) \qquad \text{in } R^n, \quad n \geq 3 \tag{1.1}$$

with

$$u(x) = O(1/|x|^m) \qquad \text{at infinity}, \quad m > 0. \tag{1.2}$$

Assume (i) *on the interval* $0 \leq u \leq u_0$, $u_0 = \max u$, $g = g_1 + g_2$ *with* $g_1 \in C^1$, g_2 *continuous and nondecreasing,* (ii) *for some*

$$\alpha > \max((n+1)/m, (2/m) + 1),$$

$g(u) = O(u^\alpha)$ *near* $u = 0$. *Then $u(x)$ is spherically symmetric about some point in R^n, and $u_r < 0$ for $r > 0$ where r is the radial coordinate about that point. Furthermore*

$$\lim_{x \to \infty} |x|^{n-2} u(x) = k > 0.$$

THEOREM 2. *Let $u(x)$ be a positive, C^2 solution of*

$$-\Delta u + m^2 u = g(u) \qquad \text{in } R^n, \qquad n \geq 2, \quad m > 0 \tag{1.3}$$

[†] Since it may prove useful, we wish to point out that in [4, Theorem 2.1], for general nonlinear equations, except in the case that $g(x) < 0$ in condition (b), it suffices that $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$.

with $u(x) \to 0$ *as* $|x| \to \infty$ *and* g *continuous,* $g(u) = O(u^\alpha)$, $\alpha > 1$ *near* $u = 0$. *On the interval* $0 \le s \le u_0 = \max u(x)$, *assume*

$$g(s) = g_1(s) + g_2(s)$$

with g_2 *nondecreasing and* $g_1 \in C^1$ *satisfying, for some* $C > 0$, $p > 1$,

$$|g_1(u) - g_1(v)| \le C|u - v|/|\log \min(u, v)|^p, \qquad 0 \le u, v \le u_0.$$

Then $u(x)$ *is spherically symmetric about some point in* R^n *and* $u_r < 0$ *for* $r > 0$, *where* r *is the radial coordinate about that point. Furthermore,*

$$\lim_{r \to \infty} r^{(n-1)/2} e^r u(r) = \mu > 0.$$

Remark. In particular, if $u > 0$ tends to zero at infinity and satisfies

$$\Delta u + f(u) = 0,$$

$f \in C^{1+\mu}$, $\mu > 0$ and $f(0) = 0$, $f'(0) < 0$ then Theorem 2 applies.

With no loss in generality we shall always take $m = 1$ in (1.3). We use the same techniques to study certain isolated singularities. More precisely, we prove

THEOREM 3. *Let* $u(x)$ *be a positive* C^2 *solution of*

$$-\Delta u = g(u) \qquad in \quad R^n - \{0\} \tag{1.4}$$

satisfying

$$u(x) = O(1/|x|^m) \qquad at\ \infty\ for\ some \quad m > 0, \tag{1.5}$$

$$u(x) \to +\infty \qquad as \quad x \to 0. \tag{1.6}$$

Furthermore, assume:

(i) $g(u)$ *is continuous, nondecreasing in* u *for* $u \ge 0$, *and for some* $\alpha > (n+1)/m$, $g(u) = O(u^\alpha)$ *near* $u = 0$,
(ii) $\underline{\lim}_{u \to +\infty} [g(u)/u^p] > 0$, *for some* $p \ge n/(n-2)$.

Then $u(x)$ *is spherically symmetric about the origin, and* $u_r < 0$ *for* $r > 0$.

A generalization, Theorem 3′, is given in Section 3, and variants of Theorem 1 are presented at the end of Section 2.

THEOREM 4. Let $u(x)$ be a positive solution of

$$\Delta u + u^{(n+2)/(n-2)} = 0 \qquad in \quad R^n - \{0, \infty\} \tag{1.7}$$

with two singularities located at the origin and at infinity. More precisely, assume

$$u(x) \to +\infty \qquad \text{as} \quad x \to 0,$$
$$|x|^{n-2}u(x) \to +\infty \qquad \text{as} \quad x \to \infty.$$

Then $u(x)$ is spherically symmetric about the origin.

A particular equation covered by Theorem 2 is

$$-\Delta u + m^2 u = \lambda u^3 \qquad \text{on } R^3, \quad \lambda > 0, \quad m > 0. \tag{1.8}$$

This equation gives rise to solitary wave solutions of the nonlinear Klein–Gordon equation. Existence of positive spherically symmetric solutions of (1.8) has been known for some time [8].

An equation covered by Theorem 3 is

$$\Delta u + u^\alpha = 0 \tag{1.9}$$

in R^n, with $(n+1)/(n-2) < \alpha$. For $(n+1)/(n-2) < \alpha < (n+2)/(n-2)$ Eq. (1.9) is known [5] to have spherically symmetric solutions satisfying (1.5) and (1.6). Certain equations in astrophysics [2] are related to (1.9) with $n = 3$ and $\alpha < 5$. See also [5] for further analysis of spherically symmetric solutions of such equations. It follows from the analysis in [5] that for $\alpha > (n+2)/(n-2)$, (1.9) does not have spherically symmetric solutions satisfying (1.5), (1.6) for $m = n - 2$; therefore Theorem 3 implies there are no positive solutions satisfying these conditions. For $\alpha = (n+2)/(n-2)$ [Eq. (1.7)], all spherically symmetric solutions are explicitly known [1, 3]. Among them there exist a two-parameter family of solutions which are singular both at the origin and at infinity. Theorem 4 asserts that these are the only solutions with two isolated singularities. Since Eq. (1.7) is conformally invariant, the two singular points could be brought to any two finite points of R^n. An example of an equation covered by Theorem 1 with $m = n - 2$ and $\alpha > (n+2)/(n-2)$ was given in [4, p. 212]. Equation (1.7) has the regular solution

$$u(x) = \left(\frac{\lambda\sqrt{n(n-2)}}{\lambda^2 + |x|^2}\right)^{(n-2)/2}, \qquad \lambda > 0, \tag{1.10}$$

and it provides an example for Theorem 1 with $m = n - 2$, and $\alpha = (n+2)/(n-2)$. It is natural to ask whether there is an equation which satisfies the conditions of Theorem 1 with $\alpha < (n+2)/(n-2)$. Here is an example, for $n > 3$; we also have an example for $n = 3$. With r as polar coordinate, and $\alpha \geq (n+1)/(n-2)$, consider the function $u(r)$ defined as follows:

$$u(r) = (1/r^{n-2}) - (a/r^s) \qquad \text{for} \quad r \geq 1,$$

where

$$s = \alpha(n-2) - 2, \qquad as = (s + 2 - n)^{-1} = [\alpha(n-2) - n]^{-1}.$$

For $0 \le r \le 1$ we define

$$u(r) = A - (b/2)r^2 + (c/4)r^4,$$

where

$$2c = n(n-2) - \frac{\alpha(n-2)}{\alpha(n-2)-n}, \qquad 2b = n^2 - 4 - \frac{\alpha(n-2)+2}{[\alpha(n-2)-n]},$$

$$A = (b/2) - (c/4) + 1 - a.$$

With these choices one verifies that $u \in C^2$ on $0 \le r < \infty$, u is positive, decreasing, and

$$u_{rr} + [(n-1)/r]u_r + g(u) = 0,$$

where g is Lipschitz continuous and

$$g(u) = O(u^\alpha) \qquad \text{near} \quad u = 0.$$

Our proofs are modeled on the proofs in [4] of similar results in balls. An important tool there was the Hopf boundary lemma, [4, Lemma (H)] and in this chapter we derive suitable forms of it for positive solutions (tending to zero at infinity) of equations in R^n. Here are two forms (in which $\partial_j = \partial/\partial x_j$). We consider only $n > 2$.

LEMMA (H_1'). *Let $u > 0$ be of class C^2 in $|x| \ge R > 0$, tend to zero at infinity, and satisfy*

$$Lu \equiv (\Delta + \textstyle\sum b_j(x)\partial_j + c(x))u \le 0 \qquad \text{in} \quad |x| \ge R, \tag{1.11}$$

with

$$b_j = O(|x|^{1-p}), \qquad c(x) = O(|x|^{-p}), \qquad p > 2. \tag{1.12}$$

Then, for some $\mu > 0$,

$$u(x) \ge \mu/|x|^{n-2}.$$

LEMMA (H_2'): *Let $u > 0$ be of class C^2 in $|x| \ge R > 0$, tend to zero at infinity, and satisfy*

$$Lu \equiv (\Delta - 1 + \textstyle\sum b_j(x)\partial_j + c(x))u \le 0 \tag{1.13}$$

with

$$b_j, c = O(|x|^{-p}), \qquad p > 1.$$

Then, for some $\mu > 0$,

$$u(x) \ge \mu e^{-|x|}/|x|^{(n-1)/2}.$$

These and some related results are proved in Section 5.

Remark. In $|x| \geq 1$ the function

$$u = 1/|x|^m, \qquad m > n - 2,$$

satisfies

$$(\Delta - m(m + 2 - n)/|x|^2)u = 0$$

showing that the condition $p > 2$ cannot be omitted in Lemma (H'_1).

A similar example concerning Lemma (H'_2) is given in Section 5.

It would be of interest in Theorems 1 and 2 to permit the nonlinear term to depend on derivatives of u. Here is an extension of Theorem 2, having a very similar proof—which we leave to the reader.

THEOREM 2′. *Let $u > 0$ be a C^2 solution of*

$$\Delta u - u + g_1(u, u_1, \ldots, u_n) + g_2(u) = 0$$

with u, $\nabla u = O(e^{-|x|})$ at infinity. Assume (i) *$g_2(s)$ is continuous and nondecreasing on the interval $0 \leq s \leq u_0 = \max u(x)$, and $g(u) = O(u^\alpha)$, $\alpha > 1$ near the origin and* (ii) *$g_1 \in C^{1+\varepsilon}$, $\varepsilon > 0$, and g_1 and its first derivatives vanish at the origin in R^{n+1}. Furthermore, for $u > 0$, g_1 is symmetric in the second argument u_1. Then u is symmetric about some hyperplane $x_1 = \lambda$ and $u_{x_1} < 0$ for $x_1 > \lambda$.*

Clearly if, in addition, $g_1(u, u_1, \ldots, u_n)$ depends only on u and $|\text{grad}\, u|^2$ one infers that u is spherically symmetric about some point and that $u_r < 0$ if that point is chosen as the origin.

