GLOBAL DYNAMICS OF REACTION-DIFFUSION SYSTEMS WITH A TIME-VARYING DOMAIN*

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Abstract. This paper is devoted to the study of the global dynamics for a large class of reaction-diffusion systems with a time-varying domain. By appealing to the theories of asymptotically autononmous and periodic semiflows, we establish the threshold type results on the long-time behavior of solutions for such a system in the cases of asymptotically bounded and periodic domains, respectively. To investigate the model system in the case of asymptotically unbounded domain, we first prove the global attractivity for nonautonomous reaction-diffusion systems with asymptotically vanishing diffusion coefficients via the method of sub- and super-solutions, and then use the comparison arguments to obtain the threshold dynamics. We also apply these analytical results to a reaction-diffusion model of Dengue fever transmission to investigate the effect of time-varying domain on the basic reproduction number. It turns out that the basic reproduction numbers with dengue fever transmission for the asymptotically bounded and unbounded domains are always less than that for the spatially homogeneous case, and under appropriate conditions, the basic reproduction numbers for asymptotically bounded and periodic domains are larger than or equal to that for the stationary bounded domain.

Key words. Reaction-diffusion systems, time-varying domain, super- and subsolutions, global stability, basic reproduction number

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1. Introduction. Reaction-diffusion equations provide a powerful tool for understanding the complex dynamics of populations in heterogeneous environments. They can be used to model the movement and interactions of individuals, ranging from elementary particles, bacteria, molecules, and cells to animal and plant populations, as well as events like epidemics or rumors. However, most reaction-diffusion models are limited to studying populations in fixed regions that are assumed to be independent of time [2, 10, 28].

In reality, habitats of almost all living organisms change frequently over time in nature. Some habitats change periodically. For example, the depth and area of many rivers and lakes vary with seasons. In summer or the rainy season, the water area generally increases, and organisms living in it can survive and move in a larger area. But in winter or the dry season, the water level drops, and the water area where the organisms move will also become smaller. Some habitats are constantly expanding, such as for Aedes mosquitoes that can transmit dengue fever, yellow fever, Zika virus and other infectious diseases. Due to global warming and frequent human activities, habitats suitable for the survival and reproduction of Aedes mosquitoes are constantly increasing their reach. Data shows that in the United States, Aedes albopictus was first discovered on August 2, 1985 in Harris County Mosquito Control Division of

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Houston, Texas, and then quickly spread to the eastern United States [17]. By 1999, Aedes albopictus had been confirmed to appear in 25 states, and by the end of 2012, it had spread from southern Texas to New Jersey and Illinois, spanning 14 latitudes [26]. These observations motivate the incorporation of time-varying domains into reaction-diffusion models to accurately capture the population dynamics in changing environments.

In many cases, habitat changes are primarily influenced by external factors such as climate, temperature, and rainfall, rather than the organisms themselves. A prototypical spatial model of the single species growth is the following Logistic reaction-diffusion equation:

$$u_t = d\Delta u + u(a - bu), \quad (x,t) \in \Omega \times (0, +\infty) \tag{1.1}$$

subject to the Dirichlet boundary condition. Recent studies have modified the fixed region Ω into a region Ω_t evolving with time, with the evolution rate of the region parameterized by $\rho(t)$, and explored the dynamics of the population density $u(x,t)$ in different evolution modes of the region. For the model (1.1), the authors of [9] assumed that the region changes T-periodically, that is, $\rho(t+T) = \rho(t)$, and concluded that when the evolution rate $\rho(t)$ is small, the periodic change of domain has a negative impact on the persistence of the species, while when $\rho(t)$ is large, the periodic change of domain has a positive impact on the persistence of the species. In another research work [19], the authors assumed that the spatial domain for model (1.1) is one-dimensional $\Omega_t = (-\rho(t), \rho(t))$, and grows exponentially in time, i.e., $\lim_{t\to\infty} \rho(t) = +\infty$ and $\lim_{t\to\infty} \frac{\dot\rho(t)}{\rho(t)} = k$. It is shown that $u(x,t)$ goes to zero when k is large, and $u(x,t)$ tends to a positive constant when k is small. Regarding the dynamics of single-population models in changing regions, moreover, the authors in [18] studied another population model:

$$u_t = d\Delta u + u(a - bu^q), \quad (x,t) \in \Omega \times (0, +\infty) \tag{1.2}$$

with the homogeneous Dirichlet boundary condition, where $q > 0$ is a constant. This research accounts for the evolving domain possessing a finite terminal size, that is, the evolution rate $\rho(t)$ satisfies $\lim_{t\to\infty} \rho(t) = \rho_\infty > 1$. They discovered that when $\rho(t)$ is small, $u(x,t)$ goes to zero, whereas when $\rho(t)$ is relatively large, $u(x,t)$ tends to a positive stable state. Furthermore, a plankton population model was investigated in [15], where the periodic evolution of the region is also considered, and it is found that the change of the depth of one-dimensional water bodies plays a crucial role in the extinction and persistence of plankton populations.

Except for the dynamics of a single species in evolving regions, some researchers have introduced the regional evolution into multi-variable reaction-diffusion equation models [32, 31, 22, 27]. For instance, in [32] and [31], the dynamics of dengue fever transmission were explored under the premise of periodic evolution and finite growth of the domain, respectively. They concluded that compared with the spread of dengue virus in a fixed region, both periodic evolution and finite growth of region will increase the basic reproduction number and thereby increase the transmission risk of dengue virus once the evolution rate is high. In [22], the periodic evolution of region was introduced into the Susceptible-Infectious-Susceptible compartment model of infectious diseases. Under appropriate conditions, the analytical relationship between the basic reproduction number R_0 and the evolution rate $\rho(t)$ was also presented. These authors found that when other epidemiological parameters remain constant, the increase of

$\rho(t)$ leads to the increase of R_0, and the decrease of $\rho(t)$ makes R_0 smaller, indicating that the evolution of region have a significant impact on the disease transmission risk.

In order to uncover the general laws governing the impact of changing regions on multiple species interaction, in this paper we will explore the dynamic behavior of a large class of reaction-diffusion systems

$$\begin{cases} v_t = D\Delta v + f(v), & x \in \Omega,\ t > 0, \\ v = 0, & x \in \partial\Omega,\ t > 0, \end{cases} \tag{1.3}$$

under the different evolving conditions of the domain Ω, where $v = (v_1, v_2, \ldots, v_m)$ represents the population densities of m interacting species. To present the basic assumptions, we need the following notations.

Let $x, y \in \mathbb{R}^m$, we write $x \geq y$ if $x - y \in \mathbb{R}^m_+$; $x > y$ if $x - y \in \mathbb{R}^m_+ \setminus \{0\}$; and $x \gg y$ if $x - y \in Int(\mathbb{R}^m_+)$. For a given $\bar{u} \gg 0$, let $\mathbb{M} := [0, \bar{u}]_{\mathbb{R}^m} = \{u \in \mathbb{R}^m : 0 \leq u \leq \bar{u}\}$. Let $\mathbb{X} := C^1_0(\bar{\Omega}_0, \mathbb{R}^m)$, $\mathbb{X}^+ := C^1_0(\bar{\Omega}_0, \mathbb{R}^m_+)$, and $\mathbb{Y} = C^1_0(\bar{\Omega}_0, [0, \bar{u}]_{\mathbb{R}^m})$. Then $(\mathbb{X}, \mathbb{X}^+)$ is an ordered Banach space. We assume that

- (A1) $f : \mathbb{M} \to \mathbb{R}^m$ is continuously differentiable, $\left(\frac{\partial f_i(u)}{\partial u_j}\right)_{m \times m}$ is cooperative, and irreducible for all $0 \leq u \ll \bar{u}$.
- (A2) For each $1 \leq i \leq m$, $f_i(u) \geq 0$ whenever $u \in \mathbb{M}$ with $u_i = 0$, $f_i(u) \leq 0$ whenever $u \in \mathbb{M}$ with $u_i = \bar{u}_i$, $f(0) = 0$, and $f(u)$ is strictly subhomogeneous on $\mathbb{M}$ in the sense that $f(\alpha u) > \alpha f(u)$, $\forall u \in \mathbb{M}$ with $u \gg 0$, and $\alpha \in (0, 1)$.

In view of the assumption (A1), we easily see that $\bar{u}$ is a super-solution of the ODE system $v' = f(v)$, and hence, it is also a super-solution of system (1.3).

Now we assume that the spatial domain Ω_t is simply connected and evolving in time according to different evolving conditions, and has a smooth boundary $\partial\Omega_t$. Let $v(x(t), t)$ be the vector-valued density of m species at position $x(t) \in \Omega_t$ and time t. Following the modeling ideas in [9, 19], we can obtain the following equation associated with (1.3) on the changing domain Ω_t

$$\frac{\partial v}{\partial t} + \nabla v \cdot \mathbf{a} + v(\nabla \cdot \mathbf{a}) = D\Delta v + f(v), \quad x \in \Omega_t,\ t > 0, \tag{1.4}$$

in which $\nabla v \cdot \mathbf{a}$ is the advection term representing the transport of material around Ω_t at a rate determined by the flow $\mathbf{a}$, and $(\nabla \cdot \mathbf{a})v$ is the dilution terms generated by local volume expansion [1, 3]. Additionally, we suppose that the matching conditions of (1.4) are the homogeneous Dirichlet boundary condition

$$v = 0 \quad \text{on } \partial\Omega_t \tag{1.5}$$

and initial condition

$$v = v_0(x) \quad \text{for } x \in \Omega_0 \text{ at } t = 0 \tag{1.6}$$

where $v_0(x)$ is a non-negative bounded continuous function and Ω_0 is the initial domain.

For simplicity, we consider a special class of evolving domain that evolves by linear isotropic deformation, which led to the following assumption:

- (A3) The evolution of domain Ω_t is given by

$$\Omega_t := \rho(t)\Omega_0 = \{\rho(t)y : y \in \Omega_0\}, \tag{1.7}$$

where $\rho : [0, \infty) \to (0, \infty)$ is the scaling factor such that $\rho(0) = 1$, and Ω_0 is a fixed domain in $\mathbb{R}^n$ with smooth boundary.

The evolution dynamics of a single population in this class of evolving domains was previously studied in [6, 9]. Following the ideas in [13, 9], we set

$$u(y,t) = v(\rho(t)y,t) \quad \text{for } y \in \Omega_0,\ t > 0,$$

and then transform the model on evolving domains into the following equivalent system in a fixed domain with nonautonomous coefficients:

$$\begin{cases} u_t = \frac{1}{\rho^2(t)} \cdot D\Delta u - \frac{n\dot{\rho}(t)}{\rho(t)} u + f(u), & y \in \Omega_0,\ t > 0, \\ u(y,t) = 0, & y \in \partial\Omega_0,\ t > 0, \\ u(y,0) = v_0(x(0)) := u_0(y), & y \in \Omega_0. \end{cases} \tag{1.8}$$

The purpose of this paper is to establish general dynamical conclusions caused by the different evolution ways of domain Ω_t. More precisely, we consider the case where Ω_t is asymptotically bounded, unbounded and periodic, respectively. In view of the equivalence between system (1.4)–(1.6) defined in $\Omega_t \times [0,+\infty)$ and the problem (1.8) defined in $\Omega_0 \times [0,+\infty)$, we will focus on the evolution dynamics of system (1.8) under different asymptotic conditions on $\rho(t)$. Our results in particular extend beyond previous works in [9, 19, 18, 15, 32, 31, 22, 27]. Here we should point out that for one-dimensional and asymptotically unbounded domains, i.e., $\Omega_t = (-\rho(t), \rho(t))$, a plus or minus characteristics of the diffusion term is determined in [19]. However, with the domain Ω_t being n-dimensional and the variable $v(x(t),t)$ being vector-valued, it is not clear how to introduce such characteristics for the diffusion term in our current case. To overcome this difficulty, we investigate a class of nonautonomous reaction-diffusion systems with the diffusion coefficients tending to zero as t goes to infinity. By constructing an innovative subsolution, we are able to establish the global dynamics for such a system (see Theorem 3.4). This result is also of its own interest. Further, we study the asymptotically periodic case by appealing to the theory of asymptotically periodic semiflows, which is more general than the periodically evolving case in previous research [9, 32].

