Propagation phenomena and reaction–diffusion systems for population dynamics in homogeneous or periodic media

Léo Girardin

Sorbonne Université

Laboratoire Jacques-Louis Lions

Phénomènes de propagation et systèmes de réaction – diffusion pour la dynamique des populations en milieu homogène ou périodique

Thèse de doctorat de Sorbonne Université

présentée et soutenue publiquement le 03/07/2018 par

Léo Girardin

pour obtenir le titre de

Docteur de Sorbonne Université

(Spécialité Mathématiques)

après avis des rapporteurs

Danielle Hilhorst
Yuan Lou

devant un jury composé de

Vincent Calvez	Directeur de thèse
Elaine C. M. Crooks	Examinatrice
Florence Débarre	Examinatrice
Adrian Lam	Examinateur
Yuan Lou	Rapporteur
Grégoire Nadin	Directeur de thèse
Benoît Perthame	Examinateur
Jean-Michel Roquejoffre	Examinateur

École doctorale de Sciences Mathématiques de Paris Centre

Résumé

Cette thèse est dédiée à l'étude des propriétés de propagation de différents systèmes de réaction – diffusion issus de la dynamique des populations.

Dans la première partie, on étudie la limite de forte compétition de systèmes compétitifs diffusifs à deux espèces. À l'aide de la ségrégation spatiale, on détermine le signe de la vitesse de l'onde progressive bistable. La généralisation aux ondes pulsatoires bistables en milieu spatialement périodique est ensuite envisagée afin d'étudier le rôle de l'hétérogénéité spatiale. Après avoir donné une condition suffisante pour l'existence de telles ondes ainsi qu'une condition suffisante pour l'existence d'états stationnaires stables susceptibles au contraire de bloquer la propagation, on montre que quand une famille d'ondes pulsatoires fortement compétitives existe, on peut établir un résultat très semblable à celui obtenu en milieu homogène.

Dans la seconde partie, des systèmes de type KPP à un nombre arbitraire d'espèces sont considérés. On étudie l'existence d'états stationnaires et d'ondes progressives, les propriétés qualitatives de ces solutions ainsi que la vitesse asymptotique de propagation de certaines solutions du problème de Cauchy. Cela résout notamment plusieurs questions ouvertes sur les systèmes de mutation – compétition – diffusion, qui constituent le prototype de système de type KPP.

Dans la troisième et dernière partie, on revient aux systèmes compétitifs diffusifs à deux espèces. Considérant cette fois-ci le cas monostable, on étudie les vitesses asymptotiques de propagation de certaines solutions du problème de Cauchy et, ce faisant, on montre l'existence de terrasses de propagation décrivant l'invasion d'un territoire inhabité par un compétiteur faible mais rapide suivie de l'invasion de ce territoire par un compétiteur fort mais lent.

Mots clés : systèmes de réaction – diffusion, phénomènes de propagation, dynamique des populations.

Abstract

This thesis is dedicated to the study of propagation properties of various reaction–diffusion systems coming from population dynamics.

In the first part, we study the strong competition limit of competition–diffusion systems with two species. Thanks to the spatial segregation, we determine the sign of the speed of the bistable traveling wave. The generalization to bistable pulsating fronts in spatially periodic media is then considered in order to study the role of spatial heterogeneity. We find a condition sufficient for the existence of such fronts as well as a condition sufficient for the existence of stable steady states which might on the contrary block the propagation. Then we show that whenever a family of strongly competing pulsating fronts exists, we can establish a result very similar to the one obtained in homogeneous media.

In the second part, systems of KPP type with any number of species are considered. We study the existence of steady states and traveling waves, the qualitative properties of these solutions as well as the asymptotic speed of spreading of certain solutions of the Cauchy problem. This settles several open questions on the prototypical KPP systems that are mutation–competition–diffusion systems.

In the third part, we go back to competition–diffusion systems with two species. Considering this time the monostable case, we study the asymptotic speeds of spreading of certain solutions of the Cauchy problem. By so doing, we show the existence of propagating terraces describing the invasion of an uninhabited territory by a weak but fast competitor followed by the invasion by a strong but slow competitor.

Keywords: reaction–diffusion systems, propagation phenomena, population dynamics.

Remerciements

Je remercie en premier lieu très chaleureusement mes directeurs de thèse Grégoire Nadin et Vincent Calvez. Tous deux m'ont tant apporté ! Ils m'ont apporté des mathématiques, évidemment, ainsi qu'un gout et une curiosité pour celles-ci et leurs applications à la biologie. Mais la réussite d'un doctorat n'est pas uniquement l'affaire d'un bon encadrement scientifique. Pour que ces longues années se déroulent bien, il faut pouvoir respirer et varier les plaisirs, ce qui exige des directeurs de thèse une écoute toute particulière. Les miens se sont acquittés de cette tâche avec brio. Ils ont été compréhensifs quand j'avais besoin de vacances, curieux quand je me suis lancé dans mon second master, disponibles quand la bureaucratie administrative me mettait des bâtons dans les roues, généreux quand il a fallu financer ma participation à des conférences, des écoles d'été et même un long stage pré-post-doctoral aux États-Unis, intéressés quand, par envie de m'échapper brièvement, j'orientais la discussion de la pause café vers des sujets n'ayant a priori pas leur place dans un laboratoire de mathématiques. Il parait que certaines thèses sont accouchées dans la douleur et la conflictualité ; quant à moi, je ne peux que me dire avec regret « déjà fini ? ». Grégoire, Vincent, vous avoir fait confiance dès 2012 a été rien de moins qu'une des meilleures décisions de ma vie. Ce fut un plaisir et un honneur que j'espère partagés !

Un immense merci à Adrian Lam, qui me fait l'honneur de participer au jury aujourd'hui. Adrian m'a accueilli pendant trois mois à Columbus, Ohio, et a été un hôte et un collaborateur exceptionnel. Un directeur de thèse suppléant, si j'ose dire, et tout à fait à la hauteur de Grégoire et Vincent. *Thank you Adrian. You managed to change a slightly worrying journey in redneck territory into a wonderful memory. I really did not expect that trip to the Niagara Falls!*

Cela m'amène naturellement à remercier vivement Yuan Lou, qui a accepté de rapporter cette thèse et me fait lui aussi l'honneur d'être présent aujourd'hui. Yuan a participé à mon accueil à Columbus avec l'hospitalité et la bonne humeur qui le caractérisent. *Thank you Yuan. Thanksgiving at your home would have been a delight and is certainly my only regret regarding my stay in Columbus!*

Je remercie chaleureusement Danielle Hilhorst qui, après m'avoir offert plusieurs fois l'opportunité de présenter mes travaux devant un public international, a accepté de superviser mon postdoctorat à l'université Paris-Sud ainsi que de rapporter cette thèse. J'espère être à la hauteur de ses attentes.

Les autres membres du jury, Elaine Crooks, Florence Débarre, Benoît Perthame et Jean-Michel Roquejoffre méritent de même chacune et chacun des remerciements appuyés. Elaine, *thank you*, pour les fructueux échanges que nous avons eu par voie électronique. Florence, pour une collaboration en cours que je crois prometteuse et surtout pour avoir accepté de participer en tant que biologiste à une soutenance de thèse de mathématiques encore peu teintées de biologie. Benoît, pour un accueil franc et amical au Laboratoire Jacques-Louis Lions et pour être, indirectement à travers Vincent, partiellement responsable du succès de ce doctorat. Jean-Michel, pour un accueil là aussi franc et amical dans les projets ERC ReaDi et ANR Nonlocal.

Après avoir remercié un « grand-père » et un « oncle – grand-oncle », il est naturel de remercier

le reste de la famille. Je pense tout particulièrement à Henri Berestycki, mon « grand-père – arrière-grand-père » qui m'a notamment autorisé à mener une passionnante enquête sociologique au sein de son équipe, ainsi qu'à François Hamel, mon « grand-père – oncle » (la famille, ce n'est plus ce que c'était. . .) qui a financé à travers le projet ANR Nonlocal mon séjour à Columbus. Je pense aussi à mes « demi-frères » et « demi-sœurs », Emeric, Juliette, Nils, Alvaro, Florian, Idriss, Ana. Je n'oublie évidemment pas les autres membres de la « famille » avec qui j'ai eu le plaisir d'échanger mais, vous en conviendrez bien volontiers, vous êtes trop nombreux pour une liste exhaustive. Alessandro et Quentin méritent cependant d'être nommés explicitement, l'un pour une collaboration qui forme un chapitre entier de cette thèse et l'autre pour avoir lancé l'idée folle du séminaire SIMBAD.

Après les « liens du sang », je passe à ceux du sol : merci à celles et ceux qui, au Laboratoire Jacques-Louis Lions, ont égayé mon doctorat ou m'ont apporté un soutien bienvenu. Catherine, Malika et Salima, pour une tenue irréprochable et accueillante du secrétariat qui me fait regretter d'avoir eu des interactions avec d'autres secrétariats. Benoît, encore une fois, pour son accueil. Ayman, pour n'avoir jamais perdu patience quand, recherchant son co-bureau, je l'interrompais dans son travail. Laurent, pour sa disponibilité et son humour jamais épuisés. Jean-François et Nina, responsables successifs de l'UE 2M216, pour m'avoir laissé une liberté pédagogique enthousiasmante. Mes co-bureaux, Thomas, Cécile, Fabien, Katia, Nicolas, Nacime, pour une ambiance excellente, studieuse et détendue simultanément, avec peu d'objets volants non-identifiés. Et évidemment, l'ensemble des doctorants, doctorantes, postdoctorants et postdoctorantes ! Malheureusement, il faut donner des noms et faire des jaloux, alors, par souci de brièveté, je vais m'en tenir à celles et ceux qui m'ont si bien accueilli au laboratoire : Cécile, Nicolas, Lucile, Guillaume, Geneviève, Antonin.

Je remercie celles et ceux qui m'ont enseigné les mathématiques de la Sup au M2. Mention spéciale pour Laurent Desvillettes, qui a été mon professeur d'analyse fonctionnelle à l'École Normale Supérieure de Cachan et est largement responsable de la suite de mon parcours puisque c'est lui qui m'a donné le gout des EDP et m'a orienté à l'époque vers Vincent Calvez pour un stage de M1. Même une fois nos parcours respectifs à l'ENS achevés, il est resté très amical et disponible et a notamment participé au comité de suivi de cette thèse, ce dont je lui sais gré.

Parler de mes années de normalien me conduit ensuite à remercier les amis cachanais, trop nombreux pour une liste exhaustive mais que je peux grossièrement grouper ainsi : la Tribu, les Machiavéliques, les Charlies (avant que ça ne soit la mode !), les Alices et les Beekeepers. Mention spéciale à Romain, Julie, Clément et Damien, qui ont eu la lourde tâche de vivre sous le même toit que moi.

J'adresse de plus mille mercis aux amis qui, en occupant sans relâche mon temps libre, m'ont fourni les exutoires nécessaires. Les intéressés se reconnaitront : on en trouve boulevard Charonne, rue d'Aubervilliers, couloir 65–66, sur des tatamis ou encore autour d'une table de jeu.

Je n'oublie pas ma véritable famille, qui m'a soutenu, toujours et partout, inconditionnellement, malgré sa peur des mathématiques et son ignorance du milieu universitaire. Merci pour cette solidarité sans faille.

Enfin, le plus grand des mercis à la meilleure d'entre nous, Lucie.

Table des matières

Introduction

Cadre général, état de l'art et objectifs

Définition mathématique des systèmes de réaction – diffusion

Un *système de réaction – diffusion* tel qu'entendu dans cette thèse est un système d'*équations aux dérivées partielles* (EDP dans la suite, *PDE* en anglais) de la forme

$$\frac{\partial \mathbf{u}}{\partial t} - \mathbf{D}\Delta_x \mathbf{u} = \mathbf{f}\left(\mathbf{u}, t, x\right),$$

où le vecteur colonne $\mathbf{u}$ est une fonction d'une variable temporelle réelle $t \in \mathbb{R}$ et d'une variable spatiale euclidienne $x \in \mathbb{R}^n$, $\frac{\partial}{\partial t}$ désigne la dérivée partielle par rapport à t (on privilégiera l'écriture compacte ∂_t dans la suite), Δ_x désigne le Laplacien spatial, c'est-à-dire la somme des dérivées partielles secondes $\frac{\partial^2}{\partial x_i^2}$ avec $x = (x_i)_{i\in\{1,\dots,n\}}$ (on privilégiera l'écriture compacte Δ dans la suite), $\mathbf{D}$ est une matrice diagonale à coefficients diagonaux strictement positifs appelée *matrice de diffusion* et $\mathbf{f}$ est une fonction possiblement non-linéaire en $\mathbf{u}$ appelée *terme de réaction*. La matrice $\mathbf{D}$ étant diagonale, l'éventuel couplage entre les équations est réalisé par le terme de réaction et n'implique aucune dérivée partielle de $\mathbf{u}$: le système est dit *faiblement couplé*. Par ailleurs, le système peut être vu comme un système d'équations de la chaleur avec second membre et est donc dit *parabolique*.

Plus généralement, le système peut ne gouverner l'évolution de $\mathbf{u}$ qu'à partir d'un certain temps initial $t_0 \in \mathbb{R}$, que jusqu'à un certain temps final $T \in \mathbb{R}$ ou encore que dans un certain domaine spatial $\Omega \subset \mathbb{R}^n$. Dans ce cas, l'ensemble de définition de $(t,x) \mapsto \mathbf{u}(t,x)$ est restreint en conséquence et l'on adjoint au système de réaction – diffusion des conditions initiales, finales ou de bord. En particulier, un problème formé d'un système de réaction – diffusion posé dans $(t_0,+\infty)\times\mathbb{R}^n$ accompagné d'une condition initiale est appelé *problème de Cauchy*. Les solutions définies sur $\mathbb{R}\times\mathbb{R}^n$ sont appelées *solutions entières*.

Dans le cas où $\mathbf{u}$ est en fait une quantité scalaire, on obtient une unique équation de réaction – diffusion de la forme

$$\partial_t u - d\Delta u = f\left(u,t,x\right).$$

Dans le cas où $\mathbf{u}$ et $\mathbf{f}$ ne dépendent pas de x, on obtient un système d'*équations différentielles ordinaires* (EDO dans la suite, *ODE* en anglais) de la forme

$$\mathbf{u}' = \mathbf{f}\left(\mathbf{u},t\right).$$

Dans le cas où $\mathbf{u}$ et $\mathbf{f}$ ne dépendent pas de t, on obtient un système d'EDP *elliptiques* faiblement couplé de la forme

$$-\mathbf{D}\Delta\mathbf{u} = \mathbf{f}\left(\mathbf{u},x\right).$$

Si **f** ne dépend que de sa variable **u**, le système est dit *posé en milieu homogène*. Dans le cas contraire, le système est dit *posé en milieu hétérogène* (spatialement ou temporellement). De la même manière, un système pourra être posé en milieu (spatialement ou temporellement) périodique, aléatoire, et ainsi de suite. Un milieu homogène est un cas (très) particulier de milieu périodique ou aléatoire.

Les systèmes de réaction – diffusion en tant que modèles de dynamique des populations

La branche de l'écologie, et donc de la biologie, qui s'intéresse à la fluctuation dans le temps du nombre d'individus au sein d'une population d'êtres vivants non-humains est la *dynamique des populations* non-humaines (ci-après simplement dynamique des populations ; la dynamique des populations humaines, qui doit prendre en compte des aspects socio-économiques, n'est plus à proprement parler de la biologie et n'est pas l'objet de cette thèse). De par son caractère quantitatif, ses origines historiques (lire à ce sujet l'ouvrage de Bacaër [9]) et sa tendance à l'abstraction, il s'agit d'une des branches les plus mathématisées de la biologie. Deux grandes familles de modèles mathématiques existent en dynamique des populations (et plus généralement en biologie) : les modèles déterministes et les modèles stochastiques. Parmi les modèles déterministes, on trouve de nombreux systèmes de réaction – diffusion.

Les systèmes de réaction – diffusion émergent en tant que modèles de dynamique des populations essentiellement de deux manières : soit à la suite des équations de réaction – diffusion scalaires, soit à la suite des systèmes d'EDO. Dans le premier cas, il s'agit de prendre en compte des couplages entre différentes populations, tandis que dans le second cas, il s'agit d'ajouter une structure spatiale au problème et de prendre en compte la dispersion des individus. À titre d'exemple et parce qu'il s'agit cependant d'exemples particulièrement importants pour la suite, la modélisation sous-jacente à une équation de réaction – diffusion particulière, l'équation de Fisher – Kolmogorov – Petrovsky – Piskunov (Fisher – KPP ou simplement KPP dans la suite),

$$\partial_t u - \Delta u = u\,(1-u)\,,$$

ainsi que la modélisation sous-jacente à un système d'EDO particulier, le système de Lotka – Volterra compétitif à deux espèces,

$$\begin{cases} u' = u\,(1-u-av) \\ v' = rv\,(1-v-bu) \end{cases},$$

où a, b et r sont des constantes strictement positives, vont maintenant être détaillées.

Les hypothèses suivantes sont communes aux deux modèles :

1. le nombre d'individus dans une population ainsi que les échelles spatiales et temporelles sont suffisamment grands pour que le nombre d'individus, qui est par essence une quantité discrète, soit correctement approché par une densité de population continue ;
2. les nouveaux-nés deviennent instantanément adultes ou, de manière équivalente, les nouveaux nés n'influencent pas la démographie et les individus ne sont comptés qu'à partir de l'âge adulte (pas de structure en âge) ;
3. si la reproduction est sexuée, la distribution des mâles et des femelles est homogène, de sorte qu'il suffit de connaitre la densité totale pour connaitre exactement la population (pas de structure sexuée).

Pour aboutir à l'équation de Fisher – KPP, on se dote d'une unique densité de population u et on suppose que :

1. la population diffuse dans l'espace avec un taux $d > 0$, ou en d'autres termes le flux de population est proportionnel au gradient de population avec un coefficient de proportionnalité $-d$;
2. en un point de l'espace donné, la partie de la variation de la densité de population $\frac{\partial_t u}{u}$ due aux naissances et aux morts est *logistique*, c'est-à-dire a la forme $r\left(1 - \frac{u}{K}\right)$ avec $r > 0$ et $K > 0$. Cette hypothèse implique la présupposition suivante, que les écologistes appellent absence d'*effet Allee* : du fait de la compétition pour les ressources entre individus, le taux de croissance de la population est une fonction strictement décroissante de la densité, positive si et seulement si $u \leq K$ et maximale en $u = 0$ où elle vaut r. Par conséquent, les constantes r et K sont respectivement appelées taux de croissance intrinsèque et capacité de charge.

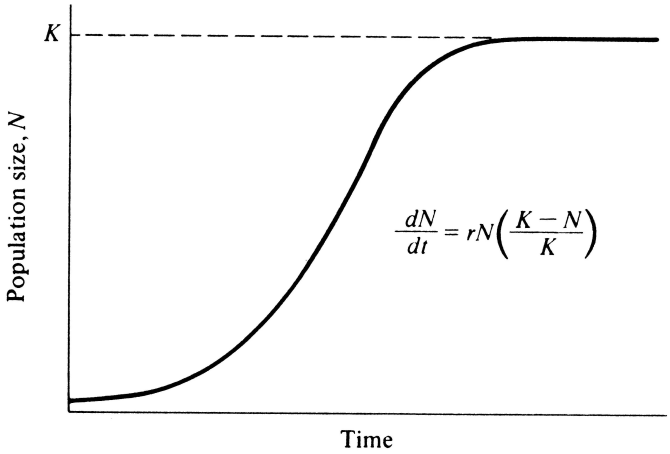


FIGURE 0.0.1 – Courbe de croissance logistique

Ces hypothèses conduisent à l'équation

$$\partial_t u - d\Delta u = r\left(1 - \frac{u}{K}\right)u.$$

En posant les quantités adimensionnelles $\tilde{u} = \frac{u}{K}$, $\tilde{t} = rt$ et $\tilde{x} = \sqrt{\frac{r}{d}}x$ puis en se débarrassant des $\sim$, on obtient bien l'équation de Fisher – KPP normalisée.

Pour aboutir au système de Lotka – Volterra compétitif, on se dote de deux densités de population u et v et on suppose que :

1. en l'absence de l'autre densité, chaque densité croît de manière logistique ;
2. quand les deux densités sont présentes, du fait de la compétition interspécifique, on retranche à chaque taux de croissance un terme supplémentaire positivement proportionnel à la densité du compétiteur (autrement dit, chaque taux de croissance est désormais une fonction affine strictement décroissante d'une certaine combinaison linéaire, à coefficients strictement positifs, des deux densités).

Ces hypothèses conduisent au système :

$$\begin{cases} u' = r_1 u\left(1 - \frac{u}{K_1} - \frac{v}{L_1}\right) \\ v' = r_2 v\left(1 - \frac{v}{K_2} - \frac{u}{L_2}\right) \end{cases}.$$

En posant $\tilde{t} = r_1 t$, $\tilde{u} = \frac{u}{K_1}$, $a = \frac{K_2}{L_1}$, $r = \frac{r_2}{r_1}$, $\tilde{v} = \frac{v}{K_2}$ et $b = \frac{K_1}{L_2}$ puis en se débarrassant des $\sim$, on obtient bien le système de Lotka – Volterra compétitif à deux espèces normalisé.

Toutes ces hypothèses faites à l'échelle de la densité de population, dite macroscopique, ont également des interprétations à l'échelle des individus, dite microscopique. Pour plus de détails sur les hypothèses microscopiques ainsi que l'histoire de la compétition entre modélisation macroscopique et modélisation microscopique, lire par exemple l'ouvrage d'Israel [98]. Discuter le bien-fondé de ces hypothèses est évidemment crucial lors de mises en application mais n'est pas l'objet de cette thèse.

En couplant de manière compétitive deux équations de Fisher – KPP, ou encore en ajoutant de la diffusion spatiale dans le système de Lotka – Volterra, on obtient finalement un premier exemple de système de réaction – diffusion : le système de compétition – diffusion de Lotka – Volterra à deux espèces,

$$\begin{cases} \partial_t u - \Delta u = u\left(1 - u - av\right) \\ \partial_t v - d\Delta v = rv\left(1 - v - bu\right) \end{cases}.$$

On précise d'ores et déjà qu'en autorisant a et b à changer de signe, on obtient deux autres importants exemples de couplage : prédation ($ab < 0$) et mutualisme ($a < 0$, $b < 0$). Plus généralement, un couplage est dit *de Lotka – Volterra* s'il a la forme $\mathbf{u} \circ (\mathbf{C}\mathbf{u})$ avec $\mathbf{C}$ une matrice carrée et $\circ$ le produit composante par composante de deux vecteurs, dit *produit de Hadamard.*

On note également que, là où les motivations de Fisher [72] et de Kolmogorov, Petrovsky et Piskunov [104] relevaient de la génétique des populations et plus précisément de problèmes de compétition entre deux allèles, la dérivation purement démographique proposée ci-dessus introduit directement l'équation de Fisher – KPP en tant qu'équation logistique diffusive. Cette dérivation plus tardive est due à Skellam [134]. L'immense majorité des études récentes de l'équation de Fisher – KPP dans la littérature de mathématiques appliquées ou d'écologie est motivée par le modèle de Skellam et non par le modèle génétique originel.

Réaction – diffusion et phénomènes de propagation

Une des principales raisons du succès des modèles de réaction – diffusion en dynamique des populations est leur capacité à décrire des invasions.

Ondes progressives, méthodes EDO, méthodes EDP

S'intéresser à des invasions à vitesse et direction constante et en milieu homogène conduit naturellement à s'intéresser aux solutions entières de la forme $\mathbf{u} : (t, x) \mapsto \boldsymbol{\varphi}\left(x \cdot e - ct\right)$, avec $e \in \mathbb{S}^{n-1}$ une direction de propagation, $c \in \mathbb{R}$ une vitesse de propagation et $\boldsymbol{\varphi}$ un profil de propagation. Les solutions de cette forme sont généralement appelées *ondes progressives* (ou plus précisément ondes progressives planes quand on considère un milieu multidimensionnel pouvant également accueillir des ondes progressives plus variées, comme les ondes progressives radiales ou coniques).

Une telle onde progressive satisfait un système d'EDO de la forme

$$-\mathbf{D}\boldsymbol{\varphi}'' - c\boldsymbol{\varphi}' = \mathbf{f}\left(\boldsymbol{\varphi}\right).$$

Grâce à cette observation, l'existence et les propriétés de telles solutions peuvent être traitées avec des méthodes issues de la littérature sur les EDO (théorème de Cauchy – Lipschitz, méthode de tir, variétés stables et instables, et ainsi de suite) ou avec des méthodes issues de la littérature sur les EDP elliptiques (théorie de Schauder, calcul variationnel, principe du maximum et de comparaison, inégalités de Harnack, et ainsi de suite).

La littérature sur les ondes progressives peut donc être scindée en deux familles, selon le type d'arguments (EDO ou EDP) utilisé. Le parti pris de cette thèse est d'utiliser autant que possible des méthodes EDP afin de pouvoir généraliser les preuves à des milieux hétérogènes convenables et afin également de pouvoir traiter dans la foulée les propriétés de propagation des problèmes de Cauchy, qui elles ne peuvent être abordées qu'avec des arguments de type EDP.

Dimension du milieu

Lors d'études d'ondes progressives en milieu homogène, on peut supposer sans perte de généralité que le milieu est unidimensionnel et que $e = +1$, ce qui simplifie les notations et sera donc systématiquement fait dans la suite. Dans ce cadre, l'unicité d'une onde progressive doit être entendue comme unicité à rotation de e et à translation de $\boldsymbol{\varphi}$ près.

Évidemment, de telles simplifications ne peuvent être réalisées dans le cadre d'études de problèmes de Cauchy avec des données initiales non-unidimensionnelles, pour lesquelles sera donc précisée la dimension du milieu.

Équations scalaires

Les recherches sur les invasions produites par des équations de réaction – diffusion ont débuté en 1937. Les résultats fondateurs sont les théorèmes suivants.

Théorème. *[104] La solution u d'un problème de Cauchy associé à l'équation de Fisher – KPP unidimensionnelle avec donnée initiale bornée positive non-nulle à support compact satisfait*

$$\lim_{t\to+\infty} \sup_{|x|>ct} u(t,x) = 0 \text{ pour toute vitesse } c > 2,$$

$$\lim_{t\to+\infty} \sup_{|x|<ct} u(t,x) = 1 \text{ pour toute vitesse } c < 2.$$

Les écologistes s'intéressent bel et bien à des invasions de population initialement confinées dans l'espace, introduites à un endroit précis. En établissant qu'une telle invasion se déroule asymptotiquement à vitesse constante, ce premier théorème montre que les solutions entières intéressantes sont en effet les ondes progressives et conduit donc à un second théorème.

Définition. Une onde progressive de vitesse positive décrivant l'invasion de 0 par 1 pour l'équation de Fisher – KPP est une onde progressive dont le profil est strictement décroissant et a pour limite en $-\infty$ et $+\infty$ 1 et 0 respectivement.

Théorème. *[104] L'équation de Fisher – KPP admet une solution sous forme d'onde progressive à vitesse $c \geq 0$ décrivant l'invasion de 0 par 1 si et seulement si $c \geq 2$. Cette solution est unique.*

En retour, il est ensuite possible de démontrer le théorème suivant.