2. THE PROOF OF THEOREM 1

We begin with some preliminary observations.

Equation (1.1) may be written in the form

$$\Delta u + c(x)u = 0, \tag{2.1}$$

where $c(x) = g(u(x))/u(x) = O(|x|^{-p})$ near infinity. Here $p = m(\alpha - 1) > 2$ by assumption. We may apply Lemma (H'_1) and infer that

$$u(x) \geq \mu|x|^{2-n}, \qquad \mu > 0. \tag{2.2}$$

It is easily seen, on the other hand, that for some constant $c > 0$,

$$u = c \int \frac{1}{|x - y|^{n-2}} f(y)\, dy, \tag{2.3}$$

where $f(y) = g(u(y)) = O(|y|^{-\alpha m})$ near infinity; recall $\alpha m > n + 1$.

For each $x \in R^n$ we use x^λ to denote the reflection of x in the hyperplane $x_1 = \lambda$. Throughout the paper, C, C_1, c, c_1, etc., denote positive constants independent of u, λ, x, and y. We write $u_j = \partial u/\partial x_j$, etc. Unless indicated otherwise, all integrals extend over R^n.

We will need some simple technical facts about integrals of the form (2.3) for f continuous and satisfying

$$f(y) = O(|y|^{-q}) \quad \text{near infinity}, \qquad q > n+1. \tag{2.4}$$

LEMMA 2.1. *Let $u \in C^2$ be given by (2.3) with f satisfying (2.4). Then*

(a) $$\lim_{|x|\to\infty} |x|^{n-2}u(x) \to c\int f(y)\,dy, \tag{2.5}$$

and

(b) $$\frac{|x|^n}{x_1} u_1(x) \to -(n-2)c\int f(y)\,dy \quad \text{as } |x| \to \infty \text{ with } x_1 \to \infty. \tag{2.6}$$

(c) *If $\lambda^i \in R \to \lambda$ and $\{x^i\}$ is a sequence of points going to infinity, with $x_1^i < \lambda^i$, then*

$$\frac{|x^i|^n}{\lambda^i - x_1^i}(u(x^i) - u(x^{i\lambda^i})) \to 2(n-2)c\int f(y)(\lambda - y_1)\,dy. \tag{2.7}$$

Postponing the proof to Appendix A, we proceed with the arguments to prove Theorem 1. In the following, u satisfies the conditions of Theorem 1 and $f = g \circ u$. Combining (2.2) and (2.5), we infer that

$$\int f(y)\,dy > 0. \tag{2.8}$$

Furthermore, using (2.6) and (2.8), we may infer that

$$u_1(x) < 0 \qquad \text{for} \quad x_1 \geq \text{some constant } \bar{\lambda}. \tag{2.9}$$

We now fix the origin so that

$$\int f(y)y_j\,dy = 0, \qquad j = 1, \ldots, n;$$

we shall prove spherical symmetry about the origin.

LEMMA 2.2. *There exists $\lambda_0 > 0$ such that $\forall \lambda \geq \lambda_0$*

$$u(x) > u(x^\lambda) \qquad \text{if} \quad x_1 < \lambda. \tag{2.10}$$

Proof. Assume not. Then $\exists \lambda^i \to +\infty$ and a sequence $\{x^i\}$, with $x_1^i < \lambda^i$ and

$$u(x^i) \leq u(z^i), \tag{2.11}$$

where $z^i = x^{i\lambda^i}$. Since $z^i \to \infty$ we have $u(z^i) \to 0$ and so, necessarily, $|x^i| \to \infty$. Furthermore, in view of (2.9), we see that $x_1^i < \bar{\lambda}$. Fix some $\lambda > 0$, $\lambda > \bar{\lambda}$. Using (2.9) once more we see that for $\lambda^i > \lambda$, i.e., for i sufficiently large, we have

$$u(x^i) \le u(z^i) \le u(x^{i\lambda}),$$

since $\bar{\lambda} < x_1^{i\lambda} < z_1^i$.

We now apply Lemma 2.1c with λ in place of λ^i and find, recalling $\int f y_1 \, dy = 0$,

$$0 \ge \frac{|x^i|^n}{\lambda - x_1^i}(u(x^i) - u(x^{i\lambda})) \to 2(n-2)c\lambda \int f(y) > 0 \tag{2.12}$$

This is a contradiction; the lemma is proved. ■

LEMMA 2.3. *Suppose condition* (2.10) *holds for some* $\lambda > 0$. *Then it holds for all* $\tilde{\lambda}$ *in a neighborhood.*

Proof. We observe first that according to [4, Lemma 2.2] and Remark 1 of Section 2.3 there (see also [4, Lemma 4.3]):

$$u_1 < 0 \quad \text{on the hyperplane} \quad x_1 = \lambda. \tag{2.13}$$

If (2.10) does not hold for all neighboring $\tilde{\lambda}$, there exists a sequence $\lambda^j \to \lambda$ and sequence of points $\{x^j\}$ with $x_1^j < \lambda^j$ such that

$$u(x^j) \le u(x^{j\lambda^j}). \tag{2.14}$$

Either a subsequence, which we again call x^j, converges to some x with $x_1 \le \lambda$ and $u(x) \le u(x^\lambda)$ or else $x^j \to \infty$. In the first case, in view of (2.10), we must have $x_1 = \lambda$—but then (2.14) implies $u_1(x) \ge 0$, contradicting (2.13). So $x^j \to \infty$. But if we now apply Lemma 2.1c exactly as in the preceding proof, using the sequence λ^i, we reach a contradiction. ■

Completion of the proof of Theorem 1. Lemmas 2.2 and 2.3 imply that the set of positive λ for which property (2.10) holds is open and includes all sufficiently large λ. Thus for all λ in some *maximal* open interval $0 \le \lambda_1 < \lambda < +\infty$, property (2.10) holds. Applying [4, Lemmas 2.2 and 4.3] and Remark 1 of Section 2.3 there, we find

$$u_{x_1} < 0 \qquad \text{on} \quad x_1 = \lambda \qquad \text{for} \quad \lambda > \lambda_1. \tag{2.15}$$

In particular

$$u_{x_1} < 0 \qquad \text{for} \quad x_1 > \lambda_1$$

and, by continuity,

$$u(x) \geq u(x^{\lambda_1}) \qquad \text{for} \quad x_1 < \lambda_1.$$

We will now show that $\lambda_1 = 0$. If $\lambda_1 > 0$, then by [4, Lemmas 2.2 and 4.3] and Remark 1 of Section 2.3 there, we either have

$$u(x) \equiv u(x^{\lambda_1}) \qquad \text{for} \quad x_1 < \lambda_1, \tag{2.16}$$

or else properties (2.10) and (2.15) hold for λ_1. The latter cannot occur, by Lemma 2.3. The former also cannot occur, because if it did, choose a sequence of $\{x^i\}$, $|x^i| \to \infty$, $x_1^i < \lambda_1$ and apply Lemma 2.1c as before to obtain a contradiction. Thus $\lambda_1 = 0$, and we have proved $u(x) \geq u(x^0)$ for $x_1 < 0$. Reversing the x_1 axis, we conclude that u is symmetric about the plane $x_1 = 0$, and $u_{x_1} < 0$ for $x_1 > 0$. Since the x_1 direction was chosen arbitrarily, we conclude that $u(x)$ is spherically symmetric about the origin and $u_r < 0$ for $r > 0$. The last assertion in Theorem 1 follows from (2.15). This completes the proof of Theorem 1. ■

Remark 2.1 Our choice of origin so that $\int f(y) y_j \, dy = 0, j = 1, \ldots, n$, was used in going from (2.7) to (2.12) in the proof of Lemma 2.3. In case $g(u)$ is nondecreasing, i.e., $g = g_2$, then Lemma 2.3 holds independently of choice of origin, i.e.,

LEMMA 2.3′. *If $g(s)$ in Theorem* 1 *is nondecreasing in s on* $0 \leq s \leq \max u$, *then the set of λ for which property* (2.10) *holds is open.*

Proof. We proceed as in the proof of Lemma 2.3. Assuming (2.10) holds for some λ, we want to show that it holds for neighboring $\tilde{\lambda}$. Assuming the contrary we find as in the proof of Lemma 2.10 that

$$0 \geq \int dy \, f(y)(\lambda - y_1) = -\int dy \, f(y^\lambda)(\lambda - y_1)$$

thus

$$0 \geq \int dy (f(y) - f(y^\lambda))(\lambda - y_1). \tag{2.17}$$

Note that $f(y) = g(u(y))$, and g is nondecreasing in u. Since u satisfies (2.10), it follows that $f(y) \geq f(y^\lambda)$ for $y_1 < \lambda$. Furthermore it is not possible that $f(y) \equiv f(y^\lambda)$, for in that case, as is easily seen, we would necessarily have $g(s) = \text{constant} = 0$ on $0 \leq s \leq \max u$, which would contradict the fact that $u > 0$. Thus the integrand in (2.17) is nonnegative and not identically zero—a contradiction. Lemma 2.3′ is proved. ■

Remark 2.2 Lemma 2.3′ will be used in the proofs of Theorems 3 and 4.

In Theorem 1 the solutions decay exactly like $c|x|^{2-n}$ near infinity. It may happen that solutions decay differently. For example, in R^3, the function

$$u = (1 + |x|^2)^{-k}, \qquad k > 0,$$

satisfies $\Delta u + g(u) = 0$ with

$$g(u) = -2k(2k-1)u^{1+1/k} + 4k(k+1)u^{1+2/k}.$$

So in this case $m = 2k$, $\alpha = 1 + 1/k$ and $m(\alpha - 1) = 2$, and of course Theorem 1 does not apply. The following is a variant of Theorem 1—which however also does not cover this example.

THEOREM 1′. *Let $u(x)$ be a positive C^2 solution of*

$$-\Delta u = g(u) \qquad \text{in } R^n, \quad n \geq 3 \tag{2.18}$$

with

$$u(x) = O(1/|x|^m) \quad \textit{at infinity}, \qquad m > 0. \tag{2.19}$$

Assume (i) $g(u) \geq 0$ *on* $0 \leq u \leq u_0$, $u_0 = \max u$, $g = g_1 + g_2$ *with* $g_1 \in C^1$, g_2 *continuous and nondecreasing*, (ii) *for some* $\alpha > (n+1)/m$, $g(u) = O(u^\alpha)$ *near* $u = 0$. *Then $u(x)$ is spherically symmetric about some point in R^n, and $u_r < 0$ for $r > 0$ where r is the radial coordinate about that point.*

Proof. As before, Lemma 2.1 applies, as does Lemma 2.3. But we have to give a new proof of Lemma 2.2. The rest of the proof of Theorem 1 then applies.

