AN INTEGRO-PDE MODEL FOR EVOLUTION OF RANDOM DISPERSAL

KING-YEUNG LAM AND YUAN LOU

ABSTRACT. We consider an integro-PDE model for a population structured by the spatial variables and a trait variable which is the diffusion rate. Competition for resource is local in spatial variables, but nonlocal in the trait variable. We focus on the asymptotic profile of positive steady state solutions. Our result shows that in the limit of small mutation rate, the solution remains regular in the spatial variables and yet concentrates in the trait variable and forms a Dirac mass supported at the lowest diffusion rate. Hastings and Dockery et al. showed that for two competing species in spatially heterogeneous but temporally constant environment, the slower diffuser always prevails, if all other things are held equal [13, 15]. Our result suggests that their findings may well hold for arbitrarily many or even a continuum of traits.

1. Introduction

In this paper, we focus on the concentration phenomena in a mutation-selection model for the evolution of random dispersal in a bounded, spatially heterogeneous and temporally constant environment. This model concerns a population structured simultaneously by a spatial variable $x \in D$ and the motility trait $\alpha \in \mathcal{A}$ of the species. Here D is a bounded open domain in $\mathbb{R}^N$, and $\mathcal{A} = [\underline{\alpha}, \overline{\alpha}]$, with $\overline{\alpha} > \underline{\alpha} > 0$, denotes a bounded set of phenotypic traits. We assume that the spatial diffusion rate is parameterized by the variable α, while mutation is modeled by a diffusion process with constant rate $\epsilon^2 > 0$. Each individual is in competition for resources with all other individuals at the same spatial location. Denoting by $u(t, x, \alpha)$ the population density of the species with trait $\alpha \in \mathcal{A}$ at location $x \in D$ and time $t > 0$, the model is given as

$$(1.1) \quad \begin{cases} u_t = \alpha \Delta u + [m(x) - \hat{u}(x,t))]\, u + \epsilon^2 u_{\alpha\alpha}, & x \in D, \alpha \in (\underline{\alpha}, \bar{\alpha}), t > 0, \\ \frac{\partial u}{\partial n} = 0, & x \in \partial D, \alpha \in (\underline{\alpha}, \bar{\alpha}), t > 0, \\ u_\alpha = 0, & x \in D, \alpha \in \{\underline{\alpha}, \overline{\alpha}\}, t > 0, \\ u(0, x, \alpha) = u_0(x, \alpha), & x \in D, \alpha \in (\underline{\alpha}, \bar{\alpha}). \end{cases}$$

Here $\Delta = \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2}$ denotes the Laplace operator in the spatial variables,

$$\hat{u}(x,t) := \int_{\underline{\alpha}}^{\overline{\alpha}} u(t, x, \alpha)\, d\alpha,$$

n denotes the outward unit normal vector on the boundary ∂D of the spatial domain D, and $\frac{\partial}{\partial n} = n \cdot \nabla$. The function $m(x)$ represents the quality of the habitat, which

Date: August 26, 2016.

2010 *Mathematics Subject Classification.* 35K57, 92D15, 92D25.

KYL and YL are partially supported by NSF grant DMS-1411476.

is assumed to be non-constant in x to reflect that the environment is spatially heterogeneous but temporally constant.

The model (1.1) can be viewed as a continuum (in trait) version of the following mutation-selection model considered by Dockery et al. [13], concerning the competition of k species with different dispersal rates but otherwise identical:

$$\begin{cases} \frac{\partial}{\partial t}u_i = \alpha_i \Delta u_i + \left[m(x) - \sum_{j=1}^k u_j\right] u_i + \epsilon^2 \sum_{j=1}^k M_{ij} u_j & \\ & \text{in } D \times (0,\infty), i = 1, ..., k, \\ \frac{\partial}{\partial n} u_i = 0 & \text{on } \partial D \times (0,\infty), i = 1, ..., k, \\ u_i(x,0) = u_{i,0}(x) & \text{in } D, i = 1, ..., k, \end{cases} \tag{1.2}$$

where $0 < \alpha_1 < \alpha_2 < ... < \alpha_k$ are constants, $m(x) \in C^2(\overline{D})$ is non-constant, M_{ij} is an irreducible real $k \times k$ matrix that models the mutation process so that $M_{ii} < 0$ for all i, and $M_{ij} \geq 0$ for $i \neq j$ and $\epsilon^2 \geq 0$ is the mutation rate.

Model (1.2) was introduced to address the question of evolution of random dispersal. In the case when there is no mutation, i.e. $\epsilon = 0$, this question was considered in [15], where it was shown that in a competition model of two species with different diffusion rates but otherwise identical, a rare competitor can invade the resident species if and only if the rare species is the slower diffuser. Dockery et al. [13] generalized the work of Hastings [15] to k species situation, and proved that no two species can coexist at equilibrium, i.e. the set of non-trivial, non-negative steady states of the system (1.2) is given by

$$\{(\theta_{\alpha_1}, 0, ..., 0), (0, \theta_{\alpha_2}, 0, ..., 0), ..., (0, ..., \theta_{\alpha_k})\},$$

where θ_α is the unique positive solution of

$$\alpha \Delta \theta + \theta(m - \theta) = 0 \quad \text{in } D, \qquad \frac{\partial \theta}{\partial n} = 0 \quad \text{on } \partial D.$$

Moreover, among the non-trivial steady states, only $(\theta_{\alpha_1}, ..., 0)$, the steady state where the slowest diffuser survives, is stable and the rest of the steady states are all unstable. Furthermore, when $k = 2$, the steady state $(\theta_{\alpha_1}, 0)$ is globally asymptotically stable among all non-negative, non-trivial solutions. Whether such a result holds for three or more species remains an interesting and important open question.

Dockery et al. [13] further inquired the effect of small mutation. More precisely, when $0 < \epsilon \ll 1$, it is shown that (1.2) has a unique steady state $\tilde{U} = (\tilde{u}_1, \tilde{u}_2, ..., \tilde{u}_N)$ in the space of non-trivial, non-negative functions, such that $\tilde{u}_i > 0$ for all i, and $\tilde{U} \to (\theta_{\alpha_1}, 0, ..., 0)$ as $\epsilon \to 0$; i.e. the system (1.2) equilibrates only when the slowest species is dominant and all other species remain at low densities.

It is natural, then, to inquire if the situation in the discrete (in trait) framework carries over to the continuum framework. The aim of this paper is to study the asymptotic behavior of steady state(s) of (1.1). Let u_ϵ be any positive steady state of (1.1), we will show that, as $\epsilon \to 0$,

$$u_\epsilon(x, \alpha) \to \delta(\alpha - \underline{\alpha}) \theta_{\underline{\alpha}}(x),$$

i.e. u_ϵ converges to a Dirac mass supported at the lowest possible trait value $\underline{\alpha}$. See Theorem 2.3 for precise descriptions of our main results.

Mutation-selection models for a continuum of trait values have been studied extensively, when the phenotypic trait is associated only with growth advantages [4, 8, 9, 12, 17, 19, 21]. See also [16] for a pure selection model. The consideration of a spatial trait is more recent [1, 2, 7, 20].

System (1.1) is also considered in an unbounded spatial domain $x \in \mathbb{R}$. A formal argument concerning the existence of an "accelerating wave" is presented in [6], which provides a theoretical explanation of the accelerating invasion front of cane toads in Australia [23]. Rigorous results are obtained when $\alpha \in \mathcal{A} = [\underline{\alpha}, \overline{\alpha}]$ more recently in [5, 24]. It can be summarized that the highest diffusion rate is selected when the underlying spatial domain is unbounded, which stands in contrast to the case of bounded spatial domains we consider in this paper, where the lowest possible diffusion rate is selected.

The rest of the paper is organized as follows: The main results are stated in Section 2. Section 3 concerns various estimates on steady states of (1.1). In Section 4 we introduce an auxiliary eigenvalue problem and a transformed problem of (2.1). The limit of $\hat{u}_\epsilon$ is determined in Section 5. In Section 6, we analyze the qualitative properties of solutions to the transformed problem. The proof of our main result is given in Section 7. Finally, the Appendices A to C establish the existence results, the smooth dependence of principal eigenvalue on coefficients as well as a Liouville-type results concerning positive harmonic functions on cylinder domains.

2. Main Results

In this paper, we consider the asymptotic behavior of positive steady states of (1.1), denoted by u_ϵ. That is, u_ϵ satisfies the following mutation-selection equation of a randomly diffusing population:

$$\begin{cases} \alpha\Delta u_\epsilon + \epsilon^2 (u_\epsilon)_{\alpha\alpha} + [m(x) - \hat{u}_\epsilon(x)]\, u_\epsilon = 0 & \text{in } \Omega := D \times (\underline{\alpha}, \bar{\alpha}), \\ \frac{\partial u_\epsilon}{\partial n} = 0 & \text{on } \partial D \times (\underline{\alpha}, \overline{\alpha}), \\ (u_\epsilon)_\alpha = 0 & \text{in } D \times \{\underline{\alpha}, \overline{\alpha}\}, \end{cases} \tag{2.1}$$

where

$$\hat{u}_\epsilon(x) = \int_{\underline{\alpha}}^{\overline{\alpha}} u_\epsilon(x, \alpha)\, d\alpha. \tag{2.2}$$

Throughout this paper, we assume

(A) $m(x)$ is a non-constant function in $C(\bar{D})$ such that $\displaystyle\int_D m(x)\, dx > 0$.

In particular, under assumption (A) it is possible for $m(x)$ to be negative somewhere in D. The existence of positive solutions to (2.1) can be stated as follows:

Theorem 2.1. *Suppose* (A) *holds, then* (2.1) *has at least one positive solution for all* $\epsilon > 0$.

We postpone the proof of Theorem 2.1 to Appendix A. For the rest of the paper we will focus on the asymptotic behavior of positive solutions of (2.1) as $\epsilon \to 0$. To this end, we define the following quantities:

Definition 2.2. (i) Let $\theta_{\underline{\alpha}}(x)$ be the unique positive solution of

$$\begin{cases} \underline{\alpha}\Delta\theta + \theta(m(x) - \theta) = 0 & \text{in } D, \\ \frac{\partial \theta}{\partial n} = 0 & \text{on } \partial D. \end{cases} \tag{2.3}$$

(ii) For each $\alpha \in [\underline{\alpha}, \overline{\alpha}]$, we denote the principal eigenvalue and principal positive eigenfunction of the following problem by $\sigma^*(\alpha)$ and $\psi^*(x,\alpha)$, respectively:

$$\begin{cases} \alpha\Delta\psi + (m(x) - \theta_{\underline{\alpha}}(x))\psi + \sigma\psi = 0 & \text{in } D, \\ \frac{\partial\psi}{\partial n} = 0 \quad \text{on } \partial D, & \text{and} \quad \int_D \psi^2\,dx = \int_D \theta_{\underline{\alpha}}^2\,dx. \end{cases} \tag{2.4}$$

(Note that by (i), $\theta_{\underline{\alpha}}(x)$ is a positive eigenfunction for (2.4) when $\alpha = \underline{\alpha}$. By uniqueness of the (normalized) principal eigenfunction, we have $\sigma^*(\underline{\alpha}) = 0$, and $\psi^*(x,\underline{\alpha}) = \theta_{\underline{\alpha}}(x)$ for $x \in D$.)

(iii) Denote by $\eta^*(s)$ the unique positive solution to

$$\begin{cases} \eta'' + (a_0 - a_1 s)\eta = 0 & \text{for } s > 0, \\ \eta'(0) = 0 = \eta(+\infty) & \text{and } \int_0^\infty \eta(s)\,ds = 1, \end{cases} \tag{2.5}$$

where a_0, a_1 are positive constants determined by $a_1 = \frac{\partial\sigma^*}{\partial\alpha}(\underline{\alpha})$ and $a_0 = (a_1)^{2/3}A_0$, where A_0 is the absolute value of the first negative zero of the derivative of the Airy function.

When $m(x) \equiv 1$, one can easily show that $u_\epsilon \equiv 1/(\overline{\alpha} - \underline{\alpha})$, i.e. there is no selection in the trait variable. Our main result shows that the outcome changes drastically when $m(x)$ is non-constant. In fact, u_ϵ concentrates at the lowest value in the trait variable, as $\epsilon \to 0$. This phenomenon is also known as spatial sorting.

Theorem 2.3. *Let u_ϵ be any positive solution of* (2.1). *Then for all $\beta > 0$, there exists $C > 0$ independent of $\epsilon > 0$ such that*

$$u_\epsilon(x,\alpha) \le C\epsilon^{-2/3} \exp\left(-\beta(\alpha - \underline{\alpha})\epsilon^{-2/3}\right) \tag{2.6}$$

in $\Omega = D \times (\underline{\alpha}, \overline{\alpha})$. Moreover, as $\epsilon \to 0$

$$\left\| \epsilon^{2/3} u_\epsilon(x,\alpha) - \theta_{\underline{\alpha}}(x)\eta^*\left(\frac{\alpha - \underline{\alpha}}{\epsilon^{2/3}}\right) \right\|_{L^\infty(\Omega)} \to 0 \tag{2.7}$$

where $\theta_{\underline{\alpha}}(x)$ and $\eta^*(s)$ are given as above. In particular, we have

$$\hat{u}_\epsilon(x) = \int_{\underline{\alpha}}^{\overline{\alpha}} u_\epsilon(x,\alpha)\,d\alpha \to \theta_{\underline{\alpha}}(x) \qquad \text{as } \epsilon \to 0. \tag{2.8}$$

As the proof of Theorem 2.3 is fairly technical, we briefly outline the main ingredients for readers, as well as to motivate the scaling $\epsilon^{2/3}$ and the Airy function $\eta^*(s)$ appearing in (2.7). Our idea is to establish the "separation of variables" formula (2.7) for u_ϵ: Let $\tau > 0$ be fixed and introduce the scaling $s = (\alpha - \bar{\alpha})/\epsilon^\tau$, we write

$$u_\epsilon(x,\alpha) = \psi_\epsilon(x,\alpha)w_\epsilon(x,s),$$

where $(\sigma_\epsilon(\alpha), \psi_\epsilon(\cdot,\alpha)$ is the the principal eigenpair of $-\alpha\Delta\psi + (\hat{u}_\epsilon - m)\psi = \sigma\psi$, subject to the zero Neumann boundary condition and the integral constraint $\int_D \psi^2 = \int_D \theta_{\underline{\alpha}}^2$. The main body of our paper is devoted to the proof of following two things: (i) As $\epsilon \to 0$, the fact that $\hat{u}_\epsilon \to \theta_{\underline{\alpha}}$ uniformly (so that $\psi_\epsilon(x, \underline{\alpha} + \epsilon^{2/3}s) \to \theta_{\underline{\alpha}}(x)$) is established in Section 5 with the help of some "rough" description of concentration of u_ϵ on the subset $D \times \{\underline{\alpha}\}$ of Ω, as well as the limit $\lim_{\epsilon\to 0}(\hat{u}_\epsilon - m)$ being non-constant; (ii) As $\epsilon \to 0$, $\tilde{w}_\epsilon(x,s) := \frac{w_\epsilon(x,s)}{\|w_\epsilon\|}$ satisfies

$$-\alpha\nabla_x \cdot (\psi_\epsilon^2 \nabla_x \tilde{w}_\epsilon) - \epsilon^{2-2\tau}(\psi_\epsilon^2 \tilde{w}_{\epsilon,s})_s + \psi_\epsilon^2 \tilde{w}_\epsilon \left[\sigma_\epsilon(\alpha) - \epsilon^2 \frac{\psi_{\epsilon,\alpha\alpha}}{\psi_\epsilon}\right] = 0.$$

Suppose one can show that

$$\tilde{w}_\epsilon(x,s) \to \eta(s), \quad \text{i.e.} \quad \nabla_x \tilde{w}_\epsilon \approx 0,$$

we may discard the terms involving derivatives with respect to x. Using the regularity of $(\sigma_\epsilon, \psi_\epsilon)$ in the variable α (see Lemma 4.1)

$$\sigma_\epsilon(\alpha) \approx \sigma_\epsilon(\underline{\alpha}) + \frac{\partial \sigma_\epsilon}{\partial \alpha}(\underline{\alpha})(\alpha - \underline{\alpha}) \quad \text{and} \quad (\psi_\epsilon)_s = \epsilon^\tau (\psi_\epsilon)_\alpha = O(\epsilon^\tau),$$

we have

$$\epsilon^{2-2\tau}\left[-\eta_{ss} + o(1)\eta_s + \eta\left(\frac{\sigma_\epsilon(\underline{\alpha})}{\epsilon^{2-2\tau}} + \frac{\partial \sigma_\epsilon}{\partial \alpha}(\underline{\alpha})\epsilon^{3\tau-2}s + O(\epsilon^{4\tau-2})\right)\right] = 0.$$

Now, if τ is taken as $2/3$, we can pass to the limit so that η satisfies a version of the Airy equation

$$-\eta'' + (A_0 + A_1 s)\eta = 0 \quad \text{for } s > 0,$$

where (see Lemma 4.1(iii) and Lemma 4.3)

$$A_1 = \lim_{\epsilon \to 0} \frac{\partial \sigma_\epsilon}{\partial \alpha}(\underline{\alpha}) > 0, \quad \text{and } A_0 = \lim_{\epsilon\to 0} \frac{\sigma_\epsilon(\underline{\alpha})}{\epsilon^{2/3}}.$$

The crucial step that $\tilde{w}_\epsilon(x,s) := \frac{w_\epsilon}{\|w_\epsilon\|}$ being asymptotically independent of x is proved in Section 6 using some key estimates established in the earlier sections.