Théorème. *[104] Soient u la solution du problème de Cauchy associé à l'équation de Fisher – KPP unidimensionnelle avec donnée initiale $\mathbf{1}_{]-\infty,0[}$ et φ_2 le profil de l'onde progressive de vitesse 2 pour cette même équation.*

Alors il existe $m : \mathbb{R} \to \mathbb{R}$ telle que, quand $t \to +\infty$,

$$m(t) = o(t),$$

$$\sup_{x\in\mathbb{R}} |u(t, x - 2t - m(t)) - \varphi_2(x)| \to 0.$$

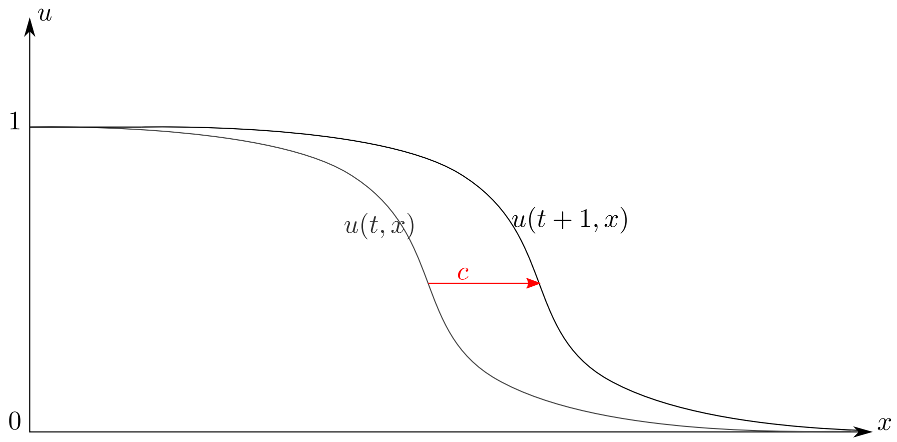


FIGURE 0.0.2 – Une onde progressive pour l'équation de Fisher – KPP

Autrement dit, à une correction $m(t)$ près, l'onde progressive de vitesse minimale correspond bien au comportement en temps long de la solution du problème de Cauchy. Le développement asymptotique de $m(t)$ a après 1937 fait l'objet de nombreuses recherches, initiées par Bramson [29, 30] avec des méthodes probabilistes. L'article récent de Hamel, Nolen, Roquejoffre et Ryzhik [90] revient sur les résultats de Bramson avec des méthodes EDP et donne quelques références bibliographiques.

Rappelant que les variables t et x sont adimensionnelles, on trouve en revenant aux variables physiques que la vitesse 2 est remplacée par $c^\star = 2\sqrt{rd}$. Grâce à la simplicité inattendue de cette formule, le modèle mathématique peut être confronté efficacement aux données empiriques.

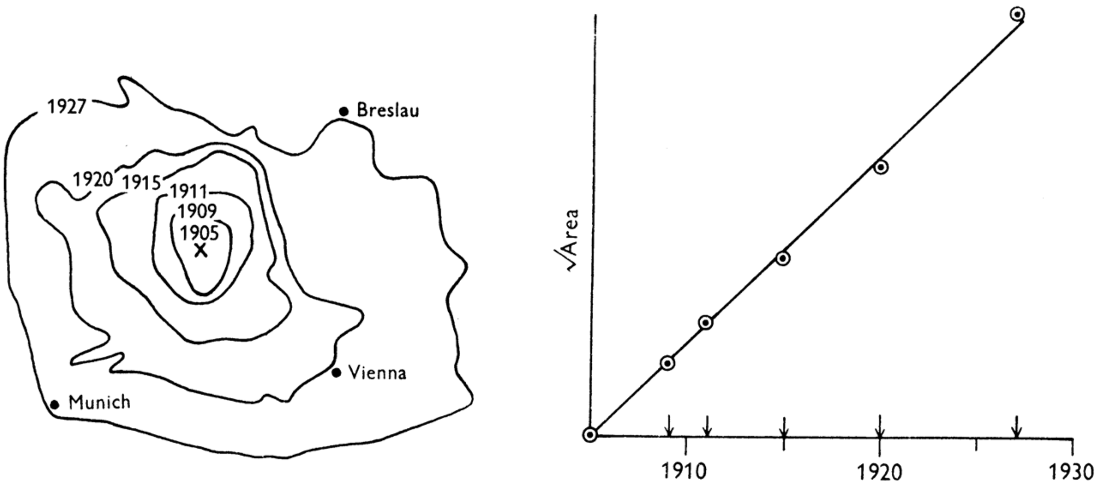


FIGURE 0.0.3 – Graphiques issus de l'article de Skellam illustrant la propagation à vitesse constante du rat musqué en Europe centrale et motivant l'utilisation de l'équation de Fisher – KPP pour la modélisation.

En particulier, cette formule ne dépend pas de la capacité de charge K. C'est une conséquence immédiate du fait que $c^\star$ est *linéairement déterminée* : elle coïncide avec la vitesse minimale d'existence de solutions positives non-nulles pour l'équation

$$-d\varphi'' - c\varphi' = r\varphi,$$

qui n'est autre que l'équation satisfaite par le profil φ d'une onde progressive de vitesse c linéarisée en 0. Cette équation, qui n'indique *a priori* que le comportement du profil là où la densité de population est encore négligeable, détermine en fait complètement la vitesse $c^\star$.

Les travaux de 1937 ont inspiré par la suite une vaste littérature sur les équations de réaction – diffusion et plus tard sur les systèmes de réaction – diffusion. En particulier, étant donné un nouveau problème de réaction – diffusion, les théorèmes ci-dessus conduisent à systématiquement poser la question d'existence d'ondes progressives ainsi que la question du lien entre la vitesse non-linéaire et la vitesse linéaire. Sans s'engager dans une impossible revue exhaustive des résultats existants à ce jour, on cite tout de même un résultat marquant de Fife et McLeod [71] qui servira par la suite.

Théorème. *[71] L'équation*

$$\partial_t u - \Delta u = u\,(u-\theta)\,(1-u)$$

avec $\theta \in \left]0,1\right[$ *admet une unique solution sous forme d'onde progressive connectant* 0 *et* 1.

De plus, la vitesse c *de cette onde progressive a le signe de* $\int_0^1 u\,(u-\theta)\,(1-u)\,du = \frac{1}{6}\left(\frac{1}{2}-\theta\right)$.

Définition. Une onde progressive connectant 0 et 1 pour l'équation

$$\partial_t u - \Delta u = u\,(u-\theta)\,(1-u)$$

est une onde progressive dont le profil a pour limite en $-\infty$ et $+\infty$ 1 et 0 respectivement.

Le signe de la vitesse est aisément obtenu par intégration par parties de l'équation satisfaite par le profil φ multipliée par φ'.

Ce résultat est fondamentalement différent de celui obtenu pour le terme de réaction $u\,(1-u)$: l'onde progressive est unique, la vitesse n'est pas linéairement déterminée et même son signe dépend des paramètres. Pour les écologistes, un terme de réaction de la forme $u\,(u-\theta)\,(1-u)$ modélise un *effet Allee*, c'est-à-dire un effet de dépendance positive en la densité : le taux de croissance $(u-\theta)\,(1-u)$ est strictement croissant en u si $u < \frac{\theta+1}{2}$.

Mathématiquement, une différence importante entre $u\,(1-u)$ et $u\,(u-\theta)\,(1-u)$ est la classification des états stationnaires constants. Dans le premier cas, les états stationnaires constants sont exactement 0 et 1 et, pour l'EDO sous-jacente,

$$u' = u\,(1-u)\,,$$

0 est instable et 1 est localement asymptotiquement stable (et même globalement attractif pour les données initiales positives non-nulles). Dans le second cas, les états stationnaires constants sont exactement 0, θ et 1 et, pour l'EDO sous-jacente,

$$u' = u\,(u-\theta)\,(1-u)\,,$$

0 et 1 sont localement asymptotiquement stables tandis que θ est instable.

Cette observation conduit à une classification des termes de réaction f réguliers, dépendant seulement de u, s'annulant en 0 et dont l'ensemble des zéros strictement positifs admet un maximum. À renormalisation près, on peut supposer que ce maximum est 1. De plus, on exclut le cas où f est strictement positive à droite de 1 afin de n'obtenir que des solutions globalement bornées et ainsi $f'(1) \le 0$.

1. S'il existe $\theta \in]0,1[$ tel que f est nulle sur $[0,\theta]$ et positive dans $]\theta,1[$, f est dite *ignition.*
2. Si 0 et 1 sont les seuls zéros positifs de f, f est dite *monostable.*
3. Si f est monostable et satisfait $f'(0)\,u \geq f(u)$ pour tout $u \in [0,1]$, f est dite *KPP.*
4. Si f admet exactement trois zéros positifs 0, θ et 1 et si $f'(0) < 0$, $f'(\theta) > 0$ et $f'(1) < 0$, f est dite *bistable.*
5. Si les zéros positifs de f sont tous isolés et s'il y en a au moins quatre, f est dite *multistable.*

Cette classification n'est pas exhaustive mais couvre cependant l'immense majorité des cas intéressants du point de vue des applications.

Le cas monostable non-KPP modélise lui aussi un effet Allee, dit faible par opposition à l'effet Allee fort du cas bistable. Le théorème de base sur ce cas est le suivant.

Théorème. *[8] Il existe $c^\star \geq f'(0)$ telle que l'équation monostable*

$$\partial_t u - \Delta u = f(u)$$

admette une solution sous forme d'onde progressive à vitesse $c \geq 0$ décrivant l'invasion de 0 par 1 si et seulement si $c \geq c^\star$. Cette solution est unique.

La question de l'égalité $c^\star = f'(0)$, c'est-à-dire de la détermination linéaire de la vitesse minimale, revêt dans ce cas une importance évidemment toute particulière. Bien que dans le cas KPP elle soit vraie, dans le cas général elle est fausse.

Systèmes

La classification des termes de réaction multidimensionnels $\mathbf{f}$ est évidemment bien plus complexe. D'un côté, il est possible de généraliser la classification scalaire (voir par exemple l'ouvrage de Volpert, Volpert et Volpert [139]), mais une telle classification n'est pas toujours appropriée pour traiter les systèmes intéressants du point de vue des applications. D'un autre côté, il est possible de suivre la classification induite par les applications, dont trois cas seraient par exemple Lotka − Volterra compétitif, Lotka − Volterra prédatif, Lotka − Volterra mutualiste, mais une telle classification échoue parfois à mettre en exergue la proximité mathématique de différents modèles.

Les deux classifications sont parfois utilisées conjointement. Cela conduit par exemple à la classification standard pour le système de Lotka − Volterra compétitif à deux espèces détaillée ci-dessous et à laquelle on se réfèrera par la suite.

1. Si $a \leq 1$ ou $b \leq 1$, le système est monostable : le système d'EDO sous-jacent possède un unique état stationnaire localement asymptotiquement stable. On distingue ensuite les sous-cas suivants :
   a) $a = b = 1$: cas dégénéré généralement écarté ;
   b) $a < 1$ et $b < 1$: cas de coexistence, l'état stable est $\left(\frac{1-a}{1-ab}, \frac{1-b}{1-ab}\right)$;
   c) $a \geq 1$, $b \leq 1$ et $a \neq b$: cas de semi-extinction, l'état stable est $(0,1)$;
   d) $b \geq 1$, $a \leq 1$ et $a \neq b$: cas de semi-extinction, l'état stable est $(1,0)$.
2. Si $a > 1$ et $b > 1$, le système est bistable : le système d'EDO sous-jacent possède exactement deux états stationnaires localement asymptotiquement stables, qui sont $(1,0)$ et $(0,1)$. On parle aussi du cas d'exclusion mutuelle.

Dans cet exemple, l'augmentation de la complexité par rapport à une équation scalaire provient de l'amenuisement des contraintes topologiques en dimension 2 et intervient déjà dans le système d'EDO sous-jacent, dépourvu de structure spatiale.

Cependant, la richesse topologique due à la dimension du système n'est pas la seule cause des difficultés rencontrées lors de l'étude des systèmes de réaction – diffusion. Des phénomènes plus subtils, tels que les instabilités de Turing [138] dues à l'interaction entre des diffusions inégales et des termes de réaction particuliers, peuvent émerger. Les systèmes concernés par ces phénomènes présentent un défaut de structure par rapport aux équations scalaires, bien souvent un manque de principe de comparaison ou de structure variationnelle. L'étude de ces systèmes, parmi lesquels se trouve la majorité des systèmes issus de la dynamique des populations, est particulièrement difficile.

En particulier, pour l'étude des ondes progressives des systèmes dépourvus de principe de comparaison, les méthodes EDO paraissent parfois incontournables. Réussir à n'employer que des méthodes EDP devient alors un enjeu en soi, puisque cela ouvre par la suite des directions de recherche difficilement abordables avec les méthodes EDO. Cela explique par exemple pourquoi les résultats sur le système de Lotka – Volterra prédatif à deux espèces

$$\begin{cases} \partial_t u - \Delta u = u\,(1-u-av) \\ \partial_t v - d\Delta v = v\,(-1+bu) \end{cases}$$

(avec $a > 0$ et $b > 0$), d'abord obtenus grâce à des méthodes EDO par Dunbar dans les années 80 [63, 64], ont été établis de nouveau avec des méthodes EDP par Fu et Tsai en 2015 [75]. Le délai de plus de trente ans illustre d'ailleurs tout à fait les difficultés auxquelles les spécialistes des méthodes EDP doivent faire face quand le principe de comparaison fait défaut.

Au contraire, les exceptionnels systèmes *monotones*, c'est-à-dire doté d'un principe de comparaison, ou variationnels peuvent bien souvent être traités comme des EDP scalaires et vérifient donc des propriétés de propagation similaires. C'est par exemple le cas du système de compétition – diffusion de Lotka – Volterra à deux espèces, qui est en effet monotone et dont les cas monostable et bistable sont fortement analogues aux cas scalaires correspondants. Cependant, dès qu'un troisième compétiteur est introduit, la monotonie du système est perdue et des phénomènes nouveaux surviennent (par exemple, sous certaines conditions, Kishimoto [102] a montré l'existence d'états stationnaires stables non-constants et Chen et Hung [95] ont montré l'inexistence d'ondes progressives).

Il faut également pointer le fait que le système suivant est à la fois monotone et variationnel et pourtant encore très incomplètement compris :

$$\begin{cases} \partial_t u - \Delta u = u\left(1-u^2-av^2\right) \\ \partial_t v - d\Delta v = rv\left(1-v^2-bu^2\right) \end{cases}.$$

Ce système très important, issu de la physique quantique et appelé système de Gross – Pitaevskii pour les condensats de Bose – Einstein à deux espèces, est l'objet d'innombrables articles. Bien qu'il puisse passer pour une simple modification variationnelle du système de compétition – diffusion de Lotka – Volterra à deux espèces, certaines questions posées par les applications et certains arguments de preuves ne peuvent se transposer d'un système à l'autre. Les littératures sur ces deux systèmes ont en fait tendance à se développer indépendamment l'une de l'autre et les rapprochements sont désormais rares (des efforts récents étant dus par exemple à Dancer, Wang et Zhang [48, 50, 51] ou Soave et Zilio [135]). Ceci est révélateur de la complexité accrue des systèmes (les termes de réaction scalaires $u\,(1-u)$ et $u\left(1-u^2\right)$ étant au contraire tous deux traités conjointement, en tant que termes de réaction KPP).

Sans plus s'attarder sur ces généralités, on présente maintenant les systèmes étudiés dans cette thèse ainsi que les résultats obtenus.

Contributions

Les contributions de cette thèse à l'étude générale des propriétés de propagation des systèmes de réaction – diffusion issus de la dynamique des populations sont de deux ordres.

1. La première direction est l'étude de questions ouvertes précédemment posées par la vaste littérature sur le système de Lotka – Volterra compétitif à deux espèces.
   a) Pour le système bistable, quand un des deux états stables envahit l'autre, duquel s'agit-il ? L'hétérogénéité spatiale est-elle susceptible de bloquer cette invasion ou de la renverser ?
   b) Pour le système monostable avec semi-extinction, l'état de semi-extinction instable est-il susceptible d'envahir un territoire inhabité avant de se faire remplacer par l'état de semi-extinction stable, et si oui quelles sont les deux vitesses en jeu ?
2. La seconde direction est l'ouverture d'un programme de recherche sur une large classe de systèmes non-monotones et non-variationnels mais analogues, par bien des aspects, à l'équation de Fisher – KPP et survenant dans de nombreux modèles de dynamique des populations. Après les vérifications d'usage (positivité et bornitude des solutions), un critère nécessaire et suffisant pour la persistance des populations est établi et les propriétés de propagation sont étudiées. Mise à part l'unicité des ondes progressives, qui reste un problème ouvert et à la solution très certainement complexe, on découvre bien des propriétés de propagation évoquant celles de l'équation de Fisher – KPP.

Sur le système de Lotka – Volterra compétitif à deux espèces

Dans toute cette sous-section, le système

$$\begin{cases} \partial_t u - \Delta u = u\,(1 - u - av) \\ \partial_t v - d\Delta v = rv\,(1 - v - bu) \end{cases} \tag{0.0.1}$$

est simplement dénoté (0.0.1).

En régime bistable avec forte compétition

Dans toute cette sous-sous-section, on suppose que (0.0.1) est bistable, c'est-à-dire que $a > 1$ et $b > 1$.

En 1982, Gardner [77] a démontré le théorème suivant.

Théorème. *[77] (0.0.1) admet une solution sous forme d'onde progressive connectant* $(0,1)$ *et* $(1,0)$.

Définition. Une onde progressive connectant $(0,1)$ et $(1,0)$ pour (0.0.1) est une onde progressive dont le profil a pour limite en $-\infty$ et $+\infty$ $(1,0)$ et $(0,1)$ respectivement. L'onde progressive est dite monotone si son profil (φ,ψ) est tel que φ et $-\psi$ sont toutes deux décroissantes et est dite strictement monotone si φ et $-\psi$ sont toutes deux strictement décroissantes.

Ce résultat a par la suite été raffiné par Kan-on [100].

Théorème. *[100] (0.0.1) admet une unique solution sous forme d'onde progressive connectant* $(0,1)$ *et* $(1,0)$.

De plus, cette onde progressive est strictement monotone et sa vitesse c satisfait $-2\sqrt{rd} < c < 2$.

Déterminer le signe de c devient alors d'une importance fondamentale, puisque cela donne le sens de l'invasion :

1. si $c < 0$, alors $(0, 1)$ envahit $(1, 0)$;
2. si $c > 0$, alors $(1, 0)$ envahit $(0, 1)$.

Autrement dit, le signe de c permet de comparer dynamiquement la stabilité de $(0, 1)$ et celle de $(1, 0)$.

Contrairement à l'équation bistable scalaire étudiée par Fife et McLeod [71], on ne peut pas ici déterminer le signe de c par une simple intégration par parties (du fait de l'absence de structure variationnelle). À vrai dire, à ce jour, aucun résultat parfaitement général n'est connu. Quand les recherches de cette thèse ont démarré, le seul résultat partiel était celui de Guo et Lin [83].

Théorème. *[83] Le signe de la vitesse c de l'unique onde progressive solution de (0.0.1) satisfait les propriétés suivantes.*

1. *Si $r = d$, alors c a le signe de $b - a$.*
2. *Si $r > d$ et $a \geq \left(\frac{r}{d}\right)^2 b$, alors $c < 0$.*
3. *Si $r < d$ et $b \geq \left(\frac{d}{r}\right)^2 a$, alors $c > 0$.*
4. *Quel que soit $\lambda > 0$, changer (d, r) en $\lambda (d, r)$ ne change pas le signe de c.*
5. *Si $r > d$, $a \geq 2$ et $b \leq 1 + \frac{d}{r}$, alors $c < 0$.*
6. *Si $r > d$, $a \geq \frac{5r}{d}$ et $(3rb - d) b \leq (4r - d) a$, alors $c < 0$.*
7. *Si $r = \frac{d}{4}$ et $(a, b) = \left(\frac{5}{4}, \frac{4}{3}\right)$, alors $c = 0$.*
8. *Si $r = \frac{d}{4}$, $a \geq \frac{5}{4}$, $b \leq \frac{4}{3}$ et $(a, b) \neq \left(\frac{5}{4}, \frac{4}{3}\right)$, alors $c < 0$.*
9. *Si $r = \frac{d}{4}$, $a \leq \frac{5}{4}$, $b \geq \frac{4}{3}$ et $(a, b) \neq \left(\frac{5}{4}, \frac{4}{3}\right)$, alors $c > 0$.*

La preuve de ce résultat repose sur les propriétés de monotonie de c par rapport à (r, a, b) établies par Kan-on [100].

Dans cette thèse, on propose une approche complètement différente pour aborder ce problème. Il s'agit de s'appuyer sur les propriétés d'un régime asymptotique particulier, appelé *régime de forte compétition* : $(r, a, b) = \left(r, k, \frac{\alpha k}{r}\right)$ avec $\alpha > 0$ et $k \to +\infty$. Ce régime correspond ainsi à la limite singulière $k \to +\infty$ du système suivant :

$$\begin{cases} \partial_t u_k - \Delta u_k = u_k (1 - u_k) - k u_k v_k \\ \partial_t v_k - d\Delta v_k = r v_k (1 - v_k) - \alpha k u_k v_k \end{cases}.$$

Dans ce régime, le théorème de Guo et Lin laisse une large zone d'incertitude. Non seulement les conditions des points 5 à 9 ne peuvent pas être remplies si k est assez grand, mais de plus les points 1 à 4 ne donnent que le résultat partiel suivant.

Corollaire. *Le signe de la vitesse c_k satisfait les propriétés suivantes.*

1. *Si $r = d$, alors c_k a le signe de $\alpha - r$.*
2. *Si $r > d$ et $d^2 \geq \alpha r$, alors $c_k < 0$.*
3. *Si $r < d$ et $\alpha r \geq d^2$, alors $c_k > 0$.*
4. *Quel que soit $\lambda > 0$, changer (d, r, α) en $\lambda (d, r, \alpha)$ ne change pas le signe de c_k.*

Par exemple, si $\alpha = r = 1$, le signe de c_k reste complètement indéterminé hormis dans le cas $d = 1$ (trivial par symétrie).

Les premiers articles sur le régime de forte compétition remontent aux années 90 et sont dus à Dancer et à ses collaborateurs [45, 46]. Le principal résultat, très générique et naturellement déduit du système, est le suivant : à la limite $k \to +\infty$, les solutions $((u_k, v_k))_{k>1}$ convergent vers une paire (u_∞, v_∞) dont les deux composantes sont positives et spatialement ségrégées, c'est-à-dire satisfaisant $u_\infty v_\infty = 0$.

Par ailleurs, pour un système à deux espèces tel que celui ci-dessus, on peut utiliser la forme particulière du couplage de Lotka – Volterra pour combiner linéairement les deux équations et obtenir

$$\partial_t (\alpha u_k - v_k) - \Delta (\alpha u_k - d v_k) = \alpha u_k (1 - u_k) - r v_k (1 - v_k).$$

Dans cette équation, la dépendance en k n'est plus qu'implicite, et on peut raisonnablement espérer que l'on puisse passer à la limite. En utilisant la relation $u_\infty v_\infty = 0$ et en notant w_∞ la limite de $(\alpha u_k - v_k)_{k>1}$, on peut identifier $\alpha u_\infty = w_\infty^+$ et $v_\infty = w_\infty^-$, où les parties positive et négative de w_∞ sont définies de sorte que $w_\infty = w_\infty^+ - w_\infty^-$, et ainsi écrire l'équation limite :

$$\partial_t w_\infty - \Delta \left(\left(\mathbf{1}_{w_\infty > 0} + d \mathbf{1}_{w_\infty < 0} \right) w_\infty \right) = w_\infty^+ \left(1 - \frac{w_\infty}{\alpha} \right) - r w_\infty^- (1 + w_\infty).$$

Sous réserve que w_∞ ne s'annule que sur un ensemble négligeable, cette équation est une équation parabolique quasilinéaire. La limite w_∞ peut donc gagner une certaine régularité et un problème de frontière libre émerge, régissant le mouvement de l'interface.

Ce raisonnement formel sera développé rigoureusement plus loin dans cette thèse. L'idée ici est simplement de montrer comment un système de deux équations couplées est réduit, en régime de forte compétition, à une unique équation quasilinéaire. Étant donné que l'on sait calculer aisément le signe de la vitesse d'une onde progressive bistable scalaire, cette réduction devrait par conséquent révéler le signe de la limite c_∞ des vitesses $(c_k)_{k>1}$.

Plus précisément, le premier résultat obtenu dans le cadre de cette thèse, en collaboration avec Grégoire Nadin, est le suivant.

Théorème. *[GN15] La famille de vitesses $(c_k)_{k>1}$ converge vers une limite $c_\infty \in \left] -2\sqrt{rd}, 2 \right[$ ayant le signe de $\alpha^2 - rd$.*

De plus, la convergence est localement uniforme par rapport à d et c_∞ est continue par rapport à d.

La continuité de c_k par rapport à d étant encore une question ouverte à ce jour, la continuité de c_∞ n'est pas une simple conséquence de la convergence localement uniforme.

Dans le cas simplifié où $\alpha = r = 1$, le signe de c_∞ est celui de $1 - d$. Autrement dit, l'espèce ayant le plus fort taux de diffusion chasse l'autre : l'important n'est pas d'être très concentré au voisinage de l'interface mais plutôt de pouvoir envoyer des éclaireurs loin dans le territoire adverse. Par conséquent, ce résultat a été nommé *« L'union ne fait pas la force »*.

Il est aisé de vérifier que notre résultat est compatible avec celui de Guo et Lin. Par exemple, si l'on suppose $r > d$ et $d^2 \geq \alpha r$, qui impliquent ensemble d'après Guo et Lin $c_k < 0$ (et donc $c_\infty \leq 0$), on a bien $\alpha^2 \leq rd$, et même $\alpha^2 < rd$:

$$\frac{\alpha^2}{rd} = \frac{\alpha^2 r^2}{dr^3} \leq \frac{d^3}{r^3} < 1.$$

En montrant qu'un fort taux de compétition interspécifique favorise l'espèce la plus mobile, notre résultat soulève plusieurs questions intéressantes. En particulier, il met en perspective un résultat célèbre de type « L'union fait la force » pour le système en milieu borné hétérogène

$$\begin{cases} \partial_t u - \Delta u = u (r(x) - u - v) & \text{dans } \Omega \\ \partial_t v - d\Delta v = v (r(x) - v - u) & \text{dans } \Omega \\ \partial_n u = \partial_n v = 0 & \text{sur } \partial\Omega \end{cases} \quad (0.0.2)$$

dû à Dockery, Hutson, Mischaikow et Pernarowski [58]. Ce renversement du résultat est-il dû avant tout à l'hétérogénéité spatiale ou l'affaiblissement considérable de la compétition interspécifique est-il également responsable ?