Proof of Lemma 2.2 *for Theorem* 1′. From (2.3) we have

$$c^{-1}(u(x) - u(x^\lambda)) = \int dy\, f(y) \left\{ \frac{1}{|x-y|^{n-2}} - \frac{1}{|x^\lambda - y|^{n-2}} \right\} = I_1 - I_2,$$

where

$$I_1 = \int_{y_1 < \lambda} dy\, f(y) \left\{ \frac{1}{|x-y|^{n-2}} - \frac{1}{|x^\lambda - y|^{n-2}} \right\},$$

$$I_2 = \int_{y_1 > \lambda} dy\, f(y) \left\{ \frac{1}{|x^\lambda - y|^{n-2}} - \frac{1}{|x-y|^{n-2}} \right\}.$$

For $r \leq s$ we use

$$(n-2)\frac{1}{s^{n-1}}(s-r) \leq \frac{1}{r^{n-2}} - \frac{1}{s^{n-2}} \leq (n-2)\frac{1}{r^{n-1}}(s-r). \tag{2.20}$$

Note that

$$|x^\lambda - y| - |x - y| = \frac{4(\lambda - x_1)(\lambda - y_1)}{|x^\lambda - y| + |x - y|}, \tag{2.21}$$

Thus

$$\begin{aligned} I_1 &\geq (n-2) \int_{y_1 < \lambda} dy\, f(y) \frac{1}{|x^\lambda - y|^{n-1}} \frac{4(\lambda - x_1)(\lambda - y_1)}{|x^\lambda - y| + |x - y|} \\ &\geq 4(n-2)(\lambda - x_1) \int_{y_1 < \lambda,\ |y| < R} dy\, f(y) \frac{1}{|x^\lambda - y|^{n-1}} \frac{\lambda - y_1}{|x^\lambda - y| + |x - y|}, \end{aligned}$$

where R is a fixed number so that $\int_{|y|<R} dy\, f(y) \geq c_0 > 0$. Using $|x^\lambda - y| \leq |x^\lambda| + |R|$, and $|x^\lambda - y| \geq |x - y|$, valid for $y_1 < \lambda$, we obtain for large λ,

$$\begin{aligned} I_1 &\geq c_1 \frac{1}{|x^\lambda|^{n-1}} \frac{\lambda - x_1}{|x^\lambda|} \int_{y_1 < \lambda,\ |y| \leq R} dy\, f(y)(\lambda - y_1) \\ &\geq c \frac{1}{|x^\lambda|^{n-1}} \frac{(\lambda - x_1)(\lambda - R)}{|x^\lambda|}. \end{aligned} \tag{2.22}$$

Next we derive an upper bound for I_2. Using (2.20) and (2.21), we obtain

$$\begin{aligned} I_2 &\leq 2(n-2)(\lambda - x_1) \int_{y_1 > \lambda} dy\, f(y) \frac{(y_1 - \lambda)}{|x^\lambda - y|^{n-1}} \frac{1}{|x^\lambda - y| + |x - y|} \\ &\leq I_2' + I_2'', \end{aligned}$$

where

$$I_2' = 2(n-2)(\lambda - x_1) \int_{y_1 > \lambda,\ |y| > 2|x^\lambda|} dy\, f(y) \frac{1}{|x^\lambda - y|^{n-1}}$$

$$I_2'' = 2(n-2)(\lambda - x_1) \int_{y_1 > \lambda,\ |y| < 2|x^\lambda|} dy\, f(y) \frac{(y_1 - \lambda)}{|x^\lambda - y|^{n-1}} \frac{1}{|x^\lambda - y| + |x - y|}.$$

To bound I_2', we use (i) $|x^\lambda - y| > |x^\lambda|$ implied by $|y| > 2|x^\lambda|$, and (ii) $f(y) = O(|y|^{-\alpha m})$ valid for large $|y|$. Thus, for large λ

$$\begin{aligned} I_2' &\leq c \frac{1}{|x^\lambda|^{n-1}} (\lambda - x_1) \int_{y_1 > \lambda,\ |y| > 2|x^\lambda|} dy\, f(y) \\ &\leq c \frac{1}{|x^\lambda|^{n-1}} (\lambda - x_1) \frac{1}{|x^\lambda|^{\alpha m - n}}, \end{aligned} \tag{2.23}$$

since $\alpha m > n$. To bound I_2'', we write

$$I_2'' \leq J_1 + J_2,$$

$$J_1 = 2(n-2)(\lambda - x_1) \int_{y_1 > \lambda,\, |y| < 2|x^\lambda|,\, |x^\lambda - y| < |x^\lambda|/2} dy\, f(y) \frac{1}{|x^\lambda - y|^{n-1}},$$

$$J_2 = 2(n-2)(\lambda - x_1) \int_{y_1 > \lambda,\, |y| < 2|x^\lambda|,\, |x^\lambda - y| > |x^\lambda|/2} dy\, f(y) \frac{|y_1 - \lambda|}{|x^\lambda - y|^n}.$$

Using $|y| > |x^\lambda|/2$, and $f(y) = O(|y|^{-\alpha m})$, for large $|y|$, we bound J_1 (for large λ) by

$$J_1 \leq c \frac{1}{|x^\lambda|^{n-1}} (\lambda - x_1) \frac{1}{|x^\lambda|^{\alpha m - n}}. \tag{2.24}$$

In the integral for J_2 we have $|x^\lambda|/2 < |x^\lambda - y| < 3|x^\lambda|$, so that

$$J_2 \leq c \frac{1}{|x^\lambda|^n} (\lambda - x_1) \int_{y_1 > \lambda} dy\, f(y)|y|$$

$$\leq c \frac{1}{|x^\lambda|^n} (\lambda - x_1) \frac{1}{\lambda^{\alpha m - n - 1}} \qquad \text{since} \quad \alpha m > n + 1.$$

Combining this with (2.23) and (2.24), we obtain

$$I_2 \leq c \frac{1}{|x^\lambda|^n} (\lambda - x_1) \frac{1}{\lambda^{\alpha m - n - 1}}. \tag{2.25}$$

Comparing (2.22) and (2.25), we see that $I_1 - I_2 > 0$ for sufficiently large λ, since $\alpha m > n + 1$. This completes the new proof of Lemma 2.2 and of Theorem 1′. ■

With the aid of a similar argument and the argument used to complete the proof of Theorem 1, one establishes

THEOREM 1″. *Let $u(x)$ be a positive C^2 solution of*

$$-\Delta u = g(|x|, u(x)) \text{ in } R^n, \qquad n \geq 3$$

with $u = O(|x|^{-m})$, $m > 0$, near infinity. Assume:

(i)′ *for $r \geq 0$ and $0 < s \leq u_0 = \max u$, $g(r, s)$ is continuous, positive, non-decreasing in s and strictly decreasing in r;*

(ii)′ *for some $\alpha > (n+1)/m$, and some constant $C > 0$,*

$$g(r, u) \leq C u^\alpha \qquad \text{for} \quad u \leq u_0.$$

Then u is spherically symmetric about the origin, and $u_r < 0$ for $r > 0$.

3. The Proofs of Theorems 3 and 4

The following proposition asserts that the singular solutions in Theorems 3 and 4 are distribution solutions.

Proposition 3.1. *Let $D = \{x \in R^n : |x| \leq R\}$, and $u(x)$ be a positive C^2 solution of*

$$-\Delta u = g(u) \qquad in \quad \Omega = D - \{0\} \tag{3.1}$$

with $u(x) \to +\infty$ as $x \to 0$.

Assume $g(u) \geq 0$ for $0 \leq u$ and

$$\lim_{u \to +\infty} \frac{g(u)}{u^p} > 0, \qquad for\ some \quad p \geq \frac{n}{n-2}. \tag{3.2}$$

Then $g(u(\cdot)) \in L_1(D)$, $u \in L_p(D)$, and u is a distribution solution of (3.1).

Proof. First we show that $g(u(\cdot)) \in L_1(D)$. Let $\zeta(u)$ be a C^∞ nonnegative function defined on $u \geq 0$ which equals 1 for $u \leq c$, vanishes for $u \geq c$ and is nonincreasing. Here $c > \max_{|x|=R} u$. Multiplying (3.1) by $\zeta(u)$ we find, using Green's theorem,

$$\int \zeta(u) g(u(x)) - \int \dot{\zeta}(u) |\nabla u|^2 = -\int_{|x|=R} \frac{\partial u}{\partial r}\, dS.$$

Since $g(u) \geq 0$ *and* $\dot{\zeta} \leq 0$ we conclude that

$$\int_{u<c} g(u(x)) \leq -\int_{|x|=R} \frac{\partial u}{\partial r}\, dS$$

Letting $c \to \infty$ we conclude that

$$\int_{|x|<R} g(u(x)) \leq \int_{|x|=R} \frac{\partial u}{\partial r}\, dS. \tag{3.3}$$

By (3.2) we see that $u \in L_p(D)$. To show that $u(x)$ is a distribution solution, we prove that for $\zeta \in C_0^\infty(D)$,

$$\int_D dx (u \Delta \zeta + \zeta g(u)) = 0. \tag{3.4}$$

Let $\tau(r)$ be a C^∞ function on $r \geq 0$ vanishing on a neighborhood of the origin and equal to 1 for $r \geq R$. Let $\tau_\varepsilon(r) = \tau(r/\varepsilon)$. It suffices to prove

$$\int_D \tau_\varepsilon(r)(u \Delta \zeta + \zeta g(u)) \to 0 \qquad \text{as} \quad \varepsilon \downarrow 0. \tag{3.5}$$

By Green's theorem and Eq. (3.1) we have

$$\int_D \tau_\varepsilon(r)(u\,\Delta\zeta + \zeta g(u)) = -\int_{|x|<\varepsilon}\left\{u\zeta\,\Delta\tau_\varepsilon + 2u\frac{\partial\zeta}{\partial x_i}\frac{\partial\tau_\varepsilon}{\partial x_i}\right\}. \tag{3.6}$$

Note that $|\Delta\tau_\varepsilon| \le c/\varepsilon^2$ and $|\partial\tau/\partial x_i| \le c/\varepsilon$. Hence

$$\begin{aligned}\int_{|x|<\varepsilon} |u\zeta\,\Delta\tau_\varepsilon| &\le c\left(\int_{|x|<\varepsilon} u^p\right)^{1/p} \varepsilon^{-2}\varepsilon^{n(1-1/p)}\\ &= o(\varepsilon^{(n-2)-n/p}) \qquad \text{as} \quad \varepsilon \downarrow 0, \quad \text{since} \quad u \in L_p,\\ &\to 0 \qquad \text{as} \quad \varepsilon \to 0;\end{aligned}$$

similarly for the other term in (3.6). Thus (3.5), and hence (3.4) are proved. ■

Proof of Theorem 3. Under the assumptions of Theorem 3, we shall prove that the set of positive λ for which property (2.10) holds contains all sufficiently large λ and is open. Using Proposition 3.1, it is easy to see that the integral equation (2.3) holds. Using condition (i) of the theorem, Lemma 2.3′ and the proof of Lemma 2.2 for Theorem 1′ hold. We now proceed as in the proof of Theorem 1 to show that $\lambda_1 = 0$. If $\lambda_1 > 0$, then either we have (2.16), or else property (2.10) holds. The former cannot occur, because $u(x) \to +\infty$ as $x \to 0$, and the latter cannot occur by Lemma 2.3′. This completes the proof as in Theorem 1. ■

A suitable adaptation of this proof yields the following generalization.