Remark 2.4. After this work is completed, the authors learned that a closely related result, under a slightly different formulation, is independently proved by B. Perthame and P.E. Souganidis under a different approach, where an intermediate trait attains the minimum diffusion rate and an interior Dirac mass is found when the mutation rate tends to zero. Apart from the distinction in our approaches, we note the following distinct features of our work: (i) A boundary concentration is found in our set-up, instead of an interior concentration in [22] which predicts different scalings in powers of ϵ; (ii) Our method does not assume the convexity of spatial domain D; (iii) Various detailed L^∞ estimates and asymptotic limits are obtained (Theorem 2.3) which paves the way to the proof of asymptotic stability and uniqueness of u_ϵ in a future paper; (iv) The key estimate of the limit $h_0(x) = \lim_{\epsilon\to 0} \hat{u}_\epsilon(x) - m(x)$ being non-constant (Lemma 3.4) reflects the effect of spatial heterogeneity, the underlying mathematical reason for the selection of small diffusion rate. See also Proposition 3.7 which makes the connection to [22, Lemma 4.3].

3. Properties of $\hat{u}_\epsilon$

In this section we establish various properties of $\hat{u}_\epsilon$. Recall that $\hat{u}_\epsilon$ is defined in (2.2).

Lemma 3.1. *There exists some positive constant $\delta_1 = \delta_1(\underline{\alpha}, \bar{\alpha}, m)$ independent of ϵ such that*

$$\delta_1 \le \hat{u}_\epsilon(x) \le 1/\delta_1 \quad \text{in } D$$

for all $\epsilon > 0$. In particular,

$$h_\epsilon(x) := \hat{u}_\epsilon(x) - m(x) \tag{3.1}$$

is bounded uniformly in $L^\infty(D)$.

Proof. The idea of the upper bound follows from [24]. Define

$$v_\epsilon(x) = \int_{\underline{\alpha}}^{\bar{\alpha}} \alpha u_\epsilon(x,\alpha)\, d\alpha. \tag{3.2}$$

Then we have

$$\underline{\alpha}\hat{u}_\epsilon(x) \le v_\epsilon(x) \le \bar{\alpha}\hat{u}_\epsilon(x) \quad \text{in } D. \tag{3.3}$$

Integrating (2.1) over α gives

$$\begin{cases} \Delta v_\epsilon(x) + (m(x) - \hat{u}_\epsilon(x))\hat{u}_\epsilon(x) = 0 & \text{in } D, \\ \frac{\partial v_\epsilon}{\partial n} = 0 & \text{on } \partial D. \end{cases} \tag{3.4}$$

Let $\max_{\bar{D}} v_\epsilon = v_\epsilon(x_0)$, then $\hat{u}_\epsilon(x_0) \le m(x_0) \le \max_{\bar{D}} m$ (see [18, Proposition 2.2]), and by (3.3),

$$\underline{\alpha} \max_{\bar{D}} \hat{u}_\epsilon \le \max_{\bar{D}} v_\epsilon = v_\epsilon(x_0) \le \bar{\alpha}\hat{u}_\epsilon(x_0) \le \bar{\alpha} \max_{\bar{D}} m.$$

Hence we deduce that $\|\hat{u}_\epsilon\|_{L^\infty(D)}$ and $\|h_\epsilon\|_{L^\infty(D)}$ are bounded uniformly in ϵ, where $h_\epsilon(x) = \hat{u}_\epsilon(x) - m(x)$ is given in (3.1).

Next, we show the lower bound of $\hat{u}_\epsilon$. By (3.3), we deduce that

$$\hat{u}_\epsilon(x) = k_\epsilon(x) v_\epsilon(x)$$

for some $k_\epsilon(x) \in L^\infty(D)$ such that $\bar{\alpha}^{-1} \le k_\epsilon(x) \le \underline{\alpha}^{-1}$. So that v_ϵ is a positive solution of

$$-\Delta v_\epsilon + h_\epsilon(x) k_\epsilon(x) v_\epsilon = 0 \quad \text{in } D, \qquad \text{and} \qquad \frac{\partial v_\epsilon}{\partial n} = 0 \quad \text{on } \partial D,$$

where we have already shown that $h_\epsilon = \hat{u}_\epsilon - m$ is uniformly bounded (in $L^\infty(D)$) in ϵ. Therefore, the Harnack inequality applies so that

$$\max_{\bar{D}} v_\epsilon \le C' \min_{\bar{D}} v_\epsilon \tag{3.5}$$

for some constant $C' > 1$ independent of ϵ. Combining with (3.3), we have

$$\underline{\alpha} \max_{\bar{D}} \hat{u}_\epsilon \le \max_{\bar{D}} v_\epsilon \le C' \min_{\bar{D}} v_\epsilon \le C'\bar{\alpha} \min_{\bar{D}} \hat{u}_\epsilon. \tag{3.6}$$

Now, if we divide (2.1) by u_ϵ and integrate by parts over $\Omega = D \times (\underline{\alpha}, \overline{\alpha})$, we obtain

$$(\overline{\alpha} - \underline{\alpha}) \int_D (\hat{u}_\epsilon - m)\, dx = \int_\Omega h_\epsilon(x)\, d\alpha dx = \int_\Omega \frac{\alpha |\nabla_x u_\epsilon|^2 + \epsilon^2 |(u_\epsilon)_\alpha|^2}{u_\epsilon^2} > 0. \tag{3.7}$$

We deduce by (3.6) and (3.7) that

$$\frac{C'\overline{\alpha}}{\underline{\alpha}} \min_{\overline{D}} \hat{u}_\epsilon \ge \max_{\overline{D}} \hat{u}_\epsilon \ge \frac{1}{|D|} \int_D \hat{u}_\epsilon(x)\, dx \ge \frac{1}{|D|} \int_D m(x)\, dx > 0.$$

This establishes the uniform lower bound of $\hat{u}_\epsilon$. □

Remark 3.2. Since $\|\hat{u}_\epsilon\|_{L^\infty(D)}$ and $\|v_\epsilon\|_{L^\infty(D)}$ are bounded uniformly in ϵ, applying elliptic L^p estimate to (3.4) implies that $\|v_\epsilon\|_{W^{2,p}(D)}$ is bounded uniformly in ϵ. In particular, there exists sequence $\epsilon_k \to 0$ such that v_{ϵ_k} converges uniformly on D.

Lemma 3.3. *There exists a constant $C > 0$ such that for any positive solution u_ϵ of* (2.1),

$$\sup_{D \times (\underline{\alpha}, \overline{\alpha})} u_\epsilon \le C\epsilon^{-1}.$$

Proof. Choose x_ϵ and α_ϵ such that the supremum of u_ϵ is attained at $(x_\epsilon, \alpha_\epsilon)$. Next, let $U_\epsilon(x,\tau) = u_\epsilon(x, \alpha_\epsilon + \epsilon\tau)$, then (extending u_ϵ to $D \times [\underline{\alpha} - \epsilon_0, \overline{\alpha} + \epsilon_0]$ by reflection across the boundary portions $D \times \{\underline{\alpha}, \overline{\alpha}\}$ if necessary) one may observe that U_ϵ satisfies a uniformly elliptic equation with uniformly bounded (in L^∞) coefficients

$$\begin{cases} \alpha\Delta_x U_\epsilon + U_{\epsilon,\tau\tau} + h_\epsilon(x) U_\epsilon = 0 & \text{in } D \times [-2,2], \\ \frac{\partial U_\epsilon}{\partial n} = 0 & \text{on } \partial D \times [-2,2], \end{cases}$$

where $\alpha = \alpha_\epsilon + \epsilon\tau$ is always bounded between $[\underline{\alpha} - \epsilon_0, \overline{\alpha}, +\epsilon_0] \subset (0, +\infty)$. Hence, we may apply the Harnack's inequality to yield a positive constant C independent of ϵ such that

$$u_\epsilon(x, \alpha_\epsilon + \epsilon\tau) \geq C u_\epsilon(x_\epsilon, \alpha_\epsilon) = C \sup_{D\times(\underline{\alpha},\overline{\alpha})} u_\epsilon$$

for all $x \in D$, $\tau \in [-1, 1]$. Hence,

$$\hat{u}_\epsilon(x) = \int_{\underline{\alpha}}^{\overline{\alpha}} u_\epsilon(x,\alpha)\, d\alpha \geq C\epsilon \sup_{D\times(\underline{\alpha},\overline{\alpha})} u_\epsilon$$

for all ϵ sufficiently small. By Lemma 3.1, we deduce that

$$\sup_{D\times(\underline{\alpha},\overline{\alpha})} u_\epsilon \leq C'\epsilon^{-1}$$

for some positive constant C'. □

By Lemma 3.1, h_ϵ is bounded in $L^\infty(D)$ uniformly in ϵ. Therefore, up to subsequences $\epsilon_j \to 0$, h_{ϵ_j} converges weakly in $L^p(D)$ for all $p > 1$. We first prove an important property of any subsequential limit h_0.

Lemma 3.4. *Let h_0 be a weak (subsequential) limit of $h_\epsilon(x)$ in $L^p(D)$ ($p > 1$) as $\epsilon \to 0$, then $h_0(x)$ is non-constant in D.*

Proof. Suppose to the contrary that for some $c \in \mathbb{R}$, $h_\epsilon(x) \rightharpoonup c$ weakly in $L^p(D)$ for all $p > 1$.

Claim 3.5. $c = 0$.

By taking $\epsilon \to 0$ in (3.7), we deduce that $c \geq 0$; i.e. for some $c \geq 0$, $\hat{u}_\epsilon = h_\epsilon + m(x) \rightharpoonup c + m(x)$ in L^p for all $p > 1$.

Next, integrating (2.1) with respect to α and then x, we obtain

$$\int_D \hat{u}_\epsilon(x)(m(x) - \hat{u}_\epsilon(x)) = 0, \tag{3.8}$$

so that

$$\int_D m(x)(m(x)+c) = \lim_{\epsilon\to 0} \int_D m(x)\hat{u}_\epsilon \geq \liminf_{\epsilon\to 0} \int_D (\hat{u}_\epsilon)^2 \geq \int_D (m(x)+c)^2, \tag{3.9}$$

where the first inequality follows from (3.8), and the second inequality from expanding $\int_D [\hat{u}_\epsilon(x) - (m(x)+c)]^2 \geq 0$ as

$$\int_D \hat{u_\epsilon}^2 \geq 2\int_D \hat{u}_\epsilon(m+c) - \int_D (m+c)^2 \to \int_D (m+c)^2.$$

Hence, by (3.9), we have $c = 0$; i.e. $h_\epsilon(x) \rightharpoonup 0$ weakly in $L^p(D)$ for all $p > 1$.

Note that we are done if $m < 0$ somewhere, since then $h_\epsilon(x) \geq -m(x) > 0$ in some open subset of D independent of ϵ, which contradicts $h_\epsilon \rightharpoonup 0$. For the general case of $m(x)$ being possibly non-negative, we continue via a blow-up argument. Let

$$C_\epsilon := \sup_{\alpha \in (\underline{\alpha}, \overline{\alpha})} \left(\frac{\sup_{x\in D} u_\epsilon(x,\alpha)}{\inf_{x \in D} u_\epsilon(x,\alpha)} \right).$$

It is enough to show that

Claim 3.6. *$C_\epsilon \searrow 1$ as $\epsilon \to 0$.*

Assuming Claim 3.6, then by definition of C_ϵ,

$$u_\epsilon(x,\alpha) \leq C_\epsilon u_\epsilon(y,\alpha) \quad \text{ for all } x, y \in D \text{ and } \underline{\alpha} < \alpha < \overline{\alpha}.$$

This gives, upon integrating over $\alpha \in (\underline{\alpha}, \overline{\alpha})$,

$$\sup_{x \in D} \hat{u}_\epsilon(x) \leq C_\epsilon \inf_{y \in D} \hat{u}_\epsilon(y).$$

Hence $\hat{u}_\epsilon(x)$ converges to a constant. But this also means that $h_\epsilon = \hat{u}_\epsilon - m(x)$ converges to a non-constant function, as $m(x)$ is non-constant. This is a contradiction.

It remains to prove Claim 3.6. Assume to the contrary that there exist some constant $c_0 > 1$, and sequences $\epsilon_k \to 0$, $\alpha_k \to \alpha_0$, $x_k, y_k \in D$ such that

$$u_{\epsilon_k}(x_k, \alpha_k) \geq c_0 u_{\epsilon_k}(y_k, \alpha_k). \tag{3.10}$$

Extend u_ϵ to $D \times [\underline{\alpha} - \epsilon_0, \overline{\alpha} + \epsilon_0]$ for some fixed ϵ_0 small by reflection on the boundary $D \times \{\underline{\alpha}, \overline{\alpha}\}$, and define

$$U_k(x,s) := \frac{u_{\epsilon_k}(x, \alpha_k + \epsilon_k s/\sqrt{\alpha_k})}{\sup_{x \in D} u_{\epsilon_k}(x, \alpha_k)} \quad \text{ in } D \times \left(\frac{\underline{\alpha} - \epsilon_0 - \alpha_k}{\epsilon_k}, \frac{\overline{\alpha} + \epsilon_0 - \alpha_k}{\epsilon_k} \right).$$

Then (3.10) says that for some $c_0 > 1$ independent of k

$$\inf_{x \in D} U_k(x,0) \leq \frac{1}{c_0} \quad \text{ for all } k. \tag{3.11}$$

Moreover, U_k satisfies

$$\begin{cases} \Delta_x U_k + \frac{\alpha_k}{\alpha_k + \epsilon_k s/\sqrt{\alpha_k}} U_{k,ss} + \frac{h_\epsilon(x)}{\alpha_k + \epsilon_k s/\sqrt{\alpha_k}} U_k = 0 & \text{ in } D \times \left(\frac{\underline{\alpha} - \epsilon_0 - \alpha_k}{\epsilon_k}, \frac{\overline{\alpha} + \epsilon_0 - \alpha_k}{\epsilon_k} \right), \\ U_k(x,s) > 0 \quad \text{ in } D \times \left(\frac{\underline{\alpha} - \epsilon_0 - \alpha_k}{\epsilon_k}, \frac{\overline{\alpha} + \epsilon_0 - \alpha_k}{\epsilon_k} \right), & \sup_{x \in D} U_k(x,0) = 1. \end{cases}$$

Since $\alpha_k \to \alpha_0 \in [\underline{\alpha}, \overline{\alpha}]$, $\epsilon_k \to 0$, the domain of U_k converges to $D \times \mathbb{R}$ as $k \to \infty$. Moveover, by the uniform boundedness of $\|h_{\epsilon_k}\|_{L^\infty(D)}$ in k (Lemma 3.1), we have for each $M > 0$ the coefficients of the equation of $U_k(x,s)$ are bounded in $L^\infty(\bar{D} \times [-M, M])$ uniformly in k. Since $\sup_{x \in D} U_k(x,0) = 1$, together with (3.11) we may apply Harnack inequality to obtain a constant $C = C(M)$ independent of k such that

$$C^{-1} \leq U_k(x,s) \leq C \quad \text{ for } x \in D \text{ and } |s| \leq M.$$

By L^p estimates (applied to $\overline{D} \times [-M, M]$ for each M), there is a subsequence U_{k_i} that converges uniformly in compact subsets of $\bar{D} \times \mathbb{R}$ to a positive solution of $\Delta_x U_0 + (U_0)_{ss} = 0$ on $D \times \mathbb{R}$. (The limiting domain is $D \times \mathbb{R}$ as $\frac{\underline{\alpha} - \epsilon_0 - \alpha_k}{\epsilon} \to -\infty$ and $\frac{\overline{\alpha} + \epsilon_0 - \alpha_k}{\epsilon} \to \infty$.) Now, we apply Proposition C.1 for positive harmonic functions on a cylinder domain, so that $U_0 \equiv c_1$ for some positive constant c_1. Since $\sup_{x \in D} U_k(x,0) = 1$ for all k, we have $c_1 = 1$. In particular, we set $s = 0$ and find a subsequence $U_{k_i}(x,0)$ converges to 1 uniformly for $x \in D$. This is in contradiction to (3.11) and proves Claim 3.6. This completes the proof. □

The following result generalizes a key estimate of [22], proved wherein via Bernstein's method under the additional assumption that D is convex. Although not needed for the rest of the paper, Proposition 3.7 enables one to follow the elegant Hamilton-Jacobi approach as in [22] to show the concentration phenomenon.