Afin d'aborder cette question, nous avons étudié le rôle de l'hétérogénéité spatiale en considérant le cas particulier, mathématiquement agréable, d'hétérogénéités spatiales périodiques unidimensionnelles. En effet, tandis que les phénomènes de propagation en milieu hétérogène général sont considérablement complexes, en milieu périodique ils se simplifient grandement et de nombreuses similitudes avec les milieux homogènes se dégagent. En particulier, la notion d'onde progressive est naturellement généralisée par celle d'*onde pulsatoire.*

Définition. Une onde pulsatoire connectant $(0,1)$ et $(1,0)$ pour

$$\begin{cases} \partial_t u - \partial_{xx} u = \mu(x) u (1-u) - kuv \\ \partial_t v - d\partial_{xx} v = \nu(x) v (1-v) - \alpha kuv \end{cases}, \tag{0.0.3}$$

où μ et ν sont deux fonctions régulières positives périodiques de même période $L > 0$, est une solution entière de la forme $(u,v) : (t,x) \mapsto (\varphi,\psi)(x-ct,x)$, avec $c \in \mathbb{R}$ une vitesse de propagation et (φ,ψ) un profil de propagation satisfaisant les propriétés suivantes :

1. φ et $-\psi$ sont toutes deux strictement décroissantes par rapport à leur première variable ;
2. φ et ψ sont toutes deux L-périodiques par rapport à leur seconde variable ;
3. les limites uniformes suivantes sont satisfaites :

$$\lim_{\xi\to-\infty} \sup_{x\in[0,L]} |(\varphi,\psi)(\xi,x) - (1,0)| = 0,$$

$$\lim_{\xi\to+\infty} \sup_{x\in[0,L]} |(\varphi,\psi)(\xi,x) - (0,1)| = 0.$$

Avant d'aller plus loin, on précise que l'on se restreint dans cet exposé introductif aux systèmes de la forme (0.0.3) mais que les résultats ci-dessous issus de [Gir17, GN18] sont en fait démontrés sous des hypothèses légèrement plus générales sur le terme de réaction. La comparaison entre (0.0.2) et (0.0.3) est bel et bien pertinente. Les hypothèses exactes seront énoncées dans les chapitres adéquats.

Les recherches sur les ondes pulsatoires en réaction – diffusion ont démarré beaucoup plus récemment que celles sur les ondes progressives (les travaux pionniers sur les ondes pulsatoires scalaires étant ceux de Gärtner, Freidlin [79] et Xin [143, 144] et remontant aux années 80 et 90). Lorsque nous nous sommes penchés sur cette question, nous avons réalisé que l'existence d'ondes pulsatoires pour le système bistable (0.0.3) n'avait encore jamais été abordée.

Une telle question peut cependant être traitée grâce au cadre théorique très général élaboré récemment par Fang et Zhao [69]. Sans entrer ici dans les détails techniques, leur conclusion générale est la suivante : les solutions sous forme d'onde progressive ou pulsatoire pour des problèmes bistables existent pourvu que tous les états stationnaires intermédiaires, respectivement constants ou périodiques, soient instables et puissent être envahis par les deux états stables extrémaux (c'est-à-dire qu'il existe des ondes monostables de vitesse au signe adéquat). C'est ainsi qu'en démontrant le résultat suivant, on déduit immédiatement l'existence d'ondes pulsatoires pour (0.0.3).

Théorème. *[Gir17] Soient $A > 0$, $B > 0$ et $\overline{L} = \pi\left(\frac{1}{\sqrt{A}} + \sqrt{\frac{d}{B}}\right)$. Supposons que $L < \overline{L}$, $\max_{[0,L]} \mu = A$ et $\max_{[0,L]} \nu = B$.*

Alors il existe $k^\star > 0$ tel que, si $k \geq k^\star$, tout état stationnaire L-périodique de coexistence pour (0.0.3) est instable et peut être envahi par les états stables $(1,0)$ et $(0,1)$.

La condition $L < \overline{L}$ permet en fait de garantir que le problème limite

$$-\Delta\left(\left(\mathbf{1}_{w_\infty>0} + d\mathbf{1}_{w_\infty<0}\right) w_\infty\right) = \mu\left(x\right) w_\infty^+ \left(1 - \frac{w_\infty}{\alpha}\right) - \nu\left(x\right) w_\infty^- \left(1 + w_\infty\right)$$

n'admette aucune solution périodique, non-nulle et changeant de signe. Elle est déduite des deux observations suivantes :

— une solution périodique, non-nulle et changeant de signe peut être vue comme une succession de solutions de problèmes de Dirichlet posés dans des intervalles de tailles strictement inférieures à L ;

— un tel problème de Dirichlet n'admet une solution positive que si l'intervalle dans lequel il est posé est suffisamment grand, la taille minimale pouvant être estimée explicitement.

Il est parfaitement naturel de se demander si ce résultat est optimal, c'est-à-dire si l'on peut construire des contre-exemples où $L < \overline{L}$ n'est pas satisfaite et où un état stationnaire périodique de coexistence stable ou ne pouvant être envahi existe. Cette question a fait l'objet d'une collaboration avec Alessandro Zilio et a conduit au résultat suivant, dont la démonstration est encore une fois basée sur le passage à la limite $k \to +\infty$.

Théorème. *[GZ18] Soient $A > 0$, $B > 0$ et $r_0 > 0$, $r_\mu > 0$ et $r_\nu > 0$ tels que $2r_0+2r_\mu+2r_\nu = 1$. On définit les deux fonctions 1-périodiques $\mu^\star$ et $\nu^\star$ satisfaisant*

$$\left(\mu^\star\right)_{|[0,1]} = A\mathbf{1}_{[0,r_\mu]} + A\mathbf{1}_{[r_\mu+2r_0+2r_\nu,1]},$$

$$\left(\nu^\star\right)_{|[0,1]} = B\mathbf{1}_{[r_\mu+r_0,r_\mu+r_0+2r_\nu]},$$

ainsi que, pour tout $L > 0$, la fonction L-périodique

$$\left(\mu_L, \nu_L\right) : x \mapsto \left(\mu^\star, \nu^\star\right)\left(\frac{x}{L}\right).$$

Il existe $\underline{L} > 0$ tel que, pour tout $L > \underline{L}$, il existe $k^\star > 1$ tel que, pour tout $k \geq k^\star$, (0.0.3) avec $(\mu,\nu) = (\mu_L,\nu_L)$ ou $(\mu,\nu) = (\mu_L+\nu_L, \mu_L+\nu_L)$ admet un état stationnaire L-périodique de coexistence linéairement stable.

De plus, pour tout $L > \underline{L}$, il existe un voisinage U_L de (μ_L,ν_L) dans la topologie de $\left(L^\infty_{L\text{-}per}\right)^2$ et un voisinage V_L de $\mu_L+\nu_L$ dans la topologie de $L^\infty_{L\text{-}per}$ tels que, pour tout $(\mu,\nu) \in U_L$ et tout $\rho \in V_L$, il existe $k^\star > 1$ tel que, pour tout $k \geq k^\star$, (0.0.3) avec (μ,ν) ou (ρ,ρ) admet un état stationnaire L-périodique de coexistence linéairement stable.

L'existence d'un tel état de coexistence empêche d'appliquer le résultat de Fang et Zhao mais pourrait même bel et bien bloquer la propagation et assurer la non-existence d'ondes pulsatoires. Pour ce faire, il faudra vérifier dans des travaux ultérieurs le signe de la vitesse de l'onde pulsatoire bistable connectant cet état à un des deux états de semi-extinction. Quoi qu'il en soit, ce résultat complète de manière intéressante un résultat de Ding, Hamel et Zhao [57] montrant que pour une classe particulière, mais naturelle, de termes de réaction, l'existence d'état de coexistence stable n'est possible que si la période n'est ni trop grande, ni trop petite.

Le contre-exemple du précédent théorème repose sur un choix particulier de (μ_L,ν_L) décrivant une situation où se succèdent périodiquement des territoires fertiles (où μ_L et ν_L sont des constantes positives) et des territoires neutres, ni fertiles ni délétères (où $\mu_L = \nu_L = 0$). L'état stationnaire obtenu dépeint alors la possibilité pour u et v de s'installer dans les territoires fertiles à numéro pair et impair respectivement. Autrement dit, ce résultat admet une interprétation biologique intéressante : une forte hétérogénéité de l'habitat est susceptible de conduire à une forte ségrégation spatiale et, sur le long terme, à une véritable spéciation. Ce résultat est ainsi, en soi, un premier complément intéressant au résultat de Dockery *et al.* : en milieu hétérogène, là où une faible compétition interspécifique ne laisse aucune chance à la coexistence, une forte compétition interspécifique peut au contraire la favoriser.

L'existence d'ondes pulsatoires sous hypothèse de haute fréquence étant néanmoins établie, la méthode développée avec Grégoire Nadin dans le cas homogène peut être appliquée au système périodique (0.0.3) sur des bases non-vides. De nouveau, le signe de la vitesse peut être déterminé grâce au problème limite.

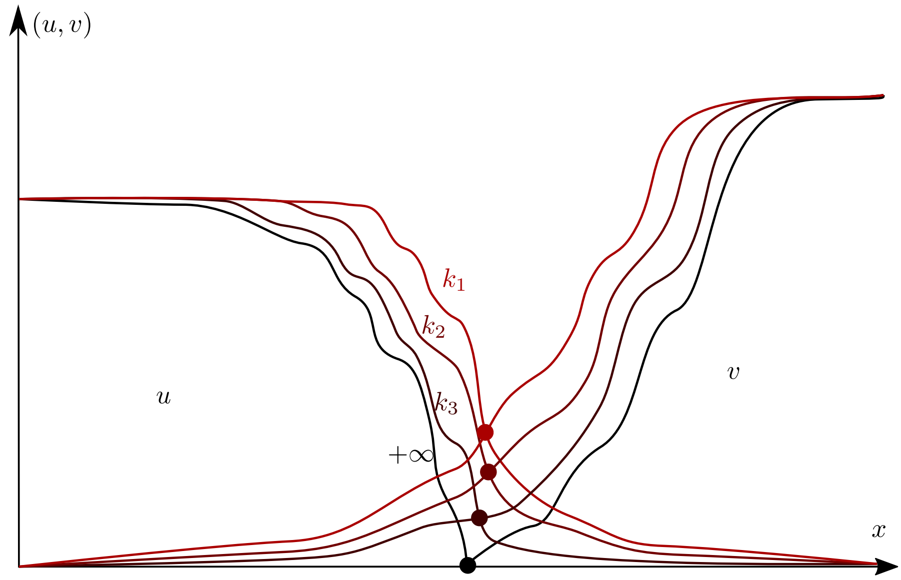


FIGURE 0.0.4 – Ségrégation d'ondes pulsatoires ($k_1 < k_2 < k_3 < +\infty$)

Théorème. *[GN18] Il existe $c_\infty \in \mathbb{R}$ tel que :*

1. *$c_\infty \in \left]-c^\star_{d,\nu}, c^\star_{1,\mu}\right[$, où $c^\star_{\delta,\rho} > 0$ est la vitesse minimale des ondes pulsatoires pour l'équation de Fisher – KPP périodique unidimensionnelle*

$$\partial_t z - \delta \partial_{xx} z = \rho(x) z (1-z) \ ;$$

2. *$c_\infty = 0$ si et seulement si $\frac{\alpha^2}{d} \in I_r$, où I_r est un intervalle fermé non-vide satisfaisant*

$$\left\{ \frac{\int_0^L \nu}{\int_0^L \mu} \right\} \subset I_r \subset \left[\frac{\min\limits_{[0,L]} \nu}{\max\limits_{[0,L]} \mu}, \frac{\max\limits_{[0,L]} \nu}{\min\limits_{[0,L]} \mu} \right] \ ;$$

3. *$c_\infty < 0$ si $\frac{\alpha^2}{d} < \min I_r$;*
4. *$c_\infty > 0$ si $\frac{\alpha^2}{d} > \max I_r$;*
5. *c_∞ est la limite, localement uniforme par rapport à d, de toute famille $(c_k)_{k \geq k^\star}$ de vitesses d'onde pulsatoire pour (0.0.3) ;*
6. *c_∞ est continue par rapport à d.*

Comme le laisse transparaitre l'énoncé ci-dessus, l'unicité de l'onde pulsatoire $(u_k, v_k) : (t, x) \mapsto (\varphi_k, \psi_k)(x - c_k t, x)$ n'a pas été rigoureusement établie lors de ces travaux dont l'objet était plutôt l'étude de la limite singulière. Toutefois, elle peut bel et bien être démontrée par des arguments de glissements très similaires à ceux qui seront présentés plus loin dans cette thèse.

Établir le théorème précédent est considérablement plus difficile qu'établir son analogue en milieu homogène. En effet, là où le profil d'une onde progressive vérifie un agréable système elliptique, le profil d'une onde pulsatoire vérifie un système elliptique dégénéré, ce qui oblige de fait à repasser régulièrement en coordonnées paraboliques (t, x) et implique notamment une disjonction des cas selon la nullité de c_∞. De plus, dans le cas $c_\infty \neq 0$, la frontière libre induite par le problème limite ne se réduit plus trivialement à un point. Une véritable étude de ce problème de frontière libre est donc nécessaire pour caractériser le profil limite. Cette étude est conduite en utilisant le principe du maximum, la monotonie en temps de la position de la frontière libre (dont la vitesse moyenne est bien c_∞) et des procédures régularisantes. Même dans le cas $c_\infty = 0$, une difficulté supplémentaire émerge du fait de la possible multiplicité des solutions du problème limite (cet obstacle ayant déjà été mis en exergue par Ding, Hamel et Zhao [57]). Cependant, ces solutions existent si et seulement si $\frac{\alpha^2}{d} \in I_r$. Un résultat d'exclusion mutuelle déduit de la caractérisation des profils quand $c_\infty \neq 0$ permet d'établir que $c_\infty \neq 0$ si et seulement si $\frac{\alpha^2}{d} \notin I_r$, ce qui permet finalement d'obtenir le signe de c_∞.

On constate qu'encore une fois, le résultat est de type « L'union ne fait pas la force ». Une telle généralisation montre que l'inversion du résultat pour (0.0.2) repose fondamentalement sur l'affaiblissement de la compétition interspécifique. Bien que l'hétérogénéité spatiale favorise le compétiteur le plus sédentaire, son effet est négligeable face à celui de la compétition, qui elle favorise au contraire le plus mobile.

Le résultat de Dockery *et al.* sur (0.0.2) est donc loin de trancher la question du lien entre taux de diffusion et avantage compétitif. D'autres recherches devront être conduites. Ceci a par la suite été confirmé par Risler [130] qui a su démontrer avec des techniques complètement différentes un résultat supplémentaire de type « L'union ne fait pas la force », cette fois-ci pour le système perturbatif en milieu homogène

$$\begin{cases} \partial_t u - \Delta u = u(1-u) - (1+\gamma) uv \\ \partial_t v - (1+\varepsilon)\Delta v = v(1-v) - (1+\gamma) uv \end{cases}$$

où $\varepsilon > 0$ et $\gamma > 0$ sont deux paramètres infinitésimaux. Contrairement au régime de forte compétition, ce système n'est qu'une modification marginale de (0.0.2) et pourtant suffit bel et bien à inverser la conclusion.

En régime monostable avec semi-extinction

Dans toute cette sous-sous-section, on suppose que (0.0.1) est monostable avec semi-extinction, c'est-à-dire que $a < 1$ et $b > 1$ (quitte à inverser les rôles de u et v) . L'état stable est donc $(1, 0)$ tandis que $(0, 1)$ est instable.

Les ondes progressives de ce système, définies comme dans le cas bistable, ont d'abord été étudiées en 1989 par Hosono [92] et Okubo, Maini, Williamson et Murray [123] sous des hypothèses restrictives sur les paramètres puis de manière générale en 1997 par Kan-on [101]. Le résultat de ce dernier, évoquant fortement le cas monostable scalaire, est le suivant.

Théorème. *[101] Il existe $c^\star \geq 2\sqrt{1-a}$ telle que (0.0.1) admette une solution sous forme d'onde progressive monotone à vitesse $c \geq 0$ décrivant l'invasion de $(0, 1)$ par $(1, 0)$ si et seulement si $c \geq c^\star$.*

Par analogie avec les équations scalaires monostables, il a été naturellement conjecturé que :

(a) 1945 (b) 2000 (c) 2010

FIGURE 0.0.5 – Invasion de l'écureuil gris dans le territoire des iles Britanniques de l'écureuil roux, classiquement modélisée par un système de la forme (0.0.1) avec $a < 1 < b$. Quadrillage fin : écureuil gris ; quadrillage moyen : zone tampon ; quadrillage grossier : écureuil roux.
Source : *Red Squirrel Survival Trust* (couleurs modifiées).

— $c^\star$ est également la vitesse de propagation asymptotique des solutions avec données initiales unidimensionnelles de la forme $(u_0, 1 - v_0) \in [0,1]^2$ avec (u_0, v_0) à support compact et $u_0 \neq 0$;

— la détermination linéaire $c^\star = 2\sqrt{1-a}$ est parfois, mais pas toujours, vérifiée.

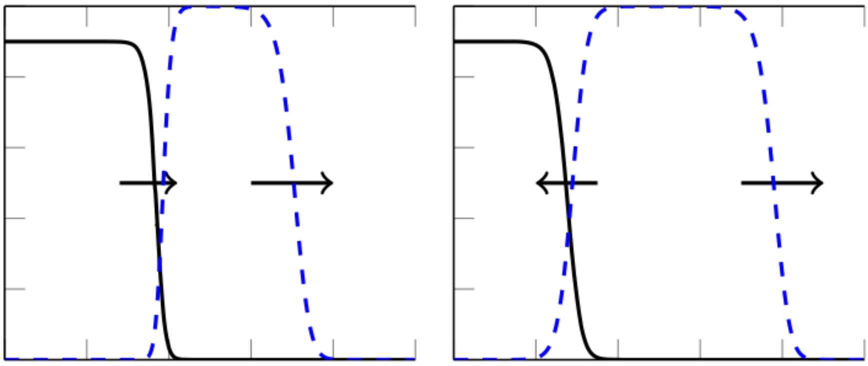
Les travaux du début des années 2000 dus à Lewis, Li et Weinberger [108, 110, 142] ont confirmé la première conjecture et ont donné des conditions suffisantes pour la détermination linéaire. Plus récemment, Huang et Han [94] sont parvenus à fournir un contre-exemple à la détermination linéaire, achevant ainsi la confirmation de la seconde conjecture. Ainsi la question de l'invasion de u dans un territoire initialement occupé par v est aujourd'hui plutôt bien comprise, même si les conditions sur les paramètres équivalentes à la détermination linéaire ne sont pas encore connues.

Au contraire, l'invasion conjointe de u et v dans un environnement initialement inhabité était un problème largement ouvert avant cette thèse. Les travaux les plus proches concernaient soit le cas bistable, traité très récemment par Carrère [35], ou le cas monostable avec coexistence, partiellement traité en 2012 par Lin et Li [112]. La conclusion générale de ces deux articles est la possibilité d'obtenir plusieurs vagues d'invasion successives :

— dans le cas bistable, si les conditions initiales sont convenables et si $2\sqrt{rd} > 2$, $(0,0)$ est envahi par $(0,1)$ qui est ensuite lui-même envahi par $(1,0)$;

— dans le cas monostable avec coexistence, si $2\sqrt{rd\,(1-b)} > 2$, $(0,0)$ est envahi par $(0,1)$ qui est ensuite lui-même suivi par une zone d'incertitude elle-même suivie par une invasion de $\left(\frac{1-a}{1-ab}, \frac{1-b}{1-ab}\right)$.

On note que, tandis que le résultat de Carrère parait optimal (la bistabilité rendant les hypothèses sur les conditions initiales nécessaires), celui de Lin et Li peut être clarifié. Dans la zone d'incertitude, assiste-t-on simplement à une invasion de $(0,1)$ par $\left(\frac{1-a}{1-ab}, \frac{1-b}{1-ab}\right)$ ou bien à une



(a) Cas $c_{(1,0)\to(0,1)} > 0$ (b) Cas $c_{(1,0)\to(0,1)} < 0$

FIGURE 0.0.6 – Simulations numériques de Carrère [35] illustrant les deux invasions successives, de vitesses respectives $2\sqrt{rd}$ et $c_{(1,0)\to(0,1)}$ (vitesse de l'onde progressive bistable connectant $(0,1)$ à $(1,0)$).

invasion de $(0,1)$ par $(1,0)$ puis à une invasion de $(1,0)$ par $\left(\frac{1-a}{1-ab}, \frac{1-b}{1-ab}\right)$?

Quoi qu'il en soit, ces vagues d'invasion successives évoquent les *terrasses de propagation* de la littérature sur les équations de réaction – diffusion scalaires. Ces solutions, décrites mathématiquement pour la première fois par Fife et McLeod [71] dans le cadre de termes de réaction multistables, nommées *systèmes d'ondes* par Volpert, Volpert et Volpert [139] et renommées suite aux travaux de Matano, Ducrot et Giletti [62], ont bénéficié récemment d'une certaine attention.

Mais le résultat de Lin et Li montre que le prisme des équations scalaires multistables ne suffit pas pour comprendre le type de terrasses qui est à l'œuvre ici. En effet, alors que seul le premier état stationnaire d'une terrasse scalaire multistable est susceptible d'être instable, Lin et Li démontrent le remplacement de l'état instable $(0,0)$ par l'état lui aussi instable $(0,1)$. Deux ondes monostables se succèdent donc. Les ondes monostables ayant typiquement une demi-droite de vitesses admissibles tandis que les ondes bistables n'ont qu'une unique vitesse admissible, l'ensemble des terrasses potentiellement engendrées par (0.0.1) est considérablement plus grand.

Il s'avère qu'adopter le point de vue des terrasses de propagation pour étudier l'invasion conjointe de u et v permet d'obtenir des résultats complets et novateurs, y compris dans le cas de données initiales à support compact. Une collaboration avec Adrian Lam a ainsi conduit aux résultats suivants.

On définit la fonction auxiliaire

$$f: \begin{array}{ccc} \left[2\sqrt{1-a}, +\infty\right[& \to & \left]2\sqrt{a}, 2\left(\sqrt{1-a}+\sqrt{a}\right)\right] \\ c & \mapsto & c - \sqrt{c^2 - 4(1-a)} + 2\sqrt{a} \end{array}.$$

Celle-ci est décroissante et bijective et satisfait en particulier

$$f(2) = 2,$$

$$f^{-1} : \tilde{c} \mapsto \frac{\tilde{c}}{2} - \sqrt{a} + \frac{2(1-a)}{\tilde{c} - 2\sqrt{a}}.$$

Dans l'énoncé qui suit, l'espace sous-jacent est unidimensionnel et $c^\star$ est bien la vitesse minimale des ondes progressives monotones décrivant l'invasion de $(0,1)$ par $(1,0)$.

Théorème. *[GL18] Soient* $u_0 \in \mathscr{C}(\mathbb{R},[0,1]) \setminus \{0\}$ *dont le support est inclus dans une demi-droite dirigée vers la gauche,* $v_0 \in \mathscr{C}(\mathbb{R},[0,1]) \setminus \{0\}$ *à support compact et* (u,v) *la solution de (0.0.1) avec données initiales* (u_0, v_0)*.*

1. *Supposons* $2\sqrt{rd} < 2$*. Alors*
$$\lim_{t\to+\infty} \sup_{x\geq 0} |v(t,x)| = 0,$$
$$\lim_{t\to+\infty} \sup_{0\leq x<(2-\varepsilon)t} |u(t,x) - 1| = 0 \text{ pour tout } \varepsilon \in \left]0,2\right[,$$
$$\lim_{t\to+\infty} \sup_{(2+\varepsilon)t<x} |u(t,x)| = 0 \text{ pour tout } \varepsilon > 0.$$
2. *Supposons* $2\sqrt{rd} \in \left]2, f(c^\star)\right[$*. Soit*
$$c_{acc} = f^{-1}\left(2\sqrt{rd}\right) = \sqrt{rd} - \sqrt{a} + \frac{1-a}{\sqrt{rd}-\sqrt{a}} \in \left]c^\star, 2\right[.$$
Alors
$$\lim_{t\to+\infty} \sup_{0\leq x<(c_{acc}-\varepsilon)t} \left(|u(t,x)-1| + |v(t,x)|\right) = 0 \text{ pour tout } \varepsilon \in \left]0, c_{acc}\right[,$$
$$\lim_{t\to+\infty} \sup_{(c_{acc}+\varepsilon)t<x<\left(2\sqrt{rd}-\varepsilon\right)t} \left(|u(t,x)| + |v(t,x)-1|\right) = 0 \text{ pour tout } \varepsilon \in \left]0, \frac{2\sqrt{rd}-c_{acc}}{2}\right[,$$
$$\lim_{t\to+\infty} \sup_{\left(2\sqrt{rd}+\varepsilon\right)t<x} \left(|u(t,x)| + |v(t,x)|\right) = 0 \text{ pour tout } \varepsilon > 0.$$
3. *Supposons* $2\sqrt{rd} \geq f(c^\star)$*. Alors*
$$\lim_{t\to+\infty} \sup_{0\leq x<(c^\star-\varepsilon)t} \left(|u(t,x)-1| + |v(t,x)|\right) = 0 \text{ pour tout } \varepsilon \in \left]0, c^\star\right[,$$
$$\lim_{t\to+\infty} \sup_{(c^\star+\varepsilon)t<x<\left(2\sqrt{rd}-\varepsilon\right)t} \left(|u(t,x)| + |v(t,x)-1|\right) = 0 \text{ pour tout } \varepsilon \in \left]0, \frac{2\sqrt{rd}-c^\star}{2}\right[,$$
$$\lim_{t\to+\infty} \sup_{\left(2\sqrt{rd}+\varepsilon\right)t<x} \left(|u(t,x)| + |v(t,x)|\right) = 0 \text{ pour tout } \varepsilon > 0.$$

Ce résultat est remarquable pour au moins trois raisons.

- Le second cas montre que l'invasion de u est accélérée ($c_{acc} > c^\star$) si celle de v est trop lente ($2\sqrt{rd} < f(c^\star)$).
- La vitesse accélérée c_{acc} est donnée par une formule algébrique explicite.
- L'énoncé ne dépend pas de la détermination linéaire de $c^\star$.

La fonction f, qui donne pour tout $c \geq 2\sqrt{1-a}$ la plus grande racine de l'équation en $\tilde{c}$
$$\tilde{c}^2 - 4\lambda(c)\tilde{c} + 4(\lambda(c)c - 1) = 0,$$
où
$$\lambda(c) = \frac{1}{2}\left(c - \sqrt{c^2 - 4(1-a)}\right),$$
apparait naturellement dans le problème.

Supposons que v envahit le territoire inhabité à vitesse $2\sqrt{rd}$ et que u chasse v à vitesse $c_2 \in \left[c^\star, 2\sqrt{rd}\right[$. Dans la zone où $v \simeq 1$, u ressemble à la queue exponentielle de l'onde progressive monostable connectant $(0,1)$ à $(1,0)$ à vitesse c_2, c'est-à-dire

$$u(t,x) \simeq \mathrm{e}^{-\lambda(c_2)(x-c_2 t)}.$$

Si l'on se place alors dans un voisinage de $x = \tilde{c}t$, avec $\tilde{c} \in \left]c_2, 2\sqrt{rd}\right[$, on ne peut observer des quantités non-négligeables qu'à condition de considérer la fonction redimensionnée

$$w : (t,x) \mapsto u(t,x)\,\mathrm{e}^{\lambda(c_2)(\tilde{c}-c_2)t}$$

plutôt que u elle-même.