THEOREM 3′. *Let $u(x)$ be a positive C^2 solution of*

$$\Delta u + g(|x|, u) = 0 \qquad \textit{in} \quad R^n\backslash\{0\}$$

satisfying

$$u(x) = O(|x|^{-m}) \ \textit{at}\ \infty \qquad \textit{for some} \quad m > 0, \qquad u(x) \to +\infty \qquad \textit{as} \quad x \to 0.$$

Assume

(i) *$g(r,u)$ is continuous, nonincreasing in r, nondecreasing in u for $r, u \ge 0$, and for some $\alpha > (n+1)/m$, $|g(r,u)| \le C|u|^\alpha$ for u small,* and

(ii) *$\underline{\lim}_{u\to+\infty} g(r,u)u^{-p} \ge c_0 > 0$ for $r < 1$ and some $p \ge n/(n-2)$. Then u is spherically symmetric about the origin and $u_r < 0$ for $r > 0$.*

The argument used in proving Theorem 3 also yields the following result:

THEOREM 5. *Let $a_1, \ldots, a_k \in R^n$ lie on a line, e.g., the x_n axis. Let $u(x)$ be a positive solution of*

$$-\Delta u = g(u) \qquad \textit{in} \quad R^n - \{a_1, a_2, \ldots, a_k\}. \tag{3.7}$$

Assume

$$u \in C^2(R^n - \{a_1, \ldots, a_k\}),$$

$$u(x) = O(1/|x|^m) \quad at\ \infty, \qquad m > 0, \tag{3.8}$$

$$u(x) \to +\infty \qquad as \quad x \to a_j, \quad j = 1, \ldots, k. \tag{3.9}$$

Assume further that conditions (i) *and* (ii) *of Theorem* 3 *hold. Then* $u(x)$ *is cylindrically symmetric about the* x_n *axis and if* r *denotes distance from the axis,* $u_r < 0$ *for* $r > 0$.

Remark. The result of Theorem 5 can be strengthened for the particular equation (1.7) and for two or three singularities. This equation is invariant under conformal transformations, i.e., if u is a solution and $x \to y$ is a conformal transformation, then the function

$$u(y) = u(x)J^{-(n-2)/2n}(x), \tag{3.10}$$

where $J(x)$ is the Jacobian of the transformation, is also a solution. If $u(x)$ has three isolated singularities, then by a conformal transformation, the three singular points may be brought to lie along a straight line. Then Theorem 5 applies and yields cylindrical symmetry about that line. For two isolated singularities, we have Theorem 4 which we prove next.

Proof of Theorem 4: Let $e_n = (0, 0, \ldots, 0, 1)$, and

$$y = \frac{x - e_n}{|x - e_n|^2} + e_n, \tag{3.11}$$

$$v(y) = |x - e_n|^{n-2} u|x|. \tag{3.12}$$

Then $v(y)$ satisfies (1.7), outside of the two singularities located at $y = 0$ and $y = e_n$, where v becomes infinite, and v is regular at infinity, i.e., $O(|y|^{2-n})$ there. Since $g(u) = u^{(n+2)/(n-2)}$ is nondecreasing. Theorem 5 applies and shows that $v(y)$ is rotationally symmetric about the y_n axis. Consequently, $u(x)$ is rotationally symmetric about the x_n axis. Since the choice of the axis was arbitrary, we conclude that $u(x)$ is spherically symmetric about the origin. ∎

4. Preliminary Results for Eq. (1.3)

We may set $m = 1$.

We will need the Green's function of $-\Delta + 1$ on R^n. It is given by

$$0 < G(|x - y|) = (|x - y|)^{-(n-2)/2} K_{(n-2)/2}(|x - y|), \tag{4.1}$$

where $K_\nu(z)$ denotes the modified Bessel function of order ν. In Appendix C we summarize the properties of K_ν used in this section. In particular, $G(r)$

satisfies

$$G(r) \leq C \frac{e^{-r}}{r^{n-2}} (1+r)^{(n-3)/2}, \tag{4.2}$$

$$G'_r/G \to -1 \quad \text{at infinity}. \tag{4.3}$$

For $n = 3$,

$$G(r) = \sqrt{\pi/2}(e^{-r}/r),$$

and the reader may carry out the computations below without using any particular properties of the Bessel functions.

First we prove that the solutions in Theorem 2 decay exponentially at ∞.

PROPOSITION 4.1. *Let $u(x) > 0$ be a C^2 solution of* (1.3), *tending to zero at infinity and assume*

$$g(u) = O(u^\alpha) \qquad \textit{for some} \quad \alpha > 1, \quad \textit{near} \quad u = 0.$$

Then

$$u(x), |\nabla u(x)| = O(e^{-|x|}/|x|^{(n-1)/2}) \quad \textit{at } \infty. \tag{4.4}$$

This result follows from the results of Kato [6, Sections 5 and 6] but we include a simple proof here.

Proof. For $r > 0$ set

$$h(r) = \int_{S^{n-1}} u(r\theta)\, d\omega(\theta).$$

Here r represents a polar coordinate and θ a point on the unit sphere; $d\omega$ is element of volume on the sphere. Since $g(u) = O(u^\alpha)$ for $\alpha > 0$, and $u \to 0$ at infinity, we see that for any $\varepsilon > 0$, $|g(u)| < \varepsilon u$ for $|x|$ sufficiently large. Hence integrating (1.3) with respect to θ we find for $r > r_0$

$$h_{rr} + [(n-1)/r]h_r > (1-\varepsilon)h. \tag{4.5}$$

Hence $r^{n-1}h_r$ is increasing. Since $h \to 0$ as $r \to \infty$ we see that $h_r \leq 0$ for $r > r_0$.

Multiplying (4.5) by $2h_r$ we find, taking $r > r_0$ from now on,

$$(h_r^2 - (1-\varepsilon)h^2)_r + 2[(n-1)/r]h_r^2 \leq 0,$$

and hence $w = h_r^2 - (1-\varepsilon)h^2$ is decreasing. Since $h \to 0$ at ∞ we must have $w \to 0$, for otherwise $w \to c^2 > 0$ which implies $h_r \to -c$—contradicting the fact that $h \to 0$. Thus

$$h_r^2 - (1-\varepsilon)h^2 \geq 0,$$

or

$$h_r + \sqrt{1-\varepsilon}\, h \leq 0.$$

Setting $a = \sqrt{1-\varepsilon}$ we have

$$e^{ar}h \text{ is decreasing}$$

and hence

$$h \leq Ce^{-ar}$$

for some constant C. Since ε was arbitrary we conclude that

$$h(r) = O(e^{-ar}), \qquad r \to \infty$$

for any $a < 1$.

It follows that for $f(y) = -u(y) + g(u(y))$,

$$\int_{|x-y|<1} |f(y)|\, dy = O(e^{-a|x|}) \quad \forall a < 1$$

so that

$$\int_{|x-y|<1} (u + |\Delta u|)\, dy = O(e^{-a|x|}). \tag{4.6}$$

By standard interior estimates we find

$$\left(\int_{|x-y|<3/4} u^p\, dy\right)^{1/p} = O(e^{-a|x|}), \qquad p = \frac{n}{n-2},$$

and hence

$$\left(\int_{|x-y|<3/4} |\Delta u|^p\, dy\right)^{1/p} = O(e^{-a|x|}).$$

Using interior estimates again we find in case $q = np/(n-2p) > 0$,

$$\left(\int_{|x-y|<1/2} u^q\, dy\right)^{1/q} \subset O(e^{-a|x|})$$

while if $n - 2p = 0$, a similar estimate holds for any L^s norm of u, $s < \infty$. If $n < 2p$, the estimate holds for the L^∞ norm.