Proposition 3.7. *Let u_ϵ be a positive solution of* (2.1). *Then there exists a constant $C > 0$ independent of ϵ such that*

$$\epsilon \left\| \frac{u_{\epsilon,\alpha}}{u_\epsilon} \right\|_{L^\infty(\Omega)} + \left\| \frac{\nabla_x u_\epsilon}{u_\epsilon} \right\|_{L^\infty(\Omega)} \le C.$$

Proof. Extend the definition of u_ϵ to $D \times (2\underline{\alpha} - \overline{\alpha}, 2\overline{\alpha} - \underline{\alpha})$ by reflecting along the boundary portions $D \times \{\underline{\alpha}, \overline{\alpha}\}$. For each $\alpha_0 \in [\underline{\alpha}, \overline{\alpha}]$, define

$$U_\epsilon(x, \tau) := u_\epsilon(x, \alpha_0 + \epsilon\tau).$$

Then U_ϵ is a positive solution to

$$A(\tau, \epsilon)\Delta_x U_\epsilon + U_{\epsilon,\tau\tau} - h_\epsilon(x) U_\epsilon = 0$$

in $D \times \big(\epsilon^{-1}(2\underline{\alpha} - \overline{\alpha} - \alpha_0), \epsilon^{-1}(2\overline{\alpha} - \underline{\alpha} - \alpha_0)\big)$ and satisfies the Neumann boundary condition on $\partial D \times \big(\epsilon^{-1}(2\underline{\alpha} - \overline{\alpha} - \alpha_0), \epsilon^{-1}(2\overline{\alpha} - \underline{\alpha} - \alpha_0)\big)$. Here $A(\tau, \epsilon)$ is a continuous function such that $\underline{\alpha} \le A(\tau) \le \overline{\alpha}$. This, together with the boundedness of $\|h_\epsilon\|_{L^\infty(D)}$ (Lemma 3.1), one may apply the Harnack inequality to $D \times (-1, 1)$ and deduce the following.

Claim 3.8. *There exists $C > 0$ independent of $\alpha_0 \in [\underline{\alpha}, \overline{\alpha}]$ and ϵ such that*

$$\sup_{D \times (-1,1)} U_\epsilon(x, \tau) \le C \inf_{D \times (-1,1)} U_\epsilon(x, \tau).$$

Next, we apply elliptic L^p estimates to U_ϵ in $D \times (-1, 1)$, so that

(3.12)
$$\sup_{x \in D} [|U_{\epsilon,\tau}(x, 0)| + |\nabla_x U_\epsilon(x, 0)|] \le C \, \|U_\epsilon\|_{L^p(D \times (-1,1))} \le C \sup_{D \times (-1,1)} U_\epsilon(x, \tau).$$

In view of Claim 3.8, we deduce for any $x \in D$,

$$|U_{\epsilon,\tau}(x, 0)| + |\nabla_x U_\epsilon(x, 0)| \le C \inf_{D \times (-1,1)} U_\epsilon \le C U_\epsilon(x, 0).$$

i.e. $\epsilon\, |u_{\epsilon,\alpha}(x, \alpha_0)| + |\nabla_x u_\epsilon(x, \alpha_0)| \le C u_\epsilon(x, \alpha_0)$ for all $x \in D$. Since C is independent of x, α_0 and ϵ, this proves Proposition 3.7. □

4. Two eigenvalue problems

4.1. **An Auxiliary Eigenvalue Problem.** Consider, for each $\alpha > 0$ and $\epsilon > 0$ the eigenvalue problem (recall $h_\epsilon(x) = \hat{u}_\epsilon(x) - m(x)$)

$$\begin{cases} -\alpha \Delta \psi + h_\epsilon \psi = \sigma \psi & \text{in } D, \\ \frac{\partial \psi}{\partial n} = 0 \quad \text{on } \partial D \quad \text{and} & \int_D \psi^2 \, dx = \int_D \theta_{\underline{\alpha}}^2 \, dx, \end{cases} \tag{4.1}$$

and denote the principal eigenvalue and positive eigenfunction by $\sigma_\epsilon(\alpha)$ and $\psi_\epsilon(x, \alpha)$, respectively. At this point, we have not shown how the two eigenvalue problems (4.1) and (2.4) are related yet.

For each $\epsilon > 0$, $\sigma_\epsilon(\alpha)$ is a smooth function of $\alpha > 0$ (Proposition B.1(ii)), and it has a Taylor expansion at $\alpha = \underline{\alpha}$:

$$\sigma_\epsilon(\alpha) = \sigma_{0,\epsilon} + \sigma_{1,\epsilon}(\alpha - \underline{\alpha}) + \sigma_{2,\epsilon}(\alpha - \underline{\alpha})^2 + O((\alpha - \underline{\alpha})^3), \tag{4.2}$$

where $\sigma_{0,\epsilon} = \sigma_\epsilon(\underline{\alpha})$ and $\sigma_{k,\epsilon} = \frac{\partial^k}{\partial \alpha^k} \sigma_\epsilon(\underline{\alpha})$.

Lemma 4.1. *Let σ_ϵ and ψ_ϵ be given as above.*

(i) *For each $k \geq 0$, $\frac{\partial^k}{\partial \alpha^k}\sigma_\epsilon(\alpha)$ is bounded uniformly in $\epsilon > 0$ and $\alpha \in [\underline{\alpha}, \overline{\alpha}]$.*

(ii) *For each $k \geq 0$ and $p > 1$, $\frac{\partial^k}{\partial \alpha^k}\psi_\epsilon(\,\cdot\,, \alpha)$ is bounded in $W^{2,p}(D)$ (and hence $C(\overline{D})$) uniformly in $\epsilon > 0$ and $\alpha \in [\underline{\alpha}, \overline{\alpha}]$.*

(iii) *There exists $c_0 > 0$ such that*

$$\liminf_{\epsilon\to 0} \frac{\partial \sigma_\epsilon}{\partial \alpha}(\alpha) \geq c_0 > 0 \quad \text{for all } \alpha \in [\underline{\alpha}, \overline{\alpha}].$$

In particular, $\liminf_{\epsilon\to 0} \sigma_{1,\epsilon} = \liminf_{\epsilon\to 0} \frac{\partial \sigma_\epsilon}{\partial \alpha}(\underline{\alpha}) > 0.$

(iv) *There exist positive constants c_1, c_2 such that for all $\epsilon > 0$,*

$$c_1 \leq \psi_\epsilon(x, \alpha) \leq c_2 \quad \text{for all } x \in D \text{ and } \alpha \in [\underline{\alpha}, \overline{\alpha}].$$

Corollary 4.2. *There exists $C > 0$ independent of ϵ such that*

$$\left\| \frac{\psi_{\epsilon,\alpha}}{\psi_\epsilon} \right\|_{L^\infty(D\times(\underline{\alpha},\overline{\alpha}))} + \left\| \frac{\psi_{\epsilon,\alpha\alpha}}{\psi_\epsilon} \right\|_{L^\infty(D\times(\underline{\alpha},\overline{\alpha}))} \leq C.$$

Proof of Lemma 4.1. By the uniform boundedness of $\|h_\epsilon\|_{L^\infty(D)}$ in ϵ (Lemma 3.1), assertions (i) and (ii) follow from Proposition B.5(i). To show (iii), it suffices to show, given any sequence $\epsilon_j \to 0$, and $\alpha_j \to \alpha_0 \in [\underline{\alpha}, \overline{\alpha}]$, $\liminf_{j\to\infty} \frac{\partial}{\partial\alpha}\sigma_{\epsilon_j}(\alpha_j) > 0$. By Lemma 3.1, we may assume without loss of generality that for some $h_0 \in L^\infty(D)$, $h_{\epsilon_j} \rightharpoonup h_0$ weakly in $L^p(D)$ for all $p > 1$. Then, in the notation of Appendix B, Proposition B.5(ii) implies that

$$\frac{\partial}{\partial\alpha}\sigma_{\epsilon_j}(\alpha_j) = \frac{\partial}{\partial\alpha}\lambda_1(\alpha_j, h_{\epsilon_j}) \to \frac{\partial}{\partial\alpha}\lambda_1(\alpha_0, h_0).$$

Since h_0 is non-constant (Lemma 3.4), Proposition B.1 implies that the last expression is positive. This proves (iii).

For (iv), suppose that along a sequence $\epsilon_j \to 0$ and $\alpha_j \to \alpha_0 > 0$, either $\inf_D \psi_{\epsilon_j}(x, \alpha_j) \to 0$ or $\sup_D \psi_{\epsilon_j}(x, \alpha_j) \to \infty$. By the uniform boundedness of $\|h_\epsilon\|_{L^\infty(D)}$ (Lemma 3.1), we may assume without loss that h_{ϵ_j} converges weakly in $L^p(D)$ for all $p > 1$. Hence by Proposition B.5(ii), $\psi_{\epsilon_j} = \varphi_1(\,\cdot\,; \alpha_j, h_{\epsilon_j})$ converges to $\varphi_1(\,\cdot\,; \alpha_0, h_0)$ uniformly in D, and the latter is a strictly positive function in $C(\overline{D})$. This is a contradiction, and proves (iv). □

4.2. **A Transformed Problem.** By the fact that $u_\epsilon(x, \alpha)$ is the principal eigenfunction, with zero as the corresponding principal eigenvalue, of the problem

$$\begin{cases} -\alpha\Delta\phi - \epsilon^2\phi_{\alpha\alpha} + h_\epsilon(x)\phi = 0 & \text{in } \Omega = D \times (\underline{\alpha}, \overline{\alpha}), \\ \frac{\partial}{\partial n}\phi = 0 \quad \text{on } \partial D \times (\underline{\alpha}, \overline{\alpha}), & \text{and} \quad \phi_\alpha = 0 \quad \text{on } D \times \{\underline{\alpha}, \overline{\alpha}\}, \end{cases} \tag{4.3}$$

we have the following variational characterization

$$0 = \inf_{\substack{\phi \,\in\, H^1(\Omega) \\ \int_\Omega \phi^2 \,=\, 1}} J_\epsilon[\phi], \tag{4.4}$$

where

$$J_\epsilon[\phi] = \int_\Omega \left[\alpha|\nabla_x\phi|^2 + \epsilon^2|\phi_\alpha|^2 + h_\epsilon\phi^2\right]. \tag{4.5}$$

Define

$$s_\epsilon = (\overline{\alpha} - \underline{\alpha})/\epsilon^{2/3}, \tag{4.6}$$

and

$$w_\epsilon(x,s) := u_\epsilon(x,\underline{\alpha}+\epsilon^{2/3}s)/\psi_\epsilon(x,\underline{\alpha}+\epsilon^{2/3}s) \quad \text{for } x\in D,\ 0\le s\le s_\epsilon, \tag{4.7}$$

where ψ_ϵ is given by (4.1). Then w_ϵ satisfies

$$\begin{cases} -\alpha\nabla_x\cdot(\psi_\epsilon^2\nabla_x w_\epsilon) - \epsilon^{2/3}(\psi_\epsilon^2 w_{\epsilon,s})_s + \psi_\epsilon^2\left[\sigma_\epsilon(\alpha) - \epsilon^2\frac{\psi_{\epsilon,\alpha\alpha}}{\psi_\epsilon}\right]w_\epsilon = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } D\times(0,s_\epsilon), \\ \frac{\partial}{\partial n}w_\epsilon = 0 \qquad \text{on } \partial D\times(0,s_\epsilon), \\ w_{\epsilon,s} = -\epsilon^{2/3}\frac{\psi_{\epsilon,\alpha}}{\psi_\epsilon}w_\epsilon \quad \text{on } D\times\{0,s_\epsilon\}. \end{cases} \tag{4.8}$$

The corresponding variational characterization can be written as

$$-\sigma_{0,\epsilon} = -\sigma_\epsilon(\underline{\alpha}) = \inf_{\phi\in H^1(D\times(0,s_\epsilon))\setminus\{0\}} \frac{\tilde{J}_\epsilon[\phi]}{\int_D\int_0^{s_\epsilon}\psi_\epsilon^2\phi^2\,dsdx} \tag{4.9}$$

where

$$\begin{aligned}\tilde{J}_\epsilon[\phi] = &\int_D\int_0^{s_\epsilon}\psi_\epsilon^2\left[\alpha|\nabla_x\phi|^2 + \epsilon^{2/3}|\phi_s|^2 + \left(\sigma_\epsilon(\alpha)-\sigma_\epsilon(\underline{\alpha}) - \epsilon^2\frac{\psi_{\epsilon,\alpha\alpha}}{\psi_\epsilon}\right)\phi^2\right]dsdx \\ &+ \epsilon^{4/3}\int_D\left[\psi_\epsilon\psi_{\epsilon,\alpha}\phi^2\right]_{s=0}^{s_\epsilon}dx.\end{aligned}$$

Lemma 4.3. $0\le -\sigma_{0,\epsilon}\le O(\epsilon^{2/3})$.

Proof. First, $\sigma_\epsilon(\underline{\alpha})$ is the principal eigenvalue (with principal eigenfunction $\tilde{\phi}(x,\alpha) = \psi_\epsilon(x,\underline{\alpha})$) of

$$-\underline{\alpha}\Delta\tilde{\phi} - \epsilon^2\tilde{\phi}_{\alpha\alpha} + h_\epsilon(x)\tilde{\phi} = \sigma\tilde{\phi} \quad \text{in } \Omega = D\times(\underline{\alpha},\overline{\alpha})$$

subject to homogeneous Neumann boundary condition, with a variation characterization analogous to (4.4) and (4.5). Since the integrand in (4.5) is monotone increasing in $\underline{\alpha}\le\alpha\le\underline{\alpha}$, $\sigma_\epsilon(\underline{\alpha})$ is necessarily less than the principal eigenvalue of (4.3), which is zero. This proves $\sigma_{0,\epsilon}=\sigma_\epsilon(\underline{\alpha})\le 0$.

For the upper estimate, we use a test function $\phi(x,s)=\eta(s)$ for (4.9), where $\eta:[0,\infty)\to[0,1]$ satisfies

$$\eta(s)>0 \quad \text{for } 0\le s\le 1, \quad \text{and} \quad \eta(s)=0 \quad \text{for } s\ge 2.$$

Then upon using $\nabla_x\eta=0$, (4.2), and

$$\int_D\left[\psi_\epsilon\psi_{\epsilon,\alpha}\eta^2\right]_{s=0}^{s_\epsilon}dx = \frac{\eta^2(0)}{2}\frac{\partial}{\partial\alpha}\left[\int_D\psi_\epsilon^2\,dx\right]_{\alpha=\underline{\alpha}} = 0,$$

(since $\eta(s_\epsilon)=0$ for ϵ small, and by normalization we have $\int_D\psi_\epsilon^2(x,\alpha)\,dx\equiv 1$ for all α), we obtain from (4.9) that

$$-\sigma_{0,\epsilon}\le\frac{\int_D\int_0^2\psi_\epsilon^2\left[\epsilon^{2/3}|\eta_s|^2 + (\sigma_{1,\epsilon}\epsilon^{2/3}s + \sigma_{2,\epsilon}\epsilon^{4/3}s^2 + O(\epsilon^2))\eta^2\right]dsdx}{\int_D\int_0^2\psi_\epsilon^2\eta^2 dsdx}.$$

The conclusion follows from Lemma 4.1(iv). $\square$

5. Uniform limit of $\hat{u}_\epsilon$.

In this section, we show that $\hat{u}_\epsilon$ converges to $\theta_{\underline{\alpha}}$ in $C(\overline{D})$. In particular, $h_\epsilon \to \theta_{\underline{\alpha}} - m$ in $C(\overline{D})$.