Or, dans un voisinage de $x = \tilde{c}t$ avec $\tilde{c} > 2\sqrt{rd}$, où $(u,v) \simeq (0,0)$, w satisfait au premier ordre

$$\partial_t w - \partial_{xx} w = (1 + \lambda(c_2)(\tilde{c} - c_2))\, w$$

et l'ansatz exponentiel $w(t,x) = \mathrm{e}^{-\Lambda(c_2,\tilde{c})(x-\tilde{c}t)}$ conduit à l'équation

$$(\Lambda(c_2,\tilde{c}))^2 - \tilde{c}\Lambda(c_2,\tilde{c}) + (1 + \lambda(c_2)(\tilde{c} - c_2)) = 0.$$

La racine minimale étant

$$\Lambda(c_2,\tilde{c}) = \frac{1}{2}\left(\tilde{c} - \sqrt{\tilde{c}^2 - 4(1 + \lambda(c_2)(\tilde{c} - c_2))}\right),$$

on déduit que $\tilde{c}$ doit précisément satisfaire

$$\tilde{c}^2 - 4\lambda(c_2)\,\tilde{c} + 4(\lambda(c_2)\,c_2 - 1) \geq 0,$$

soit $\tilde{c} \geq f(c_2)$. Prenant la limite $\tilde{c} \to 2\sqrt{rd}$, on trouve bien $2\sqrt{rd} \geq f(c_2)$.

L'essentiel de la preuve consiste donc à rendre rigoureuse cette observation à l'aide de sursolutions et sous-solutions finement construites.

Le théorème précédent est complété par deux autres qui seront présentés en détail dans le chapitre adéquat mais qui caractérisent l'ensemble des paires de vitesses possibles pour les terrasses de propagation engendrées par des données initiales exponentiellement décroissantes. La fonction f joue de nouveau un rôle essentiel et, de ce fait, l'ensemble est parfois strictement plus petit que l'ensemble maximal

$$\left\{(c_1,c_2) \in \left[2\sqrt{rd}, +\infty\right[\times [c^\star, +\infty[\ \mid c_1 > c_2\right\}.$$

Sur les systèmes KPP non-monotones

Dans cette sous-section, les inégalités vectorielles $\geq \mathbf{0}$, $> \mathbf{0}$ et $\gg \mathbf{0}$ sont respectivement comprises comme positivité de toutes les composantes, positivité de toutes les composantes avec inégalité stricte pour au moins une composante et stricte positivité de toutes les composantes. De plus, la notation $[N]$ désigne l'ensemble $\{1, \dots, N\}$.

On appelle *système KPP* un système de réaction – diffusion de la forme

$$\partial_t \mathbf{u} - \mathbf{D}\Delta\mathbf{u} = \mathbf{L}\mathbf{u} - \mathbf{c}(\mathbf{u}) \circ \mathbf{u}, \tag{0.0.4}$$

avec $\mathbf{u} \in \mathbb{R}^N$, $\mathbf{L} \in \mathbb{R}^{N\times N}$ une matrice carrée *irréductible* et *essentiellement positive* (c'est-à-dire dont seuls les termes diagonaux sont possiblement strictement négatifs) et $\mathbf{c}$ un champ de vecteurs de $\mathbb{R}^N$ vérifiant les propriétés suivantes :

1. $\mathbf{c}(\mathbf{u}) \geq \mathbf{0}$ si $\mathbf{u} \geq \mathbf{0}$, avec égalité si $\mathbf{u} = \mathbf{0}$;
2. il existe $\underline{\alpha} \geq 1$, $\delta \geq 1$ et $\underline{\mathbf{c}} \gg \mathbf{0}$ tels que, pour tous $\alpha \geq \underline{\alpha}$, $i \in [N]$ et $\mathbf{n} > \mathbf{0}$ satisfaisant $|\mathbf{n}| = 1$, l'on ait
$$\sum_{j=1}^{N} l_{i,j} n_j \geq 0 \implies \alpha^{\delta} \underline{c}_i \leq c_i(\alpha\mathbf{n}).$$

La seconde hypothèse est en particulier satisfaite si $\mathbf{c}(\mathbf{v})$ croit au moins linéairement quand $|\mathbf{v}| \to +\infty$, donc en particulier si $\mathbf{c}(\mathbf{v}) = \mathbf{C}\mathbf{v}$ avec $\mathbf{C} \gg \mathbf{0}$.

Ainsi, l'exemple typique de système KPP est le système de Lotka – Volterra compétitif avec mutations :
$$\partial_t \mathbf{u} - \mathbf{D}\Delta\mathbf{u} = \mathrm{diag}(\mathbf{r})\,\mathbf{u} + \mathbf{M}\mathbf{u} - \mathbf{C}\mathbf{u} \circ \mathbf{u},$$
où $\mathbf{r} \gg \mathbf{0}$, $\mathbf{C} \gg \mathbf{0}$ et $\mathbf{M}$ est une matrice carrée irréductible essentiellement positive satisfaisant $\sum\limits_{i=1}^{N} m_{i,j} = 0$ quel que soit $j \in [N]$, comme par exemple une matrice de la forme $\mathbf{M}_{Lap}\mathrm{diag}(\mathbf{w})$ avec $\mathbf{w} \gg \mathbf{0}$ et $\mathbf{M}_{Lap}$ le laplacien discret avec conditions de Neumann :
$$\mathbf{M}_{Lap} = \begin{pmatrix} -1 & 1 & 0 & \dots & 0 \\ 1 & -2 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & -2 & 1 \\ 0 & \dots & 0 & 1 & -1 \end{pmatrix}.$$

L'essentiel de la littérature et des travaux contemporains se concentrant sur le système de mutation – compétition – diffusion présenté ci-dessus, celui-ci suffit tout à fait pour un exposé introductif de l'état de l'art et des résultats. Toutefois, le cadre général des systèmes KPP permet d'aborder des problèmes issus d'applications fort différentes : introduction de classes d'âge dans l'équation de Fisher – KPP, système de Gross – Pitaevskii pour les condensats de Bose – Einstein à deux espèces avec couplage de Rabi, et ainsi de suite. Ces systèmes ainsi que les questions de modélisation sous-jacentes seront abordés dans les chapitres adéquats.

Les premiers travaux traitant des phénomènes de propagation pour les systèmes KPP sont dus à Freidlin [74]. À l'aide de méthodes probabilistes, ce dernier a étudié un système KPP à deux composantes admettant un état stationnaire non-nul globalement attractif. Inspirés par cette étude, Barles, Evans et Souganidis [10] ont traité en 1990 un cas beaucoup plus général à l'aide de méthodes EDP. Grâce au changement de variables WKB et à la limite de viscosité évanescente, ils ont su caractériser la vitesse de propagation des solutions du problème de Cauchy initialement à support compact. Bien que leur méthode, très utilisée aujourd'hui notamment pour traiter les modèles issus de la dynamique adaptative, n'emploie pas le cadre des ondes progressives mis en avant dans cette thèse, elle fournit donc le même genre de résultat et soulève ainsi très naturellement la question des ondes progressives.

La pertinence biologique des systèmes KPP a elle été rendue claire en 1998 par Dockery *et al.* [58] quand ils ont introduit des mutations de faible amplitude dans (0.0.2) afin de vérifier si « L'union fait la force » restait vrai dans un tel contexte. Néanmoins, à cause d'obstacles théoriques majeurs (pas de structure variationnelle, pas de principe de comparaison), un traitement plus exhaustif de ces systèmes était alors hors de portée, ce qui les a conduit à suggérer que le seul cas mathématiquement abordable était celui du système à deux composantes avec des mutations évanescentes. En effet, dans un tel cas, le système limite est précisément (0.0.1) et est donc bien mieux compris du fait du principe de comparaison.

Par conséquent, à de rares exceptions près, les recherches qui ont suivi se sont concentrées sur ce cas particulier. En 2012, Elliott et Cornell [65] ont publié une étude heuristique et numérique renouvelant l'intérêt, notamment pour la question des ondes progressives. En 2014, la question de la détermination linéaire a été posée formellement par Cosner [39]. En 2016, Griette et Raoul [82] ont montré pour la toute première fois qu'une onde progressive existe bel et bien et ont caractérisé la forme de son profil dans un régime particulier. En 2017, Morris, Börger et Crooks [115] ont établi avec des techniques encore différentes un résultat d'existence plus général et ont également obtenu un résultat sur la propagation de données initiales unidimensionnelles à support compact.

Cependant, ces résultats restent cantonnés au cas particulier du système à deux composantes avec petites mutations. Le cas général restait, avant le début de cette thèse, complètement ouvert.

L'approche employée dans cette thèse pour traiter le cas général est différente de celles précédemment employées. Elle repose sur l'observation suivante : le terme de réaction $\mathbf{Lu} - (\mathbf{Cu}) \circ \mathbf{u}$ est analogue au terme de réaction $ru - \frac{u^2}{K}$, c'est-à-dire à un terme de réaction de type KPP. En particulier, les solutions du problème linéarisé $\partial_t \mathbf{u} - \mathbf{D}\Delta\mathbf{u} = \mathbf{Lu}$ peuvent former des sur-solutions, bien que le système non-linéaire ne bénéficie pas d'un principe de comparaison.

Grâce à cette observation, les résultats suivants peuvent être démontrés.

Théorème. *[Gir18b] Toute solution positive* $\mathbf{u}$ *de (0.0.4) posé dans* $]0, +\infty[\times \mathbb{R}$ *telle que* $x \mapsto \mathbf{u}(0, x)$ *soit non-nulle satisfait* $\mathbf{u}(t, x) \gg \mathbf{0}$ *pour tout* $(t, x) \in]0, +\infty[\times \mathbb{R}$.

Théorème. *[Gir18b] Il existe une fonction* $\mathbf{g} : [0, +\infty[\to \mathbb{R}^N$ *continue, dont toutes les composantes sont croissantes et satisfaisant* $\mathbf{g}(0) \gg \mathbf{0}$ *telle que toute solution positive* $\mathbf{u}$ *de (0.0.4) posé dans* $]0, +\infty[\times \mathbb{R}$ *satisfasse*

$$\mathbf{u}(t, x) \leq \left(g_i \left(\sup_{x \in \mathbb{R}} u_i(0, x) \right) \right)_{i \in [N]} \quad \textit{pour tout } (t, x) \in [0, +\infty[\times \mathbb{R}$$

et si de plus $x \mapsto \mathbf{u}(0, x)$ *est bornée, alors*

$$\left(\limsup_{t \to +\infty} \sup_{x \in \mathbb{R}} u_i(t, x) \right)_{i \in [N]} \leq \mathbf{g}(0).$$

Dans ce qui suit, $\lambda_{PF}(\mathbf{L})$ désigne la valeur propre de Perron − Frobenius de $\mathbf{L}$ et $\mathbf{n}_{PF}(\mathbf{L})$ désigne son vecteur propre satisfaisant $\mathbf{n}_{PF}(\mathbf{L}) \gg \mathbf{0}$ et $|\mathbf{n}_{PF}(\mathbf{L})| = 1$.

Théorème. *[Gir18b] Supposons* $\lambda_{PF}(\mathbf{L}) \leq 0$. *Alors toute solution bornée positive de (0.0.4) posé dans* $]0, +\infty[\times \mathbb{R}$ *converge en temps long, uniformément en espace, vers* $\mathbf{0}$ *pourvu que l'une des conditions suivantes soit satisfaite :*

1. $\lambda_{PF}(\mathbf{L}) < 0$*, et dans ce cas la convergence est exponentielle en temps ;*
2. $\lambda_{PF}(\mathbf{L}) = 0$ *et* $\mathbf{c}(\alpha \mathbf{n}_{PF}(\mathbf{L})) > \mathbf{0}$ *pour tout* $\alpha > 0$.

Ce théorème correspond au cas dit *d'extinction* et sa preuve, relativement élémentaire, repose sur la comparaison avec la sur-solution formée par la solution du système linéarisé. Au contraire, le cas dit *de persistance*, donné par le théorème suivant, est prouvé à l'aide d'un habile jonglage entre l'instabilité de $\mathbf{0}$ et l'inégalité de Harnack établie en 2009 par Földes et Poláčik [73].

Théorème. *[Gir18b] Supposons* $\lambda_{PF}(\mathbf{L}) > 0$. *Alors il existe* $\nu > 0$ *tel que toute solution bornée positive non-nulle* $\mathbf{u}$ *de (0.0.4) posé dans* $]0, +\infty[\times \mathbb{R}$ *satisfasse, pour tout intervalle borné* $I \subset \mathbb{R}$,

$$\left(\liminf_{t \to +\infty} \inf_{x \in I} u_i(t, x) - \nu \right)_{i \in [N]} \geq \mathbf{0}.$$

De plus, il existe un état stationnaire constant positif non-nul, qui est donc à valeurs dans

$$\prod_{i=1}^{N} \left[\nu, g_i\left(0\right)\right].$$

Une fois ces théorèmes basiques établis, on peut laisser de côté le cas d'extinction, se concentrer sur le cas plus intéressant de persistance et se tourner vers les phénomènes de propagation. On constate rapidement que le comportement en temps long est difficile à caractériser plus précisément en toute généralité (en particulier, de multiples états stationnaires localement stables peuvent exister). On adopte donc une définition plus faible des ondes progressives, énoncée ci-dessous.

Définition. Une solution sous forme onde progressive pour (0.0.4) est une solution entière, positive, bornée, de la forme $\mathbf{u} : (t,x) \mapsto \mathbf{p}\left(x - ct\right)$, avec une vitesse $c \geq 0$ et un profil $\mathbf{p} \in \mathscr{C}^2\left(\mathbb{R}, \mathbb{R}^N\right)$ satisfaisant

$$\left(\liminf_{\xi \to -\infty} p_i\left(\xi\right)\right)_{i \in [N]} > \mathbf{0} \text{ et } \lim_{\xi \to +\infty} \mathbf{p}\left(\xi\right) = \mathbf{0}.$$

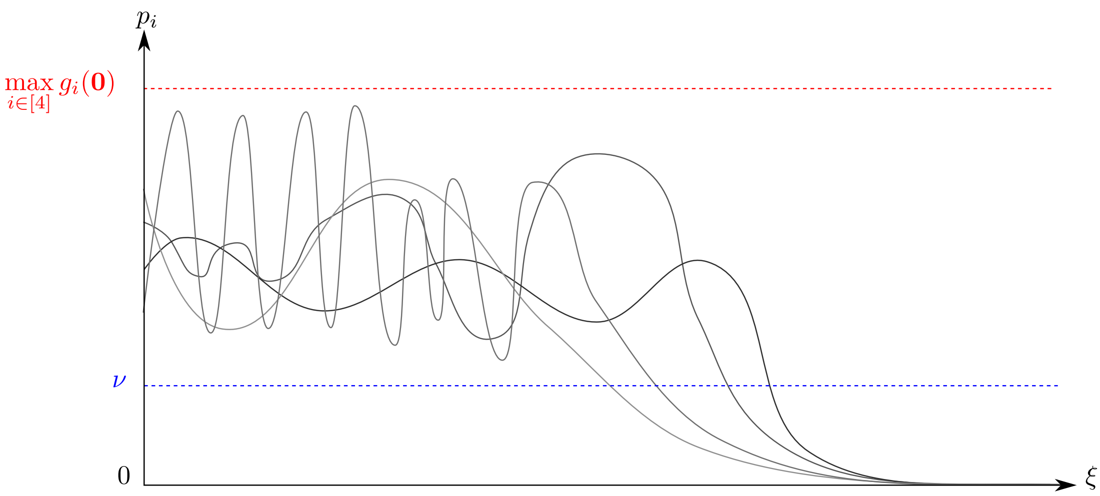


FIGURE 0.0.7 – Exemple de profil d'onde progressive pour un système KPP à quatre composantes. Le comportement à l'arrière est volontairement représenté comme non-convergent, en l'absence de résultat plus probant.

Le théorème suivant est ensuite prouvé en adaptant des idées dues à Berestycki, Nadin, Perthame et Ryzhik [19] permettant de contourner le défaut de principe de comparaison.

Théorème. *[Gir18b] Supposons* $\lambda_{PF}\left(\mathbf{L}\right) > 0$. *Soit*

$$c^\star = \min_{\mu > 0} \left(\frac{\lambda_{PF}\left(\mu^2 \mathbf{D} + \mathbf{L}\right)}{\mu}\right).$$

La quantité $c^\star$ *est strictement positive et est la vitesse minimale des ondes progressives ainsi que la vitesse de propagation des données initiales unidimensionnelles à support inclus dans une demi-droite dirigée vers la gauche, dans le sens suivant :*

1. *pour toute $c \in [0, c^\star[$, il n'existe pas de solution de (0.0.4) sous forme d'onde progressive de vitesse c ;*
2. *si $D\mathbf{c}(\mathbf{v}) \geq \mathbf{0}$ pour tout $\mathbf{v} \geq \mathbf{0}$, alors pour toute $c \geq c^\star$, il existe une solution de (0.0.4) sous forme d'onde progressive de vitesse c ;*
3. *pour tout $x_0 \in \mathbb{R}$ et toute fonction continue, bornée, positive, non-nulle $\mathbf{v}$, la solution $\mathbf{u}$ de (0.0.4) posé dans $]0, +\infty[\times \mathbb{R}$ avec donnée initiale $\mathbf{v}\mathbf{1}_{(-\infty, x_0)}$ satisfait*

$$\left(\lim_{t \to +\infty} \sup_{x > y} u_i (t, x + ct)\right)_{i \in [N]} = \mathbf{0} \text{ pour toute } c \in]c^\star, +\infty[\text{ et tout } y \in \mathbb{R},$$

$$\left(\liminf_{t \to +\infty} \inf_{x \in [-R, R]} u_i (t, x + ct)\right)_{i \in [N]} \in \mathsf{K}^{++} \text{ pour toute } c \in [0, c^\star[\text{ et tout } R > 0.$$

De plus, le profil $\mathbf{p}$ de toute onde progressive satisfait

$$\mathbf{p} \leq \mathbf{g}(0) \text{ et } \left(\liminf_{\xi \to -\infty} p_i (\xi) - \nu\right)_{i \in [N]} \geq \mathbf{0}.$$

À partir de la formule explicite donnant $c^\star$, diverses estimations peuvent également être déduites. Celles-ci seront détaillées dans le chapitre adéquat.

Tandis que les résultats ci-dessus sont démontrés en utilisant exclusivement les sur-solutions et sous-solutions classiques de la littérature sur l'équation de Fisher – KPP, l'emploi de méthodes plus variées permet de raffiner les résultats qualitatifs sur les profils d'ondes progressives.

On définit, pour toute $c \geq c^\star$, les quantités

$$\mu_c = \min \left\{ \mu > 0 \mid \frac{\lambda_{PF}\left(\mu^2 \mathbf{D} + \mathbf{L}\right)}{\mu} = c \right\},$$

$$k_c = \begin{cases} 0 & \text{si } c > c^\star, \\ 1 & \text{si } c = c^\star. \end{cases}$$

La quantité μ_c est bien définie et strictement positive (voir les chapitres adéquats).

Théorème. *[Gir18a] Pour toute onde progressive de profil $\mathbf{p}$ et de vitesse c, il existe $A > 0$ tel que, quand $\xi \to +\infty$,*

$$\begin{cases} \mathbf{p}(\xi) \sim A \xi^{k_c} e^{-\mu_c \xi} \mathbf{n}_{PF}\left(\mu_c^2 \mathbf{D} + \mathbf{L}\right), \\ \mathbf{p}'(\xi) \sim -\mu_c \mathbf{p}(\xi), \\ \mathbf{p}''(\xi) \sim \mu_c^2 \mathbf{p}(\xi). \end{cases}$$

Par conséquent, les composantes de $\mathbf{p}$ sont toutes, dans un voisinage de $+\infty$, strictement décroissantes et convexes.

De multiples preuves de ce théorème sont envisageables. Ayant fait le choix d'éviter les méthodes EDO mais ne pouvant appliquer une méthode purement EDP du fait de l'absence de principe de comparaison, on propose dans cette thèse une preuve utilisant des résultats généraux d'analyse réelle (un théorème d'Ikehara et les propriétés de la transformée de Laplace bilatérale). Il est raisonnable d'espérer que cette preuve puisse être généralisée au contexte des ondes pulsatoires en milieu spatialement périodique.

Sous des hypothèse restrictives sur les paramètres, on peut utiliser la forme de Jordan de $\mathbf{L}$ et la projection de Perron – Frobenius pour réduire le système KPP à une simple équation de Fisher – KPP. Ceci est indiqué par les deux théorèmes suivants.

Théorème. *[Gir18a] Supposons* $\lambda_{PF}(\mathbf{L}) > 0$, $\mathbf{D} = \mathbf{I}$ *ainsi que l'existence de* $b : \mathbb{R}^N \to \mathbb{R}$ *telle que, pour tout* $\mathbf{v} \geq \mathbf{0}$ *et tout* $i \in [N]$, $c_i(\mathbf{v}) = b(\mathbf{v})$ *et la fonction* $w \mapsto b(w\mathbf{e}_i + \mathbf{v})$ *est strictement croissante dans* $]0, +\infty[$.

Soient $\alpha^\star > 0$ *l'unique solution de* $b(\alpha \mathbf{n}_{PF}(\mathbf{L})) = \lambda_{PF}(\mathbf{L})$ *et* $\mathbf{v}^\star = \alpha^\star \mathbf{n}_{PF}(\mathbf{L})$.

Alors toutes les solutions bornées positives non-nulles de (0.0.4) posé dans $]0, +\infty[\times\mathbb{R}$ *convergent en temps long, localement uniformément en espace, vers* $\mathbf{v}^\star$.

Par conséquent, l'ensemble des solutions stationnaires bornées positives est exactement $\{\mathbf{0}, \mathbf{v}^\star\}$.

Théorème. *[Gir18a] Supposons que les hypothèses du théorème précédent sont toujours vérifiées.*

Pour toute $c \in [c^\star, +\infty[$, *soit* $p_c \in \mathscr{C}^2(\mathbb{R})$ *tel que* $u : (t,x) \mapsto p(x - ct)$ *est l'unique onde progressive solution de l'équation de Fisher – KPP*

$$\partial_t u - \partial_{xx} u = \lambda_{PF}(\mathbf{L}) u - b(u\mathbf{n}_{PF}(\mathbf{L})) u$$

connectant 0 *à* $\alpha^\star$ *et satisfaisant* $p_c(0) = \frac{\alpha^\star}{2}$.

Alors tout profil d'onde progressive pour (0.0.4) de vitesse c a la forme

$$\mathbf{p} : \xi \mapsto p_c(\xi - \xi_0)\, \mathbf{n}_{PF}(\mathbf{L}) \text{ avec } \xi_0 \in \mathbb{R}$$

et par conséquent, l'onde progressive de vitesse c est unique et connecte $\mathbf{0}$ *à* $\mathbf{v}^\star$.

Puisque $\mathbf{D} = \mathbf{I}$ implique $c^\star = 2\sqrt{\lambda_{PF}(\mathbf{L})}$, cette quantité est bien à la fois la vitesse minimale des ondes progressives de (0.0.4) et celle pour l'équation de Fisher – KPP scalaire apparaissant dans l'énoncé.

Enfin, pour le système à deux composantes, l'idée selon laquelle la limite de mutations évanescentes vérifie un principe de comparaison peut être rigoureusement appliquée en considérant la limite $\eta \to 0$ du système suivant :

$$\begin{cases} \partial_t u_1 - d_1 \partial_{xx} u_1 = r_1 u_1 - (c_{1,1} u_1 + c_{1,2} u_2) u_1 + \eta m_1 (u_2 - u_1) \\ \partial_t u_2 - d_2 \partial_{xx} u_2 = r_2 u_2 - (c_{2,1} u_1 + c_{2,2} u_2) u_2 + \eta m_2 (u_1 - u_2) \end{cases}.$$

On dénote $(\alpha_1, \alpha_2) = \left(\frac{r_1}{c_{1,1}}, \frac{r_2}{c_{2,2}}\right)$ les capacités de charge en absence de mutations et, si $c_{1,1}c_{2,2} \neq c_{1,2}c_{2,1}$, on dénote

$$\mathbf{v}_m = \frac{1}{c_{1,1}c_{2,2} - c_{1,2}c_{2,1}} \begin{pmatrix} r_1 c_{2,2} - r_2 c_{1,2} \\ r_2 c_{1,1} - r_1 c_{2,1} \end{pmatrix}$$

l'état de coexistence en absence de mutations. On suppose en outre que le système sans mutations est monostable, c'est-à-dire qu'il existe $i \in \{1, 2\}$ tel que

$$\frac{r_i}{r_{3-i}} > \frac{c_{i,3-i}}{c_{3-i,3-i}}.$$

L'état stable est alors

$$\mathbf{v}_s = \begin{cases} \alpha_i \mathbf{e}_i & \text{si } \frac{r_i}{r_{3-i}} \geq \frac{c_{i,i}}{c_{3-i,i}}, \\ \mathbf{v}_m & \text{si } \frac{r_i}{r_{3-i}} < \frac{c_{i,i}}{c_{3-i,i}}. \end{cases}$$

Théorème. *[Gir18a] Soient* $(\mathbf{p}_\eta)_{\eta>0}$ *et* $(c_\eta)_{\eta\geq 0}$ *telles que :*

1. *pour tout* $\eta > 0$, $(t,x) \mapsto \mathbf{p}_\eta(x - c_\eta t)$ *soit une onde progressive pour le problème avec taux de mutation* η *;*
2. $c_\eta \to c_0$ *quand* $\eta \to 0$.

Alors il existe $(\zeta_\eta)_{\eta>0}$ *telle que, quand* $\eta \to 0$, $(\xi \mapsto \mathbf{p}_\eta(\xi+\zeta_\eta), c_\eta)_{\eta>0}$ *converge à extraction près dans* $\mathscr{C}^2_{loc}(\mathbb{R},\mathbb{R}^2)\times\mathbb{R}$ *vers la paire profil – vitesse* $(\mathbf{p}, c_0)$ *d'une onde progressive pour le système sans mutations*

$$\begin{cases}\partial_t u_1 - d_1\partial_{xx}u_1 = r_1u_1 - (c_{1,1}u_1 + c_{1,2}u_2)\,u_1\\ \partial_t u_2 - d_2\partial_{xx}u_2 = r_2u_2 - (c_{2,1}u_1 + c_{2,2}u_2)\,u_2\end{cases}$$

réalisant l'une des connections suivantes :

1. $\mathbf{0}$ *à* $\mathbf{v}_s$,
2. $\alpha_{3-i}\mathbf{e}_{3-i}$ *à* $\mathbf{v}_s$,
3. $\mathbf{0}$ *à* $\alpha_i\mathbf{e}_i$ *avec* $p_{3-i}=0$.

Dans le chapitre adéquat, on présente quelques simulations numériques qui conduisent à une conjecture générale couvrant également le cas bistable. Cette conjecture permettra d'orienter efficacement les futures recherches.

Perspectives

Les perspectives de recherche future ouvertes par cette thèse sont nombreuses. On ne présentera ici que les plus intéressantes, susceptibles d'être étudiées dans un avenir très proche.

Tout d'abord, concernant la limite de forte compétition et les théorèmes de type « L'union ne fait pas la force », il pourrait être intéressant d'étudier comment des stratégies de mouvement plus complexes qu'une simple diffusion influencent le résultat. Par exemple, Potts et Petrovskii [127] ont récemment illustré numériquement qu'un résultat de type « L'union fait la force » pouvait émerger pour le système

$$\begin{cases}\partial_t u - \Delta u + s_1\nabla\cdot(u\nabla v) = u\,(1-u-av)\\ \partial_t v - d\Delta v + s_2\nabla\cdot(v\nabla u) = rv\,(1-v-bu)\end{cases},$$

où les termes $s_1\nabla\cdot(u\nabla v)$ et $s_2\nabla\cdot(v\nabla u)$ modélisent, si $s_1>0$ et $s_2>0$, un tactisme agressif poussant un compétiteur vers l'habitat de l'autre. Il serait intéressant de déterminer un régime de paramètres dans lequel une confirmation analytique rigoureuse est possible. La piste la plus évidente est le régime $s_1 = s$, $s_2 = \sigma s$, $s\to+\infty$, analogue d'une certaine manière à la limite de forte compétition. Toutefois l'existence d'ondes progressives pour un tel système, fortement couplé, est un problème réellement difficile.