Continuing in this way we finally infer that

$$u(x) = O(e^{-a|x|}), \quad \forall a < 1,$$

and hence

$$g(u) = O(e^{-b|x|}) \qquad \text{for some} \quad b > 1. \tag{4.7}$$

Next we make use of the formula

$$u(x) = \int G(|x-y|)g(u(y))\, dy$$

$$\leq C \int \frac{e^{-|x-y|}}{|x-y|^{n-2}} (1 + |x-y|)^{(n-3)/2} e^{-b|y|}\, dy,$$

by (4.2) and (4.7). Consequently

$$(1+|x|)^{(n-1)/2}e^{|x|}u(x)$$

$$\leq C\int e^{(1-b)|y|}\frac{(1+|x|)^{(n-1)/2}}{|x-y|^{n-2}}(1+|x-y|)^{(n-3)/2}\,dy \tag{4.8}$$

$$\leq C\int_{|y-x|<|x|/2}+C\int_{|y-x|>|x|/2} \tag{4.9}$$

$$=I_1+I_2.$$

Now for $|x|$ large,

$$I_1=C\int_{|y-x|<|x|/2}\leq C_1 e^{(1-b)|x|/2}|x|^n\leq C_2 \tag{4.10}$$

for some constants C_1, C_2 independent of x. For $|x|$ large, using $1+|x|\leq(1+|x-y|)(1+|y|)$ we find next

$$I_2\leq C_1\int_{R^n}e^{(1-b)|y|}(1+|x|)^{(n-1)/2}\left(\frac{1+|y|}{1+|x|}\right)^{(n-1)/2}dy\leq C_2. \tag{4.11}$$

Combining (4.8)–(4.11) we obtain the desired bound (4.4) for $u(x)$. Using interior estimates as before we obtain the same result for $|\nabla u(x)|$. The proposition is proved. ■

It follows from Proposition 4.1 that $u>0$ is an exponentially decaying solution of

$$\Delta u-u+f(x)=0,\qquad f=O(e^{-\alpha|x|}),\qquad \alpha>1. \tag{4.12}$$

We will have need of some further asymptotic results near infinity for such a solution. The solution u is given by

$$u(x)=\int G(|x-y|)f(y)\,dy \tag{4.12$'$}$$

with

$$f(y)=O(e^{-\alpha|y|}),\qquad \alpha>1. \tag{4.12$''$}$$

PROPOSITION 4.2. *Let $u>0$ be given by* (4.12′) *with f satisfying* (4.12″). *Suppose $\{x^i\}\in R^n$ is a sequence going off to infinity in R^n and*

$$\xi=\lim_i\frac{x^i}{|x^i|},\qquad \xi\equiv(\xi_1,\xi')=(\xi_1,\xi_2,\ldots,\xi_n).$$

Then

(a) $$\lim_{i\to\infty}|x^i|^{(n-1)/2}e^{|x^i|}u(x^i)=\sqrt{\frac{\pi}{2}}\int dy\,f(y)e^{\xi\cdot y}. \tag{4.13}$$

(b) *Suppose* $\xi_1 = 0$ *and that* λ^i *is a sequence of real numbers converging to a number* λ *and* $\lambda^i > x_1^i$. *Let* z^i *be the reflection of* x^i *in the plane* $x_1 = \lambda^i$, *i.e.,* $z^i = (2\lambda^i - x_1^i, x_2^i, \ldots, x_n^i)$. *Then*

$$\lim_{i\to\infty} \frac{|x^i|}{\lambda^i - x_1^i} |x^i|^{(n-1)/2} e^{|x^i|}(u(x^i) - u(z^i)) = \sqrt{2\pi} \int dy\, e^{\xi' \cdot y'} f(y)(\lambda - y_1). \quad (4.13')$$

(c) *If* $x_1^i \to +\infty$ *then*

$$\frac{|x^i|^{(n+1)/2}}{x_1^i} e^{|x^i|} u_1(x^i) \to -\sqrt{\frac{\pi}{2}} \int dy\, e^{\xi \cdot y} f(y). \quad (4.14)$$

Proposition 4.2 will be proved in Appendix B.

5. Some Forms of the Hopf Lemma

In this section we prove Lemmas (H_1') and (H_2'). In addition we shall have need of a form of the second in a half-space:

Lemma (H_2''): *Let* $u > 0$ *be of class* C^2 *in* $\{|x| \geq R\} \cap \{x_1 < 0\}$ *and continuous in the closure. Assume u vanishes on* $x_1 = 0$, *tends to zero at infinity, and satisfies a differential inequality*:

$$Lu \equiv \left(\Delta - 1 + x_1\beta_1\partial_1 + \sum_2^n b_\alpha\partial_\alpha + c(x)\right)u \leq 0 \quad (5.1)$$

with

$$x_1\beta_1, b_\alpha, c = O(|x|^{-p}) \qquad \textit{for some} \quad p > 1, \quad \alpha = 2, \ldots, n.$$

Then for some $\mu > 0$

$$u(x) \geq -\mu x_1 e^{-|x|}/|x|^{(n+1)/2}. \quad (5.2)$$

In our application we will have $\beta_1 = b_2 = \cdots = b_n = 0$. There is a similar half-space version of Lemma (H_1') which may prove useful:

Lemma (H_1''). *Let* $u > 0$ *be of class* C^2 *in* $\{|x| \geq R\} \cap \{x_1 < 0\}$ *and continuous in the closure. Assume u vanishes on* $x_1 = 0$ *and tends to zero at infinity. Assume u satisfies*

$$Lu = \left(\Delta u + x_1\beta_1\partial_1 + \sum_2^n b_\alpha\partial_\alpha + c(x)\right)u \leq 0 \quad (5.3)$$

with

$$\beta_1, c = O(|x|^{-p}), \qquad b_\alpha = O(|x|^{1-p}), \qquad \alpha = 2, \ldots, n, \quad p > 2.$$

Then for some $\mu > 0$

$$u(x) \geq -\mu x_1/|x|^n.$$

Proof of Lemma (H_1'). We know that

$$c \leq K|x|^{-p}.$$

Then, since $u \geq 0$,

$$\tilde{L}u \equiv (L - K|x|^{-p})u \leq 0,$$

and the maximum principle holds for $\tilde{L}$—for the coefficient of the zero-order term is nonpositive. We may therefore suppose that our original coefficient c is nonpositive, and we may suppose the same in the other lemmas.

Without loss of generality we may take R as large as we like.

We will use the following comparison function

$$z = (1/r^{n-2}) + (1/r^s), \tag{5.4}$$

where $r = |x|$, with

$$n - 2 < s < n - 4 + p. \tag{5.5}$$

Such a value of s exists since $p > 2$. Using (1.12), a computation yields

$$Lz \geq -Cr^{2-n-p} + s(s - 2 + n)r^{-s-2}.$$

Because of (5.5) we see that the second term dominates, for r large, and thus

$$Lz > 0 \qquad \text{for } R \text{ large.}$$

Let $t > 0$ be so small that

$$u \geq tz \qquad \text{on} \quad |x| = R.$$

Then $u_t = u - tz \geq 0$ on $|x| = R$, vanishes at infinity, and satisfies

$$Lu_t < 0 \qquad \text{in} \quad |x| > R.$$

We may therefore apply the minimum principle and conclude that $u \geq tz$ in $|x| \geq R$. The conclusion of Theorem (H_1') follows—with $\mu = t$. ∎

Proof of Lemma (H_2'). The proof is the same as the preceding, with a new comparison function:

$$z = G(r)(1 + (1/r^s)), \qquad 0 < s < p - 1; \tag{5.6}$$

here G is the Green's function for $(\Delta - 1)$ given in (4.1). According to (4.1), (4.3), and (C.3) in Appendix C, for r large,

$$G \sim \sqrt{\pi/2}(e^{-r}/r^{(n-1)/2}), \qquad -G_r = G(1 + O(1/r)),$$
$$G_{rr} = G(1 + O(1/r)). \tag{5.7}$$

A direct calculation then shows that

$$Lz \geq -C\frac{G(r)}{r^p} - \frac{2sG_r}{r^{s+1}} + \frac{G}{r^{s+2}}s(s+2-n) = G\left(\frac{2s}{r^{s+1}} - \frac{C}{r^p} - \frac{C}{r^{s+2}}\right)$$

$$> 0 \qquad \text{for } R \text{ sufficiently large,}$$

since $s + 1 < p$. The preceding argument then yields $u \geq tz$ for some $t > 0$—proving the lemma. ■

Remark. The function $u = G/r$ satisfies

$$(\Delta - 1 + c(r))u = 0$$

with $c = O(1/r)$, showing that the condition $p > 1$ may not be omitted.

Proof of Lemma (H_2''). The proof is the same, only now we work in $\Omega_R = \{|x| \geq R\} \cap \{x_1 < 0\}$, R large, and must find a corresponding comparison function. In fact the following will do:

$$w = z_1 = z_{x_1} = -\sqrt{\pi/2}(e^{-r}/r^{(n+1)/2})\, x_1[1 + O(1/r)], \tag{5.8}$$

where z is the function in (5.6). A simple calculation shows that

$$(\Delta - 1)w = \partial_1(\Delta - 1)z = -2sx_1\frac{e^{-r}}{r^{s+1+(n+1)/2}}\left(1 + O\left(\frac{1}{r}\right)\right).$$

On the other hand

$$|x_1\beta_1\partial_1 w|, |b_\alpha\partial_\alpha w|, |cw| \leq -Cx_1e^{-r}/r^{(n+1)/2+p}.$$

Since $s + 1 < p$ it follows that

$$Lw > 0 \qquad \text{in } \Omega_R, \quad \text{for } R \text{ large.}$$

Now on $|x| = R$ we have $u > cx_1$ for $c < 0$, by the Hopf lemma. Hence, for $t > 0$ sufficiently small

$$u_t = u - tw > 0 \qquad \text{on} \quad |x| = R, \quad x_1 < 0,$$

and u_t vanishes on $x_1 = 0$ and at infinity. By the maximum principle, recall that we may suppose $c(x) \leq 0$, we find

$$u \geq tw \qquad \text{in } \Omega_R, \quad R \text{ large;}$$

the desired conclusion follows from (5.8). ■

The proof of Lemma (H_1'') is the same, using the comparison function $w = z_1$ with z given by (5.4).