Recall that w_ϵ is defined in (4.7).

Lemma 5.1. *For all $\beta > 0$, there exists $C > 0$ independent of ϵ, such that*

$$w_\epsilon(x,s) \leq C\epsilon^{-1} e^{-\beta s} \qquad \text{for all } x \in D \text{ and } 0 \leq s \leq s_\epsilon,$$

where $s_\epsilon = (\overline{\alpha} - \underline{\alpha})/\epsilon^{2/3}$.

Proof. First, we derive a rough upper bound of w_ϵ from Lemma 3.3.

Claim 5.2. *There exists $C > 0$ such that*

$$\sup_{D \times (\underline{\alpha}, \overline{\alpha})} w_\epsilon \leq C\epsilon^{-1}.$$

By definition, $\sup w_\epsilon \leq (\sup u_\epsilon)/(\inf \psi_\epsilon)$, and the claim follows from the upper bound of u_ϵ (Lemma 3.3) and Lemma 4.1(iv).

Next, we construct a supersolution to prove the exponential decay.

Claim 5.3. *For each $\beta > 0$, there exists $s_0 > 0$ independent of ϵ such that*

$$\frac{\sigma_\epsilon(\alpha) - \sigma_\epsilon(\underline{\alpha})}{\epsilon^{2/3}} + \frac{\sigma_\epsilon(\underline{\alpha})}{\epsilon^{2/3}} - \epsilon^{4/3}\frac{\psi_{\epsilon,\alpha\alpha}(x,\alpha)}{\psi_\epsilon(x,\alpha)} \geq 2\beta^2 \qquad \text{for all } \alpha \in [\underline{\alpha} + \epsilon^{2/3}s_0, \overline{\alpha}].$$

To see the claim, we note that since σ_ϵ is monotone increasing in α (Proposition B.1(iii)), for $\alpha = \underline{\alpha} + \epsilon^{2/3}s$ and $s \geq s_0$,

$$\frac{\sigma_\epsilon(\alpha) - \sigma_\epsilon(\underline{\alpha})}{\epsilon^{2/3}} + \frac{\sigma_\epsilon(\underline{\alpha})}{\epsilon^{2/3}} \geq \frac{\sigma_\epsilon(\underline{\alpha} + \epsilon^{2/3}s_0) - \sigma_\epsilon(\underline{\alpha})}{\epsilon^{2/3}} + \frac{\sigma_{0,\epsilon}}{\epsilon^{2/3}}.$$

By (4.2) and Lemma 4.1(i),

$$\liminf_{\epsilon\to 0} \frac{\sigma_\epsilon(\underline{\alpha} + \epsilon^{2/3}s_0) - \sigma_\epsilon(\underline{\alpha})}{\epsilon^{2/3}} \geq (\liminf_{\epsilon \to 0} \sigma_{1,\epsilon})s_0.$$

Taking also Lemma 4.3 and Corollary 4.2 into account, we conclude that for $\alpha = \underline{\alpha} + \epsilon^{2/3}s$ and $s \geq s_0$,

$$\liminf_{\epsilon\to 0}\left[\frac{\sigma_\epsilon(\alpha) - \sigma_\epsilon(\underline{\alpha})}{\epsilon^{2/3}} + \frac{\sigma_\epsilon(\underline{\alpha})}{\epsilon^{2/3}} - \epsilon^{4/3}\frac{\psi_{\epsilon,\alpha\alpha}(x,\alpha)}{\psi_\epsilon(x,\alpha)}\right] \geq (\liminf_{\epsilon\to 0}\sigma_{1,\epsilon})s_0 - C.$$

Since $\liminf_{\epsilon\to 0}\sigma_{1,\epsilon} > 0$ by Lemma 4.1(iii), Claim 5.3 holds by choosing s_0 large.

Claim 5.4. *For each $\beta > 0$, there exists $s_0 > 0$ independent of ϵ and a supersolution*

$$\overline{W}(x,s) := \left(\sup_{\substack{x \in D \\ 0 \leq s \leq s_0}} w_\epsilon\right)[\exp(-\beta(s - s_0)) + \exp(\beta(s - (3/2)s_\epsilon))],$$

defined on $D \times (s_0, s_\epsilon)$ such that

$$\begin{cases} -\frac{\alpha}{\epsilon^{2/3}\psi_\epsilon^2}\nabla_x\cdot(\psi_\epsilon^2\nabla_x\overline{W}) - \frac{1}{\psi_\epsilon^2}[\psi_\epsilon^2\overline{W}_s]_s & \\ \quad + \left(\frac{\sigma_\epsilon(\alpha)-\sigma_\epsilon(\underline{\alpha})}{\epsilon^{2/3}} + \frac{\sigma_\epsilon(\underline{\alpha})}{\epsilon^{2/3}} - \epsilon^{4/3}\frac{\psi_{\epsilon,\alpha\alpha}}{\psi_\epsilon}\right)\overline{W} \geq 0 & \text{in } D \times (s_0, s_\epsilon), \\ \frac{\partial \overline{W}}{\partial n} = 0 & \text{on } \partial D \times (s_0, s_\epsilon), \\ \overline{W}(x, s_0) \geq w_\epsilon(x, s_0) & \text{for } x \in D, \\ \overline{W}_s(x, s_\epsilon) \geq -\epsilon^{2/3}\frac{\psi_{\epsilon,\alpha}(x,\overline{\alpha})}{\psi_\epsilon(x,\overline{\alpha})}\overline{W}(x,s_\epsilon) & \text{for } x \in D, \end{cases} \tag{5.1}$$

where $s_\epsilon = (\overline{\alpha} - \underline{\alpha})/\epsilon^{2/3}$.

To show the differential inequality, note that the term involving derivatives in x vanishes, and that by Claim 5.3 and Corollary 4.2,

$$\begin{aligned}
&-\frac{1}{\psi_\epsilon^2}\left(\psi_\epsilon^2 \overline{W}_s\right)_s + \left(\frac{\sigma_\epsilon(\alpha) - \sigma_\epsilon(\underline{\alpha})}{\epsilon^{2/3}} + \frac{\sigma_\epsilon(\underline{\alpha})}{\epsilon^{2/3}} - \epsilon^{4/3}\frac{\psi_{\epsilon,\alpha\alpha}}{\psi_\epsilon}\right)\overline{W} \\
&\geq -\overline{W}_{ss} + o(1)\overline{W}_s + 2\beta^2\overline{W} \\
&= (-\beta^2 + o(1)\beta + 2\beta^2)\overline{W} \geq 0.
\end{aligned}$$

It remains to check the boundary condition on $D \times \{s_\epsilon\}$, as the rest follows by definition. Note that $s_\epsilon \to \infty$ as $\epsilon \to \infty$, so that $\exp(-\beta(s_\epsilon - s_0)) \ll \exp(-\beta s_\epsilon/2)$. We therefore have

$$\frac{\overline{W}_s(x, s_\epsilon)}{\overline{W}(x, s_\epsilon)} + \epsilon^{2/3}\frac{\psi_{\epsilon,\alpha}(x, \overline{\alpha})}{\psi_\epsilon(x, \overline{\alpha})} = \frac{-\beta\exp(-\beta(s_\epsilon - s_0)) + \beta\exp(-\beta s_\epsilon/2)}{\exp(-\beta(s_\epsilon - s_0)) + \exp(-\beta s_\epsilon/2)} + O(\epsilon^{2/3})$$

which converges to a positive constant β uniformly for $x \in D$. This proves the claim.

Now, we claim that

$$w_\epsilon \leq \overline{W} \quad \text{in } D \times (0, s_\epsilon). \tag{5.2}$$

By definition, it is easy to see that $w_\epsilon \leq \overline{W}$ in $D \times [0, s_0]$. That the inequality holds in $D \times (s_0, s_\epsilon)$ is due to the fact that $\overline{W}$ is a strict positive supersolution of the linear problem (5.1) with homogeneous Dirichlet boundary condition on $D \times \{s_0\}$, and Neumann condition on the remaining boundary portions. Standard maximum principle applies and shows that the quotient $\frac{\overline{W} - w_\epsilon}{\overline{W}}$ is non-negative. (See, e.g. [3, p. 48].)

Finally, we obtain Lemma 5.1 by combining Claim 5.2 and (5.2). □

Lemma 5.5. *Let* v_ϵ *be given by* (3.2), *then*

$$\sup_{x \in D} |v_\epsilon(x) - \underline{\alpha}\hat{u}_\epsilon(x)| \to 0.$$

Proof. Given ϵ, take $\delta = \sqrt{\epsilon}\delta_1$, where δ_1 is given by Lemma 3.1.

$$\begin{aligned}
|v_\epsilon(x) - \underline{\alpha}\hat{u}_\epsilon(x)| &= \left|\int_{\underline{\alpha}}^{\overline{\alpha}} (\alpha - \underline{\alpha}) u_\epsilon(x, \alpha)\, d\alpha\right| \\
&\leq \delta\left|\int_{\underline{\alpha}}^{\underline{\alpha}+\delta} u_\epsilon(x, \alpha)\, d\alpha\right| + (\overline{\alpha} - \underline{\alpha})\left|\int_{\underline{\alpha}+\delta}^{\overline{\alpha}} u_\epsilon(x, \alpha)\, d\alpha\right| \\
&\leq \delta\hat{u}_\epsilon(x) + C\int_{\underline{\alpha}+\delta}^{\overline{\alpha}} \epsilon^{-1}\exp(-\beta(\alpha - \bar{\alpha})/\epsilon^{2/3})\, d\alpha \\
&\leq \sqrt{\epsilon} + o(1)
\end{aligned}$$

where we have used Lemma 4.1(iv) and Lemma 5.1 in the second last inequality. This proves the lemma. □

Proposition 5.6. $\hat{u}_\epsilon \to \theta_{\underline{\alpha}}$ *in* $C(\overline{D})$, *where* $\theta_{\underline{\alpha}}$ *is the unique positive solution of* (2.3). *In particular,* $h_\epsilon \to \theta_{\underline{\alpha}} - m$ *in* $C(\overline{D})$.

Proof. By Remark 3.2 and Lemma 5.5, we deduce that up to a subsequence $\epsilon_j \to 0$, both $\hat{u}_{\epsilon_j}$ and $v_\epsilon/\underline{\alpha}$ converges uniformly in D to some $\hat{u}_0 \in W^{2,p}(D)$. We claim that $\hat{u}_0$ is a (strong and therefore classical) solution of (2.3), i.e. for each $z(x) \in C^\infty(\bar{D})$,

$$\underline{\alpha}\int_D [\hat{u}_0 \Delta_x z + \hat{u}_0(m - \hat{u}_0)z]\, dx - \underline{\alpha}\int_{\partial D} \hat{u}_0 \frac{\partial z}{\partial n}\, dx = 0. \tag{5.3}$$

To show (5.3), multiply (3.4) by a test function $z(x)$ and integrate by parts, using the Neumann boundary condition of v_ϵ, we obtain

$$\int_D [v_\epsilon \Delta_x z + \hat{u}_{\epsilon_j}(m - \hat{u}_{\epsilon_j})z]\, dx - \int_{\partial D} v_\epsilon \frac{\partial z}{\partial n}\, dx = 0.$$

Then one can pass to the limit to obtain (5.3) by invoking Lemma 5.5. By the lower estimate in Lemma 3.1, there exists $\delta_1 > 0$ such that $\hat{u}_0(x) \geq \delta_1$ for all $x \in D$. Hence $\hat{u}_0$ is the unique positive solution of (2.3), i.e. $\hat{u}_{\epsilon_j} \rightharpoonup \theta_{\underline{\alpha}}$ in $C(\overline{D})$. Since the limit is independent of subsequences, we deduce that that $\hat{u}_\epsilon \to \theta_{\underline{\alpha}}$ as $\epsilon \to 0$ (not just along subsequences $\epsilon_j \to 0$). This proves the proposition. □

Corollary 5.7. *Let $\sigma_\epsilon(\alpha)$ and $\psi_\epsilon(x,\alpha)$ be the principal eigenvalue and eigenfunction of (4.1), and let $\sigma^*(\alpha)$ and $\psi^*(x,\alpha)$ be those of (2.4). Then as $\epsilon \to 0$, $\sigma_\epsilon \to \sigma^*$ in $C^k([\underline{\alpha},\overline{\alpha}])$ for all k, and $\psi_\epsilon(\cdot,\alpha) \to \psi^*(\cdot,\alpha)$ in $C^k([\underline{\alpha},\overline{\alpha}]; W^{2,p}(D))$. In particular,*

$$\sigma_{0,\epsilon} \to \sigma_0^* := \sigma^*(\underline{\alpha}) \quad \text{and} \quad \sigma_{1,\epsilon} \to \sigma_1^* := \frac{\partial \sigma^*}{\partial \alpha}(\underline{\alpha}) > 0. \tag{5.4}$$

Proof. Since now $h_\epsilon \to h_0 = \theta_{\underline{\alpha}} - m$ in $L^\infty(D)$, the corollary follows from Proposition B.1(ii). Since $h_0 = \theta_{\underline{\alpha}} - m$ is non-constant (Lemma 3.4), Proposition B.1(iv) asserts that $\sigma_1^* > 0$. □

6. Convergence of w_ϵ

Let w_ϵ be given by (4.7), define the normalized version $\tilde{w}_\epsilon = \tilde{w}_\epsilon(x,s)$ on $D\times[0,s_\epsilon]$ by

$$\tilde{w}_\epsilon(x,s) := \left(\frac{\int_D \theta_{\underline{\alpha}}^2\, dx}{\int_\Omega \psi_\epsilon^2 w_\epsilon^2\, dsdx} \right)^{1/2} w_\epsilon(x,s)$$

so that

$$\int_D \int_0^{s_\epsilon} \psi_\epsilon^2(x, \underline{\alpha} + \epsilon^{2/3}s)\tilde{w}_\epsilon^2(x,s)\, dsdx = \int_D \theta_{\underline{\alpha}}^2\, dx > 0. \tag{6.1}$$

Proposition 6.1. (i) *$\|\tilde{w}_\epsilon\|_{H^1(D\times(0,s_\epsilon))}$ is bounded uniformly in ϵ.*

(ii) *For any $\beta > 0$, there exists $C_1 > 0$ such that for all ϵ sufficiently small,*

$$\tilde{w}_\epsilon(x,s) \leq C_1 e^{-\beta s}$$

for all $0 \leq s \leq s_\epsilon$.

(iii) *As $\epsilon \to 0$, $\tilde{w}_\epsilon(x,s)$ converges locally uniformly in $\bar{D}\times[0,+\infty)$ to the unique positive solution of the problem*

$$\begin{cases} \tilde{\eta}_{ss} + (\tilde{a}_0 - \sigma_1^* s)\tilde{\eta} = 0 & \text{for } s \geq 0, \\ \tilde{\eta}_s(0) = 0 = \tilde{\eta}(+\infty), & \int_0^\infty \tilde{\eta}^2\, ds = 1, \end{cases} \tag{6.2}$$

where $\tilde{a}_0 = (\sigma_1^*)^{2/3} A_0$, with σ_1^* given by (5.4) and A_0 being the absolute value of the first negative root of the derivative of the Airy function.

Since $\tilde{\eta}(+\infty) = 0$ and $\tilde{w}_\epsilon(x,s) \to 0$ as $s \to +\infty$ uniformly in $x \in D$, we have in fact proved the following.

Corollary 6.2. $\|\tilde{w}_\epsilon(x,s) - \tilde{\eta}(s)\|_{L^\infty(D\times(0,s_\epsilon))} \to 0$ *as* $\epsilon \to 0$.