The remaining of this paper is organized as follows. In sections 2-4, we sequentially investigate the different long-time dynamical behaviors of system (1.8) with the three evolution trends. In section 5, as an illustrative example, we apply the obtained analytical results to a reaction-diffusion model of Dengue fever transmission for its global dynamics in terms of the basic reproduction number R_0.

2. Asymptotically bounded domain.

In this section, we suppose that the domain Ω_t is asymptotically bounded, i.e., $\rho(t)$ satisfies the following finite growing condition:

(B1) $\dot{\rho}(t) \geq 0$, $\lim_{t\to\infty} \rho(t) = \rho_\infty < \infty$, $\lim_{t\to\infty} \dot{\rho}(t) = 0$.

In order to explore the impact of the change of $\rho(t)$ on the dynamics of system (1.8), we begin with its limiting system:

$$\begin{cases} u_t = \frac{1}{\rho_\infty^2} \cdot D\Delta u + f(u), & y \in \Omega_0, t > 0, \\ u(y,t) = 0, & y \in \partial\Omega_0, t > 0, \\ u(y,0) = u_0(y), & y \in \partial\Omega_0. \end{cases} \tag{2.1}$$

Let λ_F be the principal eigenvalue of the following eigenvalue problem (see, e.g., [10, Chap. 3]):

$$\begin{cases} \frac{1}{\rho_\infty^2} \cdot D\Delta\varphi + f'(0)\varphi = \lambda\varphi, & y \in \Omega_0, \\ \varphi(y) = 0, & y \in \partial\Omega_0, \end{cases} \tag{2.2}$$

where $f'(0) = \big(\frac{\partial f_i(0)}{\partial u_j}\big)_{m\times m}$. Then we have have the following threshold type result on the global dynamics of system (2.1).

PROPOSITION 2.1. *The following statements are valid for system (2.1):*

(1) If $\lambda_F \le 0$, then $u = 0$ is globally attractive for system (2.1) in $\mathbb{Y}$.

(2) If $\lambda_F > 0$, then system (2.1) has a unique positive steady state $u_F^*(y)$, and it is globally attractive for system (2.1) in $\mathbb{Y}\backslash\{0\}$.

Proof. According to the standard theory of reaction-diffusion equation [16], it follows that for any $u_0 \in \mathbb{Y}$, system (2.1) admits a unique nonnegative solution $u(y,t,u_0)$ satisfying $u(\cdot,0,u_0) = u_0$ on its maximal existence interval $t \in [0, t_{u_0})$. By choosing $u^+ = \bar{u}, u^- = 0$, we see from [14] that $u(y,t,u_0) \in \mathbb{Y}$ for all $t \in [0, t_{u_0})$. This implies that $t_{u_0} = \infty$ for any $u_0 \in \mathbb{M}$, and solutions of system (2.1) are also ultimately bounded in $\mathbb{Y}$.

Let $Q(t)$ be the solution semiflow associated with system (2.1). Since $Q(t)$ is compact on $\mathbb{Y}$, $Q(t)$ has a global compact attractor on $\mathbb{Y}$. Note that $f(u)$ is cooperative, irreducible and strictly subhomogeneous for any $u \in \mathbb{M}$. By the arguments similar to those in [7], it follows that $Q(t)$ is a strongly monotone and strictly subhomogeneous semiflow on $\mathbb{Y}$.

Linearing system (2.1) at $u = 0$, we obtain the following linear cooperative system:

$$u_t = \frac{1}{\rho_\infty^2} \cdot D\Delta u + f'(0)u. \tag{2.3}$$

Let $U(t)$ be the linear solution semigroup on $\mathbb{X}$ generated by system (2.3) subject to the homogeneous Dirichlet boundary condition. Thus, the eigenvalue problem associated with (2.3) is problem (2.2), whose principal eigenvalue has been denoted as λ_F.

For the convenience, we use the notation Q_t instead of $Q(t)$. Then for each $t > 0$, the map Q_t is strongly monotone and strictly subhomogeneous on $\mathbb{Y}$, $Q_t(0) = 0$, and its Frechet derivative is $DQ_t(0) = U(t)$. Next, take $t = 1$ and consider the map $Q_1 := Q(1)$. Let r be the spectral radius of Q_1. According to [10, Corollary C.2.2] (see also [28, Theorem 2.3.4]), we obtain the following threshold results about the semiflow Q_t:

(i) If $r \le 1$, then $Q_t(u_0) \to 0$ for all $u_0 \in \mathbb{Y}$.

(ii) If $r > 1$, then there exists a unique equilibrium point $0 \ll u_F^* \in \mathbb{Y}$ of the semiflow such that $Q_t(u_0) \to u_F^*$ for all $u_0 \in \mathbb{Y} \setminus \{0\}$.

It remains to show that $r = e^{\lambda_F}$. Indeed, let $\varphi_F \gg 0$ be an eigenfunction of (2.3) corresponding to λ_F, then it is clear that $Q_t(\varphi_F) = e^{\lambda_F t}\varphi_F$. Hence, e^{λ_F} is an eigenvalue of Q_1 with a positive eigenfunction. It follows from the Krein-Rutman theorem (see, e.g. [10, Theorem B.3.2]) that $r(Q_1) = e^{\lambda_F}$. ∎

THEOREM 2.2. *For nonautonomous system (1.8), the following statements are valid:*

(1) If $\lambda_F \le 0$, then $u = 0$ is globally attractive for system (1.8) in $\mathbb{Y}$.

(2) If $\lambda_F > 0$, then every solution $u(y,t)$ of system (1.8) with $u(\cdot,0) \in \mathbb{Y}\backslash\{0\}$ is asymptotic to $u_F^*(y)$ uniformly for $y \in \bar{\Omega}_0$ as $t \to \infty$.

Proof. Following [31], we introduce a new time $s = \displaystyle\int_0^t \frac{1}{\rho^2(\tau)} d\tau$ for system (1.8). Since $s'(t) = \frac{1}{\rho^2(t)} > 0$, there exists an inverse transformation $t = h(s)$ and

$$\lim_{s\to\infty} t = \lim_{s\to\infty} h(s) = \infty. \tag{2.4}$$

Let $w(y,s) = u(y,t)$, then

$$u_t = w_s \cdot \frac{1}{\rho^2(t)}, \qquad \Delta u(y,t) = \Delta w(y,s),$$

so system (1.8) is translated into

$$\begin{cases} w_s = D\Delta w - n \cdot \dot{\rho}(h(s)) \cdot \rho(h(s))w + \rho^2(h(s))f(w), & y \in \Omega_0, s > 0, \\ w(y,s) = 0, & y \in \partial\Omega_0, s > 0, \\ w(y,0) = u_0(y), & y \in \Omega_0, \end{cases} \tag{2.5}$$

which implies that (2.5) has a limiting system

$$\begin{cases} w_s = D\Delta w + \rho_\infty^2 f(w), & y \in \Omega_0, s > 0, \\ w = 0, & y \in \partial\Omega_0, s > 0 \end{cases} \tag{2.6}$$

as $t \to \infty$. By the transformation $w(y,s) = u(y,t)$, it follows that system (2.6) admits the same threshold dynamics as in Proposition 2.1:

(i) If $\lambda_F \le 0$, then $\lim_{s\to\infty} w(y,s) = 0$ uniformly for $y \in \overline{\Omega}_0$.
(ii) If $\lambda_F > 0$, then $\lim_{s\to\infty} w(y,s) = w_F^*(y)$ uniformly for $y \in \overline{\Omega}_0$, where $w_F^*(y)$ is the unique positive steady state of (2.6).

Since the steady states of (2.1) and (2.6) are the same, one has $w_F^*(y) \equiv u_F^*(y)$. By the theory of asymptotically autonomous semiflows (see [20]) or the theory of chain transitive sets (see [28, Section 1.2.1] and the arguments in [12]), together with the threshold dynamics for system (2.6), we can easily obtain the following result for system (1.8):

(a) If $\lambda_F \le 0$, then any nonnegative solution $u(y,t)$ of (1.8) satisfies

$$\lim_{t\to\infty} u(y,t) = \lim_{s\to\infty} w(y,s) = 0$$

uniformly for $y \in \overline{\Omega}_0$.

(b) If $\lambda_F > 0$, then every positive solution $u(y,t)$ of (1.8) satisfies

$$\lim_{t\to\infty} u(y,t) = \lim_{s\to\infty} w(y,s) = u_F^*(y)$$

uniformly for $y \in \overline{\Omega}_0$.

This completes the proof. □

3. Asymptotically unbounded domain.

In this section, we study the dynamics of system (1.8) in the case of asymptotically unbounded domain Ω_t, which signifies that the evolution rate $\rho(t)$ meets with the following infinite growing condition:

(B2) $\dot{\rho}(t) > 0$, $\lim_{t\to\infty} \rho(t) = +\infty$, $\lim_{t\to\infty} \frac{\dot{\rho}(t)}{\rho(t)} = k \ge 0$.

We start with the global dynamics of a class of nonautonomous reaction-diffusion systems with asymptotically vanishing diffusion coefficients.

3.1. A class of nonautonomous reaction-diffusion systems.

Let Ω be a bounded smooth domain in $\mathbb{R}^n$, and Ω_η be a sequence of bounded smooth subdomains of Ω such that Ω_η is decreasing in η and $\Omega_\eta \nearrow \Omega$ as $\eta \to 0^+$. For example, we can take $\Omega_\eta = \{x \in \Omega : \ \mathrm{dist}(x, \partial\Omega) > \eta\}$.

Let $F : \mathbb{R}_+^n \to \mathbb{R}^n$ be a C^1-vector field with its Jacobian matrix $F'(y)$ being cooperative and irreducible for all $y \in \mathbb{R}_+^n$. Throughout this subsection, we always assume that

(H1) $F(0) = 0$, $s(F'(0)) := \max\{Re(\lambda) : \det(F'(0) - \lambda I) = 0\} > 0$.
(H2) The ODE system $z'(t) = F(z(t))$ has a positive equilibrium u^* such that $\lim_{t\to\infty} z(t) = u^*$ for all $z(0) \in \mathbb{R}^n_+ \setminus \{0\}$.