6. Proof of Theorem 2

Our proof is similar to that of Theorem 1 but somewhat more complicated. By Propositions 4.1 and 4.2 we see first that $u(x), |\nabla u(x)| = O(e^{-|x|})$ and $f(x) = g(u(x)) = O(e^{-\alpha|x|})$. Furthermore if $x \to \infty$, $x/|x| \to \xi$ then

$$|x|^{(n-1)/2} e^{|x|} u(x) \to \sqrt{\frac{\pi}{2}} \int dy\, f(y) e^{\xi \cdot y} \tag{6.1}$$

We have $g(u)/u = c(x) = O(e^{-\varepsilon|x|})$, $\varepsilon = \alpha - 1$. Thus we may apply Lemma (H_2') and conclude that

$$|x|^{(n-1)/2} e^{|x|} u(x) \geq \mu_0 > 0 \quad \text{near infinity.}$$

Combining with (6.1) we see that

$$\sqrt{\frac{\pi}{2}} \int dy\, f(y) e^{\xi \cdot y} \geq \mu_0 \qquad \text{for} \quad |\xi| = 1. \tag{6.2}$$

Lemma 6.1. *Under the assumptions of Theorem 2 there exists $\bar{\lambda} > 0$ such that*

$$u_1(x) < 0 \qquad \textit{for} \quad x_1 \geq \bar{\lambda}.$$

Proof. Suppose not. Then there is a sequence x^i with $x_1^i \to +\infty$ such that

$$u_1(x^i) \geq 0.$$

Applying Proposition 4.2c we find

$$-\sqrt{\frac{\pi}{2}} \int dy\, e^{\xi \cdot y} f(y) \geq 0$$

contradicting (6.2). ■

Lemma 6.2. *Under the assumptions of Theorem 2, there exists $\lambda_0 \geq \bar{\lambda}$ such that for all $\lambda \geq \lambda_0$,*

$$u(x) > u(x^\lambda) \qquad \textit{for} \quad x_1 < \lambda. \tag{6.3}$$

Proof. Suppose not. Then there exists a sequence $\lambda^i \to +\infty$ and a sequence $\{x^i\}$ with $x_1^i < \lambda^i$ such that for $z^i = x^{i\lambda^i}$

$$u(x^i) \leq u(z^i). \tag{6.4}$$

Since $|x^{i\lambda^i}| \to \infty$, so that $u(x^{i\lambda^i}) \to 0$, we see that $|x^i| \to \infty$. Furthermore, we have necessarily $x_1^i < \bar{\lambda}$, for if $x_1^i \geq \bar{\lambda}$, then

$$u(z^i) - u(x^i) = \int_{x^i}^{z^i} u_1 < 0$$

by Lemma 6.1—contradicting (6.4).

By restricting ourselves to a suitable subsequence we may suppose

$$x^i/|x^i| \to \xi, \qquad \text{and then } \xi_1 \leq 0.$$

Case 1. $\xi_1 < 0$. In this case if $d_i = |z^i| - |x^i|$ we see, omitting i,

$$d + \sqrt{|x'|^2 + x_1^2} = [(2\lambda - x_1)^2 + |x'|^2]^{1/2},$$

so that, on squaring,

$$d^2 + 2d|x_1| = 4\lambda^2 - 4\lambda x_1 \geq -4\lambda x_1.$$

Since $x_1 \sim \xi_1|x|$, and $\lambda \to +\infty$ we see that $d \to \infty$, i.e., $d_i \to \infty$.

We now apply Proposition 4.2a and infer that

$$\lim |x^i|^{(n-1)/2} e^{|x^i|} u(x^i) = \sqrt{\frac{\pi}{2}} \int f(y) e^{\xi \cdot y}$$
$$> 0 \qquad \text{by (6.2).}$$

On the other hand,

$$\lim |x^i|^{(n-1)/2} e^{|x^i|} u(z^i) = \lim (|x^i|/|z^i|)^{(n-1)/2} e^{|x^i| - |z^i|} \cdot |z^i|^{(n-1)/2} e^{|z^i|} u(z^i).$$

Since $|x^i| - |z^i| = -d_i \to -\infty$ we see again from Proposition 4.2a that this tends to zero. We infer that for Case 1, (6.4) is impossible.

Case 2. $\xi_1 = 0$. By (6.4) and Lemma 6.1 we may assert that for any fixed $\lambda \geq \bar{\lambda}$,

$$u(x^i) \leq u(z^i) \leq u(x^{i\lambda}) \qquad \text{if } \lambda^i \geq \lambda. \tag{6.5}$$

We now apply Proposition 4.2b, with λ in place of the λ^i in the proposition, to obtain (here $x^{i\lambda} = \zeta^i$)

$$0 \geq \lim \frac{|x^i|}{\lambda - x_1^i} |x^i|^{(n-1)/2} e^{|x^i|} (u(x^i) - u(\zeta^i)) = \sqrt{2\pi} \int dy\, e^{\xi' \cdot y'} f(y)(\lambda - y_1).$$

But according to (6.2) this is impossible for large λ. ■

LEMMA 6.3. *Under the conditions of Theorem* 2, *the set of* λ *for which property* (6.3) *holds is open.*

Proof. As in the proof of Lemma 2.3, we note that if property (6.3) holds then

$$u_{x_1} < 0 \qquad \text{on} \quad x_1 = \lambda. \tag{6.6}$$

Here we use the fact that $g(u) = g_1(u) + g_2(u)$ with $g_1 \in C^1$ and g_2 nondecreasing. We will show that if property (6.3) holds for some λ then it holds for any $\tilde{\lambda}$ near λ. Suppose the contrary. Then there exist sequences $\{x^i\}$, $\lambda^i \to \lambda$, $x_1^i < \lambda^i$ such that (here $z^i = x^{i\lambda^i}$)

$$u(x^i) \leq u(z^i).$$

Clearly $|x^i| \to \infty$ and we may choose a suitable subsequence so that

$$\frac{x^i}{|x^i|} \to \xi = (\xi_1, \xi'), \qquad \xi_1 \leq 0.$$

Then

$$|x^i|^{(n-1)/2} e^{|x^i|} u(x^i) \leq \left| |x^i|/|z^i| \right|^{(n-1)/2} e^{|x^i| - |z^i|} |z^i|^{(n-1)/2} e^{|z^i|} u(z^i).$$

Using the identity (2.21) we see that $|x^i| - |z^i| \to 2\lambda\xi_1$, and using Proposition 4.2a we find

$$\int f(y) e^{\xi \cdot y} \leq e^{2\lambda\xi_1} \int f(y) e^{\xi^0 \cdot y}. \tag{6.7}$$

Now we are going to use Lemma (H_2''). In the half-space $x_1 < \lambda$ set $v(x) = u(x^\lambda)$ and

$$w(x) = u(x) - v(x) > 0 \qquad \text{by (6.3)}.$$

Then

$$(\Delta - 1)w + g(u) - g(v) = 0,$$

or

$$(\Delta - 1)w + g_1(u) - g_1(v) = g_2(v) - g_2(u) \leq 0,$$

since g_2 is increasing. Using the hypotheses on g_1 we see that

$$(\Delta - 1 + c(x))w \leq 0$$

with $c(x) = O(|x|^{-p})$ near infinity, $p > 1$. In the half-space $x_1 < \lambda$, w satisfies all the conditions of Lemma (H_2'') and we may assert that for some $\mu_0 > 0$

$$\frac{|x|^{(n+1)/2} e^{|x|}}{\lambda - x_1} w(x) \geq \mu_0 > 0. \tag{6.8}$$

Suppose now $\xi_1 < 0$. If $z^i \to \infty$ is a sequence in the half-space $x_1 < \lambda$ with $z^i/|z^i| \to \xi$ then

$$\frac{|z^i|^{(n+1)/2} e^{|z^i|}}{\lambda - z_1^i} (u(z^i) - u(z^{i\lambda})) \geq \mu_0.$$

By Proposition 4.2a

$$-\frac{1}{\xi_1}\int f(y)(e^{\xi\cdot y} - e^{2\lambda\xi_1+\xi^0\cdot y}) \geq \mu_0$$

which contradicts (6.7).

Suppose, then, that $\xi_1 = 0$. According to Proposition 4.2b,

$$\int e^{\xi\cdot y} f(y)(\lambda - y_1) \leq 0. \tag{6.7'}$$

On the other hand if in the half-space $x_1 < \lambda$ we let $x \to \infty$ with $x/|x| \to \xi$, then, according to (6.8),

$$\lim \frac{|x|^{(n+1)/2} e^{|x|}}{\lambda - x_1}(u(x) - u(x^\lambda)) \geq \mu_0.$$

Applying Proposition 4.2b once more, with all $\lambda^i = \lambda$ we find

$$\sqrt{2\pi}\int e^{\xi\cdot y} f(y)(\lambda - y_1) \geq \mu_0$$

contradicting (6.7'). Lemma 6.3 is proved. ■

Completion of the proof of Theorem 2. It follows from Lemmas 6.2 and 6.3 that there is a maximal interval

$$\lambda_1 < \lambda < \infty$$

for which (6.3) holds; clearly λ_1 is finite. By [4, Lemma 4.3], and Remark 1 of Section 2.3 there, we also have

$$u_{x_1} < 0 \quad \text{for} \quad x_1 > \lambda_1.$$

By continuity we have

$$u(x) \geq u(x^{\lambda_1}) \quad \text{for} \quad x_1 < \lambda_1$$

and by the same reference in [4] we have either $\equiv$ or $>$. In view of Lemma 6.3 it cannot be the latter for then the interval would not be maximal. Thus we conclude that

$$u \text{ is symmetric about the plane } x_1 = \lambda_1 \quad \text{and} \quad u_{x_1} < 0 \quad \text{for} \quad x_1 > \lambda_1.$$

The same conclusion holds for the other coordinate directions and we conclude that u is symmetric about each of n planes, $x_j = \lambda_j$ and grad $u = 0$ only at their intersection. We may now take their intersection as the origin.

The same argument may be applied to any unit direction γ and we infer that u is symmetric about some plane $x \cdot \gamma = c(\gamma) =$ const. At the point on this plane where u achieves the maximum we have grad $u = 0$ (since the normal derivative to the plane is zero at every point of the plane). It follows that $c(\gamma) = 0$. Thus u is symmetric about every plane through the origin. In addition we also conclude that $u_r < 0$. The last inequalities in Theorem 2 follow from (6.1) and (6.2). Theorem 2 is proved. ■

APPENDIX A

We shall often make use of the identity (2.21):

$$|x^\lambda - y| - |x - y| = \frac{4(\lambda - x_1)(\lambda - y_1)}{|x^\lambda - y| + |x - y|}. \tag{A.1}$$

Proof of Lemma 2.1. Observe first that Lemma 2.1 holds in case f has compact support. Indeed in this case (2.5) and (2.6) are immediate, while (2.7) follows from

$$\frac{|x^i|^n}{\lambda^i - x_1^i}\left(\frac{1}{|x^i - y|^{n-2}} - \frac{1}{|x^i - y^{\lambda^i}|^{n-2}}\right) \to 2(n-2)(\lambda - y_1).$$

This in turn is readily verified with the aid of (A.1).

To prove the lemma for any f satisfying (2.4) we observe that f is in the Banach space $B_{q'}$ for $q' < q$, defined by the completion of C_0^∞ under the norm $\|\ \|_{q'}$:

$$\|h\|_{q'} = \sup_y \{(1 + |y|)^{q'} |h(y)|\}.$$

Let us now fix q' in $n + 1 < q' < q$.