Proof. By Lemma 4.3 and the fact that $\tilde{w}_\epsilon$ is a minimizer of (4.9), we obtain

$$\int_D \int_0^{s_\epsilon} \psi_\epsilon^2 \left[\alpha\epsilon^{-2/3}|\nabla_x \tilde{w}_\epsilon|^2 + |(\tilde{w}_\epsilon)_s|^2 + \left(\frac{\sigma_\epsilon(\alpha) - \sigma_\epsilon(\underline{\alpha})}{\epsilon^{2/3}} + O(\epsilon^{4/3})\right)\tilde{w}_\epsilon^2\right] dsdx$$
$$+ \epsilon^{2/3} \int_D \left[\psi_{\epsilon,\alpha}\psi_\epsilon \tilde{w}_\epsilon^2\right]_{s=0}^{s_\epsilon} dx \le C\left(\int_D \int_0^{s_\epsilon} \psi_\epsilon^2 \tilde{w}_\epsilon^2 \, dsdx\right) = C.$$

Claim 6.3. $\left|\int_D \left[\psi_{\epsilon,\alpha}\psi_\epsilon \tilde{w}_\epsilon^2\right]_{s=0}^{s_\epsilon} dx\right| \le C \int_D \int_0^{s_\epsilon} \psi_\epsilon^2 \left(|\nabla_x \tilde{w}_\epsilon|^2 + |\tilde{w}_{\epsilon,\alpha}|^2 + \tilde{w}_\epsilon^2\right) dsdx.$

To prove Claim 6.3, we apply the Trace Theorem, so that there is $C > 0$ independent of ϵ such that

$$\begin{aligned}\left|\int_D \left[\psi_{\epsilon,\alpha}\psi_\epsilon \tilde{w}_\epsilon^2\right]_{s=0}^{s_\epsilon} dx\right| &\le C \int_D \int_0^{s_\epsilon} \left(|\nabla_x \tilde{w}_\epsilon|^2 + |\tilde{w}_{\epsilon,\alpha}|^2 + \tilde{w}_\epsilon^2\right) dsdx \\ &\le C \int_D \int_0^{s_\epsilon} \psi_\epsilon^2 \left(|\nabla_x \tilde{w}_\epsilon|^2 + |\tilde{w}_{\epsilon,\alpha}|^2 + \tilde{w}_\epsilon^2\right) dsdx,\end{aligned}$$

where we have used $\|\psi_{\epsilon,\alpha}\psi_\epsilon\|_{L^\infty(D\times(\underline{\alpha},\overline{\alpha}))} \le C$ (Corollary 5.7) for the first inequality, and Lemma 4.1(iv) for the second inequality.

From Claim 6.3, the normalization (6.1), the estimate in the beginning of the proof, and the monotonicity of $\sigma_\epsilon(\alpha)$ in α (Proposition B.1(iv)), we have

$$(1 - C\epsilon^{2/3}) \int_D \int_0^{s_\epsilon} \psi_\epsilon^2 \left[\underline{\alpha}\epsilon^{-2/3}|\nabla_x \tilde{w}_\epsilon|^2 + |(\tilde{w}_\epsilon)_s|^2 + \tilde{w}_\epsilon^2\right] \le C \int_D \int_0^{s_\epsilon} \psi_\epsilon^2 \tilde{w}_\epsilon^2 \, dsdx = C.$$

By Lemma 4.1(iv), we deduce

$$\int_D \int_0^{s_\epsilon} \left[\epsilon^{-2/3}|\nabla_x \tilde{w}_\epsilon|^2 + |(\tilde{w}_\epsilon)_s|^2 + \tilde{w}_\epsilon^2\right] \le C, \tag{6.3}$$

which implies our assertion (i). Passing to a subsequence, $\tilde{w}_\epsilon(x,s)$ converges weakly in $H^1_{loc}(D \times [0,\infty))$ to some function $\tilde{\eta}$. of x. Moreover, as $\int_D \int_0^{s_\epsilon} |\nabla_x \tilde{w}_\epsilon|^2 \, dsdx \le C\epsilon^{2/3}$, it follows that $\nabla_x \tilde{\eta} = 0$ a.e..

We outline the rest of the proof of Proposition 6.1. First we will show (iii) except the normalization condition

$$\int_0^\infty \tilde{\eta}^2 \, ds = 1. \tag{6.4}$$

Second, we will show the estimate (ii). Finally we will use (ii) to derive (6.4) from (6.1), which completes the proof of (iii).

We claim that $\tilde{\eta}$ must satisfy the equation in (6.2). To see this claim, note that the equation for $\tilde{w}_\epsilon$ is

$$\begin{aligned} 0 = \quad & -\tfrac{\alpha}{\epsilon^{2/3}}\nabla_x \cdot (\psi_\epsilon^2 \nabla_x \tilde{w}_\epsilon) - [\psi_\epsilon^2 (\tilde{w}_\epsilon)_s]_s \\ & + \left(\tfrac{\sigma_\epsilon(\alpha) - \sigma_\epsilon(\underline{\alpha})}{\epsilon^{2/3}} + \tfrac{\sigma_\epsilon(\underline{\alpha})}{\epsilon^{2/3}} - \epsilon^{4/3}\tfrac{\psi_{\epsilon,\alpha\alpha}}{\psi_\epsilon}\right)\psi_\epsilon^2 \tilde{w}_\epsilon \end{aligned} \tag{6.5}$$

We argue via the weak formulation.

Claim 6.4. *There exists a constant* $\bar{a}_0$ *such that for each test function* $z(s)$ *that is compactly supported in* $[0,\infty)$,

$$\int_0^\infty [-z_s\tilde{\eta}_s + (\bar{a}_0 - \sigma_1^* s)z\tilde{\eta}]\, ds = 0.$$

In particular, $\tilde{\eta}$ *satisfies the equation* $\tilde{\eta}_{ss} + (\bar{a}_0 - \sigma_1^* s)\tilde{\eta} = 0$ *on* $(0,\infty)$ *and* $\tilde{\eta}_s(0) = 0$.

Multiplying (6.5) by a test function $z = z(s)$, and integrating over $x \in D$, we see that the term involving derivatives in x vanishes (by the Neumann boundary condition $\frac{\partial \tilde{w}_\epsilon}{\partial n} = 0$), and obtain

(6.6)
$$0 = -z\int_D [\psi_\epsilon^2(\tilde{w}_\epsilon)_s]_s\, dx + z\int_D \left[\frac{\sigma_\epsilon(\alpha) - \sigma_\epsilon(\underline{\alpha})}{\epsilon^{2/3}} + \frac{\sigma_\epsilon(\underline{\alpha})}{\epsilon^{2/3}} - \epsilon^{4/3}\frac{\psi_{\epsilon,\alpha\alpha}}{\psi_\epsilon}\right]\psi_\epsilon^2\tilde{w}_\epsilon\, dx.$$

Next, integrate the first term of (6.6) over $s \in [0, s_\epsilon]$, we see that

$$\begin{aligned}
&-\int_0^{s_\epsilon} z\int_D [\psi_\epsilon^2(\tilde{w}_\epsilon)_s]_s\, dxds \\
&= \int_0^{s_\epsilon}\int_D z_s\psi_\epsilon^2(\tilde{w}_\epsilon)_s\, dxds - \left[z\int_D \psi_\epsilon^2(\tilde{w}_\epsilon)_s\, dx\right]_{s=0}^{s_\epsilon} \\
&= \int_0^{s_\epsilon}\int_D z_s\psi_\epsilon^2(\tilde{w}_\epsilon)_s\, dxds + \epsilon^{2/3}\left[z\int_D \psi_\epsilon\psi_{\epsilon,\alpha}\tilde{w}_\epsilon\, dx\right]_{s=0}^{s_\epsilon},
\end{aligned}$$

where we have used the boundary condition $(\tilde{w}_\epsilon)_s = -\epsilon^{2/3}\psi_{\epsilon,\alpha}\tilde{w}_\epsilon/\psi_\epsilon$ on $D\times\{0, s_\epsilon\}$. Since $z(s)$ has compact support in $[0,\infty)$, the boundary term evaluated at $s = s_\epsilon$ vanishes, and the remaining boundary term is of order $O(\epsilon^{2/3})$ (since $\tilde{w}_\epsilon$ is bounded in $H^1(D\times(0,s_\epsilon))$ by assertion (i), and hence bounded in $L^2(D\times\{0\})$ by the Trace Theorem). Hence, we have

$$-\int_0^{s_\epsilon} z\int_D [\psi_\epsilon^2(\tilde{w}_\epsilon)_s]_s\, dxds = \int_0^{s_\epsilon}\int_D z_s\psi_\epsilon^2(\tilde{w}_\epsilon)_s\, dxds + o(1). \tag{6.7}$$

Also, in the support of $z(s)$, $(\psi_\epsilon)^2(x, \underline{\alpha} + \epsilon^{2/3}s) \to (\psi^*)^2(x,\underline{\alpha})$ uniformly, so we may use (6.7) to integrate (6.6) over $s \in [0, s_\epsilon]$ and pass to the limit to get

$$0 = \left(\int_D (\psi^*)^2(x,\underline{\alpha})\, dx\right)\left[\int_0^\infty z_s\tilde{\eta}_s\, ds + \int_0^\infty (\sigma_1^* s - \bar{a}_0)z\tilde{\eta}\, ds\right] \tag{6.8}$$

where we have used Corollary 5.7 and that $\bar{a}_0$ is a subsequential limit of $-\sigma_\epsilon(\underline{\alpha})/\epsilon^{2/3}$ (see also Lemma 4.3). This proves Claim 6.4. Next, we claim that

$$\int_0^\infty \tilde{\eta}^2\, ds < +\infty. \tag{6.9}$$

Notice that by normalization of $\tilde{w}_\epsilon$ (see (6.1)), and the uniform (in ϵ) positive upper/lower bound of ψ_ϵ (Lemma 4.1(iv)), there exists a fixed constant C_0 such that for each $M > 0$, and for all $\epsilon > 0$ sufficiently small, $\int_0^M\int_D \tilde{w}_\epsilon^2\, dxds \leq \int_0^{s_\epsilon}\int_D \tilde{w}_\epsilon^2\, dxds \leq C_0$. Letting $\epsilon \to 0$, $\int_0^M \tilde{\eta}^2\, ds \leq C_0$ for all $M > 0$. i.e. (6.9) holds.

Claim 6.5. $\tilde{\eta}$ *is a positive solution that satisfies* (6.2) *with condition* (6.4) *being replaced by* (6.9).

By Claim 6.4, $\tilde{\eta}$ satisfies

$$\tilde{\eta}_{ss} + (\bar{a}_0 - \sigma_1^* s)\tilde{\eta} = 0, \quad \tilde{\eta} \geq 0 \quad \text{on } [0, +\infty), \quad \text{and} \quad \tilde{\eta}_s(0) = 0. \tag{6.10}$$

It remains to show that $\tilde{\eta}(+\infty) = 0$, and that the subsequential limit $\bar{a}_0$ must be determined by $\tilde{a}_0$ of the proposition. By (6.10) and $\tilde{\eta} > 0$, $\tilde{\eta}_{ss} \geq 0$ for all s sufficiently large. Hence $\tilde{\eta}(+\infty)$ exists in $[0, +\infty]$. By (6.9), $\tilde{\eta}(+\infty) = 0$.

Hence, $\tilde{\eta}$ is a constant multiple of $\mathrm{Airy}((\sigma_1^*)^{1/3}s - A_0)$, where A_0 is the absolute value of the first negative root of the derivative of the Airy function $\mathrm{Airy}(x)$. In particular the subsequential limit $\bar{a}_0$ given in (6.8) is uniquely determined by $\tilde{a}_0 = (\sigma_1^*)^{2/3} A_0$ (i.e. the full limit $\lim_{\epsilon \to 0} \sigma_\epsilon(\underline{\alpha})/\epsilon^{2/3}$ exists). This shows Claim 6.5. To finish the proof of (iii) except for (6.4), it remains to establish the following.

Claim 6.6. *$\tilde{w}_\epsilon(x, s) \to \tilde{\eta}(s)$ locally uniformly in $\bar{D} \times [0, \infty)$. In particular, for each $M > 0$, $\|\tilde{w}_\epsilon\|_{L^\infty(D \times [0,M])}$ is bounded uniformly in ϵ.*

It is enough to show that for each $M > 0$,

$$\sup_{s \in [0,M]} \frac{\sup_{x \in D} \tilde{w}_\epsilon(x, s)}{\inf_{x \in D} \tilde{w}_\epsilon(x, s)} \to 1 \quad \text{as } \epsilon \searrow 0. \tag{6.11}$$

For, assuming (6.11), one can write

$$\tilde{w}_\epsilon(x, s) = \tilde{w}_\epsilon(x_0, s)(1 + \delta_\epsilon(x, s)) \quad \text{for some } x_0 \in D, \tag{6.12}$$

where $\delta_\epsilon(x, s) \to 0$ in $L^\infty_{loc}(\bar{D} \times [0, +\infty))$. Now, if we integrate (6.12) over $x \in D$, then

$$\tilde{W}_\epsilon(s) := \frac{1}{|D|} \int_D \tilde{w}_\epsilon(x, s)\, dx = \tilde{w}_\epsilon(x_0, s)(1 + \hat{\delta}_\epsilon(s)),$$

where $\hat{\delta}_\epsilon(s) \to 0$ in $L^\infty_{loc}([0, \infty))$. Since $\tilde{w}_\epsilon$ is bounded in $H^1(D \times (0, s_\epsilon))$, one can easily deduce that $\tilde{W}_\epsilon(s) \in H^1_{loc}((0, +\infty)) \subset C^{1/2}_{loc}([0, +\infty))$. Therefore, by Arzelá-Ascoli Theorem, $\tilde{W}_\epsilon(s)$ and hence $\tilde{w}_\epsilon(x_0, s)$ converges to $\tilde{\eta}(s)$ in $C_{loc}([0, \infty))$. Finally, (6.12) implies that $\tilde{w}_\epsilon(x, s) \to \tilde{\eta}(s)$ locally uniformly in $D \times [0, +\infty)$.

It remains to show (6.11) in a similar fashion as in Claim 3.6. Assume to the contrary that there exists some constant $c_0 > 1$, $\epsilon = \epsilon_k \to 0$, $s_k \to s_0 < +\infty$, such that

$$\sup_{x \in D} \tilde{w}_\epsilon(x, s_k) \geq c_0 \inf_{x \in D} \tilde{w}_\epsilon(x, s_k). \tag{6.13}$$

Similarly as before, we extend $\tilde{w}_\epsilon$ by reflection on $D \times \{0\}$ so that $\tilde{w}_\epsilon$ is defined on $D \times (-s_\epsilon, s_\epsilon)$, and define

$$W_k(x, \tau) := \frac{\tilde{w}_\epsilon(x, s_k + \epsilon^{1/3}\tau/\sqrt{\underline{\alpha}})}{\sup_{x' \in D} \tilde{w}_\epsilon(x', s_k)} \quad \text{in } D \times \left(-(s_\epsilon + s_k)\frac{\sqrt{\underline{\alpha}}}{\epsilon^{1/3}}, (s_\epsilon - s_k)\frac{\sqrt{\underline{\alpha}}}{\epsilon^{1/3}}\right).$$

Recall that s_ϵ is defined in (4.6). By the equation (6.5) satisfied by $\tilde{w}_\epsilon$, W_k satisfies

$$\left\{ \begin{aligned} &-\tfrac{\alpha}{\underline{\alpha}} \nabla_x \cdot (\psi_\epsilon^2 \nabla_x W_k) - (\psi_\epsilon^2 W_{k,\tau})_\tau \\ &\qquad + \tfrac{1}{\underline{\alpha}} \left(\sigma_\epsilon \left(\underline{\alpha} + \epsilon^{\frac{2}{3}} s_k + \epsilon \tfrac{\tau}{\sqrt{\underline{\alpha}}} \right) - \epsilon^2 \tfrac{\psi_{\epsilon,\alpha\alpha}}{\psi_\epsilon} \right) \psi_\epsilon^2 W_k = 0, \end{aligned} \right. \tag{6.14}$$

in $D \times \left(-(s_\epsilon + s_k)\frac{\sqrt{\underline{\alpha}}}{\epsilon^{1/3}}, (s_\epsilon - s_k)\frac{\sqrt{\underline{\alpha}}}{\epsilon^{1/3}}\right)$, where

$$\alpha = \alpha(\tau) = \underline{\alpha} + \left| \epsilon^{\frac{2}{3}} s_k + \epsilon \frac{\tau}{\sqrt{\underline{\alpha}}} \right|$$

and the boundary conditions

$$\begin{cases} \frac{\partial}{\partial n} W_k = 0 & \text{on} \quad \partial D \times \left(-(s_\epsilon + s_k)\frac{\sqrt{\underline{\alpha}}}{\epsilon^{1/3}}, (s_\epsilon - s_k)\frac{\sqrt{\underline{\alpha}}}{\epsilon^{1/3}} \right) \\ W_{k,\tau} = -\frac{\epsilon}{\sqrt{\underline{\alpha}}} \frac{\psi_{\epsilon,\alpha}}{\psi_\epsilon} W_k & \text{on } D \times \left\{ -(s_\epsilon + s_k)\frac{\sqrt{\underline{\alpha}}}{\epsilon^{1/3}}, (s_\epsilon - s_k)\frac{\sqrt{\underline{\alpha}}}{\epsilon^{1/3}} \right\}. \end{cases}$$

Since $s_\epsilon \to +\infty$ as $\epsilon \to 0$ and that s_k remains bounded, we see in particular that the domain of W_k tends to $D \times \mathbb{R}$ as $k \to \infty$.