Les résultats obtenus avec Adrian Lam sur les propriétés de propagation de système de Lotka – Volterra compétitif monostable pourraient être renforcés en prouvant la convergence localement uniforme des solutions vers des ondes progressives. Des avances ou retards à la Bramson [30, 29] sont évidemment attendus et posent de remarquables obstacles. Il pourrait également être intéressant de se tourner vers la classification des solutions entières du système, dans laquelle les terrasses de propagation que nous avons exhibées joueront un rôle central.

Ensuite, concernant les systèmes KPP, la limite de mutations évanescentes pour le système à deux composantes et la conjecture susmentionnée méritent une étude plus approfondie. La principale difficulté lors du passage à la limite est de trouver une normalisation correcte : où centrer l'observation pour que la limite ne soit pas un état stationnaire constant ? Là où le théorème cité ci-dessus utilise une normalisation connue du cas monostable, le cas bistable semble bien plus mystérieux.

Toujours au sujet des systèmes KPP, il parait également important d'étudier la possible généralisation des théorèmes réduisant le système KPP à une équation scalaire : est-il possible de supprimer l'hypothèse $\mathbf{D} = \mathbf{I}$? Est-il possible de changer $c_i(\mathbf{v}) = b(\mathbf{v})$ en $c_i(\mathbf{v}) = b(\mathbf{v})\, a_i$ avec $\mathbf{a} \gg \mathbf{0}$? Naïvement, toutes les réponses paraissent envisageables à ce stade. Des simulations numériques devront donc être réalisées.

Enfin, une autre direction de recherche sur les systèmes KPP serait la généralisation des résultats obtenus dans cette thèse aux milieux périodiques unidimensionnels et aux ondes pulsatoires. Une telle généralisation résoudrait de nombreuses questions laissées en suspens par Alfaro et Griette [2].

Introduction (English version)

General setting, state of the art and goals

Mathematical definition of a reaction–diffusion system

A *reaction–diffusion system* as understood in this thesis is a system of *partial differential equations* (*PDEs* hereafter) of the form

$$\frac{\partial \mathbf{u}}{\partial t} - \mathbf{D}\Delta_x \mathbf{u} = \mathbf{f}\left(\mathbf{u}, t, x\right),$$

where the column vector $\mathbf{u}$ is a function of a real time variable $t \in \mathbb{R}$ and of a Euclidean space variable $x \in \mathbb{R}^n$, $\frac{\partial}{\partial t}$ denotes the partial derivative with respect to t (the compact notation ∂_t will be often used hereafter), Δ_x denotes the spatial Laplacian, that is the sum of the second order partial derivatives $\frac{\partial^2}{\partial x_i^2}$ with $x = (x_i)_{i\in\{1,...,n\}}$ (the compact notation Δ will be often used hereafter), $\mathbf{D}$ is a diagonal matrix with positive diagonal entries referred to as the *diffusion matrix* and $\mathbf{f}$ is a function possibly non-linear with respect to $\mathbf{u}$ referred to as the *reaction term*. The matrix $\mathbf{D}$ being diagonal, any coupling between the equations is due to the reaction term and involves no partial derivatives of $\mathbf{u}$: the system is referred to as *weakly coupled*. Furthermore, the system can be understood as a system of heat equations with internal heat generation and is therefore referred to as *parabolic*.

More generally, the system can govern the evolution of $\mathbf{u}$ starting from some initial time $t_0 \in \mathbb{R}$, until some final time $T \in \mathbb{R}$ or in some spatial domain $\Omega \subset \mathbb{R}^n$. In such a case, the domain of definition of $(t, x) \mapsto \mathbf{u}(t, x)$ is accordingly restricted and the reaction–diffusion system is supplemented with initial conditions, final conditions or boundary conditions. In particular, a problem formed of a reaction–diffusion system set in $(t_0, +\infty) \times \mathbb{R}^n$ supplemented with an initial condition is referred to as a *Cauchy problem*. Solutions defined in $\mathbb{R} \times \mathbb{R}^n$ are referred to as *entire solutions*.

Provided $\mathbf{u}$ is actually a scalar quantity, we obtain a single reaction–diffusion equation of the form

$$\partial_t u - d\Delta u = f\left(u, t, x\right).$$

Provided $\mathbf{u}$ and $\mathbf{f}$ do not depend on x, we obtain a system of *ordinary differential equations* (*ODEs* herefater) of the form

$$\mathbf{u}' = \mathbf{f}\left(\mathbf{u}, t\right).$$

Provided $\mathbf{u}$ and $\mathbf{f}$ do not depend on t, we obtain a system of weakly coupled *elliptic* PDEs of the form

$$-\mathbf{D}\Delta \mathbf{u} = \mathbf{f}\left(\mathbf{u}, x\right).$$

If $\mathbf{f}$ depends only on its variable $\mathbf{u}$, the system is referred to as *set in homogeneous media*. The contrary case is referred to as *set in (spatially or temporally) heterogeneous media*. Similarly, a system can be set in (spatially or temporally) periodic media, random media, and so on. A homogeneous medium is a (very) particular case of periodic or random medium.

Reaction–diffusion systems as population dynamic models

The branch of ecology, and therefore of biology, concerned with the evolution in time of the number of individuals in a population of non-human lifeforms is the non-human *population dynamics* (hereafter simply population dynamics; human population dynamics, which takes into account socio-economical aspects, is not proper biology and is not the subject of this thesis). Due to its qualitative side, its historical origins (read on this subject the book by Bacaër [9]) and its inclination for abstraction, it is one of the most mathematized branches of biology. Two main families of mathematical models exist in population dynamics (and more generally in biology): deterministic models and stochastic models. Among deterministic models, we find a lot of reaction–diffusion systems.

Reaction–diffusion systems arise as population dynamic models mainly in two ways: either as a refinement of scalar reaction–diffusion equations or as a refinement of ODE systems. In the former case, the point is to account for coupling between different populations, whereas in the latter case, the point is to introduce a spatial structure in the problem and to take into account the dispersal of individuals. Let us now present the underlying modeling for one example of reaction–diffusion equation, the Fisher–Kolmogorov–Petrovsky–Piskunov (Fisher–KPP or simply KPP hereafter),

$$\partial_t u - \Delta u = u\,(1-u)\,,$$

as well as that for one example of ODE system, the Lotka–Volterra system of two competitive species,

$$\begin{cases} u' = u\,(1-u-av) \\ v' = rv\,(1-v-bu) \end{cases},$$

where a, b and r are positive constants. These two examples will turn out to be very important examples hereafter.

The assumptions shared by the two models are:

1. the number of individuals in a population and the spatial and temporal scales are so large that the number of individuals, which is really a discrete quantity, is correctly approximated by a continuous population density;
2. newborns become instantly adults or, equivalently, newborns do not influence the demography and only adult individuals are counted (no age structure);
3. if the reproduction of individuals is sexual, the distribution of males and females is homogeneous, so that it suffices to know the total density to know exactly the population (no sexual structure).

In order to obtain the Fisher–KPP equation, we consider one population density u and we assume the following:

1. the population is diffusing in space with a rate $d > 0$, or in other words the population flux is proportional to the population gradient with a proportionality constant $-d$;
2. at a given point in space, the part of the variation of the population density $\frac{\partial_t u}{u}$ due to births and deaths is *logistic*, that is has the form $r\left(1-\frac{u}{K}\right)$ with $r > 0$ and $K > 0$. This assumption implies the following presupposition, referred to by ecologists as *absence of Allee effect*: because of the competition for resources between individuals, the growth rate of the population is a decreasing function of the density, nonnegative if and only if $u \leq K$ and maximized at $u = 0$ where it equals $r > 0$. Consequently, the constants r and K are respectively referred to as intrinsic growth rate and carrying capacity.

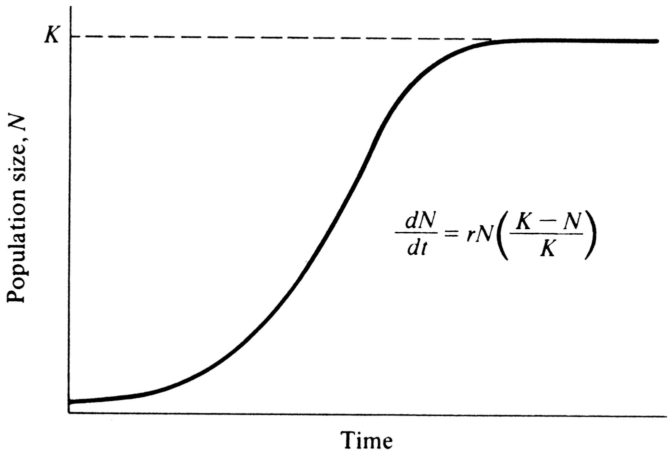


Figure 0.0.1 – Logistic growth curve

These assumptions lead to the equation

$$\partial_t u - d\Delta u = r\left(1 - \frac{u}{K}\right)u.$$

Setting dimensionless quantities $\tilde{u} = \frac{u}{K}$, $\tilde{t} = rt$ and $\tilde{x} = \sqrt{\frac{r}{d}}x$ and then getting rid of the $\sim$, we obtain indeed the normalized Fisher–KPP equation.

In order to obtain the competitive Lotka–Volterra system, we consider two population densities u and v and we assume the following:

1. in the absence of the other density, each density grows logistically;
2. when both densities are present, because of the interspecific competition, we subtract from each growth rate an additional term positively proportional to the density of the competitor (in other words, each growth rate is now a decreasing affine function of some linear combination, with positive coefficients, of the two densities).

These assumptions lead to the system

$$\begin{cases} u' = r_1 u\left(1 - \frac{u}{K_1} - \frac{v}{L_1}\right) \\ v' = r_2 v\left(1 - \frac{v}{K_2} - \frac{u}{L_2}\right) \end{cases}.$$

Setting $\tilde{t} = r_1 t$, $\tilde{u} = \frac{u}{K_1}$, $a = \frac{K_2}{L_1}$, $r = \frac{r_2}{r_1}$, $\tilde{v} = \frac{v}{K_2}$ and $b = \frac{K_1}{L_2}$ and then getting rid of the $\sim$, we obtain indeed the normalized Lotka–Volterra system of two competitive species.

All these assumptions done at the population density scale, referred to as the macroscopic scale, can also be interpreted at the individual scale, referred to as the microscopic scale. More details on the microscopic assumptions and on the history of the competition between macroscopic modeling and microscopic modeling can be found for instance in the book by Israel [98] (in French). Of course, discussing the assumptions is crucial when applying the models but this is not the point of this thesis.

By coupling competitively two Fisher–KPP equations, or by introducing spatial diffusion in the Lotka–Volterra system, we finally get a first example of reaction–diffusion system: the two-species competition–diffusion Lotka–Volterra system,

$$\begin{cases} \partial_t u - \Delta u = u\left(1 - u - av\right) \\ \partial_t v - d\Delta v = rv\left(1 - v - bu\right) \end{cases}.$$

We point out right now that by letting a and b have any arbitrary sign, we obtain two other important types of coupling: predative ($ab < 0$) and mutualistic ($a < 0$, $b < 0$). More generally, a coupling is *of Lotka–Volterra type* if it has the form $\mathbf{u} \circ (\mathbf{Cu})$ with $\mathbf{C}$ a square matrix and $\circ$ the *Hadamard product*, namely component-by-component product.

We emphasize also that, although the motivations of Fisher [72] and of Kolmogorov, Petrovsky and Piskunov [104] came from population genetics and more precisely from problems of competition between two alleles, the above purely demographic derivation directly presents the Fisher–KPP equation as the logistic equation with diffusion. This later derivation is due to Skellam [134]. Nowadays, most studies of the Fisher–KPP equation in the applied mathematics literature or in the ecology literature are motivated by the model of Skellam and not by the original genetic model.

Reaction–diffusion and propagation phenomena

One of the main reasons of the success of reaction–diffusion models in population dynamics is their ability to describe invasions.

Traveling waves, ODE methods, PDE methods

Investigating invasions with constant speed and direction in homogeneous media naturally brings forth entire solutions of the form $\mathbf{u} : (t,x) \mapsto \boldsymbol{\varphi}(x \cdot e - ct)$, with $e \in \mathbb{S}^{n-1}$ a direction of propagation, $c \in \mathbb{R}$ a speed of propagation and $\boldsymbol{\varphi}$ a profile of propagation. Such solutions are generally referred to as *traveling waves* (or, more precisely, traveling plane waves, when having in mind multidimensional media in which more various traveling waves, like radial or conical traveling waves, can exist).

Such a traveling wave satisfies an ODE system of the form

$$-\mathbf{D}\boldsymbol{\varphi}'' - c\boldsymbol{\varphi}' = \mathbf{f}(\boldsymbol{\varphi}).$$

Thanks to this observation, the existence and the properties of such solutions can be addressed with methods coming from the ODE literature (Picard–Lindelöf theorem, shooting method, stable and unstable manifolds, and so on) or with methods coming from the elliptic PDE literature (Schauder theory, variational calculus, maximum principle and comparison principle, Harnack inequalities, and so on).

The literature on traveling waves can therefore be separated into two families, according to the type of arguments (ODE or PDE) that are used. In this thesis, we try to use as much as possible PDE methods, in order to be able to generalize the proofs to suitable heterogeneous media and also in order to be able to study concurrently the propagation properties of the Cauchy problem, which truly require PDE arguments.

Dimension of the medium

When studying traveling waves in a homogeneous medium, we can assume without loss of generality that this medium is one-dimensional and that $e = +1$. These assumptions simplify the notations and therefore will be always assumed hereafter. In this setting, the uniqueness of a traveling wave is understood as uniqueness up to rotation of e and up to translation of $\boldsymbol{\varphi}$.

Of course, such simplifications cannot be performed anymore when studying Cauchy problems with non-one-dimensional initial data. For such problems, the dimension of the medium will always be precised.

Scalar equations

Research on invasions produced by reaction–diffusion equations started in 1937. The founding results are the following theorems.

Theorem. *[104] The solution u of a Cauchy problem associated with the one-dimensional Fisher–KPP equation with bounded nonnegative nonzero compactly supported initial data satisfies*

$$\lim_{t\to+\infty} \sup_{|x|>ct} u(t,x) = 0 \text{ for all speeds } c > 2,$$

$$\lim_{t\to+\infty} \sup_{|x|<ct} u(t,x) = 1 \text{ for all speeds } c < 2.$$

Ecologists are indeed interested in invasions of initially spatially confined populations, introduced at some precise place. By establishing that such an invasion occurs asymptotically at constant speed, this first theorem shows that the relevant entire solutions are indeed the traveling waves and therefore leads to a second theorem.

Definition. A traveling wave with nonnegative speed describing the invasion of 0 by 1 for the Fisher–KPP equation is a traveling wave whose profile is decreasing and has limits 1 and 0 at $-\infty$ and $+\infty$ respectively.

Theorem. *[104] The Fisher–KPP equation admits a traveling wave solution with speed $c \geq 0$ describing the invasion of 0 by 1 if and only if $c \geq 2$. This solution is unique.*

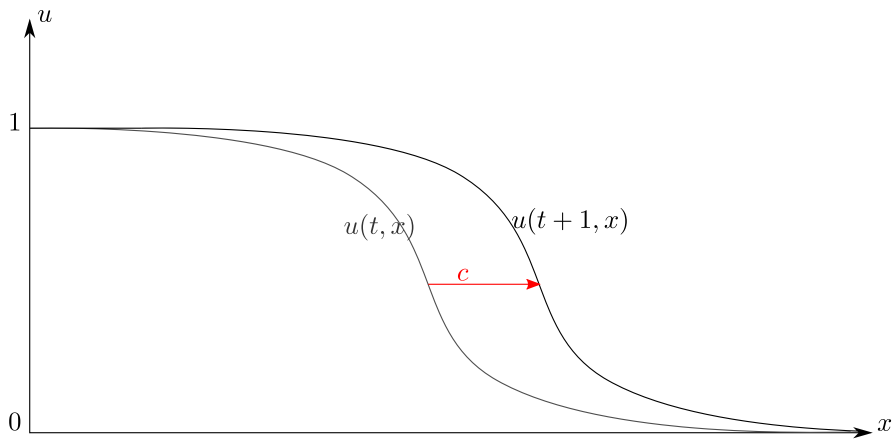


Figure 0.0.2 – A traveling wave for the Fisher–KPP equation

In return, it is then possible to prove the following theorem.

Theorem. *[104] Let u be the solution of the Cauchy problem associated with the one-dimensional Fisher–KPP equation with initial condition $\mathbf{1}_{(-\infty,0)}$ and φ_2 be the profile of the traveling wave with speed 2 for this equation.*

Then there exists $m : \mathbb{R} \to \mathbb{R}$ such that, as $t \to +\infty$,

$$m(t) = o(t),$$

$$\sup_{x\in\mathbb{R}} |u(t, x - ct - m(t)) - \varphi_2(x)| \to 0.$$

In other words, up to a correction $m(t)$, the traveling wave with minimal speed corresponds indeed to the long-time behavior of the solution of the Cauchy problem. The asymptotic expansion of $m(t)$ became after 1937 an important research subject, first considered by Bramson [30, 29] with probabilistic methods. The recent paper by Hamel, Nolen, Roquejoffre and Ryzhik [90] offers a PDE proof of the main result of Bramson and provides some bibliographical references.

Recalling that t and x are dimensionless variables, we find by going back to the physical variables that the speed 2 is replaced by $c^\star = 2\sqrt{rd}$. Thanks to the unexpected simplicity of this formula, the mathematical model can be efficiently compared to empirical data.

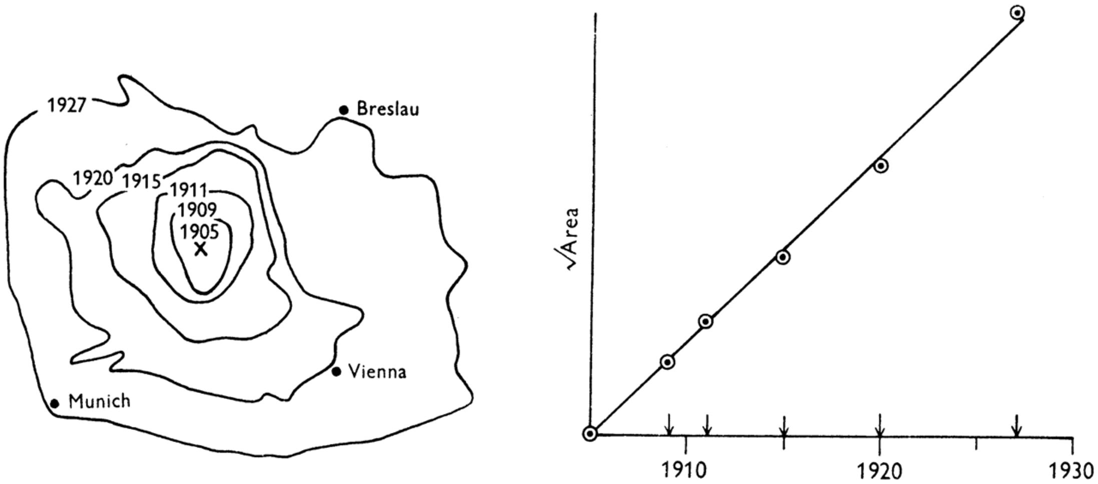


Figure 0.0.3 – Graphics from the article by Skellam illustrating the propagation at constant speed of the muskrat in central Europe and motivating the use of the Fisher–KPP equation as model.

In particular, this formula does not depend on the carrying capacity K. This is an immediate consequence of the fact that $c^\star$ is *linearly determined*: it coincides with the minimal speed of existence of positive solutions for the equation

$$-d\varphi'' - \varphi' = r\varphi,$$

which is actually the equation satisfied by the profile φ of a traveling wave with speed c linearized at 0. This equation, which *a priori* governs only the behavior of the profile where the population density is negligible, determines in fact completely the speed $c^\star$.

The vast literature on reaction–diffusion equations, and then systems, developed after 1937 is largely influenced by the founding papers on the Fisher–KPP equation. Especially, given a new reaction–diffusion problem, the existence of traveling waves as well as the relation between the nonlinear speed and the linear one are nowadays systematically investigated. Without trying to provide an exhaustive overview of existing results, we cite nevertheless an important result of Fife and McLeod [71] that will be used hereafter.

Theorem. *[71] The equation*

$$\partial_t u - \Delta u = u(u-\theta)(1-u)$$

with $\theta \in (0,1)$ *admits a unique traveling wave solution connecting* 0 *and* 1.

Furthermore, the speed c *of this traveling wave has the sign of* $\int_0^1 u(u-\theta)(1-u)\,du = \frac{1}{6}\left(\frac{1}{2}-\theta\right)$.

Definition. A traveling wave connecting 0 and 1 for the equation

$$\partial_t u - \Delta u = u\left(u-\theta\right)\left(1-u\right)$$

is a traveling wave whose profile converges to 1 and 0 at $-\infty$ and $+\infty$ respectively.

The sign of the speed is easily deduced by integration by parts of the equation satisfied by the profile φ multiplied by φ'.

This result is deeply different from the one obtained for the reaction term $u\left(1-u\right)$: the traveling wave is unique, the speed is not linearly determined and even its sign depends on the parameters. For ecologists, a reaction term of the form $u\left(u-\theta\right)\left(1-u\right)$ models an *Allee effect*, that is an effect of positive dependency on the density: the growth rate $\left(u-\theta\right)\left(1-u\right)$ is increasing with respect to u if $u < \frac{\theta+1}{2}$.

Mathematically, an important difference between $u\left(1-u\right)$ and $u\left(u-\theta\right)\left(1-u\right)$ is the classification of constant stationary states. In the former case, the constant stationary states are exactly 0 and 1 and, for the underlying ODE,

$$u' = u\left(1-u\right),$$

0 is unstable and 1 is locally asymptotically stable (and actually globally attractive for positive initial data). In the latter case, the constant stationary states are exactly 0, θ and 1 and, for the underlying ODE,

$$u' = u\left(u-\theta\right)\left(1-u\right),$$

0 and 1 are locally asymptotically stable whereas θ is unstable.

This observation leads to a classification of the reaction terms f which are regular, depend only on u, vanish at 0 and whose set of positive zeros admits a maximum. Up to a renormalization, we can assume that this maximum is 1. Moreover we exclude the case where f is positive on the right of 1 so that all solutions are globally bounded and hence $f'\left(1\right) \leq 0$.

1. If there exists $\theta \in (0,1)$ such that f is zero on $[0,\theta]$ and positive in $(\theta,1)$, f is referred to as *ignition type*.
2. If 0 and 1 are the only two nonnegative zeros of f, f is referred to as *monostable*.
3. If f is monostable and satisfies $f'\left(0\right)u \geq f\left(u\right)$ for all $u \in [0,1]$, f is referred to as *KPP type*.
4. If f admits exactly three nonnegative zeros 0, θ and 1 and if $f'\left(0\right) < 0$, $f'\left(\theta\right) > 0$ and $f'\left(1\right) < 0$, f is referred to as *bistable*.
5. If the nonnegative zeros of f are all isolated and if there exist at least four of them, f is referred to as *multistable*.

This classification is not exhaustive but covers most of the cases interesting from the application viewpoint.

The monostable non-KPP case models also an Allee effect, referred to as weak by opposition to the strong Allee effect of the bistable case. The basic theorem on this case follows.

Theorem. *[8] Il existe $c^\star \geq f'\left(0\right)$ such that the monostable equation*

$$\partial_t u - \Delta u = f\left(u\right)$$

admits a traveling wave solution with speed $c \geq 0$ describing the invasion of 0 by 1 if and only if $c \geq c^\star$. This solution is unique.

Addressing the equality $c^\star = f'\left(0\right)$, that is the linear determinacy of the minimal speed, is of course highly important in such a case. Although it is true in the KPP case, in general it fails.

Systems

The classification of multidimensional reaction terms $\mathbf{f}$ is of course much more complex. On one hand, it is possible to generalize the scalar classification (see for instance the book by Volpert, Volpert and Volpert [139]), but such a classification is not always appropriate to deal with systems interesting from the application viewpoint. On the other hand, it is possible to follow the classification induced by applications, which would contain for instance the three cases competitive Lotka–Volterra, predative Lotka–Volterra, mutualistic Lotka–Volterra, but such a classification sometimes fails to exhibit the mathematical links between different models.

The two classifications are sometimes used concurrently. This leads for instance to the standard classification for the Lotka–Volterra system of two competitive species, detailed now and to which we will frequently refer hereafter.

1. If $a \leq 1$ or $b \leq 1$, the system is monostable: the underlying ODE system admits a unique locally asymptotically stable stationary state. The following subcases are then distinguished:
   a) $a = b = 1$: degenerated case usually discarded;
   b) $a < 1$ and $b < 1$: coexistence case, the stable state is $\left(\frac{1-a}{1-ab}, \frac{1-b}{1-ab}\right)$;
   c) $a \geq 1$, $b \leq 1$ and $a \neq b$: semi-extinction case, the stable state is $(0, 1)$;
   d) $b \geq 1$, $a \leq 1$ and $a \neq b$: semi-extinction case, the stable state is $(1, 0)$.
2. If $a > 1$ and $b > 1$, the system is bistable: the underlying ODE system admits exactly two locally asymptotically stable stationary states, which are $(1, 0)$ and $(0, 1)$. This case is also referred to as the mutual exclusion case.

In this example, the increased complexity compared to a scalar equation is due to the loss of topological constraints in dimension 2 and is already effective in the underlying ODE system, without spatial structure.

However, the topological freedom due to the dimension of the system is not the only obstacle encountered when studying reaction–diffusion systems. More delicate phenomena, such as Turing instabilities [138] due to interactions between unequal diffusions and special reaction terms, can arise. The systems concerned by these phenomena have a default of structure compared to scalar equations, usually a default of comparison principle or variational structure. The study of these systems, among which we find the vast majority of systems coming from population dynamics, is especially difficult.

In particular, to investigate the traveling wave solutions of systems devoid of comparison principle, the ODE methods sometimes seem unavoidable. Managing to use only PDE methods becomes then a goal in itself, as these are required for some subsequent research directions. This explains for instance why the results on the predator–prey Lotka–Volterra system

$$\begin{cases} \partial_t u - \Delta u = u\,(1 - u - av) \\ \partial_t v - d\Delta v = v\,(-1 + bu) \end{cases}$$

(with $a > 0$ and $b > 0$), first obtained by Dunbar using ODE methods in the 80s [63, 64], were recovered with PDE methods by Fu and Tsai in 2015 [75]. The delay of more than thirty years illustrates perfectly the difficulties that PDE specialists have to overcome when the comparison principle is missing.

On the contrary, the scarce *monotone systems*, that is satisfying a comparison principle, or variational systems can quite often be dealt with similarly to scalar PDEs and satisfy therefore similar propagation properties. This is for instance the case of the two-species competition–diffusion Lotka–Volterra system, which is monotone and whose monostable and bistable cases are strongly analogous to the corresponding scalar cases. Nevertheless, as soon as a third competitor is introduced, the monotony of the system is lost and new phenomena arise (for instance, under certain conditions, Kishimoto [102] showed the existence of stable non-constant stationary states and Chen and Hung [95] showed the nonexistence of traveling waves).