Note first that the right-hand sides of (2.5), (2.6), and (2.7) are bounded linear functionals on $B_{q'}$. To prove the lemma we have only to show that the following functionals

$$j_x(f) = |x|^{n-2} u(x),$$

$$k_x(f) = \frac{|x|^n}{x_1} u_1(x), \qquad x_1 > 1,$$

$$l_i(f) = \frac{|x^i|^n}{\lambda^i - x_1^i}(u(x^i) - u(x^{i\lambda^i}))$$

are all uniformly bounded linear functionals on $B_{q'}$. A standard argument then yields the result.

For example let us prove (2.7) using the uniform boundedness of the l_i. Let $l(f)$ denote the linear functional defined on the right-hand side of (2.7). Suppose $f_j \in C_0^\infty$ converges in $B_{q'}$ to f. Then

$$l_i(f) - l(f) = l_i(f - f_j) + (l_i - l)f_j + l(f_j - f),$$

so that

$$|l_i(f) - l(f)| \leq C\|f - f_j\|_{q'} + |(l_i - l)f_j| + C\|f_j - f\|_{q'}.$$

Given $\varepsilon > 0$, choose j so large that $C\|f - f_j\|_{q'} < \varepsilon/3$ and then choose i so large that $|(l_i - l)f_j| < \varepsilon/3$. Hence $|(l_i - l)f| < \varepsilon$ and the desired result follows. C will denote various constants.

Consider first

$$|j_x(f)| = c\left|\int \frac{|x|^{n-2}}{|x-y|^{n-2}} f(y)\right|$$
$$\leq c\|f\|_{q'} \int \frac{|x|^{n-2}}{|x-y|^{n-2}} \frac{1}{(1+|y|)^{q'}}.$$

Clearly

$$\int_{|y-x|<|x|/2} \leq C$$

while

$$\int_{|y-x|>|x|/2} \leq C \int \frac{1}{(1+|y|)^{q'}} \leq C.$$

It follows that

$$|j_x(f)| \leq C\|f\|_{q'}.$$

Next, for $x_1 > 1$,

$$|k_x(f)| \leq C\|f\|_{q'} \int \frac{|x_1 - y_1|}{x_1} \frac{|x|^n}{|x-y|^n} \frac{1}{(1+|y|)^{q'}}. \tag{A.2}$$

Now

$$\int_{|x-y|<|x|/2} \leq C$$

and

$$\int_{|x-y|>|x|/2} \leq C \int \frac{1}{(1+|y|)^{q'-1}} \leq C,$$

since $|x_1 - y_1|/x_1 \leq 1 + |y_1|$. The desired uniform boundedness of k_x follows.

To bound the functionals l_i is more tedious. We shall use the inequalities: For $0 < r \leq s$

$$(n-2)\frac{1}{s^{n-1}}(s-r) \leq \frac{1}{r^{n-2}} - \frac{1}{s^{n-2}} \leq (n-2)\frac{1}{r^{n-1}}(s-r). \tag{A.3}$$

Let

$$K = K(x; y; \lambda) = \frac{|x|^n}{\lambda - x_1} \left| \frac{1}{|x - y|^{n-2}} - \frac{1}{|x^\lambda - y|^{n-2}} \right|,$$

then

$$K \le C_1 |x|^n \frac{|\lambda - y_1|}{|x - y| + |x^\lambda - y|} \max\{|x - y|^{1-n}, |x^\lambda - y|^{1-n}\}. \tag{A.4}$$

Indeed, for $y_1 < \lambda$ we have [using (A.3)]

$$K \le (n-2) \frac{|x|^n}{\lambda - x_1} \frac{1}{|x - y|^{n-1}} \frac{4(\lambda - x_1)(\lambda - y_1)}{|x^\lambda - y| + |x - y|},$$

similarly, for $y_1 > \lambda$, and (A.4) is proved. Therefore

$$|l_i(f)| \le C_1 |x^i|^n \int dy\, f(y) \frac{|\lambda^i - y_1|}{|x^i - y| + |x^{i\lambda^i} - y|} \max\{|x^i - y|^{1-n}, |x^{i\lambda^i} - y|^{1-n}\}.$$

We wish to prove

$$|l_i(f)| \le C\|f\|_{q'} \quad \text{with} \quad C \text{ independent of } i. \tag{A.5}$$

Let

$$A_1 = c|x|^n \int_{|y-x|<|x|/2,\, y_1<\lambda} dy\, f(y) \frac{\lambda - y_1}{|x - y| + |x^\lambda - y|} |x - y|^{1-n},$$

$$A_2 = c|x|^n \int_{|y-x^\lambda|<|x^\lambda|/2,\, y_1>\lambda} dy\, f(y) \frac{y_1 - \lambda}{|x - y| + |x^\lambda - y|} |x^\lambda - y|^{1-n},$$

$$A_3 = c|x|^n \int_{|y-x|>|x|/2,\, y_1<\lambda} dy\, f(y) \frac{\lambda - y_1}{|x - y| + |x^\lambda - y|} |x - y|^{1-n},$$

$$A_4 = c|x|^n \int_{|y-x^\lambda|>|x^\lambda|/2,\, y_1>\lambda} dy\, f(y) \frac{y_1 - \lambda}{|x - y| + |x^\lambda - y|} |x^\lambda - y|^{1-n}.$$

To prove (A.5) it suffices to prove with some constant C independent of x but which may depend on λ:

$$\sum A_j \le C\|f\|_{q'}. \tag{A.6}$$

With various constants C (which may depend on λ) we have

$$\begin{aligned} A_1 &\le c|x|^n \|f\|_{q'} \int_{|y-x|<|x|/2,\, y_1<\lambda} dy\, |x - y|^{1-n} (1 + |y|)^{-q'} \\ &\le C\|f\|_{q'} |x|^{n+1} (1 + |x|)^{-q'} \\ &\le C\|f\|_{q'} \quad \text{since} \quad q' > n + 1. \end{aligned}$$

A similar estimate holds for A_2. Next

$$\begin{aligned} A_3 &\le C\|f\|_{q'}|x|^n \int_{|y-x|>|x|/2,\, y_1<\lambda} dy\, |x-y|^{-n}(1+|y|)^{1-q'} \\ &\le C\|f\|_{q'} \int_{R^n} dy\,(1+|y|)^{1-q'} \\ &\le C\|f\|_{q'} \qquad \text{since} \quad q' > n+1. \end{aligned}$$

The same argument applies to A_4. Combining these estimates we obtain (A.6) and hence (A.5).

Lemma 2.1 is proved. ■

Appendix B. The Proof of Proposition 4.2

We will make use of the following consequence of (C.5) in Appendix C:

$$G(r) - G(s) = -\int_r^s d\xi\, G'(\xi) = \int_r^s d\xi\, \frac{K_{n/2}(\xi)}{\xi^{(n-2)/2}}.$$

By (C.1), we have for $r \le s$

$$\frac{K_{n/2}(s)}{s^{(n-2)/2}}(s-r) \le G(r) - G(s) \le \frac{K_{n/2}(r)}{r^{(n-2)/2}}(s-r). \tag{B.1}$$

First we prove (4.13), (4.13′), and (4.14) for functions f with compact support. For y in a compact set K, if $x \to \infty$ with $x/|x| \to \xi$, we see from (C.3) that

$$\begin{aligned} \lim |x|^{(n-1)/2} e^{|x|} G(|x-y|) &= \sqrt{\frac{\pi}{2}} \lim \left(\frac{|x|}{|x-y|}\right)^{(n-1)/2} e^{|x|-|x-y|} \\ &= \sqrt{\frac{\pi}{2}}\, e^{\xi \cdot y}, \end{aligned} \tag{B.2}$$

since

$$|x| - |x-y| = y \cdot \frac{x}{|x|} + O(|x|^{-1}) \qquad \text{uniformly for } y \text{ in } K.$$

Using (B.2) we see that (4.13) holds for continuous functions f with compact support.

To verify (4.13′) for such functions we will use the formula [see (C.2)]

$$G(z) = \sqrt{\pi/2}\, e^{-|z|} H_{(n-2)/2}(|z|),$$

where

$$H_{(n-2)/2}(|z|) = |z|^{(1-n)/2} \frac{1}{\Gamma[(n-1)/2]} \int_0^\infty e^{-t} t^{(n-3)/2} \left(1 + \frac{t}{2|z|}\right)^{(n-3)/2} dt. \tag{B.3}$$

We wish to prove that for y in a compact set K if $x_i \to \infty$ with $x_i/|x_i| \to \xi$ and $\xi_1 = 0$, and $x_1^i < \lambda^i$, $\lambda^i \to \lambda$, then

$$G_i = \frac{|x^i|^{(n+1)/2}}{\lambda^i - x_1^i} e^{|x^i|}(G(|x^i - y|) - G(|x^i - y^{\lambda^i}|)) \to 2\sqrt{\frac{\pi}{2}}(\lambda - y_1)e^{\xi' \cdot y'} \tag{B.4}$$

uniformly for y in K. We have

$$\begin{aligned}\sqrt{\frac{2}{\pi}} G_i = {} & \frac{|x^i|^{(n+1)/2}}{\lambda^i - x_1^i} e^{|x^i|} e^{-|x^i - y|}(1 - e^{|x^i - y| - |x^i - y^{\lambda^i}|}) H_{(n-2)/2}(|x^i - y^{\lambda^i}|) \\ & + \frac{|x^i|^{(n+1)/2}}{\lambda^i - x_1^i} e^{|x^i|} e^{-|x^i - y|} |H_{(n-2)/2}(|x^i - y|) - H_{(n-2)/2}(|x^i - y^{\lambda^i}|)|.\end{aligned} \tag{B.5}$$

Using (A.1) we see that

$$\begin{aligned}\frac{|x^i|}{\lambda^i - x_1^i}(1 - e^{|x^i - y| - |x^i - y^{\lambda^i}|}) &= \frac{|x^i|}{\lambda^i - x_1^i}\left(1 - \exp \frac{4(\lambda^i - x_1^i)(y_1 - \lambda^i)}{|x^i - y| + |x^i - y^{\lambda^i}|}\right) \\ &\to 2(\lambda - y_1),\end{aligned}$$

because $x_1^i/|x^i| \to 0$. Since $|x^i|/|x^{i\lambda^i}| \to 1$ and $|z|^{(n-1)/2} H_{(n-2)/2}(|z - y|) \to 1$ as $z \to \infty$, the first product in the right-hand side of (B.5) tends to $2(\lambda - y_1)e^{\xi \cdot y}$.