By the Dancer-Hess connecting orbit theorem (see, e.g., [28]), it follows that system $z'(t) = F(z(t))$ admits a connecting orbit $\alpha : \mathbb{R} \to (0,\infty)^n$ such that

$$\alpha(-\infty) = 0, \quad \alpha(+\infty) = u^*, \quad \alpha'(t) \gg 0 \text{ in } \mathbb{R}^n, \quad \forall t \in \mathbb{R}.$$

Further, the Perron-Frobenius theorem implies that $\bar{r} := s(F'(0))$ is the principal eigenvalue of $F'(0)$, that is, there exists a vector $\bar{v} \gg 0$ in $\mathbb{R}^n$ such that $F'(0)\bar{v} = \bar{r}\bar{v}$.

Let $D = diag(d_1, d_2, \ldots, d_n)$ with $d_i > 0$ for all $1 \le i \le n$. Clearly, $d := \max\{d_1, d_2, \ldots, d_n\} > 0$. Let $\epsilon(t)$ be a continuous and positive function on $[0,\infty)$ such that $\lim_{t\to\infty} \epsilon(t) = 0$. We consider the following non-autonomous reaction-diffusion system

$$\begin{cases} \partial_t u = \epsilon(t) D\Delta u + F(u) & \text{in } \Omega \times (0,\infty), \\ u = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x) \ge 0 & \text{on } \Omega. \end{cases} \tag{3.1}$$

To study the asymptotic behavior of solutions of system (3.1), we need the following two lemmas on its subsolutions.

Let μ_0 and $\phi(x)$ be the principal eigenvalue and eigenfunction of the Laplacian on Ω, that is,

$$\Delta\phi + \mu_0\phi = 0 \quad \text{in } \Omega, \quad \text{and} \quad \phi = 0 \quad \text{on } \partial\Omega.$$

Note that $\mu_0 > 0$ and $\phi \gg 0$ in $C^1_0(\bar{\Omega})$.

LEMMA 3.1. *There exists $\delta_0 > 0$ such that for any $\delta \in (0, \delta_0]$, the function*

$$\underline{u}_\delta(x) := \delta\phi(x)\bar{v}$$

where $\bar{v}$ is the positive eigenvector of $F'(0)$, is a subsolution of system (3.1) in Ω whenever $\epsilon(t) \in (0, \bar{r}/(2d\mu_0))$.

Proof. We fix a real number $k > 0$ such that $F'(0) + kI$ has positive diagonal entries, and let $\epsilon_0 = \frac{\bar{r}}{2(\bar{r}+k)}$. By an elementary analysis (see, e.g., (6.10) in [5]), there exists a vector $\hat{v} \gg 0$ in $\mathbb{R}^n$ such that

$$F(y) \ge F'(0)y - \epsilon_0[F'(0)y + ky], \quad \forall y \in [0, \hat{v}]_{\mathbb{R}^n}.$$

Now we choose a sufficiently small $\delta_0 > 0$ such that $\underline{u}_{\delta_0}(x) \in [0, \hat{v}]_{\mathbb{R}^n}$ for all $x \in \bar{\Omega}$. It then follows that for any $\delta \in (0, \delta_0]$, we have

$$\begin{aligned} \epsilon(t)D\Delta\underline{u}_\delta + F(\underline{u}_\delta) &\ge \epsilon(t)D\Delta\underline{u}_\delta + [(1-\epsilon_0)F'(0) - \epsilon_0 kI]\underline{u}_\delta \\ &= -\epsilon(t)\mu_0 D\underline{u}_\delta + [(1-\epsilon_0)\bar{r} - \epsilon_0 k]\,\underline{u}_\delta \\ &\ge [-\epsilon(t)d\mu_0 + (1-\epsilon_0)\bar{r} - \epsilon_0 k]\,\underline{u}_\delta \\ &= \left[-\epsilon(t)d\mu_0 + \frac{\bar{r}}{2}\right]\underline{u}_\delta \gg 0 \text{ in } \mathbb{R}^n, \ \forall x \in \Omega, \end{aligned}$$

provided that $\epsilon(t) \in (0, \bar{r}/(2d\mu_0))$. ∎

Since $\alpha(-\infty)=0$, there exists $\underline{\tau}=\underline{\tau}(\delta,\eta)\in\mathbb{R}$ such that

$$\alpha(\underline{\tau})\leq \underline{u}_\delta(x),\quad \forall x\in\overline{\Omega}_\eta. \tag{3.2}$$

Recall that $\overline{\Omega}_{2\eta}\subset\Omega_\eta\subset\overline{\Omega}_\eta\subset\Omega$. We choose a smooth cut-off function $\rho_\eta:\Omega\to[0,1]$ satisfying

$$\rho_\eta(x)=0 \quad \text{in } \Omega\setminus\Omega_\eta, \quad \text{and} \quad \rho_\eta(x)=1 \quad \text{in } \Omega_{2\eta}. \tag{3.3}$$

LEMMA 3.2. *Given $\delta\in(0,\delta_0)$, $\eta>0$, $\underline{\tau},\bar{\tau}\in\mathbb{R}$ such that $\underline{\tau}<\bar{\tau}$ and (3.2) hold, there exist two positive numbers $\beta=\beta(\underline{\tau},\bar{\tau})$ and $\hat{\varepsilon}=\hat{\varepsilon}(\eta,\underline{\tau},\bar{\tau})$ such that for any $t_0\in\{\bar{t}\geq 0:\ \epsilon(t)\in(0,\hat{\varepsilon}],\forall t\geq\bar{t}\}$, the function*

$$\underline{w}(x,t):=\max\left\{\underline{u}_\delta(x),\alpha(\ln\rho_\eta(x)+\bar{\tau}-(\bar{\tau}-\underline{\tau})e^{-\beta(t-t_0)})\right\}$$

is a generalized subsolution of system (3.1) in $\Omega\times[t_0,+\infty)$.

Proof. For any given $\eta>0$ and $\bar{\tau}>\underline{\tau}$, we define four real numbers

$$c_1=c_1(\underline{\tau},\bar{\tau})=\max_{1\leq i\leq n}\left\{\max_{s\in[\underline{\tau},\bar{\tau}]}\frac{\alpha_i'(s)}{\alpha_i(s)}\right\}>0,$$

$$c_2=c_2(\underline{\tau},\bar{\tau})=\min_{1\leq i\leq n}\left\{\min_{s\in[\underline{\tau},\bar{\tau}]}\frac{\alpha_i'(s)}{2\alpha_i(s)}\right\}>0,$$

$$c_3=c_3(\eta,\underline{\tau},\bar{\tau})=\sup_{\tau\in[\underline{\tau},\bar{\tau}]}\left\{\sup_{\{x:\ \ln\rho_\eta(x)\geq(\underline{\tau}-\bar{\tau})\}}|\Delta(\alpha(\ln\rho_\eta(x)+\tau))|\right\}>0,$$

and

$$\hat{\varepsilon}=\hat{\varepsilon}(\eta,\underline{\tau},\bar{\tau})=\min\left\{\frac{\bar{r}}{2d\mu_0},\min_{1\leq i\leq n}\left\{\frac{\min_{s\in[\underline{\tau},\bar{\tau}]}\alpha_i'(s)}{2dc_3}\right\}\right\}>0.$$

Let $t_0\in\{\bar{t}\geq 0:\ \epsilon(t)\in(0,\hat{\varepsilon}],\forall t\geq\bar{t}\}$ be given. It then follows that for all (x,τ) such that $\ln\rho_\eta(x)+\tau\geq\underline{\tau}$ and $\tau\in[\underline{\tau},\bar{\tau}]$, there hold

$$\frac{\partial}{\partial\tau}\alpha(\ln\rho_\eta(x)+\tau)\leq c_1\alpha(\ln\rho_\eta(x)+\tau) \tag{3.4}$$

and

$$\begin{aligned}
&\epsilon(t)D\Delta(\alpha(\ln\rho_\eta(x)+\tau))+F(\alpha(\ln\rho_\eta(x)+\tau))\\
&=\epsilon(t)D\Delta(\alpha(\ln\rho_\eta(x)+\tau))+\alpha'(\ln\rho_\eta(x)+\tau)\\
&\geq -\varepsilon(t)dc_3\mathbf{e}+\frac{1}{2}\min_{s\in[\underline{\tau},\bar{\tau}]}\alpha'(s)+c_2\alpha(\ln\rho_\eta(x)+\tau)\\
&\geq c_2\alpha(\ln\rho_\eta(x)+\tau),\ \forall t\geq t_0,
\end{aligned} \tag{3.5}$$

where $\mathbf{e}\in\mathbb{R}^n$ with $e_i=1$ for all $1\leq i\leq n$. Let $\delta\in(0,\delta_0]$, $\beta=\frac{c_2}{c_1(\bar{\tau}-\underline{\tau})}$ and $\underline{w}(x,t)$ be defined as in the statement of Lemma 3.2. We first claim that the function $\alpha(\ln\rho_\eta(x)+\tau(t))$ with

$$\tau(t):=\bar{\tau}-(\bar{\tau}-\underline{\tau})e^{-\beta(t-t_0)}, \tag{3.6}$$

is a subsolution of system (3.1) in the domain

$$\mathcal{D} := \{(x,t) \in \Omega \times [t_0, +\infty) : \quad \alpha(\ln \rho_\eta(x) + \tau(t)) \geq \underline{u}_\delta(x)\}.$$

Indeed, using equations (3.3), (3.6) together with the monotonicity of $\alpha(\cdot)$, we easily see that $\ln \rho_\eta(x) + \tau(t) \leq \overline{\tau}$ in $\Omega \times [t_0, +\infty)$. In view of (3.2), it follows that for any $(x,t) \in \mathcal{D}$, there hold

$$x \in \Omega_\eta, \quad \ln \rho_\eta(x) + \tau(t) \in [\underline{\tau}, \overline{\tau}], \quad \tau'(t) \leq (\overline{\tau} - \underline{\tau})\beta. \tag{3.7}$$

This, together with estimates (3.4) and (3.5), implies that

$$\frac{\partial}{\partial t}\alpha(\ln \rho_\eta(x) + \tau(t)) \leq \epsilon(t) D\Delta(\alpha(\ln \rho_\eta(x) + \tau(t)) + F(\alpha(\ln \rho_\eta(x) + \tau(t))$$

for all $(x,t) \in \mathcal{D}$. Thus, $\underline{w}_1(x,t) := \alpha(\ln \rho_\eta(x) + \tau(t))$ is a subsolution in $\mathcal{D}$.

By virtue of Lemma 3.1, we see that $\underline{w}_2(x,t) := \underline{u}_\delta(x)$ is a subsolution in $\Omega \times [t_0, \infty)$. It follows that $\underline{w}(x,t) = \max\{\underline{w}_1(x,t), \underline{w}_2(x,t)\}$ is a generalized subsolution of system (3.1) in $\Omega \times [t_0, +\infty)$ (see [10]). ∎

Remark 3.3. In Lemma 3.2, we can also choose $\tau(t) = \underline{\tau} + \frac{c_2}{c_1}(t - t_0)$ for $t \in [t_0, t_0 + \frac{c_1}{c_2}(\overline{\tau} - \underline{\tau}))$, and $\tau(t) = \overline{\tau}$ for $t \geq t_0 + \frac{c_1}{c_2}(\overline{\tau} - \underline{\tau})$. This gives an alternative subsolution for the parabolic problem since $\tau'(t) \leq \frac{c_2}{c_1}$ for all $t \geq t_0$.