Claim 6.7. *For each $M > 0$, $\sigma_\epsilon \left(\underline{\alpha} + s_k \epsilon^{\frac{2}{3}} + \epsilon \frac{\tau}{\sqrt{\underline{\alpha}}} \right) \to 0$ as $\epsilon \to 0$, uniformly for $\tau \in [-M, M]$.*

By Lemma 4.1(i), σ_ϵ are bounded in $C^1([\underline{\alpha}, \overline{\alpha}])$ uniformly in ϵ. Hence we may write

$$\left| \sigma_\epsilon \left(\underline{\alpha} + s_k \epsilon^{\frac{2}{3}} + \epsilon \frac{\tau}{\sqrt{\underline{\alpha}}} \right) \right| \le |\sigma_\epsilon(\underline{\alpha})| + C \left| \epsilon^{2/3} s_k + \epsilon \frac{\tau}{\sqrt{\underline{\alpha}}} \right|$$

and conclude that $\sigma_\epsilon \left(\underline{\alpha} + s_k \epsilon^{\frac{2}{3}} + \epsilon \frac{\tau}{\sqrt{\underline{\alpha}}} \right)$ goes to zero by Lemma 4.3, and boundedness of s_k, τ. This proves Claim 6.7.

Since the coefficients of (6.14) are bounded in $L^\infty_{loc}(\overline{D} \times \mathbb{R})$ uniformly in k, Harnack inequality, and the normalization condition $\sup_{x \in D} W_k(x, 0) = 1$ implies that W_k are bounded in $L^\infty_{loc}(\overline{D} \times \mathbb{R})$ uniformly in k. Hence we may apply elliptic L^p estimates similarly as in Claim 3.6 to conclude that a subsequence of W_k converges in $L^\infty_{loc}(\overline{D} \times \mathbb{R})$ to a positive solution of $(\psi_0(x, \underline{\alpha}))^{-2} \nabla_x \cdot (\psi_0^2(x, \underline{\alpha}) \nabla_x W) + W_{\tau\tau} = 0$ in $D \times \mathbb{R}$. (Here we used Claim 6.7.) Now, we apply the following Liouville Theorem, whose proof is exactly analogous to Proposition C.1 and is skipped.

Proposition 6.8. *Let $\psi(x)$ be a smooth positive function defined in $\bar{D}$, then every positive solution W to $\psi^{-2} \nabla_x \cdot (\psi^2 \nabla_x W) + W_{tt} = 0$ in $D \times \mathbb{R}$, subject to Neumann boundary condition on $\partial D \times \mathbb{R}$, is necessarily a constant.*

So that by normalization $\sup_{x \in D} W_k(x, 0) = 1$, $W_k(x, 0) \to 1$ uniformly in D. This contradicts (6.13) and proves (6.11). This establishes Claim 6.6. Claims 6.6 and 6.5 establish assertion (iii) except for condition (6.4).

Next, we proceed to show the estimate in (ii). By the preceding argument in the proof of Lemma 5.1, specifically the construction of supersolution $\overline{W}$ in Claim 5.4, we can show that for all $\beta > 0$, there exists $s_0 > 0$ such that

$$\tilde{w}(x, s) \le \left(\sup_{\substack{x \in D \\ 0 \le s \le s_0}} \tilde{w}_\epsilon \right) [\exp(-\beta(s - s_0)) + \exp(\beta(s - (3/2)s_\epsilon))]$$

for $x \in D$ and $s \in [s_0, s_\epsilon)$. Then (ii) follows from Claim 6.6, as the expression inside paranthesis is bounded uniformly in ϵ. We do not repeat the details.

For (iii), it remains to show (6.4). By assertion (ii), and that

$$\psi_\epsilon(x, \underline{\alpha} + \epsilon^{2/3} s) \to \psi^*(x, \underline{\alpha}) \quad \text{and} \quad \tilde{w}_\epsilon(x, s) \to \tilde{\eta}(s) \tag{6.15}$$

in $L^\infty_{loc}(\bar{D} \times [0, \infty))$ (by Lemma 4.1(iv) and Claim 6.6 resp.), we may pass to the limit in (6.1) to obtain

$$\int_D \theta_{\underline{\alpha}}^2 \, dx = \int_D \int_0^{s_\epsilon} \psi_\epsilon^2(x, \underline{\alpha} + \epsilon^{2/3} s) \tilde{w}_\epsilon^2(x, s) \, ds dx \to \int_D (\psi^*)^2(x, \underline{\alpha}) \, dx \int_0^\infty \tilde{\eta}^2 \, ds$$

Upon noting that (see Definition 2.2(ii))

$$\psi^*(x,\underline{\alpha}) = \theta_{\underline{\alpha}}(x) \quad \text{in } D, \tag{6.16}$$

the proof is completed. □

7. Proof of Theorem 2.3

Proof of Theorem 2.3. First, we note that by Proposition 5.6,

$$\epsilon^{2/3}\int_D\int_0^{s_\epsilon}\psi_\epsilon w_\epsilon\,dsdx = \int_D\int_{\underline{\alpha}}^{\overline{\alpha}} u_\epsilon(x,\alpha)\,d\alpha dx = \int_D \hat{u}_\epsilon\,dx \to \int_D \theta_{\underline{\alpha}}\,dx \tag{7.1}$$

as $\epsilon\to 0$. Furthermore, by (6.15), (6.16) and the estimate of Proposition 6.1(ii),

$$\int_D\int_0^{s_\epsilon}\psi_\epsilon\tilde{w}_\epsilon\,dsdx \to \int_D\psi^*(x,\underline{\alpha})\,dx\int_0^\infty\tilde{\eta}(s)\,ds = \int_D\theta_{\underline{\alpha}}(x)\,dx\int_0^\infty\tilde{\eta}(s)\,ds. \tag{7.2}$$

By the definition of w_ϵ and $\tilde{w}_\epsilon$, there is a function $\Gamma(\epsilon)$ such that

$$w_\epsilon(x,s) = \Gamma(\epsilon)\tilde{w}_\epsilon(x,s). \tag{7.3}$$

By (7.1) and (7.2), we have

$$\lim_{\epsilon\to 0}\epsilon^{2/3}\Gamma(\epsilon) = \left(\int_0^\infty\tilde{\eta}\,ds\right)^{-1}. \tag{7.4}$$

Hence, by (7.3) and Corollary 6.2,

$$\left\|\epsilon^{2/3}w_\epsilon(x,(\alpha-\underline{\alpha})/\epsilon^{2/3}) - \left(\int_0^\infty\tilde{\eta}\,ds\right)^{-1}\tilde{\eta}\left(\frac{\alpha-\underline{\alpha}}{\epsilon^{2/3}}\right)\right\|_{L^\infty(\Omega)} \to 0.$$

By the fact that $\left(\int_0^\infty\tilde{\eta}\,ds\right)^{-1}\tilde{\eta}(s) = \eta^*(s)$ where η^* is given in Definition 2.2(iii), we also have

$$\left\|\epsilon^{2/3}w_\epsilon(x,(\alpha-\underline{\alpha})/\epsilon^{2/3}) - \eta^*\left(\frac{\alpha-\underline{\alpha}}{\epsilon^{2/3}}\right)\right\|_{L^\infty(\Omega)} \to 0.$$

Using Lemma 4.1(iv), we have

$$\left\|\epsilon^{2/3}u_\epsilon(x,\alpha) - \psi_\epsilon(x,\alpha)\eta^*\left(\frac{\alpha-\underline{\alpha}}{\epsilon^{2/3}}\right)\right\|_{L^\infty(\Omega)} \to 0. \tag{7.5}$$

By the fact that $\eta^*(s)\le Ce^{-\beta s}$ for some $C,\beta>0$, (6.15) and (6.16), we have

$$\left\|(\psi_\epsilon(x,\alpha)-\theta_{\underline{\alpha}}(x))\eta^*\left(\frac{\alpha-\underline{\alpha}}{\epsilon^{2/3}}\right)\right\|_{L^\infty(\Omega)} \to 0. \tag{7.6}$$

And (2.7) follows from (7.5) and (7.6). □

8. Acknowledgement

We are grateful to Professor Benoit Perthame for stimulating discussions and for bring reference [22] to our attention, and also Professor Fan-Hua Lin for a comment that simplified the proof of Proposition C.1. KYL thanks the hospitality of National Center for Theoretical Sciences, National Tsing-Hua University, Taiwan, and the Institute for Mathematical Sciences, Renmin University of China, where parts of this work are completed.

Appendix A. Existence Results

In this section we show the existence of positive solution to (2.1). For this purpose, we fix positive parameters ϵ and $\overline{\alpha} > \underline{\alpha}$, and denote (in this section only) the principal eigenvalue and eigenfunction of the following problem by μ_1 and ϕ_1.

$$
\begin{cases} \alpha\Delta_x\phi + \epsilon^2\phi_{\alpha\alpha} + m(x)\phi + \mu\phi = 0 & \text{in } \Omega := D \times (\underline{\alpha}, \overline{\alpha}), \\ \frac{\partial\phi}{\partial n} = 0 \quad \text{on } \partial D \times (\underline{\alpha}, \overline{\alpha}), \\ \phi_\alpha = 0 \quad \text{on } D \times \{\underline{\alpha}, \overline{\alpha}\}. \end{cases} \tag{A.1}
$$

Theorem A.1. *If $\mu_1 \geq 0$, then the equation (2.1) has no positive steady-state. If $\mu_1 < 0$, then the equation (2.1) has at least one positive steady-state.*

Proof. First, we prove the non-existence result. Suppose $\mu_1 \geq 0$ and let u be a non-negative solution of (2.1). Multiply (2.1) by the principal eigenfunction ϕ_1 of (A.1), and integrate by parts, we obtain

$$
\mu_1 \int_D \int_{\underline{\alpha}}^{\overline{\alpha}} \phi_1^2 \, d\alpha dx + \int_D \int_{\underline{\alpha}}^{\overline{\alpha}} u\hat{u}\phi_1 \, d\alpha dx = 0.
$$

Since $\mu_1 \geq 0$, both terms are non-negative, and both must be identically zero. i.e. $u \equiv 0$.

For the existence result, we consider, for $\tau \in [0, 1]$, the positive solutions of

$$
\begin{cases} \alpha\Delta u + \epsilon^2(u)_{\alpha\alpha} + (m(x) - \tau\hat{u} - (1-\tau)u)u = 0 & \text{in } D \times (\underline{\alpha}, \bar{\alpha}), \\ \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial D \times (\underline{\alpha}, \overline{\alpha}), \quad (u)_\alpha = 0 \quad \text{in } D \times \{\underline{\alpha}, \overline{\alpha}\}. \end{cases} \tag{A.2}
$$

Here we recall that $\hat{u} = \int_{\underline{\alpha}}^{\overline{\alpha}} u(x, \alpha) \, d\alpha$. It remains to show the following claim, from which existence of positive solution to (2.1) follows by a standard topological degree argument, as the existence of a unique, linearly stable positive solution to (A.2) when $\tau = 0$ is standard.

Claim A.2. *For some $\delta > 0$ independent of $\tau \in [0, 1]$, any positive solution u of (A.2) satisfies*

$$
\delta < \|u\|_{L^1(\Omega)} < 1/\delta.
$$

For the upper bound, one can integrate (A.2) over Ω to get

$$
\begin{aligned}
\int_D \hat{u} m \, dx &= \tau \int_D \hat{u}^2 \, dx + (1-\tau) \int_D \int_{\underline{\alpha}}^{\overline{\alpha}} u^2 \, d\alpha dx \\
&\geq \left(\tau + \frac{1-\tau}{\overline{\alpha} - \underline{\alpha}} \right) \int_D \hat{u}^2 \, dx \\
&\geq c_0 \int_D \hat{u}^2 \, dx \\
&\geq \frac{c_0}{|D|} \left(\int_D \hat{u} \, dx \right)^2 = \frac{c_0}{|D|} \|u\|_{L^1(\Omega)}^2,
\end{aligned}
$$

from which the upper bound follows.

For the lower bound, let $u = v\phi_1$, where $\phi_1 > 0$ is the principal eigenfunction corresponding to the principal eigenvalue $\mu_1 < 0$ of (A.1). Moreover, if we normalize ϕ_1 by $\int_\Omega \phi_1^2 = 1$, then $\sup_\Omega \phi_1$ and $\inf_\Omega \phi_1$ are fixed positive constants independent of τ, as (A.1) is independent of τ. The equation for v can be written as

$$
\begin{cases} \alpha\nabla_x \cdot (\phi_1^2 \nabla_x v) + \epsilon^2(\phi_1^2 v_\alpha)_\alpha + \phi_1^2 v(-\mu_1 - \tau\hat{u}_\epsilon - (1-\tau)u) = 0 & \text{in } \Omega, \\ \frac{\partial v}{\partial n} = 0 \quad \text{on } \partial D \times (\underline{\alpha}, \overline{\alpha}), \quad v_\alpha = 0 \quad \text{on } D \times \{\underline{\alpha}, \overline{\alpha}\}. \end{cases}
$$

Hence, if we divide by v and integrate by parts, we have

$$\int_\Omega \phi_1^2(\mu_1 + \tau\hat{u} + (1-\tau)u)\,d\alpha dx = \int_\Omega \phi_1^2 \frac{\alpha|\nabla_x v|^2 + \epsilon^2|v_\alpha|^2}{v^2}\,d\alpha dx > 0.$$

Hence we have

$$\left(\sup_\Omega \phi_1\right)^2 [\tau(\overline{\alpha} - \underline{\alpha}) + (1-\tau)]\|u\|_{L^1(\Omega)} > -\mu_1 \int_\Omega \phi_1^2\,d\alpha dx = -\mu_1 > 0.$$

Since μ_1 and $\sup_\Omega \phi_1$ are independent of τ, we have

$$\|u\|_{L^1(\Omega)} \geq \frac{-\mu_1}{(\sup_\Omega \phi_1)^2[\tau(\overline{\alpha} - \underline{\alpha}) + (1-\tau)]}.$$

Since the latter term is bounded from below uniformly in $\tau \in [0,1]$, the claim is proved. □

Corollary A.3. *If $\int_D m(x)\,dx > 0$, then for any $\epsilon > 0$,* (2.1) *has at least one positive solution.*

Proof. Divide the equation (A.1) by the principal eigenfunction ϕ_1 and integrate by parts over Ω, we get

$$\int_D \int_{\underline{\alpha}}^{\overline{\alpha}} \frac{\alpha|\nabla_x \phi_1|^2 + \epsilon^2|\phi_{1,\alpha}|^2}{\phi_1^2}\,d\alpha dx + \int_D \int_{\underline{\alpha}}^{\overline{\alpha}} (m(x) + \mu_1)\,d\alpha dx = 0.$$

Hence for all $\epsilon > 0$,

$$\mu_1 \leq -\frac{1}{|D|}\int_D m(x)\,dx < 0,$$

and the existence of positive solution of (2.1) follows from Theorem A.1. □

Appendix B. Eigenvalue problems with diffusion parameter α and weight function $h(x)$

For each $\alpha > 0$ and $h \in L^\infty(D)$, let $\lambda_1 = \lambda_1(\alpha, h) \in \mathbb{R}$ and $\varphi(x) = \varphi_1(x;\alpha,h)$ be the normalized principal eigenvalue and principal eigenfunction of the following problem.

$$\text{(B.1)} \qquad \begin{cases} -\alpha\Delta_x\varphi + h\varphi = \lambda\varphi & \text{in } D, \\ \frac{\partial\varphi}{\partial n} = 0 \quad \text{on } \partial D, & \int_D \varphi^2\,dx = 1. \end{cases}$$

We shall state and prove a number of properties of λ_1 and φ_1, and its dependence on the parameters α and h, some of which is folklore among specialists.