To conclude the proof of (B.4) we must show that the other term in (B.5) tends to zero. This follows rather easily from the implicit formula (B.3) with the aid of (A.1). Namely, using (A.1) one shows first that

$$\frac{|x^i|^{(n+1)/2}}{\lambda^i - x_1^i}\left|\frac{1}{|x^i - y|^{(n-1)/2}} - \frac{1}{|x^i - y^{\lambda^i}|^{(n-1)/2}}\right| \le C \frac{|\lambda^i - y_1|}{|x^i - y| + |x^i - y^{\lambda^i}|} \to 0.$$

Next one establishes, with the aid of (A.1) that

$$\int_0^\infty e^{-t} t^{(n-3)/2} \frac{|x^i|}{\lambda^i - x_1^i}\left[\left(1 + \frac{t}{2|x^i - y|}\right)^{(n-3)/2} - \left(1 + \frac{t}{2|x^i - y^{\lambda^i}|}\right)^{(n-3)/2}\right] dt$$
$$\to 0 \quad \text{as} \quad i \to \infty.$$

These yield the desired result and (B.4) is proved.

To prove (4.14) for functions with compact support we must prove that for y in a compact set K if $x \to \infty$ with $x_1 \to +\infty$, $x/|x| \to \xi$, then uniformly in K,

$$J(x, y) = \frac{|x^i|^{(n+1)/2}}{x_1} e^{|x|} G'(|x - y|) \frac{x_1 - y_1}{|x - y|} \to -\sqrt{\frac{\pi}{2}} e^{\xi \cdot y}.$$

Now

$$G'(r) = \sqrt{\frac{\pi}{2}} \frac{e^{-r}}{r^{(n-1)/2}}\left(-1 + O\left(\frac{1}{r}\right)\right),$$

and so

$$J(x, y) = -\sqrt{\frac{\pi}{2}} \frac{|x|^{(n+1)/2}}{|x-y|^{(n+1)/2}} e^{|x|-|x-y|} \frac{x_1 - y_1}{x_1} \left(1 + O\left(\frac{1}{|x-y|}\right)\right),$$

and one obtains the desired limit immediately since, as we noted earlier, $|x| - |x - y| \to \xi \cdot y$.

We have thus proved (4.13), (4.13′), and (4.14) for continuous functions f with compact support. To establish these for our function $f(y) = O(e^{-\alpha|y|})$ we define for each γ in $1 < \gamma < \alpha$ the Banach space B_γ with norm

$$\|h\|_\gamma = \sup\{e^{\gamma|y|}|h(y)|\}.$$

Now choose a sequence of functions f_j with compact support so that $\|f - f_j\|_{\alpha'} \to 0$ as $j \to \infty$, where $1 < \alpha' < \alpha$. As in Lemma 2.1, the proofs will be completed if we prove that the functionals involved in these equations are bounded on $B_{\alpha'}$. We consider first (4.13′) [Eq. (4.13) is simpler and can be treated in a similar way]. Let

$$l_i(f) = \int dy\, f(y) J(x^i; y; \lambda^i), \tag{B.6}$$

where

$$J = J(x; y; \lambda) = [|x|/(\lambda - x_1)](|x|)^{(n-1)/2} e^{|x|}\{G(|x-y|) - G(|x-y^\lambda|)\}.$$

For $y_1 < \lambda$, we use (B.1) to bound

$$|J| \le \frac{|x|}{\lambda - x_1} (|x|)^{(n-1)/2} e^{|x|} \frac{K_{n/2}(|x-y|)}{|x-y|^{(n-2)/2}} \frac{4(\lambda - x_1)(\lambda - y_1)}{|x-y| + |x-y^\lambda|}$$

$$\le C|x|^{(n+1)/2} e^{|y|} \frac{\lambda - y_1}{|x-y| + |x-y^\lambda|} \frac{(1 + |x-y|)^{(n-1)/2}}{|x-y|^{n-1}} = J_1$$

by (C.4). Similarly, for $y_1 > \lambda$,

$$|J| \le c|x|^{(n+1)/2} e^{|y|} \frac{y_1 - \lambda}{|x-y| + |x-y^\lambda|} \frac{(1 + |x-y^\lambda|)^{(n-1)/2}}{|x-y|^{n-1}} = J_2.$$

Setting

$$A_1 = \int_{y_1 < \lambda,\ |y-x| < |x|/2} J_1 f, \qquad A_2 = \int_{y_1 > \lambda,\ |y-x^\lambda| < |x^\lambda|/2} J_2 f$$

$$A_3 = \int_{y_1 < \lambda,\ |y-x| > |x|/2} J_1 f, \qquad A_4 = \int_{y_1 > \lambda,\ |y-x^\lambda| > |x^\lambda|/2} J_2 f,$$

we wish to show

$$\sum_1^4 A_i \le C\|f\|_{\alpha'}$$

with C independent of x, but it may depend on λ. This will yield the desired result

$$|l_i(f)| \le C\|f\|_{\alpha'}$$

and will then complete the proof of Proposition 4.2b.

We have

$$A_1 \le C\|f\|_{\alpha'}|x|^{(n+1)/2} \int_{|y-x|<|x|/2} e^{(1-\alpha')|y|} \frac{(1+|x-y|)^{(n-1)/2}}{|x-y|^{n-1}} dy$$
$$\le C\|f\|_{\alpha'}|x|^{(n+1)/2} e^{(1-\alpha')|x|/2}|x|^{(n+1)/2} \le C\|f\|_{\alpha'}.$$

A similar estimate holds for A_2. Next

$$A_3 \le C\|f\|_{\alpha'}|x|^{(n+1)/2} \int_{|y-x|>|x|/2} \frac{(|\lambda|+|y|)e^{(1-\alpha')|y|}}{|x-y|^{(n+1)/2}} dy$$
$$\le C\|f\|_{\alpha'} \int (|\lambda|+|y|)e^{(1-\alpha')|y|} dy \le C\|f\|_{\alpha'}$$

with a similar result for A_4. Proposition 4.2b is proved. As we remarked, the completion of the proof of Proposition 4.2a is simpler.

To complete the proof of Proposition 4.2c we show that for $x_1 > 1$ the linear functionals

$$k_x(f) = \frac{|x|^{(n+1)/2}}{x_1} e^{|x|}u_1(x)$$

are uniformly bounded in $B_{\alpha'}$. This follows from the inequality

$$\int \frac{|x|^{(n+1)/2}}{x_1} e^{|x|}|G_r(|x-y|)| \frac{|x_1-y_1|}{|x-y|} e^{-\alpha'|y|} dy \le C$$

for $x_1 > 1$. From (C.5) we have

$$G_r(x) = -K_{n/2}/r^{(n-2)/2},$$

so we want to prove the uniform boundedness in $x_1 > 1$ of

$$\int \frac{|x|^{(n+1)/2}}{|x-y|^{n/2}} |K_{n/2}(|x-y|)| \left|1 - \frac{y_1}{x_1}\right| e^{|x|-\alpha'|y|} dy.$$

By the estimate in (C.4), it suffices to prove

$$J = \int \frac{|x|^{(n+1)/2}}{|x-y|^n} (1+|x-y|)^{(n-1)/2} \left|1 - \frac{y_1}{x_1}\right| e^{|x|-|x-y|-\alpha'|y|} dy \le C.$$

Since $|x| - |x-y| \le |y|$ and $1 + |x-y| \le (1+|x|)(1+|y|)$ we find

$$J \le C \int \frac{|x|^n}{|x-y|^n} \frac{|x_1-y_1|}{x_1} (1+|y|)^{(n-1)/2} e^{(1-\alpha')|y|} dy.$$

The last integral is bounded by some constant times the integral occurring in (A.2), which we proved was bounded.

The proof is complete. ■

APPENDIX C

In this appendix we list the properties of the modified Bessel functions K_ν [7] we used in Sections 4 and 6. Let $\nu \geq 0$ be half an integer and $z > 0$. Then

$$K_\nu(z) \text{ is positive and decreasing in } z. \tag{C.1}$$

The most convenient integral representation for our purpose is

$$K_\nu(z) = \frac{1}{\Gamma(\nu + \frac{1}{2})} \sqrt{\frac{\pi}{2}} \frac{e^{-z}}{\sqrt{z}} \int_0^{+\infty} e^{-t} t^{\nu - 1/2} \left(1 + \frac{1}{2}\frac{t}{z}\right)^{\nu - \frac{1}{2}} dt \tag{C.2}$$

where Γ is the gamma function. In case $\nu - \frac{1}{2} = k$ is an integer this reduces to

$$K_\nu = \sqrt{\frac{\pi}{2}} \frac{e^{-z}}{z^{1/2}} \sum_{j=0}^{k} \frac{\Gamma(k+1+j)}{\Gamma(k+1-j)} (2z)^{-j}.$$

Asymptotic expansion for large z if $\nu =$ integer:

$$K_\nu(z) = \sqrt{\frac{\pi}{2}} \frac{e^{-z}}{\sqrt{z}} \left\{ \sum_{j=0}^{M-1} \frac{\Gamma(\nu + \frac{1}{2} + j)}{j!\,\Gamma(\nu + \frac{1}{2} - j)} \frac{1}{(2z)^j} + O\left(\frac{1}{|z|^M}\right) \right\}. \tag{C.3}$$

The remainder after M terms does not exceed the $(M+1)$-term in absolute value, and is of the same sign provided that $M \geq \nu - \frac{1}{2}$.

As $z \to 0$

$$K_\nu(z) \sim \begin{cases} \frac{1}{2}\Gamma(\nu)(\frac{1}{2}z)^{-\nu}, & \nu > 0, \\ -\log z, & \nu = 0. \end{cases} \tag{C.4}$$

Thus for $\nu > 0$, $z \geq 0$, $K_\nu(z) \leq C(e^{-z}/z^\nu)(1+z)^{\nu - \frac{1}{2}}$.

$$K'_\nu(z) = -K_{\nu-1}(z) - \frac{\nu}{z} K_\nu(z) = \frac{\nu}{z} K_\nu(z) - K_{\nu+1}(z) \tag{C.5}$$

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Added in proof: We have just learned that Lemma (H_1') is almost the same as the corollary on page 528 of N. Meyers and J. Serrin, The exterior Dirichlet problem for second-order elliptic partial differential equations, *J. Math. and Mech.* 9 (1960), 513–538.