THEOREM 3.4. *Each non-negative and non-trivial solution $u(x,t)$ of system (3.1) satisfies $\lim_{t\to\infty} u(x,t) = u^*$ uniformly for x in any compact subset of Ω.*

Proof. To proceed, we fix an arbitrary non-negative and non-trivial solution $u(x,t)$ of system (3.1). By the strong maximum principle and the Hopf boundary lemma, there exists $\delta \in (0, \delta_0]$ such that

$$u(x,1) \geq \underline{u}_\delta(x) \quad \text{in } \Omega.$$

Since the latter is a strict subsolution, it follows that

$$u(x,t) \geq \underline{u}_\delta(x) \quad \text{in } \Omega \times [1, \infty).$$

Next, let $\eta > 0$ be arbitrarily fixed and $\underline{\tau}$ be chosen according to (3.2). We further define β and $\hat{\epsilon}$ as in Lemma 3.2. Since $\lim_{t\to\infty} \epsilon(t) = 0$, there exists $t_0 \geq 1$ such that $\epsilon(t) \in (0, \hat{\epsilon}]$ for all $t \in [t_0, +\infty)$. Let $\underline{w}(x,t)$ be defined as in Lemma 3.2 with this specific $t_0 \geq 1$. By the choice of $\underline{\tau}$, it follows that $\underline{w}(x,t_0) = \underline{u}_\delta(x)$, $\forall x \in \Omega$. Thus, the comparison principle implies that

$$u(x,t) \geq \underline{w}(x,t), \quad \forall x \in \Omega,\ t \in [t_0, +\infty),$$

and hence,

$$u(x,t) \geq \alpha(\ln \rho_\eta(x) + \tau(t)) = \alpha(\tau(t)), \quad \forall x \in \Omega_{2\eta},\ t \in [t_0, +\infty),$$

where we used the fact that $\rho_\eta(x) = 1$ in $\Omega_{2\eta}$. Since $\lim_{t\to\infty} \tau(t) = \overline{\tau}$, it follows that

$$\liminf_{t\to\infty} \inf_{x\in\Omega_{2\eta}} u(x,t) \geq \alpha(\overline{\tau}).$$

Letting $\overline{\tau} \to +\infty$, we have

$$\liminf_{t\to\infty} \inf_{x\in\Omega_{2\eta}} u(x,t) \geq u^* \quad \text{for each } \eta > 0. \tag{3.8}$$

Let $\bar{u}(t)$ be the unique solution of the ODE system $z'(t) = F(z(t))$ with $\bar{u}(0) = \max_{x\in\bar{\Omega}} u(x, 0) \in \mathbb{R}^n_+ \setminus \{0\}$. Then the comparison theorem gives rise to $u(x, t) \le \bar{u}(t)$, $\forall x \in \Omega$, $t \ge 0$. In view of the hypothesis (H2), we have $\lim\limits_{t\to\infty} \bar{u}(t) = u^*$. It follows that

$$\limsup_{t\to\infty} u(x, t) \le u^*, \quad \text{uniformly for } x \in \Omega. \tag{3.9}$$

Now two inequalities (3.8) and (3.9) imply that $\lim\limits_{t\to\infty} u(x, t) = u^*$ uniformly for in $\Omega_{2\eta}$. Since $\eta > 0$ is arbitrary, and $\Omega_{2\eta} \nearrow \Omega$ as $\eta \to 0$, it follows that this convergence is uniform on each compact subset of Ω. ∎

Next, we study the evolution dynamics of system (1.8) with asymptotically unbounded domain.

3.2. The dynamics of system (1.8) with expanding domains. Recall that the stability modulus of a square $m \times m$ matrix A is defined as $s(A) := \max\{\mathrm{Re}\,\lambda : \lambda \text{ is an eigenvalue of } A\}$.

Under hypothesis (B2), we set $\frac{\dot{\rho}}{\rho} = k$ in (1.8) to obtain the following kinetic system:

$$\begin{cases} \frac{du}{dt} = \bar{f}(u) := -nku + f(u), \\ u(0) = u_0 \in [0, \bar{u}]_{\mathbb{R}^m}. \end{cases} \tag{3.10}$$

As a straightforward consequence of [29, Corollary 3.2], we have the following threshold type result for system (3.10).

Proposition 3.5. *Assume that $\bar{f}$ satisfies (A1) and (A2), let $\lambda_I = s(f'(0) - nkI)$, where I represents $m \times m$ identity matrix. Then the following statements are valid:*

(1) If $\lambda_I \le 0$, then $u = 0$ is globally asymptotically stable for system (3.10) on $[0, \bar{u}]_{\mathbb{R}^m}$, where $\bar{f}'(0) = \left(\frac{\partial \bar{f}_i(0)}{\partial u_j}\right)_{m\times m}$.

(2) If $\lambda_I > 0$, then system (3.10) has a positive equilibrium u_I^*, and u_I^* is globally stable for system (3.10) in $[0, \bar{u}]_{\mathbb{R}^m} \backslash \{0\}$.

Now we are ready to address the global dynamics of system (1.8).

Theorem 3.6. *The following statements are valid:*

(1) If $\lambda_I \le 0$, then every solution $u(y, t)$ of system (1.8) with $u(\cdot, 0) \in \mathbb{Y}$ satisfies $\lim_{t\to\infty} u(y, t) = 0$ uniformly for $y \in \bar{\Omega}$.

(2) If $\lambda_I > 0$, then every solution $u(y, t)$ of system (1.8) with $u(\cdot, 0) \in \mathbb{Y}\backslash\{0\}$ satisfies $\lim_{t\to\infty} u(y, t) = u_I^*$ uniformly for y in any compact subset of Ω_0, where u_I^* is the unique positive equilibrium of (3.10).

Proof. (1) Consider the nonautonomous ODE system

$$\frac{du}{dt} = -\frac{n\dot{\rho}(t)}{\rho(t)} u + f(u). \tag{3.11}$$

Since (3.10) is the limiting system of (3.11), by appealing to the theory of asymptotically autonomous semiflows or the theory of chain transitive sets (see [28]), we have the following threshold type results for system (3.11):

(i) If $\lambda_I \le 0$, then every nonnegative solution $u(t)$ of (3.11) satisfies $\lim_{t\to\infty} u(t) = 0$.

(ii) If $\lambda_I > 0$, then every positive solution $u(t)$ of (3.11) satisfies $\lim_{t\to\infty} u(t) = u_I^*$.

Since every solution of (3.11) is a super-solution of system (1.8), it follows from the above statement (i) and the comparison argument that the conclusion (1) holds true.

(2) Recalling system (1.8), we now set $\lambda_\varepsilon = s(f'(0)-n(k+\varepsilon)I)$. Since $\lim_{\varepsilon\to 0^+}\lambda_\varepsilon = \lambda_I$, it follows that $\exists\ \varepsilon_0 > 0$ such that $\lambda_\varepsilon > 0$ for $\forall\varepsilon \in (0,\varepsilon_0)$, and the ODE system $u'(t) = -n(k+\varepsilon)u + f(u)$ has a globally stable positive equilibrium u^*_ε for any $\varepsilon \in (0,\varepsilon_0)$. It is easy to see that

$$lim_{\varepsilon\to 0^+} u^*_\varepsilon = u^*_I. \tag{3.12}$$

Let $u(y,t)$ be a nontrivial and nonnegative soltuion of (1.8), and $\varepsilon \in (0,\varepsilon_0)$ be given. Since $\lim_{t\to\infty}\frac{\rho'(t)}{\rho(t)} = k \geq 0$, there exists $T_\varepsilon > 0$ such that

$$\frac{\rho'(t)}{\rho(t)} < k+\varepsilon, \quad \forall t \geq T_\varepsilon.$$

Thus, $u(y,t)$ satisfies

$$u_t \geq \frac{1}{\rho^2(t)}\cdot D\Delta u - n(k+\varepsilon)u + f(u), \quad \forall\ t \geq T_\varepsilon. \tag{3.13}$$

Let $z(y,t)$ be the unique solution of the reaction diffusion system:

$$\begin{cases} U_t = \frac{1}{\rho^2(t)}\cdot D\Delta U - n(k+\varepsilon)U + f(U), & y \in \Omega_0, t > T_\varepsilon, \\ U(y,t) = 0, & y \in \partial\Omega_0, t > T_\varepsilon, \end{cases} \tag{3.14}$$

with $U(y,T_\varepsilon) = u(y,T_\varepsilon), \forall\ y \in \Omega_0$. By the comparison principle, it follows that

$$u(y,t) \geq z(y,t), \quad \forall y \in \Omega_0, t \geq T_\varepsilon. \tag{3.15}$$

Since $\lim_{t\to\infty}\frac{1}{\rho^2(t)} = 0$, it follows from Theorem 3.4 that

$$\lim_{t\to\infty} z(y,t) = u^*_\varepsilon \quad \text{in } C_{loc}(\Omega_0).$$

In view of (3.15), it suffices to show that

$$\liminf_{t\to\infty} u(y,t) \geq u^*_\varepsilon \quad \text{uniformly for } y \text{ in compact subset of } \Omega_0.$$

Recalling (3.12), we may let $\varepsilon \to 0$ to deduce

$$\liminf_{t\to\infty} u(y,t) \geq u^*_I \quad \text{uniformly for } y \text{ in compact subset of } \Omega_0.$$

By arguing similarly with small negative ε, we can likewise show that $\limsup_{t\to\infty} u(y,t) \leq u^*_I$ uniformly for y in Ω_0. Hence, $u(\cdot,t) \to u^*_I$ uniformly for y in any compact subset of Ω_0. ∎

4. Asymptotically periodic domain. In this section, we explore the global dynamics in the case where Ω_t is asymptotically periodic. Accordingly, we assume that

(B3) $\lim_{t\to\infty}(\rho(t)-\rho_0(t)) = 0$, and $\lim_{t\to\infty}(\dot\rho(t)-\dot\rho_0(t)) = 0$, where ρ_0 is a positive T-periodic function for some $T > 0$.

We first consider an auxiliary periodic system:

$$
(4.1)\qquad \begin{cases} u_t = \frac{1}{\rho_0^2(t)} \cdot D\Delta u - \frac{n\dot{\rho}_0(t)}{\rho_0(t)} u + f(u), & y \in \Omega_0, t > 0, \\ u(y,t) = 0, & y \in \partial\Omega_0, t > 0, \\ u(y,0) = u_0(y), & y \in \Omega_0. \end{cases}
$$

Let λ_T be the principal eigenvalue of the $T-$periodic parabolic eigenvalue problem:

$$
(4.2)\qquad \begin{cases} \varphi_t = \frac{1}{\rho_0^2(t)} \cdot D\Delta \varphi - \frac{n\dot{\rho}_0(t)}{\rho_0(t)} \varphi + f'(0)\varphi + \lambda\varphi, & y \in \Omega_0, t > 0, \\ \varphi(y,t) = 0, & y \in \partial\Omega_0, t > 0, \\ \varphi(y,t+T) = \varphi(y,t), & y \in \Omega_0. \end{cases}
$$

Then we have the following threshold dynamics of system (4.1) in terms of λ_T.