Proposition B.1.
(i) *For each $\alpha > 0$ and $h \in L^\infty(D)$, the problem* (B.1) *has a principal eigenvalue λ_1 which is simple, and the corresponding eigenfunction φ_1 can be chosen positive and uniquely determined by the constraint $\int_D \varphi_1^2\,dx = 1$.*
(ii) *For each $p > 1$, the mapping $(\alpha, h) \mapsto (\lambda_1, \varphi_1(\cdot))$ is smooth from $\mathbb{R}_+ \times L^\infty(D)$ to $\mathbb{R} \times W^{2,p}_{\mathcal{N}}(D)$, where $W^{2,p}_{\mathcal{N}}(D) = \{\phi \in W^{2,p}(D) : \frac{\partial\phi}{\partial n} = 0$ on $\partial D\}$.*
(iii) *If $h \in L^\infty(D)$ is non-constant, then $\frac{\partial\lambda_1}{\partial\alpha}(\alpha, h) > 0$ for all $\alpha > 0$.*

Proof. Part (i) is well-known. See, e.g. [14, Section 8.12]. In particular, the principal eigenvalue is given by the variational characterization

$$\text{(B.2)} \qquad \lambda_1(\alpha, h) = \inf_{\varphi \in C^1(\overline{D})\setminus\{0\}} \frac{\int_D (\alpha|\nabla_x\varphi|^2 + h\varphi^2)\,dx}{\int_D \varphi^2\,dx}.$$

Fix $p > N$ (N being the dimension of D). Consider the following mapping $F : W^{2,p}_{\mathcal{N}}(D) \times \mathbb{R} \times \mathbb{R}_+ \times L^\infty(D) \to L^p(D) \times \mathbb{R}$, given by

$$F(\varphi, \lambda, \alpha, h) = (\alpha \Delta_x \varphi - h\varphi + \lambda\varphi, \int_D \varphi^2 \, dx - 1),$$

Then for each $\alpha > 0$ and $h \in L^\infty(D)$, the principal eigenpair $(\varphi_1(\cdot;\alpha, h), \lambda_1(\alpha, h))$ of (B.1) satisfies

$$F(\varphi_1(\,\cdot\,;\alpha, h), \lambda_1(\alpha, h), \alpha, h) = (0, 0). \tag{B.3}$$

Assertion (ii) follows from the following claim, in view of the Implicit Function Theorem and the smooth dependence of the operator F on α and h.

Claim B.2. *For each fixed $\alpha > 0$ and $h \in L^\infty(D)$,*

$$D_{(\varphi,\lambda)} F(\varphi_1(\,\cdot\,;\alpha, h), \lambda_1(\alpha, h), \alpha, h) \;\;:\;\; W^{2,p}_{\mathcal{N}} \times \mathbb{R} \;\to\; L^p(D) \times \mathbb{R}$$

is a bijection.

We shall follow the arguments in the proof of [10, Lemma 2.1]. Suppose for some $(\Phi, t) \in W^{2,p}_{\mathcal{N}} \times \mathbb{R}$, $D_{(\varphi,\lambda)} F(\varphi_1(\,\cdot\,;\alpha, h), \lambda_1(\alpha, h), \alpha, h)[\Phi, t] = (0, 0)$, i.e.

$$\alpha \Delta_x \Phi - h\Phi + \lambda_1 \Phi + t\varphi_1 = 0 \quad \text{in } D, \quad \text{and} \quad \frac{\partial \Phi}{\partial n} = 0 \quad \text{on } \partial D, \tag{B.4}$$

and

$$2 \int_D \Phi \varphi_1 \, dx = 0, \tag{B.5}$$

where $\lambda_1 = \lambda(\alpha, h)$ and $\varphi_1 = \varphi(x;\alpha, h)$. The Fredholm alternative implies that $\int_D t\varphi_1^2 \, dx = 0$, i.e. $t = 0$. Hence $\Phi = c\varphi_1$ for some constant c (as $\lambda_1 = \lambda_1(\alpha, h)$ is a simple eigenvalue). But then (B.5) implies that $c = 0$. This shows that the kernel of $D_{(\varphi,\lambda)} F(\varphi_1(\,\cdot\,;\alpha, h), \lambda_1(\alpha, h), \alpha, h)$ is trivial. Now let $(f, q) \in L^p(D) \times \mathbb{R}$ be given, we need to solve for (Φ, t) in

$$\alpha \Delta_x \Phi - h\Phi + \lambda_1 \Phi + t\varphi_1 = f \quad \text{in } D, \quad \text{and} \quad \frac{\partial \Phi}{\partial n} = 0 \quad \text{on } \partial D, \tag{B.6}$$

and

$$2 \int_D \Phi \varphi_1 \, dx = q. \tag{B.7}$$

Set $t = (\int_D f\varphi_1 \, dx)/(\int_D \varphi_1^2 \, dx)$, then $\int_D (f - t\varphi_1)\varphi_1 \, dx = 0$, so (B.6) has solution of the form $\Phi = s\varphi_1 + \Phi^\perp$ where $\Phi^\perp \in W^{2,p}_{\mathcal{N}}(D)$ is unique and satisfies $\int_D \Phi^\perp \varphi_1 \, dx = 0$. Finally, if we set $s = q/(2\int_D \varphi_1^2 \, dx)$ then (Φ, t) solves (B.6) and (B.7). This proves Claim B.2, which implies assertion (ii).

For (iii), we differentiate (B.1) with respect to α,

$$\begin{cases} -\alpha \Delta_x \frac{\partial \varphi_1}{\partial \alpha} + h \frac{\partial \varphi_1}{\partial \alpha} - \lambda_1 \frac{\partial \varphi_1}{\partial \alpha} = \Delta_x \varphi_1 + \frac{\partial \lambda_1}{\partial \alpha} \varphi_1 & \text{in } D, \\ \frac{\partial}{\partial n} \frac{\partial \varphi_1}{\partial \alpha} = 0 \quad \text{on } \partial D, \quad \text{and} \quad \int_D \frac{\partial \varphi_1}{\partial \alpha} \varphi_1 \, dx = 0. \end{cases} \tag{B.8}$$

Multiply (B.8) by φ_1 and integrate by parts, we have $\frac{\partial \lambda_1}{\partial \alpha} \int_D \varphi_1^2 \, dx = \int_D |\nabla_x \varphi_1|^2 \, dx$. Since $h(x)$ is non-constant in x, $\varphi_1 = \varphi_1(\,\cdot\,;\alpha, h)$ is non-constant in x and this implies that $\frac{\partial \lambda_1}{\partial \alpha} > 0$. This proves (iii). $\square$

First, we show that λ_1 and φ_1 are continuous with respect to the weak topology of $\mathbb{R}_+ \times \cap_{p>1} L^p(D)$.

Lemma B.3. *Let $\lambda_1(\alpha,h)$ and $\varphi_1(\,\cdot\,;\alpha,h)$ be the principal eigenpair of* (B.1)*.*

(i) *For each $p>1$, there exists $C_0'=C_0'(p,M,\underline{\alpha},\overline{\alpha},\partial D)$ such that*
$$|\lambda_1(\alpha,h)|+\|\varphi_1(\,\cdot\,;\alpha,h)\|_{W^{2,p}(D)}\le C_0'$$
provided $\alpha\in[\underline{\alpha},\overline{\alpha}]$ and $\|h\|_{L^\infty(D)}\le M$.

(ii) *If $\alpha_j\to\alpha_0\in[\underline{\alpha},\overline{\alpha}]$, $\sup_{j\ge 0}\|h_j\|_{L^\infty(D)}<+\infty$ and $h_j\rightharpoonup h_0$ in $L^p(D)$ for all $p>1$, then as $j\to\infty$, $\lambda_1(\alpha_j,h_j)\to\lambda_1(\alpha_0,h_0)$ and $\varphi_1(\,\cdot\,;\alpha_j,h_j)\rightharpoonup\varphi_1(\,\cdot\,;\alpha_0,h_0)$ weakly in $W^{2,p}(D)$ for all $p>1$.*

Proof. By (B.2), $\lambda_1:=\lambda_1(\alpha,h)$ forms a bounded sequence in $[-\|h\|_{L^\infty(D)},\|h\|_{L^\infty(D)}]$. The L^p estimate (for $p>N$) applied to (B.1) and interpolation inequality together imply
$$\|\varphi_1\|_{W^{2,p}(D)}\le C\|\varphi_1\|_{L^p(D)}\le\frac{1}{2}\|\varphi_1\|_{W^{2,p}(D)}+C\|\varphi_1\|_{L^2(D)},$$
where C is a generic constant, depending on $\|h\|_{L^\infty(D)},\underline{\alpha},\overline{\alpha}$ and the domain D that changes from line to line. This proves (i).

For (ii), let $\alpha_j\to\alpha_0\in[\underline{\alpha},\overline{\alpha}]$ and h_j be a uniformly bounded sequence in $L^\infty(D)$ and $h_j\rightharpoonup h_0$ weakly in $L^p(D)$. Denote $\lambda_{1,j}=\lambda_1(\alpha_j,h_j)$ and $\varphi_{1,j}=\varphi_1(\,\cdot\,;\alpha_j,h_j)$. By assertion (i), there are subsequences $\lambda_{1,j'}$ and $\varphi_{1,j'}$ such that $\lambda_1(\alpha_j,h_j)\to\tilde\lambda$ and $\varphi_1(\,\cdot\,;\alpha_j,h_j)\rightharpoonup\tilde\varphi$ weakly in $W^{2,p}(D)$, for some $\tilde\lambda\in\mathbb{R}$ and $\tilde\varphi\in W^{2,p}(D)$. Take $\alpha=\alpha_{j'}$, $h=h_{j'}$ in (B.1), and pass to the weak limit $j'\to\infty$, we deduce
$$\begin{cases}-\alpha_0\Delta_x\tilde\varphi+h_0\tilde\varphi=\tilde\lambda\tilde\varphi & \text{in } D,\\ \frac{\partial\tilde\varphi}{\partial n}=0\quad\text{on }\partial D, & \int_D\tilde\varphi^2\,dx=1.\end{cases}$$
Hence $(\tilde\varphi,\tilde\lambda)$ is an eigenpair of (B.1) when $\alpha=\alpha_0$, $h=h_0$ and such that $\tilde\varphi\ge 0$. Moreover, $\tilde\varphi$ is non-trivial, as $\int_D\tilde\varphi^2\,dx=1$. By uniqueness of principal eigenpair, it follows that $\tilde\lambda=\lambda_1(\alpha_0,h_0)$ and $\tilde\varphi=\varphi_1(\,\cdot\,;\alpha_0,h_0)$. Since the limit is independent of subsequence, we deduce that the full sequence $\lambda_1(\alpha_j,h_j)\to\lambda(\alpha_0,h_0)$ and $\varphi_1(\,\cdot\,;\alpha_j,h_j)\rightharpoonup\varphi_1(\,\cdot\,;\alpha_0,h_0)$ weakly in $W^{2,p}(D)$. This proves the assertion (ii). $\square$

Next, we show the following uniform estimate of $(D_{(\varphi,\lambda)}F)^{-1}$.

Lemma B.4. *There exists $C_2=C_2(M,\underline{\alpha},\overline{\alpha},D)$ such that for any $\alpha\in[\underline{\alpha},\overline{\alpha}]$ and $\|h\|_{L^\infty(D)}\le M$, if*
$$D_{(\varphi,\lambda)}F(\varphi_1(\,\cdot\,;\alpha,h),\lambda_1(\alpha,h),\alpha,h)[\Phi,t]=(f,q),$$
i.e. (B.6) *and* (B.7) *hold with $\lambda_1=\lambda(\alpha,h)$ and $\varphi_1=\varphi_1(x;\alpha,h)$, then*
$$|t|+\|\Phi\|_{W^{2,p}(D)}\le C_2(|q|+\|f\|_{L^p(D)}).\tag{B.9}$$

Proof. Let $M>0$ be given. Suppose to the contrary that there are $\alpha_j\in[\underline{\alpha},\overline{\alpha}]$, h_j, Φ_j, t_j, q_j, f_j such that
$$\sup_j\|h_j\|_{L^\infty(D)}\le M,\quad |t_j|+\|\Phi_j\|_{W^{2,p}(D)}\to\infty,\quad |q_j|+\|f_j\|_{L^p(D)}\le 1.\tag{B.10}$$
Without loss of generality, we may assume $\alpha_j\to\alpha_0\in[\underline{\alpha},\overline{\alpha}]$ and for some $h_0\in L^\infty(D)$, $h_j\rightharpoonup h_0$ weakly in $L^p(D)$. Denote
$$\lambda_{1,j}=\lambda_1(\alpha_j,h_j),\quad\text{and}\quad\varphi_{1,j}=\varphi_1(\cdot;\alpha_j,h_j)\quad\text{for } j\in\mathbb{N}\cup\{0\}.$$
The above arguments ensure that
$$\Phi_j=\Phi_j^\perp+q_j/(2\int_D\varphi_{1,j}^2\,dx)\varphi_{1,j},\quad\text{and}\quad t_j=\int_D f_j\varphi_{1,j}\,dx/\int_D\varphi_{1,j}^2\,dx$$

where $\Phi_j^\perp$ is the unique solution of (B.6) subject to the constraint $\int_D \Phi_j^\perp \varphi_{1,j}\,dx = 0$. By the normalization $\int_D \varphi_{1,j}^2\,dx = 1$, we have

$$\Phi_j = \Phi_j^\perp + \frac{q_j}{2}\varphi_{1,j} \quad \text{and} \quad t_j = \int_D f_j\varphi_{1,j}\,dx. \tag{B.11}$$

Since we have shown that $|\lambda_{1,j}|$ and $\|\varphi_{1,j}\|_{W^{2,p}(D)}$ remain bounded uniformly in j, (B.10) and (B.11) imply that $\|\Phi_j^\perp\|_{W^{2,p}(D)} \to \infty$. Apply L^p estimate to the equation of $\Phi_j^\perp$, which is

$$\begin{cases} \alpha_j\Delta_x\Phi_j^\perp - h_j\Phi_j^\perp + \lambda_{1,j}\Phi_j^\perp = f_j - (\int_D f_j\varphi_{1,j}\,dx)\varphi_{1,j} & \text{in } D,\\ \frac{\partial\Phi_j^\perp}{\partial n} = 0 \quad \text{on } \partial D, \quad \text{and} \quad \int_D \Phi_j^\perp\varphi_{1,j}\,dx = 0. \end{cases} \tag{B.12}$$

Using the boundedness of $\varphi_{1,j}$ in $W^{2,p}(D)$ and hence in $L^\infty(D)$, we have

$$\begin{aligned} \|\Phi_j^\perp\|_{W^{2,p}(D)} &\leq C\left[\|\Phi_j^\perp\|_{L^p(D)} + \|f_j - \left(\int_D f_j\varphi_{1,j}\,dx\right)\varphi_{1,j}\|_{L^p(D)}\right]\\ &\leq C(\|\Phi_j^\perp\|_{L^\infty(D)} + \|f_j\|_{L^p(D)}). \end{aligned}$$

Hence we must have $\|\Phi_j^\perp\|_{L^\infty(D)} \to \infty$ as $j\to\infty$. Define $\tilde\Phi_j := \Phi_j^\perp/\|\Phi_j^\perp\|_{L^\infty(D)}$, then $\tilde\Phi_j$ satisfies

$$\begin{cases} \alpha_j\Delta_x\tilde\Phi_j - h_j\tilde\Phi_j + \lambda_{1,j}\tilde\Phi_j = \tilde f_j & \text{in } D,\\ \frac{\partial\tilde\Phi_j}{\partial n} = 0 \quad \text{on } \partial D, \quad \int_D \tilde\Phi_j\varphi_{1,j}\,dx = 0, \quad \text{and} \quad \sup_D\tilde\Phi_j = 1, \end{cases}$$

where $\tilde f_j = [f_j - (\int_D f_j\varphi_{1,j}\,dx)\varphi_{1,j}]/\|\Phi_j^\perp\|_{L^\infty(D)}$ converges to zero in $L^p(D)$ as $j\to\infty$. By L^p estimates, $\tilde\Phi_j$ is bounded uniformly in $W^{2,p}(D)$. Hence, there is a subsequence $\tilde\Phi_{j'}$ that converges, weakly in $W^{2,p}(D)$ and strongly in $C^1(\overline{D})$, to some function $\tilde\Phi_0$. By normalization $\sup_D\tilde\Phi_0 = \lim_{j'}\left(\sup_D\tilde\Phi_j\right) = 1$. Moreover, $\tilde\Phi_0$ satisfies (using Lemma B.3(ii))

$$\begin{cases} \alpha_0\Delta_x\tilde\Phi_0 - h_0\tilde\Phi_0 + \lambda_1(\alpha_0,h_0)\tilde\Phi_0 = 0 & \text{in } D,\\ \frac{\partial\tilde\Phi_0}{\partial n} = 0 \quad \text{on } \partial D, \quad \text{and} \quad \int_D\tilde\Phi_0\varphi_{1,0}\,dx = 0. \end{cases}$$

Since $\tilde\Phi_0$ is non-negative, Proposition B.1(i) implies $\tilde\Phi_0 = c\varphi_1(\,\cdot\,;\alpha_0,h_0) = c\varphi_{1,0}$ but the integral constraint implies that $c = 0$. i.e. $\tilde\Phi_0 = 0$. This is a contradiction to $\sup_D\tilde\Phi_0 = 1$. This proves (B.9). $\square$

Proposition B.5. *Let $\lambda_1(\alpha,h)$ and $\varphi_1(\,\cdot\,;\alpha,h)$ be the principal eigenpair of* (B.1).