We also point out that the following system is both monotone and variational and is however still incompletely understood:

$$\begin{cases} \partial_t u - \Delta u = u\left(1 - u^2 - av^2\right) \\ \partial_t v - d\Delta v = rv\left(1 - v^2 - bu^2\right) \end{cases}.$$

This very important system, coming from quantum physics and referred to as the Gross–Pitaevskii system for two-component Bose–Einstein condensates, is the subject of countless articles. Although it could be described as a mere variational modification of the two-species competition–diffusion Lotka–Volterra system, some problems coming from the applications as well as some proof arguments cannot be transferred from one system to the other. The literatures on these two systems have actually a tendency to grow independently and links are nowadays rare (recent efforts being due for instance to Dancer, Wang and Zhang [50, 51] or Soave and Zilio [135]). This illustrates very well the increased intricacy of systems (the scalar reaction terms $u(1-u)$ and $u\left(1-u^2\right)$ are on the contrary handled concurrently, as KPP reaction terms).

Without further lingering, let us now present the systems studied in this thesis as well as the obtained results.

Contributions

The contributions of this thesis to the general study of propagation properties of reaction–diffusion systems coming from population dynamics are twofold.

1. First, we study open questions previously raised by the vast literature on the two-species competition–diffusion Lotka–Volterra system.
   a) For the bistable system, when one stable state invades the other, which one is it? Is it possible to block or revert the invasion by introducing spatial heterogeneity?
   b) For the monostable system with semi-extinction, is there a possibility of invasion of an uninhabited territory by the unstable semi-extinct state followed by a replacement of this state by the stable semi-extinct state, and if yes what are the two speeds involved?
2. Second, we initiate investigations on a large class of non-monotone and non-variational systems closely related to the Fisher–KPP equation and arising in several population dynamic models. After the standard verifications (positivity and boundedness of the solutions), we prove a necessary and sufficient criterion for the population persistence and we study the propagation properties. Apart from the uniqueness of the traveling waves, which remains an open problem and is likely very complex, we find indeed propagation properties reminiscent of those satisfied by the Fisher–KPP equation.

On the Lotka–Volterra system of two competitive species

In this whole subsection, the system

$$\begin{cases} \partial_t u - \Delta u = u(1 - u - av) \\ \partial_t v - d\Delta v = rv(1 - v - bu) \end{cases} \tag{0.0.1}$$

is simply denoted (0.0.1).

In strongly competing bistable regime

In this whole subsubsection, we assume that (0.0.1) is bistable, that is $a > 1$ and $b > 1$.

In 1982, Gardner [77] showed the following theorem.

Theorem. *[77] (0.0.1) admits a traveling wave solution connecting* $(0,1)$ *and* $(1,0)$.

Definition. A traveling wave connecting $(0,1)$ and $(1,0)$ for (0.0.1) is a traveling wave whose profile converges to $(1,0)$ and $(0,1)$ at $-\infty$ and $+\infty$ respectively. The traveling wave is referred to as monotonic if its profile (φ, ψ) is such that φ and $-\psi$ are both non-increasing and is referred to as strictly monotonic if φ and $-\psi$ are both decreasing.

This result was then improved by Kan-on [100].

Theorem. *[100] (0.0.1) admits a unique traveling wave solution connecting* $(0,1)$ *and* $(1,0)$.
Furthermore, this traveling wave is strictly monotonic and its speed c satisfies $-2\sqrt{rd} < c < 2$.

Determining the sign of c becomes then very important, since it gives the direction of the invasion:

1. if $c < 0$, then $(0,1)$ invades $(1,0)$;
2. if $c > 0$, then $(1,0)$ invades $(0,1)$.

In other words, the sign of c gives a criterion to compare dynamically the stability of $(0,1)$ and that of $(1,0)$.

Contrarily to the scalar bistable equation studied by Fife and McLeod [71], here the sign of c cannot be determined simply by integration by parts (because of the lack of variational structure). Actually, to this day, no completely general result is known. When the research for this thesis started, the only partial result was the one proved by Guo and Lin [83].

Theorem. *[83] The sign of the speed c of the unique traveling wave solution of (0.0.1) satisfies the following properties.*

1. *If* $r = d$, *then c has the sign of* $b - a$.
2. *If* $r > d$ *and* $a \geq \left(\frac{r}{d}\right)^2 b$, *then* $c < 0$.
3. *If* $r < d$ *and* $b \geq \left(\frac{d}{r}\right)^2 a$, *then* $c > 0$.
4. *For any* $\lambda > 0$, *replacing* (d, r) *by* $\lambda(d, r)$ *does not change the sign of c.*
5. *If* $r > d$, $a \geq 2$ *and* $b \leq 1 + \frac{d}{r}$, *then* $c < 0$.
6. *If* $r > d$, $a \geq \frac{5r}{d}$ *and* $(3rb - d)\, b \leq (4r - d)\, a$, *then* $c < 0$.
7. *If* $r = \frac{d}{4}$ *and* $(a, b) = \left(\frac{5}{4}, \frac{4}{3}\right)$, *then* $c = 0$.
8. *If* $r = \frac{d}{4}$, $a \geq \frac{5}{4}$, $b \leq \frac{4}{3}$ *and* $(a, b) \neq \left(\frac{5}{4}, \frac{4}{3}\right)$, *then* $c < 0$.
9. *If* $r = \frac{d}{4}$, $a \leq \frac{5}{4}$, $b \geq \frac{4}{3}$ *and* $(a, b) \neq \left(\frac{5}{4}, \frac{4}{3}\right)$, *then* $c > 0$.

The proof of this result relies upon the monotonicities of c with respect to (r, a, b) established by Kan-on [100].

In this thesis, we adopt a completely different viewpoint. Our idea is to exploit the properties of a particular asymptotic regime, known as *strong competition regime*: $(r, a, b) = \left(r, k, \frac{\alpha k}{r}\right)$ with $\alpha > 0$ and $k \to +\infty$. This regime corresponds to the singular limit $k \to +\infty$ of the following system:

$$\begin{cases} \partial_t u_k - \Delta u_k = u_k (1 - u_k) - k u_k v_k \\ \partial_t v_k - d\Delta v_k = r v_k (1 - v_k) - \alpha k u_k v_k \end{cases}.$$

In this regime, the theorem of Guo and Lin leaves a large area of uncertainty. On one hand, the conditions of the points 5 to 9 cannot be satisfied if k is large enough, and on the other hand, the points 1 to 4 only give the following partial result.

Corollary. *The sign of the speed c_k satisfies the following properties.*

1. *If $r = d$, then c_k has the sign of $\alpha - r$.*
2. *If $r > d$ and $d^2 \geq \alpha r$, then $c_k < 0$.*
3. *If $r < d$ and $\alpha r \geq d^2$, then $c_k > 0$.*
4. *For any $\lambda > 0$, replacing (d, r, α) by $\lambda (d, r, \alpha)$ does not change the sign of c_k.*

For instance, if $\alpha = r = 1$, the sign of c_k remains completely unknown, except in the case $d = 1$ (trivial by symmetry).

The first papers on the strong competition regime were published in the 90s and are due to Dancer and his collaborators [45, 46]. The main result, very generic and naturally deduced from the system, is the following: as $k \to +\infty$, the solutions $((u_k, v_k))_{k>1}$ converge toward a pair (u_∞, v_∞) whose components are nonnegative and spatially segregated, that is satisfying $u_\infty v_\infty = 0$.

Moreover, for a two-species system such as the one above, we can use the particular form of the Lotka–Volterra coupling to linearly combine the two equations and obtain

$$\partial_t (\alpha u_k - v_k) - \Delta (\alpha u_k - d v_k) = \alpha u_k (1 - u_k) - r v_k (1 - v_k).$$

In this equation, the dependency on k is only implicit, and it seems reasonable to try to pass to the limit. Using the relation $u_\infty v_\infty = 0$ and denoting w_∞ the limit of $(\alpha u_k - v_k)_{k>1}$, we can identify $\alpha u_\infty = w_\infty^+$ and $v_\infty = w_\infty^-$, where the positive and negative parts of w_∞ are defined so that $w_\infty = w_\infty^+ - w_\infty^-$, and by so doing we obtain the limiting equation:

$$\partial_t w_\infty - \Delta \left(\left(\mathbf{1}_{w_\infty > 0} + d \mathbf{1}_{w_\infty < 0} \right) w_\infty \right) = w_\infty^+ \left(1 - \frac{w_\infty}{\alpha} \right) - r w_\infty^- (1 + w_\infty).$$

Provided w_∞ is null only on a negligible set, this equation is a parabolic quasilinear equation. The regularity of w_∞ can therefore be enhanced and a free boundary problem governing the motion of the interface arises.

This formal argument will be rigorously developed later in this thesis. Here, the idea is simply to show how a two-component system is reduced, in the strong competition limit, to a single quasilinear equation. Since the sign of a scalar bistable traveling wave is easily determined, this reduction should consequently reveal the sign of the limit c_∞ of the speeds $(c_k)_{k>1}$.

More precisely, the first result obtained in this thesis, in collaboration with Grégoire Nadin, is the following.

Theorem. *[GN15] The family of speeds $(c_k)_{k>1}$ converges to a limit $c_\infty \in \left(-2\sqrt{rd}, 2\right)$ having the sign of $\alpha^2 - rd$.*

Furthermore, the convergence is locally uniform with respect to d and c_∞ is continuous with respect to d.

The continuity of c_k with respect to d being still an open question, the continuity of c_∞ is not a mere consequence of the locally uniform convergence.

In the simplified case where $\alpha = r = 1$, the sign of c_∞ is that of $1-d$. In other words, the species having the strongest diffusion rate chases the other: what matters is not the concentration near the interface but rather the ability to send individuals far away in the territory of the competitor. Consequently, we named this result "Unity is not strength".

It is easily verified that our result is compatible with that of Guo and Lin. Assuming for instance $r > d$ and $d^2 \geq \alpha r$, which, according to Guo and Lin, imply together $c_k < 0$ (and thus $c_\infty \leq 0$), we find indeed $\alpha^2 \leq rd$, and even $\alpha^2 < rd$:

$$\frac{\alpha^2}{rd} = \frac{\alpha^2 r^2}{dr^3} \leq \frac{d^3}{r^3} < 1.$$

By showing that a strong interspecific competition rate favors the more mobile species, our result raises several interesting ensuing questions. In particular, it brings forth perspective to a celebrated result of the opposite type, "Unity is strength", for the system in heterogeneous bounded media

$$\begin{cases} \partial_t u - \Delta u = u\,(r\,(x) - u - v) & \text{in } \Omega \\ \partial_t v - d\Delta v = v\,(r\,(x) - v - u) & \text{in } \Omega \\ \partial_n u = \partial_n v = 0 & \text{on } \partial\Omega \end{cases} \tag{0.0.2}$$

due to Dockery, Hutson, Mischaikow and Pernarowski [58]. Is this inversion of the result mainly caused by the spatial heterogeneity, or is the relatively weak interspecific competition responsible as well?

In order to address this question, we studied the role of the spatial heterogeneity by considering the special case, mathematically comfortable, of one-dimensional spatially periodic heterogeneities. Indeed, although propagation phenomena in general heterogeneous media are substantially complex, in periodic media they become much simpler and several similarities with homogeneous media arise. In particular, the notion of traveling wave is naturally generalized by that of *pulsating front*.

Definition. A pulsating front connecting $(0,1)$ and $(1,0)$ for

$$\begin{cases} \partial_t u - \partial_{xx} u = \mu\,(x)\,u\,(1-u) - kuv \\ \partial_t v - d\partial_{xx} v = \nu\,(x)\,v\,(1-v) - \alpha kuv \end{cases}, \tag{0.0.3}$$

where μ and ν are two regular positive periodic functions with period $D > 0$, is an entire solution of the form $(u,v) : (t,x) \mapsto (\varphi,\psi)\,(x - ct, x)$, with $c \in \mathbb{R}$ a propagation speed and (φ,ψ) a propagation profile satisfying the following properties:

1. φ and $-\psi$ are both decreasing with respect to their first variable;
2. φ and ψ are both L-periodic with respect to their second variable;
3. the following uniform limits hold true:

$$\lim_{\xi \to -\infty} \sup_{x \in [0,L]} |(\varphi,\psi)\,(\xi,x) - (1,0)| = 0,$$

$$\lim_{\xi \to +\infty} \sup_{x \in [0,L]} |(\varphi,\psi)\,(\xi,x) - (0,1)| = 0.$$

Before going any further, let us precise that this introductory presentation only considers systems of the form (0.0.3) but the forthcoming results taken from [Gir17, GN18] are actually proved under slightly less restrictive assumptions on the reaction term. The comparison between (0.0.2) and (0.0.3) is relevant indeed. The exact assumptions will be stated in the adequate chapters.

Research on pulsating fronts in reaction–diffusion started much more recently than that on traveling waves (pioneering works on scalar pulsating fronts are due to Gärtner, Freidlin [79] and Xin [144, 143] and go back to the 80s and 90s). When we looked at this question, we realized that the existence of pulsating fronts for the bistable system (0.0.3) had never been addressed before.

Such a question can however be addressed with the general theoretical framework elaborated recently by Fang and Zhao [69]. Omitting the technical details, their general conclusion is as follows: traveling waves or pulsating fronts for bistable problems exist provided all intermediate stationary states, respectively constant or periodic, are unstable and invadable by the two extremal stable states (namely, there exist monostable waves with speed of adequate sign). Consequently, the following result immediately implies the existence of pulsating fronts for (0.0.3).

Theorem. *[Gir17] Let* $A>0$*,* $B>0$ *and* $\overline{L}=\pi\left(\frac{1}{\sqrt{A}}+\sqrt{\frac{d}{B}}\right)$*. Assume* $L<\overline{L}$*,* $\max\limits_{[0,L]}\mu=A$ *and* $\max\limits_{[0,L]}\nu=B$*.*

Then there exists $k^\star>0$ *such that, if* $k\geq k^\star$*, any L-periodic coexistence stationary state for (0.0.3) is unstable and invadable by the stable states* $(1,0)$ *and* $(0,1)$*.*

The condition $L<\overline{L}$ is in fact sufficient to guarantee that the limiting problem

$$-\Delta\left(\left(\mathbf{1}_{w_\infty>0}+d\mathbf{1}_{w_\infty<0}\right)w_\infty\right)=\mu\left(x\right)w_\infty^+\left(1-\frac{w_\infty}{\alpha}\right)-\nu\left(x\right)w_\infty^-\left(1+w_\infty\right)$$

admits no periodic nonzero sign-changing solution. It is deduced from the following two observations:

- a periodic nonzero sign-changing solution can be understood as a juxtaposition of solutions of Dirichlet problems set in intervals of size smaller than L;
- such a Dirichlet problem is solvable only if the interval in which it is set is large enough, and the minimal size can be explicitly estimated.

It is perfectly natural to wonder whether this result is optimal, that is if it is possible to construct counter-examples where $L<\overline{L}$ is not satisfied and where a stable or invadable periodic coexistence stationary state exists. This question was the object of a collaboration with Alessandro Zilio and lead to the following result, whose proof is again based on the limit $k\to+\infty$.

Theorem. *[GZ18] Let* $A>0$*,* $B>0$ *and* $r_0>0$*,* $r_\mu>0$ *and* $r_\nu>0$ *such that* $2r_0+2r_\mu+2r_\nu=1$*. We define two 1-periodic functions* $\mu^\star$ *and* $v^\star$ *by*

$$\left(\mu^\star\right)_{|[0,1]}=A\mathbf{1}_{[0,r_\mu]}+A\mathbf{1}_{[r_\mu+2r_0+2r_\nu,1]},$$

$$\left(\nu^\star\right)_{|[0,1]}=B\mathbf{1}_{[r_\mu+r_0,r_\mu+r_0+2r_\nu]},$$

as well as, for all $L>0$*, the L-periodic function*

$$\left(\mu_L,\nu_L\right):x\mapsto\left(\mu^\star,\nu^\star\right)\left(\frac{x}{L}\right).$$

There exists $\underline{L}>0$ *such that, for all* $L>\underline{L}$*, there exists* $k^\star>1$ *such that, for all* $k\geq k^\star$*, (0.0.3) with* $(\mu,\nu)=(\mu_L,\nu_L)$ *or* $(\mu,\nu)=(\mu_L+\nu_L,\mu_L+\nu_L)$ *admits a linearly stable L-periodic coexistence stationary state.*

Furthermore, for all $L>\underline{L}$*, there exists a neighborhood* U_L *of* (μ_L,ν_L) *in the topology of* $\left(L^\infty_{L\text{-per}}\right)^2$ *and a neighborhood* V_L *of* $\mu_L+\nu_L$ *in the topology of* $L^\infty_{L\text{-per}}$ *such that, for all* $(\mu,\nu)\in U_L$ *and all* $\rho\in V_L$*, there exists* $k^\star>1$ *such that, for all* $k\geq k^\star$*, (0.0.3) with* (μ,ν) *or* (ρ,ρ) *admits a linearly stable L-periodic coexistence stationary state.*

Because of the existence of such a coexistence state, the result of Fang and Zhao cannot be applied. In fact, the coexistence state might even block the propagation and ensure the nonexistence of pulsating fronts. To establish such a blocking, we will have to verify in ulterior works the sign of the speed of the bistable pulsating front connecting this state to one of the two semi-extinct states. Anyways, this result completes interestingly a result of Ding, Hamel and Zhao [57] showing that for a particular but natural class of reaction terms, the existence of stable coexistence state is possible only if the period is neither too large nor too small.

The counter-example of the preceding theorem relies upon a particular choice of (μ_L,ν_L) describing a situation where fertile territories (where μ_L and ν_L are positive constants) are periodically separated by neutral territories neither fertile nor deleterious (where $\mu_L=\nu_L=0$).

The stationary state obtained subsequently describes the possibility for u and v to settle in the fertile territories respectively evenly numbered and oddly numbered. In other words, this result admits an interesting biological interpretation: a strong heterogeneity of the habitat can lead to a strong spatial segregation and eventually to speciation. This result is therefore in itself a first interesting complement to the result of Dockery *et al.*: in heterogeneous media, although a weak competition leaves no chance to coexistence, a strong competition can on the contrary favor it.

The existence of pulsating fronts under the high frequency assumption $L < \overline{L}$ being in any cases established, the method developed with Grégoire Nadin in the homogeneous setting can be applied meaningfully to the periodic system (0.0.3). Again, the sign of the speed can be determined thanks to the limiting problem.

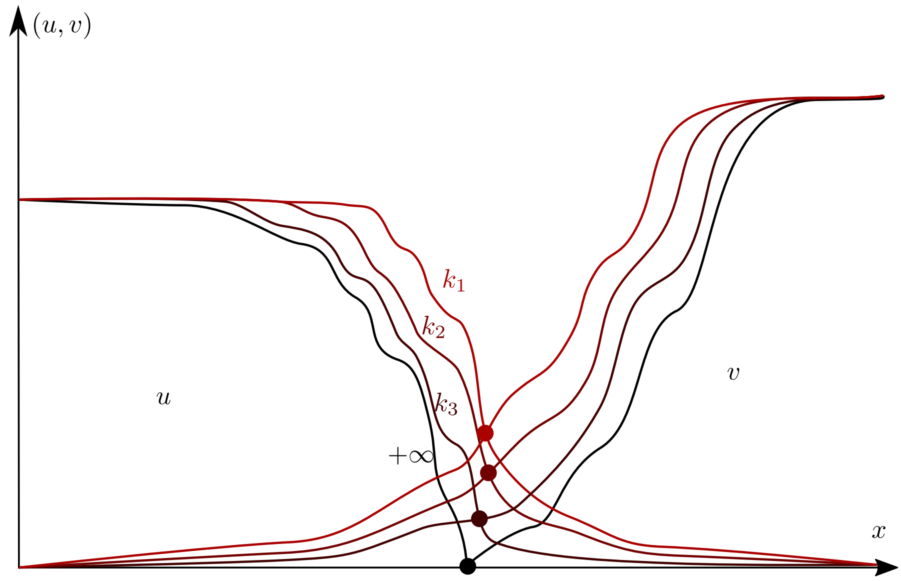


Figure 0.0.4 – Segregation of pulsating fronts ($k_1 < k_2 < k_3 < +\infty$)

Theorem. *[GN18] There exists $c_\infty \in \mathbb{R}$ such that:*

1. $c_\infty \in \left(-c^\star_{d,\nu}, c^\star_{1,\mu}\right)$, *where $c^\star_{\delta,\rho} > 0$ is the minimal speed of pulsating fronts for the one-dimensional periodic Fisher–KPP equation*

$$\partial_t z - \delta \partial_{xx} z = \rho(x) z (1-z);$$

2. $c_\infty = 0$ *if and only if* $\frac{\alpha^2}{d} \in I_r$, *where I_r is a nonempty closed interval satisfying*

$$\left\{ \frac{\int_0^L \nu}{\int_0^L \mu} \right\} \subset I_r \subset \left[\frac{\min_{[0,L]} \nu}{\max_{[0,L]} \mu}, \frac{\max_{[0,L]} \nu}{\min_{[0,L]} \mu} \right];$$

3. $c_\infty < 0$ *if* $\frac{\alpha^2}{d} < \min I_r$;
4. $c_\infty > 0$ *if* $\frac{\alpha^2}{d} > \max I_r$;

5. *c_∞ is the limit, locally uniform with respect to d, of any family $(c_k)_{k\geq k^\star}$ of speeds of pulsating fronts for (0.0.3);*

6. *c_∞ is continuous with respect to d.*

As hinted by the above statement, the uniqueness of the pulsating front $(u_k, v_k) : (t,x) \mapsto (\varphi_k, \psi_k)(x - c_k t, x)$ was not established rigorously during these works, whose object was really the study of the singular limit. Nevertheless, it can be proved indeed by sliding arguments very similar to those presented later in this thesis.

The proof of the previous theorem is especially more difficult than its homogeneous counterpart. Indeed, although the profile of a traveling wave satisfies a comfortable elliptic system, the profile of a pulsating front satisfies a degenerate elliptic system. Hence it is necessary to go back and forth between the parabolic coordinates (t,x) and the traveling coordinates (ξ, x) and this implies in particular a distinction of cases according to the nullity of c_∞. Moreover, in the case $c_\infty \neq 0$, the free boundary induced by the limiting problem is not trivially reduced to a point anymore. An involved study of this free boundary problem is required to characterize properly the limiting profile. This study is performed using the maximum principle, the monotonicity in time of the free boundary position (whose average speed is indeed c_∞) and regularizing procedures. Even in the case $c_\infty = 0$, we have to face an additional difficulty: the possible multiplicity of the solutions of the limiting problem (this fact was already pointed out by Ding, Hamel and Zhao [57]). However, these solutions exist if and only if $\frac{\alpha^2}{d} \in I_r$. A mutual exclusion result deduced from the characterization of the profiles when $c_\infty \neq 0$ yields then that $c_\infty \neq 0$ if and only if $\frac{\alpha^2}{d} \notin I_r$, which finally leads to the sign of c_∞.

Once again, the result is of "Unity is not strength" type. Such a generalization shows that the inversion of the result for (0.0.2) relies heavily upon the weakening of the interspecific competition. Although spatial heterogeneity favors the less mobile competitor, its effect is negligible compared to that of competition, which favors on the contrary the more mobile one.

The result of Dockery *et al.* on (0.0.2) is consequently not the end of the story between diffusion rates and competitive advantage. Further research will have to be done. This was later on confirmed by Risler [130] who was able to prove with completely different methods a new result of "Unity is not strength" type, this time for the perturbative system in homogeneous media

$$\begin{cases} \partial_t u - \Delta u = u(1-u) - (1+\gamma)uv \\ \partial_t v - (1+\varepsilon)\Delta v = v(1-v) - (1+\gamma)uv \end{cases}$$

where $\varepsilon > 0$ and $\gamma > 0$ are two infinitesimal parameters. Contrarily to the strong competition regime, this system is just a marginal modification of (0.0.2) and still suffices to invert the conclusion.

In monostable regime with semi-extinction

In this whole subsubsection, we assume that (0.0.1) is monostable with semi-extinction, that is $a < 1$ and $b > 1$ (up to changing the roles of u and v). Therefore the stable state is $(1,0)$ whereas $(0,1)$ is unstable.

The traveling waves of this system, defined as in the bistable case, were first studied in 1989 by Hosono [92] and Okubo, Maini, Williamson and Murray [123] under restrictive assumptions on the parameters and then without such assumptions in 1997 by Kan-on [101]. The result of Kan-on, strongly reminiscent of the scalar monostable setting, is the following.

Theorem. *[101] There exists $c^\star \geq 2\sqrt{1-a}$ such that (0.0.1) admits a monotonic traveling wave solution with speed $c \geq 0$ describing the invasion of $(0,1)$ by $(1,0)$ if and only if $c \geq c^\star$.*

(a) 1945 (b) 2000 (c) 2010

Figure 0.0.5 – Invasion of the gray squirrel in the territory of the red squirrel in the British Isles, classically modeled as a system of the form (0.0.1) with $a < 1 < b$.
Fine grid: gray squirrel; intermediate grid: buffer; coarse grid: red squirrel.
Source : *Red Squirrel Survival Trust* (modified colors).

By analogy with the scalar monostable equations, it was naturally conjectured that:

— $c^\star$ is also the asymptotic speed of propagation of the solutions with one-dimensional initial data of the form $(u_0, 1 - v_0) \in [0,1]^2$ with (u_0, v_0) compactly supported and $u_0 \neq 0$;

— the linear determinacy $c^\star = 2\sqrt{1-a}$ is sometimes, but not always, verified.

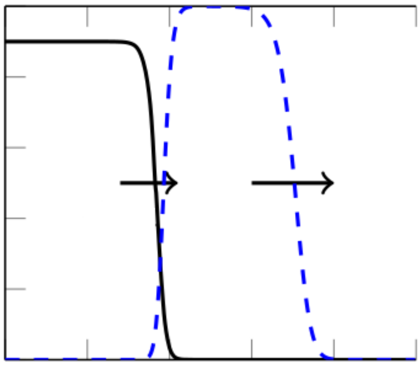
The subsequent works of the beginning of the 2000s by Lewis, Li and Weinberger [108, 110, 142] confirmed the first conjecture and gave sufficient conditions for linear determinacy. More recently, Huang and Han [94] achieved the construction of a linear determinacy counter-example which ends the confirmation of the second conjecture. Hence the question of the invasion of u in a territory initially occupied by v is nowadays well understood, even though necessary and sufficient conditions on the parameters for linear determinacy are not yet known.

On the contrary, the concurrent invasion of u and v in a territory initially uninhabited was a problem mostly open before this thesis. Related works are concerned either with the bistable case, very recently treated by Carrère [35], or with the monostable case with coexistence, partially treated in 2012 by Lin and Li [112]. The general conclusion of these two articles is the possibility to obtain successive invasion waves:

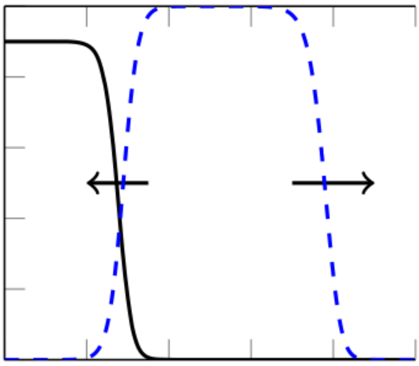
— in the bistable case, if the initial values are suitable and if $2\sqrt{rd} > 2$, $(0,0)$ is invaded by $(0,1)$ which is itself invaded subsequently by $(1,0)$;

— in the monostable case with coexistence, if $2\sqrt{rd\,(1-b)} > 2$, $(0,0)$ is invaded by $(0,1)$ which is itself followed by an area of uncertainty itself followed by an invasion of $\left(\frac{1-a}{1-ab}, \frac{1-b}{1-ab}\right)$.