PROPOSITION 4.1. *The following statements are valid:*

(1) If $\lambda_T \geq 0$, then $u = 0$ is globally attractive for system (4.1) with $u(\cdot, 0) \in \mathbb{Y}$ in the sense that $\lim_{t\to\infty} u(y,t) = 0$ uniformly for $y \in \bar{\Omega}_0$.

(2) If $\lambda_T < 0$, then system (4.1) has a unique positive periodic solution $u_T^*(y,t)$, and every solution $u(y,t)$ of system (4.1) with $u(\cdot,0) \in \mathbb{Y}\backslash\{0\}$ satisfies $\lim_{t\to\infty}(u(y,t) - u_T^*(y,t)) = 0$ uniformly for $y \in \bar{\Omega}_0$.

Proof. Using the standard theory of periodic reaction-diffusion systems, one can easily obtain the global existence, uniqueness and boundedness of solutions to system (4.1).

Note that $\bar{u}$ is a sup-solution of the ODE system $u' = f(u)$. We can define the solution map $G(t) : \mathbb{Y} \to \mathbb{Y}$ by

$$
G(t)(u_0)(y) = u(y,t,u_0), \forall u_0 \in \mathbb{Y}, y \in \bar{\Omega}_0.
$$

Then $\{G(t)\}_{t\geq 0}$ is a T-periodic semiflow, and $G := G(T) : \mathbb{Y} \to \mathbb{Y}$ is the Poincare map associated with system (4.1). Since that $-\frac{n\dot{\rho}_0(t)}{\rho_0(t)} u + f(u)$ is cooperative, irreducible and strictly subhomogeneous, it follows that $G(t)$ is a strongly monotone and strictly subhomogeneous on $\mathbb{Y}$ for each $t > 0$. Thus, the arguments similar to those for [28, Theorem 3.1.5] give rise to the desired threshold result for system (4.1) in term of λ_T. ∎

Next, we use the theory of asymptotically periodic semifows (see [28]) to lift the threshold dynamics of periodic system (4.1) to nonautonomous system (1.8) .

THEOREM 4.2. *Assume that $\int_0^\infty \left(\frac{1}{\rho^2(\tau)} - \frac{1}{\rho_0^2(\tau)} \right) d\tau$ converges. Then the following statements are valid:*

(1) If $\lambda_T \geq 0$, then every solution $u(y,t)$ of system (1.8) with $u(\cdot,0) \in \mathbb{Y}$ converges to zero uniformly for $y \in \bar{\Omega}_0$ as $t \to \infty$.

(2) If $\lambda_T < 0$, then every solution $u(y,t)$ of system (1.8) with $u(\cdot,0) \in \mathbb{Y}\backslash\{0\}$ satisfies $\lim_{t\to\infty}(u(y,t) - u_T^*(y,t)) = 0$ uniformly for $y \in \bar{\Omega}_0$.

Proof. Let $c_0 = \int_0^\infty \left(\frac{1}{\rho^2(\tau)} - \frac{1}{\rho_0^2(\tau)} \right) d\tau$, and $s = \int_0^t \frac{1}{\rho^2(\tau)} d\tau - c_0$. Since $s'(t) = \frac{1}{\rho^2(t)} \geq \delta_0, \forall t \geq 0$, for some $\delta_0 > 0$, it follows that $s(\infty) = \infty$. Thus, the inverse function $t = h(s)$ exists and $h(\infty) = \infty$. Set $w(y,s) = u(y,h(s))$, $s \geq -c_0$. Then system (1.8) is equivalent to

$$\begin{cases} \frac{\partial w}{\partial s} = D\Delta w - n \cdot \dot{\rho}(h(s))\rho(h(s))w + \rho^2(h(s))f(w), & y \in \Omega_0, s > -c_0, \\ w = 0, & y \in \partial\Omega_0, s > -c_0. \end{cases} \tag{4.3}$$

Let $t = h_0(s)$ be the inverse of the function $s = \int_0^t \frac{1}{\rho_0^2(\tau)} d\tau$. Then we have

$$\int_0^{h(s)} \frac{1}{\rho^2(\tau)} d\tau - c_0 = \int_0^{h_0(s)} \frac{1}{\rho_0^2(\tau)} d\tau, \ \forall s \geq \max\{-c_0, 0\},$$

and hence,

$$\lim_{s\to\infty} \int_{h(s)}^{h_0(s)} \frac{1}{\rho_0^2(\tau)} d\tau = \lim_{s\to\infty} \left(\int_0^{h(s)} \left(\frac{1}{\rho^2(\tau)} - \frac{1}{\rho_0^2(\tau)} \right) d\tau - c_0 \right) = 0.$$

This implies that $\lim_{s\to\infty}(h(s) - h_0(s)) = 0$. Letting $s \to \infty$, we have the following limiting system of (4.3) under the assumption (B3):

$$\begin{cases} \frac{\partial w}{\partial s} = D\Delta w - n \cdot \dot{\rho}_0(h_0(s))\rho_0(h_0(s))w + \rho_0^2(h_0(s))f(w), & y \in \Omega_0, s > 0, \\ w = 0, & y \in \partial\Omega_0, s > 0. \end{cases} \tag{4.4}$$

Let $T_1 = \int_0^T \frac{1}{\rho_0^2(\tau)} d\tau$. Since

$$s + T_1 = \int_0^{h_0(s)} \frac{1}{\rho_0^2(\tau)} d\tau + \int_{h_0(s)}^{h_0(s)+T} \frac{1}{\rho_0^2(\tau)} d\tau = \int_0^{h_0(s)+T} \frac{1}{\rho_0^2(\tau)} d\tau, \ \forall s \geq 0,$$

it follows that $h_0(s + T_1) = h_0(s) + T$ for all $s \geq 0$, and hence, (4.4) is T_1-periodic system for new time s.

To proceed, we divide the proof into three steps.

Step 1. For every $w_0 \in \mathbb{Y}$ and every $\tau \geq -c_0$, there exists a unique regular solution $\phi(s, \tau, w_0)$ of (4.3) satisfying $\phi(\tau, \tau, w_0) = w_0$ with its maximal interval of existence $I^+(\tau, w_0) \subset [\tau, \infty)$, and $\phi(s, \tau, w_0)$ is globally defined, provided that there is an $L^\infty-$bound on $\phi(s, \tau, w_0)$. Further, for any $w_0 \in \mathbb{Y}$, we let $\phi_0(s, \tau, w_0)$ be the unique solution of T_1-periodic system (4.4) with $\phi_0(\tau, \tau, w_0) = w_0$, and define $P(s)w_0 = \phi_0(s, 0, w_0)$ for all $s \geq 0$.

Let

$$\bar{f}(y, s, w) = -n \cdot \dot{\rho}(h(s))\rho(h(s))w + \rho^2(h(s))f(w),$$

and

$$\bar{f}_0(y, s, w) = -n \cdot \dot{\rho}_0(h_0(s))\rho_0(h_0(s))w + \rho_0^2(h_0(s))f(w).$$

Obviously, $\lim_{s\to\infty} |\bar{f}(y, s, w) - \bar{f}_0(y, s, w)| = 0$ uniformly for $y \in \overline{\Omega}_0$ and w in any bounded set of $\mathbb{R}^m$.

Since solutions of (4.3) and (4.4) are uniformly bounded in $\mathbb{Y}$, it follows from [28, Proposition 3.2.1] that for any given positive integer k and real number $r > 0$,

$$\lim_{n\to\infty} \|\phi(s + nT_1, nT_1, w_0) - P(s)w_0)\|_{\mathbb{X}} = 0$$

uniformly for $s \in [0, kT_1]$ and $\|w\| \leq r$. In particular, for any $w_0 \in \mathbb{Y}$, $\gamma^+(w_0) = \{\phi(nT_1, 0, w_0) : n \geq 0\}$ is precompact in $\mathbb{Y}$, and ϕ is an asymptotically periodic semiflow with limit periodic semiflow $\{P(s)\}_{s\geq 0}$.

Step 2. From Step 1, it follows that for any $\tau \geq 0$, $\phi(s, \tau, w_0)$ and $\phi_0(s, \tau, w_0)$ exist globally on $[\tau, \infty)$ and are uniformly bounded in $\mathbb{Y}$. Let $P_n(w_0) := \phi(nT_1, 0, w_0)$, $\forall w_0 \in \mathbb{Y}, n \geq 0$. In view of the conclusions in Step 1, we see that omega limit set $\omega(w_0)$ of $\gamma^+(w_0)$ exists. By [28, Theorem 3.2.1], it suffices to prove that $\lim_{n\to\infty} P_n(w_0) = 0$ for any $w_0 \in \mathbb{Y}$ in case (1), and $\lim_{n\to\infty} P_n(w_0) = u_T^*(\cdot, 0)$ for any $w_0 \in \mathbb{Y}\backslash\{0\}$ in case (2), respectively. Note that $P_n : \mathbb{Y} \to \mathbb{Y}, n \geq 0$, is an asymptotically autonomous discrete process with limit discrete semiflow $S^n : \mathbb{Y} \to \mathbb{Y}, n \geq 0$, where $S = P(T_1)$ is the Poincaré map associated with periodic system (4.4). Thus, [28, Lemma 1.2.2] implies that for any $w_0 \in \mathbb{Y}$, $\omega(w_0)$ is a chain transitive set for $S : \mathbb{Y} \to \mathbb{Y}$.

In case (1) where $\lambda_T \geq 0$, combining Proposition 4.1, one can acquire that 0 is a globally asymptotically stable fixed point of S, and hence, $\mathrm{W}^s(0) = \mathbb{Y}$, where $\mathrm{W}^s(0)$ is the stable set of 0 for S in $\mathbb{Y}$. It is easy to see that $\omega(w_0) \cap \mathbb{Y} \neq \emptyset$. Hence, $\omega(w_0) = 0$ due to [28, Theorem 1.2.1], which implies $\lim_{n\to\infty} P_n(w_0) = 0$.

Step 3. In case (2) where $\lambda_T < 0$, using Proposition 4.1 again, we deduce that $u_T^*(\cdot, 0)$ is a globally asymptotically stable fixed point of S in $\mathbb{Y}\backslash\{0\}$, so $\mathrm{W}^s(u_T^*(\cdot, 0)) = \mathbb{Y}\backslash\{0\}$, where $\mathrm{W}^s(u_T^*(\cdot, 0))$ is the stable set of $u_T^*(\cdot, 0)$ for S. Motivated by the proof of [28, Proposition 3.2.3], we have the following claim.

Claim. $\widetilde{\mathrm{W}}^s(0) \cap (\mathbb{Y}\backslash\{0\}) = \emptyset$.