(i) *For each k, there exists $C_k' = C_k'(M,\underline{\alpha},\overline{\alpha},D)$ such that*

$$\sum_{j=0}^{k}\left|\frac{\partial^j}{\partial\alpha^j}\lambda_1(\alpha,h)\right| + \left\|\frac{\partial^j}{\partial\alpha^j}\varphi_1(\,\cdot\,;\alpha,h)\right\|_{W^{2,p}(D)} \leq C_k' \tag{B.13}$$

provided $\alpha\in[\underline{\alpha},\overline{\alpha}]$ and $\|h\|_{L^\infty(D)}\leq M$.

(ii) *If $\sup_{j\geq 0}\|h_j\|_{L^\infty(D)} < +\infty$ and $h_j \rightharpoonup h_0$ in $L^p(D)$ for all $p>1$, then for each $k\geq 0$,*

$$\frac{\partial^k}{\partial\alpha^k}\lambda_1(\cdot,h_j) \to \frac{\partial^k}{\partial\alpha^k}\lambda_1(\cdot,h_0) \quad \text{in } C_{loc}([0,\infty)) \quad \text{as } j\to\infty.$$

Moreover, given $k \geq 0$, $p > 1$, and sequence $\alpha_j \to \alpha_0 > 0$,

$$\frac{\partial^k}{\partial \alpha^k}\varphi_1(\,\cdot\,;\alpha_j,h_j) \rightharpoonup \frac{\partial^k}{\partial \alpha^k}\varphi_1(\,\cdot\,;\alpha_0,h_0) \quad \text{weakly in } W^{2,p}(D) \text{ as } j \to \infty.$$

Proof. Assertions (i) and (ii) for the case $k = 0$ are exactly Lemma B.3. We first prove assertion (i) for $k = 1$, differentiate the relation (B.3) with respect to α,

$$D_{(\varphi,\lambda)}F\left[\frac{\partial}{\partial \alpha}\varphi_1(\,\cdot\,;\alpha,h), \frac{\partial}{\partial \alpha}\lambda_1(\alpha,h)\right] = -D_\alpha F, \tag{B.14}$$

where the partial derivatives of F are evaluated at $(\varphi_1(\,\cdot\,;\alpha,h), \lambda_1(\alpha,h), \alpha, h)$. By (B.2), we may write

$$(\varphi_1', \lambda_1') = (D_{(\varphi,\lambda)}F)^{-1}[-D_\alpha F] = (D_{(\varphi,\lambda)}F)^{-1}(-\Delta_x\varphi_1, 0)$$

and deduce by Lemma B.4 that

$$|\lambda_1'| + \|\varphi_1'\|_{W^{2,p}(D)} \leq C_2\|\Delta_x\varphi_1\|_{L^p(D)} \leq C_2\|\varphi_1\|_{W^{2,p}(D)} \leq C.$$

i.e. assertion (i) holds for $k = 1$. We argue inductively for $k > 1$. Suppose (i) holds for $k = K - 1$. We can write

$$D_{(\varphi,\lambda)}F\left[\frac{\partial^K}{\partial \alpha^K}\varphi_1(\,\cdot\,;\alpha,h), \frac{\partial^K}{\partial \alpha^K}\lambda_1(\alpha,h)\right] = \mathfrak{F}_K(\alpha,h) \tag{B.15}$$

where

$$\mathfrak{F}_K(\alpha,h) := \left(-K\frac{\partial^{K-1}}{\partial \alpha^{K-1}}(-\Delta_x\varphi_1) - \sum_{k=1}^{K-1}\begin{pmatrix} K \\ k \end{pmatrix}\frac{\partial^k}{\partial \alpha^k}\lambda_1\frac{\partial^{K-k}}{\partial \alpha^{K-k}}\varphi_1\ ,\ 0\right) \tag{B.16}$$

By the form of $\mathfrak{F}_K$, we can deduce the following result.

Claim B.6. $\|\mathfrak{F}_K(\alpha,h)\|_{L^\infty(D)} \leq C\sum_{k=0}^{K-1}\left(\left|\frac{\partial^k}{\partial \alpha^k}\lambda_1\right| + \left\|\frac{\partial^k}{\partial \alpha^k}\varphi_1\right\|_{W^{2,p}(D)}\right)$.

By the induction assumption (i.e. (i) holds for $k = K-1$) we have $\|\mathfrak{F}_K\|_{L^p(D)} \leq C(M, \underline{\alpha}, \overline{\alpha}, D)$. Hence we may apply Claim B.4 to (B.15) to conclude the assertion (i) for the case K. This induction argument proves (i).

By Lemma B.3(ii), it remains to prove assertion (ii) for case $k \geq 1$. Let $\alpha_j \to \alpha_0 \in [\underline{\alpha}, \overline{\alpha}]$ and h_j be a uniformly bounded sequence in $L^\infty(D)$ and $h_j \rightharpoonup h_0$ weakly in $L^p(D)$. Denote $\lambda_{1,j} = \lambda_1(\alpha_j,h_j)$ and $\varphi_{1,j} = \varphi_1(\,\cdot\,;\alpha_j,h_j)$. By assertion (i), there are subsequences $\lambda_{1,j'}$ and $\varphi_{1,j'}$ such that for all $k \geq 0$, $\frac{\partial^k}{\partial \alpha^k}\lambda_1(\alpha_{j'},h_{j'}) \to \tilde{\lambda}_k$ and $\frac{\partial^k}{\partial \alpha^k}\varphi_1(\,\cdot\,;\alpha_{j'},h_{j'}) \rightharpoonup \tilde{\varphi}_k$ weakly in $W^{2,p}(D)$, for some $\tilde{\lambda}_k \in \mathbb{R}$ and $\tilde{\varphi}_k \in W^{2,p}(D)$. Passing to the limit in (B.14), we deduce that

$$D_{(\varphi,\lambda)}F\left[\tilde{\varphi}_1, \tilde{\lambda}_1\right] = -D_\alpha F, \tag{B.17}$$

where the partial derivatives of F are evaluated at $(\varphi_1(\,\cdot\,;\alpha_0,h_0), \lambda_1(\alpha_0,h_0), \alpha_0, h_0)$. Since we also have

$$D_{(\varphi,\lambda)}F\left[\frac{\partial}{\partial \alpha}\varphi_1(\,\cdot\,;\alpha_0,h_0), \frac{\partial}{\partial \alpha}\lambda_1(\alpha_0,h_0)\right] = -D_\alpha F, \tag{B.18}$$

where the partial derivatives of F are evaluated at $(\varphi_1(\,\cdot\,;\alpha_0,h_0), \lambda_1(\alpha_0,h_0), \alpha_0, h_0)$, we may invert $D_{(\varphi,\lambda)}F$ in both (B.17) and (B.18), and conclude that

$$\tilde{\varphi}_1 = \frac{\partial}{\partial \alpha}\varphi_1(\,\cdot\,;\alpha_0,h_0) \quad \text{and} \quad \tilde{\lambda}_1 = \frac{\partial}{\partial \alpha}\lambda_1(\alpha_0,h_0).$$

Since the limit is determined independent of the subsequence, we conclude assertion (ii) for the case $k = 1$.

Again, we may argue inductively for $k > 1$. Suppose (ii) is proved for $k = 1, ..., K - 1$. The following can be easily observed from (B.16).

Claim B.7. *If assertion (ii) holds for $k = 1, ..., K - 1$, then*

$$\mathfrak{F}_K(\alpha_j, h_j) \rightharpoonup \mathfrak{F}_K(\alpha_0, h_0)$$

weakly in $L^p(D)$, where $\mathfrak{F}_K$ is defined in (B.16).

Based on Claim B.7, and the assertion (ii) for the cases $k = 1, ..., K-1$, we may pass to the limit in (B.15). Together with the uniform boundedness of $[D_{(\varphi,\lambda)}F]^{-1} : L^p(D) \to W^{2,p}(D)$ (Lemma B.4), this implies $\frac{\partial^K}{\partial \alpha^K}\lambda_1(\alpha_j, h_j) \to \frac{\partial^K}{\partial \alpha^K}\lambda_1(\alpha_0, h_0)$ and

$$\frac{\partial^K}{\partial \alpha^K}\varphi_1(\cdot; \alpha_j, h_j) \rightharpoonup \varphi_1(\cdot; \alpha_0, h_0) \text{ in } W^{2,p}(D).$$

Thus assertion (ii) follows by induction on k. □

Appendix C. Liouville Theorem for Positive Harmonic Functions in Cylinder Domain

We give a proof of the Liouville-type theorem for positive harmonic functions in cylinder domains, since we cannot locate a proper reference for this result.

Proposition C.1. *Let $k \in \mathbb{N}$, D be a bounded smooth domain in $\mathbb{R}^N$ and u be a non-negative harmonic function on $\Omega := D \times \mathbb{R}^k \subset \mathbb{R}^{N+k}$, so that $\frac{\partial u}{\partial n} = 0$ on $\partial D \times \mathbb{R}^k$. Then u is necessarily a constant.*

Proof. Let $x \in D$, $y \in \mathbb{R}^k$ and let $u(x, y)$ be a non-negative harmonic function on $\Omega = D \times \mathbb{R}^k$, subject to Neumann boundary condition on $\partial D \times \mathbb{R}^k$. By subtracting a positive constant from u, we may assume that $\inf_\Omega u = 0$.

Harnack inequality says that there is a constant $C > 1$ such that for all $y' \in \mathbb{R}^k$, we have

$$\sup_{x\in D, |y-y'|<2} u \le C \inf_{x\in D, |y-y'|<2} u.$$

Define $v(y) = \frac{1}{|D|}\int_D u(x', y)\, dy$, then v is a harmonic function on $\mathbb{R}^k$ and must be equal to a non-negative constant v_0. Hence for each $y' \in \mathbb{R}^k$, there exists $x' \in \bar{D}$ such that $u(x', y') = v_0$. It follows that for each $y' \in \mathbb{R}^k$,

$$v_0 \le C \inf_{x\in D, |y-y'|<2} u(x, y).$$

Taking infimum in $y' \in \mathbb{R}^k$, it follows that from $\inf_\Omega u = 0$ that $v_0 = 0$. Hence,

$$\frac{1}{|D|}\int_D u(x, y)\, dx = v(y) = v_0 = 0$$

for all $y \in \mathbb{R}^k$. i.e. $u \equiv 0$ in Ω. □

References

1. M. Alfaro, J. Coville and G. Raoul, *Travelling waves in a nonlocal reaction-diffusion equation as a model for a population structured by a space variable and a phenotypical trait*, Comm. Partial Differential Equations, 38 (2013), pp. 2126-2154.
2. A. Arnold, L. Desvillettes and C. Prévost, *Existence of nontrivial steady states for populations structured with respect to space and a continuous trait*, Comm. Pure Appl. Anal., 11 (2012), pp. 83-96.
3. H. Berestycki, L. Nirenberg and S.R.S. Varadhan, *The principal eigenvalue and maximum principle for second-order elliptic operators in general domains*, Comm. Pure Appl. Math., 47 (1994), pp. 47-92.
4. N. Bessonov, N. Reinberg and V. Volpert, *Mathematics of Darwin's Diagram,* Math. Model. Nat. Phenom., 9 (2014), pp. 5-25.
5. E. Bouin, V. Calvez, *Travelling waves for the cane toads equation with bounded traits,* Nonlinearity, 27 (2014), pp. 2233-2253.
6. E. Bouin, V. Calvez, N. Meunier, S. Mirrahimi, B. Perthame, G. Raoul, and R. Voituriez, *Invasion fronts with variable motility: phenotype selection, spatial sorting and wave acceleration*, C. R. Math. Acad. Sci. Paris, 350 (2012), pp. 761-766.
7. E. Bouin and S. Mirrahimi, *A Hamilton-Jacobi limit for a model of population structured by space and trait*, Comm. Math. Sci., 13 (2015), pp. 1431-1452.
8. R. Bürger, *Perturbations of positive semigroups and applications to population genetics*, Math. Z., 197 (1988), pp. 259-272.
9. À. Calsina and S. Cuadrado, *Small mutation rate and evolutionarily stable strategies in infinite dimensional adaptive dynamics*, J. Math. Biol., 48 (2004), pp. 135-159.
10. R.S. Cantrell and C. Cosner, *On the steady-state problem for the Volterra-Lotka competition model with diffusion,* Houston J. Math., 13 (1987), pp. 337-352.
11. O. Diekmann, *A beginner's guide to adaptive dynamics,* in Mathematical modeling of population dynamics, Rudnicki, R. (Ed.), Banach Center Publications 63 (2004), pp. 47-86.
12. O. Diekmann, P.-E. Jabin, S. Mischler, and B. Perthame, *The dynamics of adaptation: an illuminating example and a Hamilton-Jacobi approach*, Theo. Pop. Biol., 67 (2005), pp. 257-271.
13. J. Dockery, V. Hutson, K. Mischaikow and M. Pernarowski, *The evolution of slow dispersal rates: A reaction-diffusion model*, J. Math. Biol., 37 (1998), pp. 61-83.
14. D. Gilbarg and N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Grundlehren der Mathematischen Wissenschaften 224, Springer-Verlag, Berlin, 1983.
15. A. Hastings, *Can spatial variation alone lead to selection for dispersal?,* Theo. Pop. Biol., 24 (1983), pp. 244-251.
16. P.-E. Jabin and G. Raoul, *Selection dynamics with competition,* J. Math. Biol., 63 (2011), pp. 493-517.
17. A. Lorz, S. Mirrahimi, and B. Perthame, *Dirac mass dynamics in multidimensional nonlocal parabolic equations*, Comm. Partial Differential Equations, 36 (2011), pp. 1071-1098.
18. Y. Lou and W.-M. Ni, *Diffusion, self-diffusion and cross-diffusion*, J. Differential Equations, 131 (1996), pp. 79-131.
19. P. Magal and G.F. Webb, *Mutation, selection, and recombination in a model of phenotypic evolution*, Discre. Contin. Dyn. Syst., 6 (2000), pp. 221-236.
20. S. Mirrahimi and B. Perthame, *Asymptotic analysis of a selection model with space*, J. Math. Pures Appl., 104 (2015), pp. 1108-1118.
21. B. Perthame and G. Barles, *Dirac concentrations in Lotka-Volterra parabolic PDEs*, Indiana Univ. Math. J., 57 (2008), pp. 3274-3301.
22. B. Perthame and P.E. Souganidis, *Rare mutations limit of a steady state dispersion trait model*, arXiv:1505.03420v1, 2015.
23. B.L. Phillips, G.P. Brown, J.K. Webb, and R. Shine, *Invasion and the evolution of speed in toads*, Nature, 439 (2006), pp. 803-803.
24. O. Turanova, *On a model of a population with variable motility*, Math. Models Methods Appl. Sci., 25 (2015), pp 1961-2014.

Department of Mathematics, Columbus, OH, United States.
E-mail address: `lam.184@math.ohio-state.edu`

Institute for Mathematical Sciences, Renmin University of China, and Department of Mathematics, Ohio State University
E-mail address: `lou@math.ohio-state.edu`