We notice that, although the result of Carrère seems optimal (because of the bistability, assumptions on the initial values are necessary), the one of Lin and Li could be clarified. In the uncertainty area, is there simply an invasion of $(0,1)$ by $\left(\frac{1-a}{1-ab}, \frac{1-b}{1-ab}\right)$ or is there an invasion of $(0,1)$ by $(1,0)$ followed by an invasion of $(1,0)$ by $\left(\frac{1-a}{1-ab}, \frac{1-b}{1-ab}\right)$?



(a) Case $c_{(1,0)\to(0,1)} > 0$



(b) Case $c_{(1,0)\to(0,1)} < 0$

Figure 0.0.6 – Numerical simulations by Carrère [35] illustrating the two successive invasions, with respective speeds $2\sqrt{rd}$ and $c_{(1,0)\to(0,1)}$ (speed of the bistable traveling wave connecting $(0,1)$ to $(1,0)$).

Anyways, these successive invasion waves are reminiscent of the *propagating terraces* of the literature on scalar reaction–diffusion equations. These solutions, first mathematically described by Fife and McLeod [71] in the multistable setting, named systems of waves by Volpert, Volpert and Volpert [139] and renamed after the works of Matano, Ducrot and Giletti [62], attracted some attention recently.

But the result of Lin and Li shows that the prism of multistable scalar equations does not suffice to grasp the kind of terraces involved here. Indeed, although only the first stationary state of a multistable scalar terrace is possibly unstable, Lin and Li show that the unstable state $(0,0)$ is replaced by the unstable state $(0,1)$. One monostable wave follows another monostable wave. Since monostable waves have typically a half-line of admissible speeds whereas bistable waves have a unique admissible speed, the set of all terraces possibly generated by (0.0.1) is considerably larger.

Actually, when studying the concurrent invasion of u and v, the propagation terrace viewpoint is highly fruitful and leads to complete and new results, enlightening also compactly supported initial data. Together with Adrian Lam, we proved the following results.

We define the auxiliary function

$$f: \begin{array}{ccc} \left[2\sqrt{1-a}, +\infty\right) & \to & \left(2\sqrt{a}, 2\left(\sqrt{1-a}+\sqrt{a}\right)\right] \\ c & \mapsto & c - \sqrt{c^2 - 4(1-a)} + 2\sqrt{a} \end{array}.$$

It is decreasing and bijective and satisfies in particular

$$f(2) = 2,$$

$$f^{-1}: \tilde{c} \mapsto \frac{\tilde{c}}{2} - \sqrt{a} + \frac{2(1-a)}{\tilde{c} - 2\sqrt{a}}.$$

In the following statement, the underlying space is one-dimensional and $c^\star$ is indeed the minimal speed of monotonic traveling waves describing the invasion of $(0,1)$ by $(1,0)$.

Theorem. *[GL18] Let $u_0 \in \mathscr{C}(\mathbb{R},[0,1]) \setminus \{0\}$ with support included in a left half-line and $v_0 \in \mathscr{C}(\mathbb{R},[0,1]) \setminus \{0\}$ with compact support. Let (u,v) be the solution of (0.0.1) with initial data (u_0, v_0).*

1. *Assume* $2\sqrt{rd} < 2$. *Then*

$$\lim_{t\to+\infty} \sup_{x\geq 0} |v(t,x)| = 0,$$

$$\lim_{t\to+\infty} \sup_{0\leq x<(2-\varepsilon)t} |u(t,x)-1| = 0 \text{ for each } \varepsilon\in(0,2),$$

$$\lim_{t\to+\infty} \sup_{(2+\varepsilon)t<x} |u(t,x)| = 0 \text{ for each } \varepsilon > 0.$$

2. *Assume* $2\sqrt{rd}\in(2, f(c^\star))$ *and define*

$$c_{acc} = f^{-1}\left(2\sqrt{rd}\right) = \sqrt{rd}-\sqrt{a}+\frac{1-a}{\sqrt{rd}-\sqrt{a}} \in (c^\star, 2).$$

Then

$$\lim_{t\to+\infty} \sup_{0\leq x<(c_{acc}-\varepsilon)t} (|u(t,x)-1|+|v(t,x)|) = 0 \text{ for each } \varepsilon\in(0,c_{acc}),$$

$$\lim_{t\to+\infty} \sup_{(c_{acc}+\varepsilon)t<x<\left(2\sqrt{rd}-\varepsilon\right)t} (|u(t,x)|+|v(t,x)-1|) = 0 \text{ for each } \varepsilon\in\left(0,\frac{2\sqrt{rd}-c_{acc}}{2}\right),$$

$$\lim_{t\to+\infty} \sup_{\left(2\sqrt{rd}+\varepsilon\right)t<x} (|u(t,x)|+|v(t,x)|) = 0 \text{ for each } \varepsilon > 0.$$

3. *Assume* $2\sqrt{rd}\geq f(c^\star)$. *Then*

$$\lim_{t\to+\infty} \sup_{0\leq x<(c^\star-\varepsilon)t} (|u(t,x)-1|+|v(t,x)|) = 0 \text{ for each } \varepsilon\in(0,c^\star),$$

$$\lim_{t\to+\infty} \sup_{(c^\star+\varepsilon)t<x<\left(2\sqrt{rd}-\varepsilon\right)t} (|u(t,x)|+|v(t,x)-1|) = 0 \text{ for each } \varepsilon\in\left(0,\frac{2\sqrt{rd}-c^\star}{2}\right),$$

$$\lim_{t\to+\infty} \sup_{\left(2\sqrt{rd}+\varepsilon\right)t<x} (|u(t,x)|+|v(t,x)|) = 0 \text{ for each } \varepsilon > 0.$$

This result is remarkable for at least three reasons.

— The second case shows that the invasion of u is accelerated ($c_{acc} > c^\star$) if that of v is too slow ($2\sqrt{rd} < f(c^\star)$).

— The accelerated speed c_{acc} is given by an explicit algebraic formula.

— The statement does not depend on the linear determinacy of $c^\star$.

The function f, which associates with each $c\geq 2\sqrt{1-a}$ the largest root of the equation in $\tilde{c}$

$$\tilde{c}^2 - 4\lambda(c)\tilde{c} + 4(\lambda(c)c-1) = 0,$$

where

$$\lambda(c) = \frac{1}{2}\left(c-\sqrt{c^2-4(1-a)}\right),$$

appears naturally in the problem.

Assume that v invades the uninhabited territory at speed $2\sqrt{rd}$ and that u chases v at some speed $c_2\in\left[c^\star, 2\sqrt{rd}\right)$. In the area where $v\simeq 1$, u looks like the exponential tail of the monostable traveling wave connecting $(0,1)$ to $(1,0)$ at speed c_2, that is

$$u(t,x)\simeq \mathrm{e}^{-\lambda(c_2)(x-c_2 t)}.$$

Accordingly, in a neighborhood of $x = \tilde{c}t$ with $\tilde{c} \in \left(c_2, 2\sqrt{rd}\right)$, we can observe non-negligible quantities only if we consider the rescaled function

$$w : (t,x) \mapsto u(t,x)\,\mathrm{e}^{\lambda(c_2)(x-c_2t)}$$

instead of u itself.

Yet, in a neighborhood of $x = \tilde{c}t$ with $\tilde{c} > 2\sqrt{rd}$, where $(u,v) \simeq (0,0)$, w satisfies at the first order

$$\partial_t w - \partial_{xx} w = (1 + \lambda(c_2)(\tilde{c} - c_2))\, w$$

whence the exponential ansatz $w(t,x) = \mathrm{e}^{-\Lambda(x-\tilde{c}t)}$ leads to the equation

$$(\Lambda(c_2,\tilde{c}))^2 - \tilde{c}\Lambda(c_2,\tilde{c}) + (1 + \lambda(c_2)(\tilde{c} - c_2)) = 0.$$

The minimal zero of this equation being

$$\Lambda(c_2,\tilde{c}) = \frac{1}{2}\left(\tilde{c} - \sqrt{\tilde{c}^2 - 4(\lambda(c_2)(\tilde{c} - c_2) + 1)}\right),$$

we deduce then that $\tilde{c}$ has to satisfy

$$\tilde{c}^2 - 4\lambda(c_2)\tilde{c} + 4(\lambda(c_2)c_2 - 1) \geq 0,$$

that is $\tilde{c} \geq f(c_2)$. Passing to the limit $\tilde{c} \to 2\sqrt{rd}$, we find indeed $2\sqrt{rd} \geq f(c_2)$.

The idea of the proof is to use delicately constructed super-solutions and sub-solutions to change this heuristic argument into a rigorous one.

Two other theorems complete the preceding one and will be presented in detail in the adequate chapter. They characterize the set of admissible pairs of speeds for the propagating terraces generated by exponentially decaying initial data. A crucial role is played by the function f again and, accordingly, the set is sometimes smaller than the maximal set

$$\left\{(c_1,c_2) \in \left[2\sqrt{rd}, +\infty\right) \times [c^\star, +\infty) \;\mid\; c_1 > c_2\right\}.$$

On non-monotone KPP systems

In this subsection, the vector inequalities $\geq \mathbf{0}$, $> \mathbf{0}$ and $\gg \mathbf{0}$ are respectively understood as nonnegativity of all components, nonnegativity of all components with at least one positive component and positivity of all components. Moreover, the notation $[N]$ denotes the set $\{1,\dots,N\}$.

A *KPP system* is a reaction–diffusion system of the form

$$\partial_t \mathbf{u} - \mathbf{D}\Delta\mathbf{u} = \mathbf{L}\mathbf{u} - \mathbf{c}(\mathbf{u}) \circ \mathbf{u}, \tag{0.0.4}$$

with $\mathbf{u} \in \mathbb{R}^N$, $\mathbf{L} \in \mathbb{R}^{N\times N}$ an *irreducible* and *essentially nonnegative* (that is with nonnegative off-diagonal entries) square matrix and $\mathbf{c}$ a vector field in $\mathbb{R}^N$ satisfying:

1. $\mathbf{c}(\mathbf{u}) \geq \mathbf{0}$ if $\mathbf{u} \geq \mathbf{0}$, with equality if $\mathbf{u} = \mathbf{0}$;
2. there exist $\underline{\alpha} \geq 1$, $\delta \geq 1$ and $\underline{\mathbf{c}} \gg \mathbf{0}$ such that, for all $\alpha \geq \underline{\alpha}$, $i \in [N]$ and $\mathbf{n} > \mathbf{0}$ satisfying $|\mathbf{n}| = 1$, we have

$$\sum_{j=1}^{N} l_{i,j} n_j \geq 0 \implies \alpha^\delta \underline{c}_i \leq c_i(\alpha\mathbf{n}).$$

The second assumption is satisfied for instance if $\mathbf{c}(\mathbf{v})$ grows at least linearly as $|\mathbf{v}| \to +\infty$ and is therefore satisfied if $\mathbf{c}(\mathbf{v}) = \mathbf{C}\mathbf{v}$ with $\mathbf{C} \gg \mathbf{0}$.

Thus the prototypical example of KPP system is the competitive Lotka–Volterra system with mutations:

$$\partial_t \mathbf{u} - \mathbf{D}\Delta\mathbf{u} = \operatorname{diag}(\mathbf{r})\,\mathbf{u} + \mathbf{M}\mathbf{u} - \mathbf{C}\mathbf{u} \circ \mathbf{u},$$

where $\mathbf{r} \gg \mathbf{0}$, $\mathbf{C} \gg \mathbf{0}$ and $\mathbf{M}$ is an irreducible essentially nonnegative square matrix satisfying $\sum_{i=1}^{N} m_{i,j} = 0$ for all $j \in [N]$. Important examples of such matrices $\mathbf{M}$ are matrices of the form $\mathbf{M}_{Lap}\operatorname{diag}(\mathbf{w})$ with $\mathbf{w} \gg \mathbf{0}$ and $\mathbf{M}_{Lap}$ the discrete Laplacian with Neumann conditions:

$$\mathbf{M}_{Lap} = \begin{pmatrix} -1 & 1 & 0 & \dots & 0 \\ 1 & -2 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & -2 & 1 \\ 0 & \dots & 0 & 1 & -1 \end{pmatrix}.$$

Since most contemporary works focus on the above mutation–competition–diffusion system, it is sufficient for an introductory presentation of the state of the art and of the results. Nevertheless, the general framework of KPP systems contains problems coming from much more various applications: Fisher–KPP equation with age classes, Gross–Pitaevskii system for two-component Bose–Einstein condensates with Rabi coupling, and so on. These systems and the underlying modeling questions will be addressed in the adequate chapters.

The first works on propagation phenomena for KPP systems are due to Freidlin [74]. Thanks to probabilistic methods, he studied a two-component KPP system admitting a globally attractive nontrivial stationary state. Inspired by this work, Barles, Evans and Souganidis [10] studied in 1990 a much more general case with PDE methods. Thanks to the WKB change of variable and to the vanishing viscosity limit, they were able to characterize the asymptotic speed of propagation of solutions of the Cauchy problem with compactly supported initial data. Although their method, nowadays commonly used in particular in mathematical adaptive dynamics, does not use the framework of traveling waves highlighted in this thesis, it provides the same kind of results and therefore motivates naturally the problem of traveling waves.

The biological relevance of KPP systems was made clear in 1998 by Dockery *et al.* [58] when they introduced mutations of small amplitude in (0.0.2) in order to verify whether "Unity is strength" would remain true in such a context. However, due to important theoretical obstacles (no variational structure, no comparison principle), a more exhaustive treatment of these systems was out of reach, and it lead them to the suggestion that the only mathematically tractable case was that of the two-component system with vanishing mutations. Indeed, in such a case, the limiting system is exactly (0.0.1) and is therefore much more understood thanks to the comparison principle.

Consequently, up to rare exceptions, the subsequent research focused on this particular case. In 2012, Elliott and Cornell [65] resurrected the interest for the traveling wave problem with a heuristic and numerical study. In 2014, the linear determinacy question was raised formally by Cosner [39]. In 2016, Griette and Raoul [82] showed for the very first time the existence of a traveling wave and characterized the shape of its profile in a particular regime. In 2017, Morris, Börger et Crooks [115] established with different techniques a more general existence result and also obtained a result on the propagation of one-dimensional compactly supported initial data.

However, all these results are only concerned with the two-component system with small mutations. The general case remained, before this thesis, completely open.

The approach of this thesis to deal with this general case is different from the ones previously used. It relies upon the following observation: the reaction term $\mathbf{Lu} - (\mathbf{Cu}) \circ \mathbf{u}$ is analogous to the reaction term $ru - \frac{u^2}{K}$, that is to a KPP type reaction term. In particular, the solutions of the linearized system $\partial_t \mathbf{u} - \mathbf{D}\Delta\mathbf{u} = \mathbf{Lu}$ can be used as super-solutions, even though the nonlinear system does not satisfy a comparison principle.

Thanks to this observation, the following results can be proved.

Theorem. *[Gir18b] Any positive solution $\mathbf{u}$ of (0.0.4) set in $(0,+\infty)\times\mathbb{R}$ such that $x \mapsto \mathbf{u}(0,x)$ is nonzero satisfies $\mathbf{u}(t,x) \gg \mathbf{0}$ for all $(t,x) \in (0,+\infty)\times\mathbb{R}$.*

Theorem. *[Gir18b] There exists a continuous function $\mathbf{g} : [0,+\infty) \to \mathbb{R}^N$ whose components are all nondecreasing and satisfying $\mathbf{g}(0) \gg \mathbf{0}$ such that any nonnegative solution $\mathbf{u}$ of (0.0.4) set in $(0,+\infty)\times\mathbb{R}$ satisfies*

$$\mathbf{u}(t,x) \leq \left(g_i \left(\sup_{x\in\mathbb{R}} u_i(0,x) \right) \right)_{i\in[N]} \quad \text{for all } (t,x) \in [0,+\infty)\times\mathbb{R}$$

and furthermore if $x \mapsto \mathbf{u}(0,x)$ is bounded then

$$\left(\limsup_{t\to+\infty} \sup_{x\in\mathbb{R}} u_i(t,x) \right)_{i\in[N]} \leq \mathbf{g}(0).$$

In what follows, $\lambda_{PF}(\mathbf{L})$ denotes the Perron–Frobenius eigenvalue of $\mathbf{L}$ and $\mathbf{n}_{PF}(\mathbf{L})$ denotes its eigenvector satisfying $\mathbf{n}_{PF}(\mathbf{L}) \gg \mathbf{0}$ and $|\mathbf{n}_{PF}(\mathbf{L})| = 1$.

Theorem. *[Gir18b] Assume $\lambda_{PF}(\mathbf{L}) \leq 0$. Then all bounded nonnegative solutions of (0.0.4) set in $(0,+\infty)\times\mathbb{R}$ vanish asymptotically in time, uniformly in space, provided one of the following conditions is satisfied:*

1. *$\lambda_{PF}(\mathbf{L}) < 0$, and in such a case the convergence is exponential in time;*
2. *$\lambda_{PF}(\mathbf{L}) = 0$ and $\mathbf{c}(\alpha\mathbf{n}_{PF}(\mathbf{L})) > \mathbf{0}$ for all $\alpha > 0$.*

This theorem corresponds to the so-called *extinction case* and its proof, relatively straightforward, relies upon the comparison with the super-solution obtained with the linearized system. On the contrary, the so-called *persistence case*, corresponding to the following theorem, requires a delicate proof using the instability of $\mathbf{0}$ and the Harnack inequality established in 2009 by Földes and Poláčik [73].

Theorem. *[Gir18b] Assume $\lambda_{PF}(\mathbf{L}) > 0$. Then there exists $\nu > 0$ such that any bounded positive solution $\mathbf{u}$ of (0.0.4) set in $(0,+\infty)\times\mathbb{R}$ satisfies, for any bounded interval $I \subset \mathbb{R}$,*

$$\left(\liminf_{t\to+\infty} \inf_{x\in I} u_i(t,x) - \nu \right)_{i\in[N]} \geq \mathbf{0}.$$

Furthermore, there exists a constant positive stationary state, which is consequently valued in

$$\prod_{i=1}^{N} [\nu, g_i(0)].$$

Once these basic theorems are established, we can focus on the more interesting persistence case and consider propagation phenomena. It is then quickly noticed that a more precise characterization of the long-time behavior seems out of reach in full generality (in particular, multiple locally stable stationary states can exist). Consequently we opt for a weaker definition of traveling wave, stated now.

Definition. A traveling wave solution for (0.0.4) is an entire solution which is positive, bounded and of the form $\mathbf{u} : (t,x) \mapsto \mathbf{p}(x-ct)$, with a speed $c \geq 0$ and a profile $\mathbf{p} \in \mathscr{C}^2\left(\mathbb{R}, \mathbb{R}^N\right)$ satisfying

$$\left(\liminf_{\xi \to -\infty} p_i(\xi)\right)_{i\in[N]} > \mathbf{0} \text{ and } \lim_{\xi\to+\infty} \mathbf{p}(\xi) = \mathbf{0}.$$

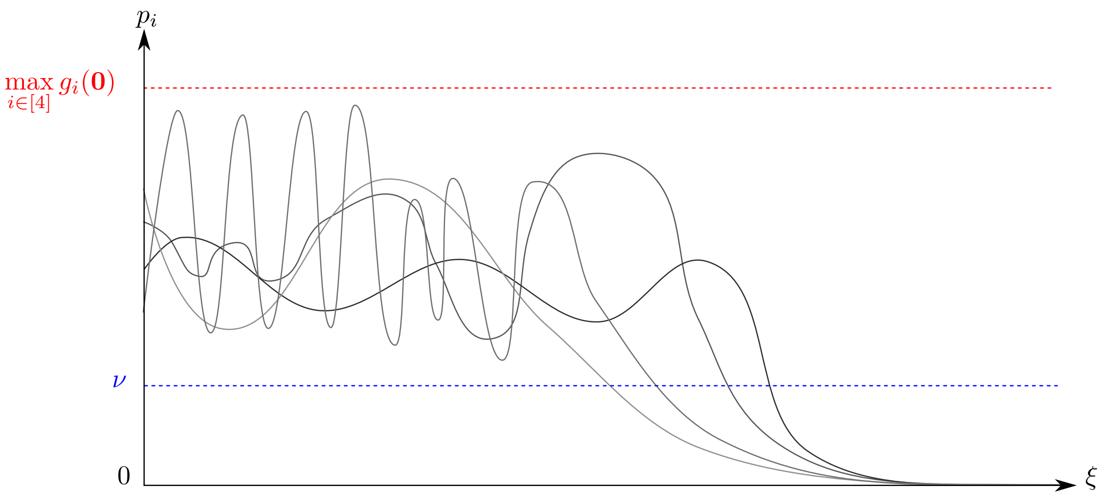


Figure 0.0.7 – Example of traveling wave profile for a four-component KPP system. The behavior at the back is voluntarily represented as non-convergent, in absence of more convincing result.

The forthcoming theorem is then proved by adapting ideas of Berestycki, Nadin, Perthame and Ryzhik [19] to overcome the default of comparison principle.

Theorem. *Assume* $\lambda_{PF}(\mathbf{L}) > 0$*. Let*

$$c^\star = \min_{\mu>0}\left(\frac{\lambda_{PF}\left(\mu^2\mathbf{D}+\mathbf{L}\right)}{\mu}\right).$$

The quantity $c^\star$ is positive and is the minimal wave speed as well as the asymptotic speed of propagation of one-dimensional initial data whose support is included in a left half-line, in the following sense:

1. *for all $c \in [0, c^\star)$, there exists no traveling wave solution of (0.0.4) with speed c;*
2. *if $D\mathbf{c}(\mathbf{v}) \geq \mathbf{0}$ for all $\mathbf{v} \geq \mathbf{0}$, then for all $c \geq c^\star$, there exists a traveling wave solution of (0.0.4) with speed c;*
3. *for all $x_0 \in \mathbb{R}$ and all bounded nonnegative nonzero functions $\mathbf{v}$, the solution $\mathbf{u}$ of (0.0.4) set in $(0,+\infty)\times\mathbb{R}$ with initial data $\mathbf{v}\mathbf{1}_{(-\infty,x_0)}$ satisfies*

$$\left(\lim_{t\to+\infty}\sup_{x\in(y,+\infty)} u_i(t,x+ct)\right)_{i\in[N]} = \mathbf{0} \textit{ for all } c \in (c^\star,+\infty) \textit{ and all } y \in \mathbb{R},$$

$$\left(\liminf_{t\to+\infty}\inf_{x\in[-R,R]} u_i(t,x+ct)\right)_{i\in[N]} \in \mathsf{K}^{++} \textit{ for all } c \in [0,c^\star) \textit{ and all } R > 0.$$

Furthermore, all profiles $\mathbf{p}$ *satisfy*

$$\mathbf{p} \leq \mathbf{g}(0) \quad and \quad \left(\liminf_{\xi\to-\infty} p_i(\xi) - \nu\right)_{i\in[N]} \geq \mathbf{0}.$$

From the explicit formula giving $c^\star$, various estimations can also be deduced. They will be detailed in the adequate chapter.

Although the above results are proved by using exclusively classical super-solutions and sub-solutions of the KPP literature, refined qualitative results on the profiles can be achieved by resorting to more various methods.

We define, for all $c \geq c^\star$, the quantities

$$\mu_c = \min\left\{\mu > 0 \mid \frac{\lambda_{PF}\left(\mu^2\mathbf{D} + \mathbf{L}\right)}{\mu} = c\right\},$$

$$k_c = \begin{cases} 0 & \text{if } c > c^\star, \\ 1 & \text{if } c = c^\star. \end{cases}$$

The quantity μ_c is well-defined and positive (see the adequate chapters).

Theorem. *[Gir18a] For any traveling wave solution of profile* $\mathbf{p}$ *and speed* c*, there exists* $A > 0$ *such that, as* $\xi \to +\infty$*,*

$$\begin{cases} \mathbf{p}(\xi) \sim A\xi^{k_c}e^{-\mu_c\xi}\mathbf{n}_{PF}\left(\mu_c^2\mathbf{D} + \mathbf{L}\right), \\ \mathbf{p}'(\xi) \sim -\mu_c\mathbf{p}(\xi), \\ \mathbf{p}''(\xi) \sim \mu_c^2\mathbf{p}(\xi). \end{cases}$$

Consequently, all the components of $\mathbf{p}$ *are, in a neighborhood of* $+\infty$*, decreasing and strictly convex.*

Multiple proofs of this theorem exist. Since we chose to avoid ODE methods but since we also cannot apply a pure PDE method because of the default of comparison principle, we suggest in this thesis a proof using general results of real analysis (a Ikehara theorem as well as properties of the bilateral Laplace transform). It is reasonable to hope that this proof can be generalized to the context of pulsating fronts in spatially periodic media.

Under restrictive assumptions on the parameters, we can use the Jordan form of $\mathbf{L}$ and the Perron–Frobenius projection to reduce the KPP system to a simple KPP equation. This is indicated by the following two theorems.

Theorem. *[Gir18a] Assume* $\lambda_{PF}(\mathbf{L}) > 0$*,* $\mathbf{D} = \mathbf{I}$ *and the existence of* $b : \mathbb{R}^N \to \mathbb{R}$ *such that, for all* $\mathbf{v} \geq \mathbf{0}$ *and all* $i \in [N]$*,* $c_i(\mathbf{v}) = b(\mathbf{v})$ *and the function* $w \mapsto b(w\mathbf{e}_i + \mathbf{v})$ *is increasing in* $(0, +\infty)$*.*

Let $\alpha^\star > 0$ *be the unique solution of* $b(\alpha\mathbf{n}_{PF}(\mathbf{L})) = \lambda_{PF}(\mathbf{L})$ *and define* $\mathbf{v}^\star = \alpha^\star\mathbf{n}_{PF}(\mathbf{L})$*.*

Then all positive classical solutions of (0.0.4) set in $(0, +\infty)$ *converge asymptotically in time, locally uniformly in space, to* $\mathbf{v}^\star$*.*

Consequently, the set of bounded nonnegative stationary solutions is exactly $\{\mathbf{0}, \mathbf{v}^\star\}$*.*

Theorem. *[Gir18a] Assume the assumptions of the preceding theorem are still satisfied.*

For all $c \in [c^\star, +\infty)$*, let* $p_c \in \mathscr{C}^2(\mathbb{R})$ *such that* (p_c, c) *is the unique traveling wave solution of the KPP equation*

$$\partial_t u - \partial_{xx} u = \lambda_{PF}(\mathbf{L})\, u - b\left(u\mathbf{n}_{PF}(\mathbf{L})\right) u$$

connecting 0 *to* $\alpha^\star$ *and satisfying* $p_c(0) = \frac{\alpha^\star}{2}$*.*

Then any profile of traveling wave solution of (0.0.4) with speed c has the form

$$\mathbf{p} : \xi \mapsto p_c (\xi - \xi_0)\, \mathbf{n}_{PF} (\mathbf{L}) \text{ with } \xi_0 \in \mathbb{R}$$

and, consequently, the traveling wave with speed c is unique and connects $\mathbf{0}$ *to* $\mathbf{v}^\star$.

Since $\mathbf{D} = \mathbf{I}$ implies $c^\star = 2\sqrt{\lambda_{PF}(\mathbf{L})}$, this quantity is indeed both the minimal wave speed of (0.0.4) and of the KPP equation appearing in the statement.