Indeed, we assume, by contradiction, that there exists a $w_0 \in \widetilde{\mathrm{W}}^s(0) \cap (\mathbb{Y}\backslash\{0\})$, that is, $w_0 \in \mathbb{Y}\backslash\{0\}$ and $P_n(w_0) \to 0$ as $n \to \infty$. Clearly, $w(s) := \phi(s, 0, w_0)$ satisfies $w(s) \gg 0$ in $\mathbb{X}$ for all $s > 0$. It then follows from [28, Theorem 3.2.1] that $\lim_{s\to\infty} \|w(s)\|_{\mathbb{X}} = 0$, and hence, $\lim_{s\to\infty} \|w(s)\|_{C(\overline{\Omega}_0)} = 0$. As in the proof of Lemma 3.1, we fix a real number $k > 0$ such that $f'(0) + kI$ has positive diagonal entries. It then follows from the arguments for (6.10) in [5] that for any $\epsilon \in (0, 1)$, there exists a vector $v_\epsilon \gg 0$ in $\mathbb{R}^n$ such that

$$f(y) \geq f'(0)y - \epsilon(f'(0) + kI)y = (1 - \epsilon)(f'(0) + kI)y - ky, \quad \forall y \in [0, v_\epsilon]_{\mathbb{R}^n}.$$

Define

$$\begin{aligned} A(s, \epsilon) = &-n[\dot{\rho}_0(h_0(s))\rho_0(h_0(s)) + \epsilon]I + \\ &(\rho_0^2(h_0(s)) - \epsilon)(1 - \epsilon)(f'(0) + kI) - (\rho_0^2(h_0(s)) + \epsilon)kI, \end{aligned}$$

and let λ_ϵ be the principal eigenvalue of T_1-periodic parabolic eigenvalue problem associated with the T_1-periodic linear system:

$$\begin{cases} \frac{\partial w}{\partial s} = D\Delta w + A(s, \epsilon)w, & y \in \Omega_0, s > 0, \\ w = 0, & y \in \partial\Omega_0, s > 0. \end{cases} \tag{4.5}$$

Since $\lim_{\epsilon\to 0} \lambda_\epsilon = \frac{T}{T_1}\lambda_T < 0$, we can fix a small number $\epsilon \in (0, 1)$ such that $\lambda_\epsilon < 0$. Thus, there exists $N = N(\epsilon) > 0$ such that

$$|\dot{\rho}(h(s))\rho(h(s)) - \dot{\rho}_0(h_0(s))\rho_0(h_0(s)| < \epsilon, \quad |\rho^2(h(s)) - \rho_0^2(h_0(s))| < \epsilon,$$

and $w(s) \in [0, v_\epsilon]_{\mathbb{R}^n}$ for all $s \geq NT_1$. It then follows that $w(y, s) := w(s)(y)$ satisfies

$$\frac{\partial w}{\partial s} \geq D\Delta w + A(s, \epsilon)w, \quad \forall y \in \Omega_0, s \geq NT_1.$$

Let $\psi^\epsilon(y, s)$ be the positive eigenfunction corresponding to the principal eigenvalue

λ_ϵ, that is,

$$\begin{aligned}&\frac{\partial\psi^\epsilon}{\partial s} = D\Delta\psi^\epsilon + A(s,\epsilon)\psi^\epsilon + \lambda_\epsilon\psi^\epsilon \quad \text{in } \Omega_0\times(0,\infty),\\ &\psi^\epsilon = 0 \quad \text{on } \partial\Omega_0\times(0,\infty),\\ &\psi^\epsilon(y, s+T_1) = \psi^\epsilon(y,s).\end{aligned}$$

Since $w(\cdot, NT_1) \gg 0$ in $\mathbb{Y}$, there exists $\delta = \delta(\epsilon, w_0) > 0$ such that $w(\cdot, NT_1) \geq \delta\psi^\epsilon(\cdot, NT_1) = \delta\psi^\epsilon(\cdot, 0)$. By the standard comparison theorem, we get

$$w(\cdot, s) \geq \delta e^{-\lambda_\epsilon(s-NT_1)}\psi^\epsilon(\cdot, s-NT_1) \geq \delta\psi^\epsilon(\cdot, s), \quad \forall s \geq NT_1.$$

In particular, $w(\cdot, nT_1) \geq \delta\psi^\epsilon(\cdot, 0), \forall n \geq N$, which contradicts the assumption that $\lim_{n\to\infty} w(\cdot, nT_1) = 0$ in $\mathbb{Y}$. This proves the above claim.

Consequently, for any $w_0 \in \mathbb{Y}\backslash\{0\}$, one has $\omega(w_0) \cap (\mathbb{Y}\backslash\{0\}) \neq \emptyset$, which means that $\omega(w_0) \cap \mathrm{W}^s(u_T^*(\cdot, 0)) \neq \emptyset$. By [28, Theorem 1.2.1], it then follows that for any $w_0 \in \mathbb{Y}\backslash\{0\}$, $\omega(w_0) = u_T^*(\cdot, 0)$, which leads to $\lim_{n\to\infty} \|P_n(w_0) - u_T^*(\cdot, 0)\|_{\mathbb{X}} = 0$. □

Remark 4.3. Under the homogeneous Neumann boundary condition, the threshold type results Theorem 2.2 (in terms of λ_F), Theorem 3.6 (in terms of λ_I) and Theorem 4.2 (in terms of λ_T) also hold true, and three principal eigenvalues λ_F, λ_I and λ_T are determined, respectively, by two different autonomous ODE systems and one periodic ODE system. Indeed, the proofs of Theorems 2.2 and 4.2 are based on a dynamical systems approach, so they are valid for the homogeneous Neumann, Robin type and Dirichlet boundary conditions. Regarding Theorem 3.6, if the homogeneous Neumann boundary condition is imposed, it is much easier to construct the sub-solution for the reaction-diffusion system as they are given by exact solutions of the kinetic system. In such a case, the convergence in Theorem 3.6 can also be made uniform on $\bar{\Omega}_0$ up to the boundary.

5. An application. Dengue fever (DF) is a common mosquito-borne disease with Aedes being the primary vector. Owing to its growing serious threat to human health, the mathematical modeling of Dengue fever has drawn increasing attentions [6, 8, 4, 23, 21]. The authors of [31] proposed the following ordinary differential model of Dengue fever:

$$(5.1) \qquad \begin{cases} S_H'(t) = \mu_h N_H - \frac{\beta_H b}{N_H+m} S_H I_V + \gamma_H I_H - \mu_H S_H, \\ I_H'(t) = \frac{\beta_H b}{N_H+m} S_H I_V - \gamma_H I_H - \mu_H I_H, \\ S_V'(t) = A - \frac{\beta_V b}{N_H+m} S_V I_H - \mu_V S_V, \\ I_V'(t) = \frac{\beta_V b}{N_H+m} S_V I_H - \mu_V I_V, \\ S_H(0) > 0, I_H(0) \geq 0, S_V(0) > 0, I_V(0) \geq 0, \end{cases}$$

where the variables and parameters are described in Table 1:

Table 1 Description of variables and parameters of model (5.1)

Parameters	Description
S_H	the density of the susceptible class in the human population at time t
I_H	the density of the infectious class in the human population at time t
N_H	the total population size of human
S_V	the density of the susceptible class in the mosquito population
I_V	the density of the infectious class in the mosquito population
m	population density of other alternative hosts (such as pets)
μ_h	the birth rate of human
μ_H	the death rate of human
γ_H	the recovery rate of human
A, μ_V	the recruitment and the per capita mortality rate of mosquitoes, respectively
b	the biting rate of mosquitoes, namely, the average number of bites per mosquito per day
β_H	the contact transmission probability from infectious mosquitoes to susceptible humans
β_V	the contact transmission probability from infectious humans to susceptible mosquitoes

In model (5.1), the equations on S_H and I_H satisfy

$$\begin{cases} (S_H + I_H)'(t) = \mu_h N_H - \mu_H(S_H + I_H), \\ (S_H + I_H)(0) = S_H(0) + I_H(0) > 0, \end{cases}$$

which leads to $\lim_{t\to\infty}(S_H + I_H)(t) = \frac{\mu_h N_H}{\mu_H}$. Using the similar manner for the equations on S_V and I_V, we see that $\lim_{t\to\infty}(S_V + I_V)(t) = \frac{A}{\mu_V}$. Accordingly, a simplified dengue fever ODE model is as follows

$$\begin{cases} I_H'(t) = \frac{\beta_H b}{N_H+m}\big(\frac{\mu_h N_H}{\mu_H} - I_H\big)I_V - \gamma_H I_H - \mu_H I_H, \\ I_V'(t) = \frac{\beta_V b}{N_H+m}\big(\frac{A}{\mu_V} - I_V\big)I_H - \mu_V I_V, \\ 0 \le I_H(0) \le \frac{\mu_h N_H}{\mu_H}, \quad 0 \le I_V(0) \le \frac{A}{\mu_V}. \end{cases} \tag{5.2}$$

By the method of next generation matrix, we easily obtain the basic reproduction number of ODE system (5.1) or (5.2) as follows

$$\mathcal{R}_0 = \sqrt{\frac{\frac{\beta_H b}{N_H+m}\frac{\mu_h N_H}{\mu_H} \cdot \frac{\beta_V b}{N_H+m}\frac{A}{\mu_V}}{(\gamma_H + \mu_H)\mu_V}}. \tag{5.3}$$

In [32, 31], the authors considered the following dengue fever model in the asymptotically bounded and periodically evolving domain, respectively,

$$\begin{cases} \frac{\partial I_H}{\partial t} + \mathbf{a}\cdot\nabla I_H + I_H(\nabla\cdot\mathbf{a}) = d_H\Delta I_H + \frac{\beta_H b}{N_H+m}\big(\frac{\mu_h N_H}{\mu_H} - I_H\big)I_V - \gamma_H I_H - \mu_H I_H & \text{in } \Omega_t, \\ \frac{\partial I_V}{\partial t} + \mathbf{a}\cdot\nabla I_V + I_V(\nabla\cdot\mathbf{a}) = d_V\Delta I_V + \frac{\beta_V b}{N_H+m}\big(\frac{A}{\mu_V} - I_V\big)I_H - \mu_V I_V & \text{in } \Omega_t, \end{cases} \tag{5.4}$$

with the homogeneous Dirichlet boundary condition

$$I_H(x(t), t) = I_V(x(t), t) = 0 \quad \text{on } \partial\Omega_t, \tag{5.5}$$

and the initial condition

$$I_H = I_{H,0}(x),\ I_V = I_{V,0}(x) \quad \text{for } x \in \Omega_0 \text{ at } t = 0, \tag{5.6}$$

where $I_{H,0}(x) \le \frac{\mu_h N_H}{\mu_H}$ and $I_{V,0}(x) \le \frac{A}{\mu_V}$ are positive, continuous functions, and Ω_0 is the initial domain. According to Lagrangian transformation, we can write $I_H(x(t),t)$ and $I_V(x(t),t)$ as

$$I_H(x(t),t) = I_H(x_1(t), x_2(t), \cdots, x_n(t), t) = u(y_1, y_2, \cdots, y_n, t),$$
$$I_V(x(t),t) = I_V(x_1(t), x_2(t), \cdots, x_n(t), t) = v(y_1, y_2, \cdots, y_n, t).$$

Under the isotropic hypothesis (A3), system (5.4)-(5.6) is further transformed into
(5.7)
$$\begin{cases} u_t - \frac{d_H}{\rho^2(t)}\Delta u = \frac{\beta_H b}{N_H+m}\big(\frac{\mu_h N_H}{\mu_H} - u\big)v - (\gamma_H + \mu_H + \frac{n\dot\rho(t)}{\rho(t)})u, & y \in \Omega_0, t > 0,\\ v_t - \frac{d_V}{\rho^2(t)}\Delta v = \frac{\beta_V b}{N_H+m}\big(\frac{A}{\mu_V} - v\big)u - (\mu_V + \frac{n\dot\rho(t)}{\rho(t)})v, & y \in \Omega_0, t > 0,\\ u(y,t) = v(y,t) = 0, & y \in \partial\Omega_0, t > 0,\\ u(y,0) = I_{H,0}(x(0)) \le \frac{\mu_h N_H}{\mu_H}, \quad v(y,0) = I_{V,0}(x(0)) \le \frac{A}{\mu_V}, & y \in \Omega_0. \end{cases}$$

Obviously, if $\rho(t) \equiv 1$, then system (5.7) reduces to the traditional reaction-diffusion problem:
(5.8)
$$\begin{cases} u_t - d_H\Delta u = \frac{\beta_H b}{N_H+m}\big(\frac{\mu_h N_H}{\mu_H} - u\big)v - (\gamma_H + \mu_H)u, & y \in \Omega_0, t > 0,\\ v_t - d_V\Delta v = \frac{\beta_V b}{N_H+m}\big(\frac{A}{\mu_V} - v\big)u - \mu_V v, & y \in \Omega_0, t > 0,\\ u(y,t) = v(y,t) = 0, & y \in \partial\Omega_0, t > 0,\\ u(y,0) = I_{H,0}(x(0)) \le \frac{\mu_h N_H}{\mu_H}, \quad v(y,0) = I_{V,0}(x(0)) \le \frac{A}{\mu_V}, & y \in \Omega_0. \end{cases}$$

whose basic reproduction number has been analogously presented in [30, Theorem 2.3], that is,

$$\widehat{\mathcal{R}}_0 = \sqrt{\frac{\frac{\beta_H b}{N_H+m}\frac{\mu_h N_H}{\mu_H}\cdot\frac{\beta_V b}{N_H+m}\frac{A}{\mu_V}}{(d_H\lambda^* + \gamma_H + \mu_H)(d_V\lambda^* + \mu_V)}}, \tag{5.9}$$

where λ^* is the principal eigenvalue of the eigenvalue problem

$$-\Delta\psi = \lambda\psi, y \in \Omega_0, \quad \psi(y) = 0, y \in \partial\Omega_0. \tag{5.10}$$

In the rest of this section, we apply the analytical results in Sections 2–4 to reveal the long-time behaviors of system (5.7) under the different evolution trends of spatial domain.

5.1. The DF model in asymptotically bounded domain. In this subsection, the rate $\rho(t)$ in (5.7) is assumed to satisfy (B1). We first address the limiting system of system (5.7):
(5.11)
$$\begin{cases} u_t - \frac{d_H}{\rho_\infty^2}\Delta u = \frac{\beta_H b}{N_H+m}\big(\frac{\mu_h N_H}{\mu_H} - u\big)v - (\gamma_H + \mu_H)u, & y \in \Omega_0, t > 0,\\ v_t - \frac{d_V}{\rho_\infty^2}\Delta v = \frac{\beta_V b}{N_H+m}\big(\frac{A}{\mu_V} - v\big)u - \mu_V v, & y \in \Omega_0, t > 0,\\ u(y,t) = v(y,t) = 0, & y \in \partial\Omega_0, t > 0,\\ u(y,0) = I_{H,0}(x(0)) \le \frac{\mu_h N_H}{\mu_H}, \quad v(y,0) = I_{V,0}(x(0)) \le \frac{A}{\mu_V}, & y \in \Omega_0. \end{cases}$$

Let σ_F be the principal eigenvalue of the eigenvalue problem

$$\begin{cases} \frac{d_H}{\rho_\infty^2}\Delta\phi + \frac{\beta_H b}{N_H+m}\frac{\mu_h N_H}{\mu_H}\psi - (\gamma_H+\mu_H)\phi = \sigma\phi, & y\in\Omega_0, \\ \frac{d_V}{\rho_\infty^2}\Delta\psi + \frac{\beta_V b}{N_H+m}\frac{A}{\mu_V}\phi - \mu_V\psi = \sigma\psi, & y\in\Omega_0, \\ \phi(y)=\psi(y)=0, & y\in\partial\Omega_0. \end{cases} \tag{5.12}$$

By Proposition 2.1, we have the corresponding conclusion for system (5.11) in terms of σ_F.

Proposition 5.1. *The following statements are valid:*

(1) If $\sigma_F \le 0$, then the disease-free equilibrium $(0,0)$ is globally attractive for any nonnegative solution of (5.11) with $(u(y,0),v(y,0)) \in [0,\frac{\mu_h N_H}{\mu_H}]\times[0,\frac{A}{\mu_V}]$.

(2) If $\sigma_F > 0$, then system (5.11) has a unique positive epidemic equilibrium $(u_F^*(y),v_F^*(y))$, which is globally attractive for any positive solution of (5.11) with $(u(y,0),v(y,0)) \in [0,\frac{\mu_h N_H}{\mu_H}]\times[0,\frac{A}{\mu_V}]\backslash\{(0,0)\}$.

Following [25], we can employ the method of the next infection operator to define the basic reproduction number $R_0^{F(\rho)}$ of system (5.7) via (5.11) and (5.12). Additionally, the explicit expression of $R_0^{F(\rho)}$ is as follows (see, e.g., [31, Theorem 3.1])):

$$R_0^{F(\rho)} = \sqrt{\frac{\frac{\beta_H b}{N_H+m}\frac{\mu_h N_H}{\mu_H}\cdot\frac{\beta_V b}{N_H+m}\frac{A}{\mu_V}}{[\frac{d_H\lambda^*}{\rho_\infty^2}+\gamma_H+\mu_H][\frac{d_V\lambda^*}{\rho_\infty^2}+\mu_V]}}. \tag{5.13}$$

where λ^* is defined by (5.10). As a consequence of [28, Theorem 11.3.3], we have the following observation.

Lemma 5.2. $\mathrm{sign}(R_0^{F(\rho)}-1) = \mathrm{sign}(\sigma_F)$.

Thus, Theorem 2.2 and Lemma 5.2 give rise to the following result.

Theorem 5.3. *The nonautonomous system (5.7) admits the following threshold dynamics:*

(1) If $R_0^{F(\rho)} \le 1$, then $(0,0)$ is globally attractive for system (5.7) in $[0,\frac{\mu_h N_H}{\mu_H}]\times[0,\frac{A}{\mu_V}]$.

(2) If $R_0^{F(\rho)} > 1$, then $(u_F^*(y),v_F^*(y))$ is globally attractive for system (5.7) in $[0,\frac{\mu_h N_H}{\mu_H}]\times[0,\frac{A}{\mu_V}]\backslash\{(0,0)\}$.

5.2. The DF model in asymptotically unbounded domain. In this subsection, we assume that the rate $\rho(t)$ in (5.7) satisfies condition (B2). We start with the limiting ODE system

$$\begin{cases} \frac{du}{dt} = \frac{\beta_H b}{N_H+m}\big(\frac{\mu_h N_H}{\mu_H}-u\big)v - (\gamma_H+\mu_H+nk)u, & t>0, \\ \frac{dv}{dt} = \frac{\beta_V b}{N_H+m}\big(\frac{A}{\mu_V}-v\big)u - (\mu_V+nk)v, & t>0, \\ u(0)=u_0\le\frac{\mu_h N_H}{\mu_H}, \quad v(0)=v_0\le\frac{A}{\mu_V} \end{cases} \tag{5.14}$$

and let σ_I be the principal eigenvalue of the matrix

$$\begin{pmatrix} -(\gamma_H+\mu_H+nk) & \frac{\beta_H b}{N_H+m}\frac{\mu_h N_h}{\mu_H} \\ \frac{\beta_V b}{N_H+m}\frac{A}{\mu_V} & -(\mu_V+nk) \end{pmatrix}.$$

A straightforward computation shows that

$$\sigma_I = -(\gamma_1+\gamma_2)+\sqrt{(\gamma_1+\gamma_2)^2-4(\gamma_1\gamma_2-\beta_1\beta_2)} \tag{5.15}$$

where

$$\gamma_1=\gamma_H+\mu_H+nk, \qquad \gamma_2=\mu_V+nk,$$
$$\beta_1=\tfrac{\beta_H b}{N_H+m}\tfrac{\mu_h N_h}{\mu_H}, \qquad \beta_2=\tfrac{\beta_V b}{N_H+m}\tfrac{A}{\mu_V}.$$

As a straightforward consequence of Proposition 3.5, we have the following threshold type result on the dynamics of system (5.14).

PROPOSITION 5.4. *The following statements are valid for system (5.14):*

(1) If $\sigma_I\le 0$, then the disease free equilibrium $(0,0)$ of (5.14) is globally asymptotically stable in $[0,\frac{\mu_h N_H}{\mu_H}]\times[0,\frac{A}{\mu_V}]$.

(2) If $\sigma_I>0$, then system (5.14) has a unique epidemic equilibrium (u_I^*,v_I^*), and it is also globally stable in $[0,\frac{\mu_h N_H}{\mu_H}]\times[0,\frac{A}{\mu_V}]\backslash\{(0,0)\}$.

By using the method of next generation matrix [24] and the classical formula of basic reproduction number $R_0=r(FV^{-1})$, we can calculate out the number of model (5.14) as follows

$$R_0^{I(\rho)}=\sqrt{\frac{\beta_1\beta_2}{\gamma_1\gamma_2}}=\sqrt{\frac{\frac{\beta_H b}{N_H+m}\cdot\frac{\mu_h N_H}{\mu_H}\cdot\frac{\beta_V b}{N_H+m}\cdot\frac{A}{\mu_V}}{(\gamma_H+\mu_H+nk)(\mu_V+nk)}}. \tag{5.16}$$

With (5.15) and (5.16), one can easily verify that $\operatorname{sign}(R_0^{I(\rho)}-1)=\operatorname{sign}\sigma_I$. This, together with Theorem 3.6, implies the following result on the long-time behavior of system (5.7).

THEOREM 5.5. *The nonautonomous system (5.7) admits the following threshold dynamics:*

(1) If $R_0^{I(\rho)}\le 1$, then every solution $(u(y,t),v(y,t))$ of (5.7) satisfies $lim_{t\to\infty}(u(y,t),v(y,t))=(0,0)$ uniformly for $y\in\bar{\Omega}_0$, whenever $(u(\cdot,0),v(\cdot,0))\in[0,\frac{\mu_h N_H}{\mu_H}]\times[0,\frac{A}{\mu_V}]$.

(2) If $R_0^{I(\rho)}>1$, then every positive solution $(u(y,t),v(y,t))$ of (5.7) satisfies $lim_{t\to\infty}(u(y,t),v(y,t))=(u_I^*,v_I^*)$ uniformly for y in any compact subset of Ω_0, whenever $(u(\cdot,0),v(\cdot,0))\in[0,\frac{\mu_h N_H}{\mu_H}]\times[0,\frac{A}{\mu_V}]\backslash\{(0,0)\}$.

5.3. The DF model in asymptotically periodic domain. In this subsection,we assume that the rate $\rho(t)$ in (5.7) satisfies condition (B3). We first consider the auxiliary periodic system:

(5.17)
$$\begin{cases} u_t-\frac{d_H}{\rho_0^2(t)}\Delta u=\frac{\beta_H b}{N_H+m}\big(\frac{\mu_h N_H}{\mu_H}-u\big)v-(\gamma_H+\mu_H)u-\frac{n\dot\rho_0(t)}{\rho_0(t)}u, & y\in\Omega_0, t>0,\\ v_t-\frac{d_V}{\rho_0^2(t)}\Delta v=\frac{\beta_V b}{N_H+m}\big(\frac{A}{\mu_V}-v\big)u-\mu_V v-\frac{n\dot\rho_0(t)}{\rho_0(t)}v, & y\in\Omega_0, t>0,\\ u(y,t)=v(y,t)=0, & y\in\partial\Omega_0, t>0. \end{cases}$$

Let σ_T be the principal eigenvalue of the T-periodic parabolic eigenvalue problem:

$$\begin{cases} \phi_t - \frac{d_H}{\rho_0^2(t)}\Delta\phi = \frac{\beta_H b}{N_H+m}\frac{\mu_h N_H}{\mu_H}\psi - (\gamma_H+\mu_H)\phi - \frac{n\dot{\rho}_0(t)}{\rho_0(t)}\phi + \sigma\phi, & y\in\Omega_0, t>0,\\ \psi_t - \frac{d_V}{\rho_0^2(t)}\Delta\psi = \frac{\beta_V b}{N_H+m}\frac{A}{\mu_V}\phi - \mu_V\psi - \frac{n\dot{\rho}_0(t)}{\rho_0(t)}\psi + \sigma\psi, & y\in\Omega_0, t>0,\\ \phi(y,t)=\psi(y,t)=0, & y\in\partial\Omega_0, t>0,\\ \phi(y,t+T)=\phi(y,t), \psi(y,t+T)=\psi(y,t), & y\in\Omega_0, \end{cases}$$

In view of Proposition 4.1, we have the following the threshold dynamics for system (5.17).

Proposition 5.6. *For the periodic system (5.17), the following statements are valid:*

(1) If $\sigma_T \geq 0$, then the disease-free periodic solution $(0,0)$ of system (5.17) is globally attractive for nonnegative solution $(u(y,t),v(y,t))$ of system (5.17) in $[0,\frac{\mu_h N_H}{\mu_H}]\times[0,\frac{A}{\mu_V}]$, that is, $\lim_{t\to\infty}(u(y,t),\, v(y,t)) = (0,0)$ uniformly for $y\in\bar{\Omega}_0$.

(2) If $\sigma_T<0$, then system (5.17) admits a unique periodic epidemic equilibrium $(u_T^*(y,t), v_T^*(y,t))$, and every positive solution $(u(y,t),v(y,t))$ of it satisfies $\lim_{t\to\infty}(u(y,t)-u_T^*(y,t), v(y,t)-v_T^*(y,t)) = (0,0)$ uniformly for $y\in\bar{\Omega}_0$.

Following [11] with $\tau=0$, we define the basic reproduction number for system (5.7). We linearize problem (5.17) at $(0,0)$ to obtain

$$\begin{cases} \frac{\partial \mathbf{z}}{\partial t} - d(t)\Delta\mathbf{z} = \beta\cdot\mathbf{z} - \gamma(t)\mathbf{z}, & (y,t)\in\Omega_0\times(0,+\infty),\\ \mathbf{z}=0, & (y,t)\in\partial\Omega_0\times(0,+\infty), \end{cases}$$

where

$$\mathbf{z} = \begin{pmatrix} u \\ v \end{pmatrix}, \quad d(t) = \begin{pmatrix} \frac{d_H}{\rho_0^2(t)} & 0 \\ 0 & \frac{d_V}{\rho_0^2(t)} \end{pmatrix},$$

and

$$\beta = \begin{pmatrix} 0 & \frac{\beta_H b}{N_H+m}\frac{\mu_h N_H}{\mu_H} \\ \frac{\beta_V b}{N_H+m}\frac{A}{\mu_V} & 0 \end{pmatrix}, \quad \gamma(t) = \begin{pmatrix} \gamma_H+\mu_H+\frac{n\dot{\rho}_0(t)}{\rho_0(t)} & 0 \\ 0 & \mu_V + \frac{n\dot{\rho}_0(t)}{\rho_0(t)} \end{pmatrix}.$$

Let $W(t,s)$ be the evolution operator of the linear system:

$$\begin{cases} \frac{\partial \mathbf{z}}{\partial t} - d(t)\Delta\mathbf{z} = -\gamma(t)\mathbf{z}, & (y,t)\in\Omega_0\times(0,+\infty),\\ \mathbf{z}=0, & (y,t)\in\partial\Omega_0\times(0,+\infty). \end{cases}$$

Thanks to the standard semigroup theory, there exist the positive constants M and c_0 such that

$$\|W(t,s)\| \leq Me^{-c_0(t-s)}, \quad \forall\ t\geq s, t,s\in\mathbb{R}.$$

Furthermore, we assume that $\eta=(\eta_1,\eta_2)\in C_T$ is the density distribution of the infected human and infected mosquitoes at the spatial location $y\in\Omega_0$ and time s. We then define the linear operator $\mathcal{L}: C_T\times C_T \mapsto C_T\times C_T$,

$$\mathcal{L}(\eta)(t) := \int_{-\infty}^{t} W(t,s)\beta\cdot\eta(\cdot,s)ds = \int_0^\infty W(t,t-\tau)\beta\cdot\eta(\cdot,t-\tau)d\tau,$$

which is the next infection operator, and we know that the spectral radius of $\mathcal{L}$ is just the basic reproduction number $R_0^{T(\rho)}$ of problem (5.7), that is,

$$R_0^{T(\rho)} = r(\mathcal{L}). \tag{5.18}$$

Meanwhile, owing to [32, Theorem 2.2] and [11, Theorems 3.7 and 3.8], respectively, one has the estimated value of $R_0^{T(\rho)}$ and the relationship between $R_0^{T(\rho)}$ and σ_T as follows.

LEMMA 5.7. $\operatorname{sign}(R_0^{T(\rho)} - 1) = \operatorname{sign}(\sigma_T)$.

Remark 5.8. The basic reproduction number $R_0^{T(\rho)}$ meets with

$$R_0^{T(\rho)} \geq \sqrt{\frac{\frac{\beta_H b}{N_H+m}\frac{\mu_h N_H}{\mu_H} \cdot \frac{\beta_V b}{N_H+m}\frac{A}{\mu_V}}{(d_H\lambda^* \cdot \frac{1}{T}\int_0^T \frac{1}{\rho^2(t)}dt + \gamma_H + \mu_H)(d_V\lambda^* \frac{1}{T}\int_0^T \frac{1}{\rho^2(t)}dt + \mu_V)}}, \tag{5.19}$$

where λ^* is defined as in (5.10).

As a consequence of Lemma 5.7 and Theorem 4.2, we have the following result on the global dynamics of system (5.7).

THEOREM 5.9. *The nonautonomous model (5.7) admits the following threshold dynamics:*

(1) If $R_0^{T(\rho)} \leq 1$, then as $t \to \infty$, every nonnegative solution $(u(y,t), v(y,t))$ of (5.7) converges to the disease-free periodic solution $(0,0)$ of (5.17) uniformly for $y \in \bar{\Omega}_0$.

(2) If $R_0^{T(\rho)} > 1$, then every positive solution $(u(y,t), v(y,t))$ of (5.7) satisfies $\lim_{t\to\infty}(u(y,t) - u_T^*(y,t), v(y,t) - v_T^*(y,t)) = (0,0)$ uniformly for $y \in \bar{\Omega}_0$.

5.4. Discussion. In this section, we have used the analytical results in Sections 2–4 to study a reaction-diffusion model of Dengue fever transmission where the domain is time-varying and is given by $\Omega_t = \rho(t)\Omega_0$ with some fixed bounded smooth domain Ω_0. Our results illustrated the impact of domain evolution on the global dynamics of such a system. The threshold dynamics determining the spread of Dengue fever is characterized by the basic reproduction numbers of the respective models, given by $\mathcal{R}_0$ in (5.3) (for the ordinary differential equations model) and $\widehat{\mathcal{R}}_0$ in (5.9) (for the case of $\rho(t) \equiv 1$), respectively.

Moreover, we consider three different domain evolution scenarios: (i) asymptotically bounded domain where $\rho(t)$ increases from 1 to a finite value $\rho(\infty)$ (Section 2); (ii) asymptotically unbounded domain where $\dot{\rho}(t)/\rho(t) \approx k$ for some constant $k > 0$ (Section 3); and (iii) asymptotically periodic domain where $\rho(t)$ is asymptotic to a positive T-periodic function $\rho_0(t)$ (Section 4). The corresponding basic reproduction numbers are given by (5.13), (5.16) and (5.18), respectively. To describe our result, we recall that $\mathcal{R}_0$ (resp. $\widehat{\mathcal{R}}_0$) is the basic reproduction number of the ordinary differential equation model (5.2) (resp. of the partial differential equation model (5.8) with fixed domain Ω_0).

Firstly, for the case of asymptotically bounded domain, we deduce that the basic reproduction number $R_0^{F(\rho)}$ satisfies

$$\widehat{\mathcal{R}}_0 \leq R_0^{F(\rho)} < \mathcal{R}_0,$$

owing to (5.3), (5.9) and (5.13). Next, since Ω_t is assumed to be nondecreasing, this suggests that the basic reproduction number of Dengue fever is increased with increased domain size. This is consistent with our finding in Section 3 for the case of asymptotically unbounded domains. In this case, the basic reproduction number $R_0^{I(\rho)}$ of the evolving domain Ω_t can again be bounded from above by the corresponding number $\mathcal{R}_0$ of the ordinary differential equation model (5.2), owing to (5.2) and (5.16), that is,

$$R_0^{I(\rho)} < \mathcal{R}_0.$$

Note that $\mathcal{R}_0$ may be regarded as the basic reproduction number for $\Omega = \mathbb{R}^n$. Also, by Proposition 5.4 and Theorem 5.5, we see that the long-time dynamics of the solution of (5.7) respect to the local uniform topology is consistent with that of another ordinary differential equation model (5.14). Finally, in Section 5.3 we studied the case of asymptotically periodic domains and obtained an estimate for the basic reproduction number $R_0^{T(\rho)}$, which is given in (5.19). Thanks to Remark 5.8, we show that

$$\widehat{\mathcal{R}}_0 \leq R_0^{T(\rho)}$$

when a type of harmonic mean value $\bar{\rho} := \frac{1}{T}\int_0^T \frac{1}{\rho^2(t)}\,dt \leq 1$.

Due to the independence of $d_H\lambda^*$, $d_V\lambda^*$ and nk, as well as the lack of an explicit expression for $R_0^{T(\rho)}$, we are unable to analytically determine the relationship between $R_0^{I(\rho)}$ and $\widehat{\mathcal{R}}_0$, and the one between $R_0^{T(\rho)}$ and $\mathcal{R}_0$.

The global dynamics of time-varying domains equipped with more general boundary conditions, such as the Robin boundary condition, is also an interesting problem for future investigation. In conclusion, both the theoretical findings in Sections 2–4 and the specific example in this section fully demonstrate that the domain evolution may have a significant impact on the global dynamics of the model systems.

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