Finally, for the two-component system, the idea according to which the vanishing mutation limit satisfies a comparison principle can be rigorously applied by considering the limit $\eta \to 0$ of the following system:

$$\begin{cases} \partial_t u_1 - d_1 \partial_{xx} u_1 = r_1 u_1 - (c_{1,1} u_1 + c_{1,2} u_2) u_1 + \eta m_1 (u_2 - u_1) \\ \partial_t u_2 - d_2 \partial_{xx} u_2 = r_2 u_2 - (c_{2,1} u_1 + c_{2,2} u_2) u_2 + \eta m_2 (u_1 - u_2) \end{cases}.$$

We denote $(\alpha_1, \alpha_2) = \left(\frac{r_1}{c_{1,1}}, \frac{r_2}{c_{2,2}}\right)$ the carrying capacities in absence of mutations and, if $c_{1,1}c_{2,2} \neq c_{1,2}c_{2,1}$, we denote

$$\mathbf{v}_m = \frac{1}{c_{1,1}c_{2,2} - c_{1,2}c_{2,1}} \begin{pmatrix} r_1 c_{2,2} - r_2 c_{1,2} \\ r_2 c_{1,1} - r_1 c_{2,1} \end{pmatrix}$$

the coexistence state in absence of mutations. We assume further that the system without mutations is monostable, that is there exists $i \in \{1, 2\}$ such that

$$\frac{r_i}{r_{3-i}} > \frac{c_{i,3-i}}{c_{3-i,3-i}}.$$

Thus the stable state is

$$\mathbf{v}_s = \begin{cases} \alpha_i \mathbf{e}_i & \text{if } \frac{r_i}{r_{3-i}} \geq \frac{c_{i,i}}{c_{3-i,i}}, \\ \mathbf{v}_m & \text{if } \frac{r_i}{r_{3-i}} < \frac{c_{i,i}}{c_{3-i,i}}. \end{cases}$$

Theorem. *[Gir18a] Let* $(\mathbf{p}_\eta)_{\eta>0}$ *and* $(c_\eta)_{\eta\geq 0}$ *such that:*

1. *for all* $\eta > 0$, $(t, x) \mapsto \mathbf{p}_\eta (x - c_\eta t)$ *is a traveling wave solution of the problem with mutation rate* η*;*
2. $c_\eta \to c_0$ *as* $\eta \to 0$.

Then there exists $(\zeta_\eta)_{\eta>0}$ *such that, as* $\eta \to 0$, $(\xi \mapsto \mathbf{p}_\eta (\xi + \zeta_\eta), c_\eta)_{\eta>0}$ *converges up to extraction in* $\mathscr{C}^2_{loc}(\mathbb{R}, \mathbb{R}^2) \times \mathbb{R}$ *to the profile – speed pair* $(\mathbf{p}, c_0)$ *of a traveling wave solution of the system without mutations*

$$\begin{cases} \partial_t u_1 - d_1 \partial_{xx} u_1 = r_1 u_1 - (c_{1,1} u_1 + c_{1,2} u_2) u_1 \\ \partial_t u_2 - d_2 \partial_{xx} u_2 = r_2 u_2 - (c_{2,1} u_1 + c_{2,2} u_2) u_2 \end{cases}$$

achieving one of the following connections:

1. $\mathbf{0}$ *to* $\mathbf{v}_s$*,*
2. $\alpha_{3-i}\mathbf{e}_{3-i}$ *to* $\mathbf{v}_s$*,*
3. $\mathbf{0}$ *to* $\alpha_i \mathbf{e}_i$ *with* $p_{3-i} = 0$.

In the adequate chapter, we present some numerical simulations leading to a general conjecture covering also the bistable case. This conjecture will help to guide efficiently the future research.

Perspectives

Research opportunities opened by this thesis are numerous. Here we will only present the more interesting leads, susceptible of being studied in the very near future.

First, regarding the strong competition limit and the theorems of "Unity is not strength" type, it could be relevant to study how more complex movement strategies influence the outcome. For instance, Potts and Petrovskii [127] recently illustrated numerically that a result of "Unity is strength" type could arise for the system

$$\begin{cases} \partial_t u - \Delta u + s_1 \nabla \cdot (u \nabla v) = u\,(1 - u - av) \\ \partial_t v - d\Delta v + s_2 \nabla \cdot (v \nabla u) = rv\,(1 - v - bu) \end{cases},$$

where the terms $s_1 \nabla \cdot (u \nabla v)$ and $s_2 \nabla \cdot (v \nabla u)$ model, if $s_1 > 0$ and $s_2 > 0$, an aggressive taxis pushing one competitor toward the habitat of the other. It would be interesting to determine a parameter regime in which an analytical confirmation is possible. The more obvious lead is the regime $s_1 = s$, $s_2 = \sigma s$, $s \to +\infty$, analogous in some sense to the strong competition limit. However the existence of traveling waves for such a system, strongly coupled, is a very difficult problem.

Results obtained with Adrian Lam on the propagation properties of the monostable competitive Lotka–Volterra system might be enhanced by proving the locally uniform convergence of the solutions to traveling waves. Positive or negative delays of Bramson type [30, 29] are obviously expected and are remarkable obstacles. Another interesting lead is the classification of the entire solutions of the system, in which the propagating terraces we exhibited will play a crucial role.

Next, regarding KPP systems, the vanishing mutation limit for the two-component system and the aforementioned conjecture deserve a closer investigation. The main difficulty when trying to pass to the limit is to find an appropriate normalization; where should the observation be centered so that the limit is not a constant stationary state? The above theorem uses a known normalization of the monostable case but the bistable case seems much more mysterious.

On KPP systems again, it seems important as well to study the possible generalization of the theorems reducing the KPP system to a scalar equation: is it possible to remove the assumption $\mathbf{D} = \mathbf{I}$? Is it possible to change $c_i(\mathbf{v}) = b(\mathbf{v})$ into $c_i(\mathbf{v}) = b(\mathbf{v})\,a_i$ with $\mathbf{a} \gg \mathbf{0}$? Naively, all conclusions seem reasonable at this point. Thus numerical simulations will have to be performed.

Finally, another research direction on KPP systems would be the generalization of the results of this thesis to one-dimensional periodic media and pulsating fronts. Such a generalization would settle several questions left open by Alfaro and Griette [2].

Systèmes de compétition – diffusion fortement compétitifs à deux espèces

« "Pas de compétition ! La compétition est toujours nuisible à l'espèce et il y a de nombreux moyens de l'éviter". Telle est la tendance de la nature, non pas toujours pleinement réalisée, mais toujours présente. »

(P. Kropotkine)

Chapitre 1

Ondes progressives pour des systèmes diffusifs et fortement compétitifs : mobilité relative et vitesse d'invasion

Résumé

Le but de ce chapitre est de déterminer l'envahisseur dans un système de compétition – diffusion de Lotka – Volterra à deux espèces, dans le cas particulier de solutions sous forme d'onde progressive en milieu homogène. La question est très difficile en toute généralité mais deux cas asymptotiques semblent dignes d'intérêt : faible compétition interspécifique et forte compétition interspécifique. Ici, on étudie le second cas et on obtient une conclusion sans équivoque : l'espèce qui chasse l'autre est, à une constante multiplicative près, la plus diffusive.

Ce chapitre, co-écrit avec Grégoire Nadin, a fait l'objet d'une publication sous le titre *Traveling waves for diffusive and strongly competitive systems : relative motility and invasion speed* dans *European Journal of Applied Mathematics* [GN15].

1.1 Introduction

Competitive reaction–diffusion systems have been widely studied in the last few years. These mathematical models are motivated by numerous applications: ecology, chemistry, genetics, etc. In general, the mathematical formulations of this problem are, for some spatial domain Ω (not necessarily bounded), some $n \in \mathbb{N}$ and some positive constants $(d_i, r_i, a_i, k_{i,j})_{i,j\in\{1,\dots,n\}}$:

$$\forall i \in \{1,\dots,n\} \quad \partial_t u_i = d_i \Delta_x u_i + u_i \left(r_i - a_i u_i - \sum_{j\neq i} k_{i,j} u_j \right) \text{ in } \Omega \times (0,+\infty). \qquad (1.1.1)$$

One tough question is how their solutions and, when they exist, the long-time steady states, depend on the diffusion rates $(d_i)_{i\in\{1,\dots,n\}}$. Asymptotically, how do the species (if we see these as continuous approximations of some population-dynamics problems) represented by the densities $(u_i)_{i\in\{1,\dots,n\}}$ share the domain Ω? Basically, in the neighborhood of any spatial point x, two cases may occur: either only one species persists (exclusion case) or two or more persist (coexistence case). In the exclusion case, the only persistent species is called invading species. A priori, all the parameters participate in the determination of this invader: number of species n, heterogeneity of Ω, boundedness of Ω, boundary conditions, intrinsic growth rates $(r_i)_{i\in\{1,\dots,n\}}$, interspecific competition rates $(k_{i,j})_{i,j\in\{1,\dots,n\}}$, intraspecific competition rates $(a_i)_{i\in\{1,\dots,n\}}$ and of course diffusion rates $(d_i)_{i\in\{1,\dots,n\}}$.

The dependency on diffusion rates is a very open general problem. Previous works show clearly that a very general result is for the moment unachievable and that we have to consider in each study a specific case for the other parameters of the problem. A key work in this area is the paper by Dockery et al. [58]. They proved that, when Ω is bounded, heterogeneous, with Neumann boundary conditions and when $k_{i,j} = 1$ for all $i, j \in \{1,\dots,n\}$, the less motile species – that is the one with the lower diffusion rate – is the invading species. Their result relies fundamentally on the heterogeneity, the basic idea being that each species loses the individuals trying to invade unfavorable areas while, in favorable areas, the competition helps the more concentrated one, that is the less diffusive one.

We leave the extension of Dockery's result for different $(k_{i,j})_{i,j\in\{1,\dots,n\}}$ to others and wonder if a similar result can be obtained in homogeneous domains (bounded or not).

Actually, it is quite tough to guess heuristically what could happen in homogeneous domains. Indeed, on one hand, the more diffusive species might be able to ignore its competitors long enough and invade the whole territory while eliminating the competitors slowly. On the other hand, the more concentrated species – that is the less diffusive one – might benefit from the maxim "unity is strength" and eliminate slowly the dispersed competitors and, asymptotically, invade the domain. It is well-known that diffusion tends to bring unexpected results. In any case, if something can revert the invasion, we expect it to be the competition. With this in mind, we decide to focus first on the infinite competition limit which should amplify the effects of competition.

Many papers limit their study to the case $n = 2$ (and so will we) because then the system becomes monotonic and is therefore much simpler to study than the general case. We will not use the monotonicity explicitly but it will be the underlying mechanism behind many results.

When $n = 2$, the PDE system can be rewritten:

$$\begin{cases} \partial_t u = d_1 \Delta_x u + u (r_1 - a_1 u - k_1 v) & \text{in} \quad \Omega \times (0,+\infty) \\ \partial_t v = d_2 \Delta_x v + v (r_2 - a_2 v - k_2 u) & \text{in} \quad \Omega \times (0,+\infty). \end{cases}$$

When there is no diffusion at all, this system becomes an ODE system. Then, the steady state $(u,v)=(0,1)$ (resp. $(u,v)=(1,0)$) is stable when $\frac{k_1 r_2}{r_1 a_2}>1$ (resp. $\frac{k_2 r_1}{r_2 a_1}>1$), unstable when $\frac{k_1 r_2}{r_1 a_2}<1$ (resp. $\frac{k_2 r_1}{r_2 a_1}>1$). Our interest lies in the bistable case and more precisely in the so-called "strong competition case" where $\frac{k_1 r_2}{r_1 a_2}$ and $\frac{k_2 r_1}{r_2 a_1}$ are much larger than 1. In the monostable case, only one species is a "strong" competitor.

The infinite competition limit ($k_1 \to +\infty$ and $\frac{k_1}{k_2}$ constant) has been studied by Dancer et al. in 1999 in the case of bounded domains with Neumann boundary conditions [47] (they also investigated Dirichlet conditions five years later [42]). They obtained a free boundary Stefan problem and, under regularity assumptions, a spatial segregation with an explicit condition on the interface. In 2007, Nakashima and Wakasa [120] studied the generation of interfaces for such systems and obtained a similar free boundary condition.

It is worth mentioning that the spatial segregation in multi-dimensional domains for elliptic PDE yields highly non-trivial issues. It can be either approached as a free boundary problem (Caffarelli [31], Dancer [47] and references therein) or as an optimal partition problem (Conti [38] and references therein), but in both cases it is really a problem in itself, which requires additional assumptions on the initial conditions and a lot of work.

Therefore, our interest goes to unbounded homogeneous domains. Reaction–diffusion studies in such domains usually conjecture the existence of propagation fronts and, when their existence can be rigorously proved, derive from them some information on the dynamics of the system and the long-time steady state. Here, it is important to recall that the main underlying assumption with propagation fronts is that, when the initial conditions are well-chosen, the solutions of the PDE asymptotically behave like the traveling wave solution. We refer to Gardner [77] for such results for finite k. We will not treat this aspect of the problem in this paper but will indeed investigate traveling wave solutions.

A straightforward consequence of the traveling wave approach is that it reduces the multi-dimensional $\Omega \times (0,+\infty)$ to $\mathbb{R}$. The problem becomes one-dimensional, that is an ODE problem, and thus all the free boundary issues vanish. Our hope is to find a similar spatial segregation limit, with an explicit condition on the interface connecting the invasion speed of the traveling wave to the diffusion rates. We know from Gardner [77] and Kan-On [100] that the invasion speed is constant and bounded by the Fisher–KPP speeds [104] of the species. Can we use the infinite competition limit to derive its sign and therefore know which species invades the other? Will unity be strength?

It is important to remark that the invasion speed is not linearly determined here. Actually, a linearization near $(0,1)$ or $(1,0)$ yields no condition on the invasion speed and the linearized speed cannot be defined as usual. As far as we know, the linear determinacy for competition–diffusion systems is useful only with a specific class of monostable problems (Huang [94], Lewis [108]).

In the next section, we fully pose the problem, enunciate our final result and recall that the problem is well-posed. The third and main section is dedicated to a compactness result and the convergence to a limit problem which is similar in many ways to the one Dancer et al. obtained. Eventually, the last section exhibits the relation between the speed and the diffusion rates.

1.2 Formulation of the problem and main theorem

In this first section, we present the PDE problem studied in this article, give its ecological interpretation and enunciate our main result. We also check quickly that the problem is well-posed.

1.2.1 Model

1.2.1.1 Reaction–diffusion system

We first consider the following one-dimensional Lotka–Volterra competition–diffusion problem:

$$\begin{cases} \partial_t \mu = d_1 \partial_{xx} \mu + \mu (r_1 - a_1 \mu - k_1 \rho) & \text{in} \quad \mathbb{R} \times (0, +\infty) \\ \partial_t \rho = d_2 \partial_{xx} \rho + \rho (r_2 - a_2 \rho - k_2 \mu) & \text{in} \quad \mathbb{R} \times (0, +\infty) . \end{cases}$$

where $d_1, d_2, r_1, r_2, a_1, a_2, k_1, k_2$ are positive constants with ecological meaning (diffusion rates, intrinsic growth rates, intraspecific competition rates, interspecific competition rates). We assume, without loss of generality, that $\frac{k_2 a_2}{r_2^2} \geq \frac{k_1 a_1}{r_1^2}$.

Let $k = \frac{k_1 r_2}{a_2 r_1} > 0$, $\alpha = \frac{k_2 a_2 r_1}{k_1 a_1 r_2} > 0$, $d = \frac{d_2}{d_1} > 0$, $r = \frac{r_2}{r_1} > 0$ and

$$(u_k, v_k) : (x,t) \mapsto \left(\frac{a_1}{r_1} \mu \left(\sqrt{\frac{d_1}{r_1}} x, \frac{1}{r_1} t \right), \frac{a_2}{r_2} \rho \left(\sqrt{\frac{d_1}{r_1}} x, \frac{1}{r_1} t \right) \right).$$

We get:

$$\begin{cases} \partial_t u_k = \partial_{xx} u_k + u_k (1 - u_k) - k u_k v_k & \text{in} \quad \mathbb{R} \times (0, +\infty) \\ \partial_t v_k = d \partial_{xx} v_k + r v_k (1 - v_k) - \alpha k u_k v_k & \text{in} \quad \mathbb{R} \times (0, +\infty) . \end{cases}$$

As soon as $k > 1$ (which will always be assumed thereafter), $\frac{\alpha k}{r} > 1$, that is the system is bistable. Indeed, the free assumption $\frac{k_2 a_2}{r_2^2} \geq \frac{k_1 a_1}{r_1^2}$ we made earlier ensures that $\frac{\alpha}{r} \geq 1$.

A priori, the parameters k, α, d and r can take any positive value. Let $\mathcal{P}(k, \alpha, d, r)$ denote this generic PDE problem. Our interest lies in the limit, as $k \to +\infty$, of the set of problems $\{\mathcal{P}(k, \alpha, d, r)\}_{k>1}$ (associated with a given (α, d, r)) (hence the notation u_k and v_k).

Moreover, going back to the initial parameters, this means that we actually consider a larger class of ecological problems than just $k_1 \to +\infty$ and $\frac{k_1}{k_2}$ constant. Indeed, the only restrictions are that $\frac{d_2}{d_1}$, $\frac{r_2}{r_1}$ and
$\frac{k_2 a_2}{k_1 a_1}$ are fixed along the whole class. For example, the limit $k \to +\infty$ may correspond to:

- k_2 proportional (with a fixed constant along the whole class) to k_1 and $k_1 \to +\infty$ with a_1 and a_2 fixed (along the whole class);
- $k_1 \to +\infty$ and a_1 proportional to $\frac{1}{k_1}$ with a_2 and k_2 fixed;
- a_2 proportional to a_1 and $a_1 \to 0$ with k_1 and k_2 fixed.

1.2.1.2 Traveling wave system

Searching for a traveling wave of the variable $\xi = x - c_k t$, where $c_k \in \mathbb{R}$ is the unknown invasion speed, the problem is eventually rewritten as:

$$\begin{cases} -u_k'' - c_k u_k' = u_k (1 - u_k) - k u_k v_k & \text{in} \quad \mathbb{R} \\ -d v_k'' - c_k v_k' = r v_k (1 - v_k) - \alpha k u_k v_k & \text{in} \quad \mathbb{R} \\ u_k(-\infty) = 1, \ u_k(+\infty) = 0 & \\ v_k(-\infty) = 0, \ v_k(+\infty) = 1 & \\ u_k' < 0 & \text{in} \quad \mathbb{R} \\ v_k' > 0 & \text{in} \quad \mathbb{R}. \end{cases} \tag{1.2.1}$$

It is well-known that natural selection tends to differentiate the niches of competing species. The traveling wave solution corresponds to the case where u_k lives essentially in the left half-space while v_k lives essentially in the right half-space. In such a situation, it seems obvious that one species might chase the other and invade the abandoned territory. The whole point of this article is to determine this species, or equivalently, the sign of the invasion speed. Indeed,

1. $c_k > 0$ if and only if u_k chases v_k;
2. $c_k < 0$ if and only if v_k chases u_k.

Of course, we aim to find a result depending on the value of d. Thus in the following pages, when we focus on the dependency of c_k on d, we write $c_{k,d}$; otherwise, when d is fixed, we simply write c_k.

1.2.2 "Unity is not strength" theorem

Our main result follows.

Theorem 1.1. *$(d \mapsto c_{k,d})_{k>1}$ converges locally uniformly in $(0, +\infty)$ to a continuous function $d \mapsto c_{\infty,d}$ valued in $\left(-2\sqrt{rd}, 2\right)$ and which has exactly the sign of $\alpha^2 - rd$, that is:*

1. *$c_{\infty,d} = 0$ if $d = \frac{\alpha^2}{r}$;*
2. *$c_{\infty,d} \in (0, 2)$ if $d \in \left(0, \frac{\alpha^2}{r}\right)$;*
3. *$c_{\infty,d} \in \left(-2\sqrt{rd}, 0\right)$ if $d > \frac{\alpha^2}{r}$.*

Remark. This result is profoundly unexpected! It does not suffice to compare d to 1 or α to 1. v can lose even if r is large and u can lose even if α is large, for example. This should yield interesting insight into ecological applications.

1.2.3 Well-posedness and regularity of the problem

Theorem 1.2. *For any $k > 1$, there exists a unique c_k such that there exist solutions u_k and v_k of the problem (1.2.1). It is needed that $c_k \in \left(-2\sqrt{rd}, 2\right)$, $u_k \in \mathcal{C}^\infty(\mathbb{R})$ and $v_k \in \mathcal{C}^\infty(\mathbb{R})$. We can moreover assume exactly one of the following normalization hypotheses:*

1. *$u_k(0) = v_k(0)$,*
2. *$u_k(0) = \frac{1}{2}$,*
3. *$v_k(0) = \frac{1}{2}$,*

and if we do so, u_k and v_k are unique.

Proof. The well-posedness and the bounds for c_{k} are proven by Gardner in [77] and also by Kan-On in [100] (actually, Gardner only showed $c_k \in \left[-2\sqrt{rd}, 2\right]$ but Kan-On showed indeed $c_k \in \left(-2\sqrt{rd}, 2\right)$ which will be important in the end). It is worth mentioning that their papers actually proved that the problem is well-posed without any monotonicity condition and that the monotonicity is indeed needed.

Since $u_k, v_k \in L^\infty(\mathbb{R})$ and $u_k', v_k' \in L^1(\mathbb{R})$, the regularity just follows from $W^{k,p}$-estimates and Sobolev injections. □

Remark. The minimal and maximal speeds $-2\sqrt{rd}$ and 2 are the invasion speeds of respectively v_k when $u_k = 0$ and u_k when $v_k = 0$. This is a well-known result from Fisher and Kolmogorov, Petrovsky and Piskunov [72, 104].

1.3 Limit problem

Here we show that (u_k), (v_k) and (c_k) converge when $k \to +\infty$ and formulate the limit problem.

1.3.1 Existence of limit points

First, (c_k) is relatively compact and therefore, by the Bolzano–Weierstrass theorem, has a limit point $c \in \left[-2\sqrt{rd}, 2\right]$.

If $c \leq 0$, we fix for any $k > 1$ the normalization $u_k(0) = \frac{1}{2}$. On the contrary, if $c > 0$, we fix for any $k > 1$ $v_k(0) = \frac{1}{2}$. This choice will be explained later on. In either case, this implies that the functions $k \mapsto u_k$ and $k \mapsto v_k$ are well-defined.

Proposition 1.3. *For any $i \geq 1$, let $K_i = [-i, i]$. (u_k) and (v_k) are relatively compact in $\mathcal{C}(K_i)$.*

Proof. Our aim here is to use the Ascoli–Arzela theorem. To that end, let us show that each u_k is Hölder-continuous with a constant independent of k.

There exists a positive function $\chi \in \mathcal{D}(\mathbb{R})$ such that $\chi(x) = 0$ if $x \notin [-i-1, i+1]$ and $\chi(x) = 1$ if $x \in [-i, i]$.

For any $k > 1$, if we multiply the equation defining u_k by $u_k\chi$ and then integrate, we get:

$$\int \left(-u_k'' u_k \chi - c_k u_k' u_k \chi\right) = \int u_k^2 \chi - \int u_k^2 \left(u_k + k v_k\right) \chi.$$

The third term is obviously negative. An integration by parts yields:

$$\int u_k'^2 \chi - \int \frac{u_k^2}{2} \chi'' + c_k \int \frac{u_k^2}{2} \chi' \leq \int u_k^2 \chi.$$

Finally, since $\int u_k'^2 \chi \geq \int_{-i}^{i} u_k'^2$ and $\|u_k\|_{L^\infty} \leq 1$, we have:

$$\|u_k'\|_{L^2(K_i)}^2 \leq \int \left(\chi + \frac{|c_k|}{2} |\chi'| + \frac{1}{2} |\chi''|\right).$$

Then we use the Ascoli–Arzela theorem: the family (u_k) is bounded in $L^\infty(K_i)$ and uniformly equicontinuous in K_i therefore it is relatively compact in $\mathcal{C}(K_i)$.

The exact same proof works for (v_k). □

It is now clear, by a standard diagonal extraction argument, that there exists a subsequence of (u_k) (resp. (v_k)) which converges locally uniformly to a limit point u (resp. v).

1.3.2 Properties of the limit points

c, u and v are actually unique and true limits as will be proven later on. For the moment, let us just consider extracted convergent subsequences, still denoted by (c_k), (u_k) and (v_k).

Lemma 1.4. $uv = 0$.

Proof. Multiplying by a test function $\varphi \in \mathcal{D}(\mathbb{R})$ and integrating the equation for u_k yields:

$$\begin{aligned} k \left| \int u_k v_k \varphi \right| &\leq \int u_k \left(1 - u_k\right) |\varphi| + |c_k| \int u_k |\varphi'| + \int u_k |\varphi''| \\ &\leq C \|\varphi\|_{W^{2,1}(\mathbb{R})}. \end{aligned}$$

Hence $u_k v_k \to 0$ in $\mathcal{D}'(\mathbb{R})$.

Since $u_k v_k \to uv$ locally uniformly, we get indeed $uv = 0$. □

Remark. This kind of result is usually referred to as a segregation property. There is a lot of similar results in the literature.

Lemma 1.5. *We have*

$$-\alpha u'' + dv'' - \alpha c u' + cv' = \alpha u (1-u) - rv(1-v)$$

in $\mathcal{D}'(\mathbb{R})$.

Proof. Multiply the equation for u_k by α and subtract from it the one for v_k. The left-hand side converges trivially in $\mathcal{D}'(\mathbb{R})$. The right-hand side converges by dominated convergence. □

Lemma 1.6. $u, v \in \mathcal{C}(\mathbb{R})$ *and* $\alpha u - dv \in \mathcal{C}^1(\mathbb{R})$.

Proof. The continuity of u and v is immediate thanks to the continuity of each u_k and v_k and the locally uniform convergence.

Let $a, b \in \mathbb{R}$ such that $a < b$ and $I_a : C([a,b]) \to C([a,b])$ defined by $I_a(f) : x \mapsto \int_a^x f$. By continuity of u and v, it is quite obvious that the function

$$\alpha c u - cv + I_a(\alpha u(1-u) - rv(1-v)) - (\alpha c u(a) - cv(a))$$

is continuous. But, thanks to the previous lemma, it is also equal in $\mathcal{D}'((a,b))$ to $-\alpha u' + dv'$ up to an additive constant. Therefore $-\alpha u' + dv'$ is a well-defined function of $\mathcal{C}([a,b])$. □

Lemma 1.7. *u and v have finite limits at $\pm\infty$. Also,*

$$0 \leq \lim_{+\infty} u \leq \lim_{-\infty} u \leq 1$$

and

$$0 \leq \lim_{-\infty} v \leq \lim_{+\infty} v \leq 1.$$

Proof. By locally uniform convergence, u and v are monotone, respectively non-increasing and non-decreasing, and satisfy $0 \leq u, v \leq 1$. □

Lemma 1.8. *u and v cannot vanish simultaneously on a non-empty compact set.*

Proof. Once again, we consider a non-empty compact set $[a,b]$. By monotonicity, if $u_{|[a,b]} = 0$, then $u_{|[a,+\infty)} = 0$. Similarly, $v_{|(-\infty,b]} = 0$. Thus, in $\mathcal{D}'((-\infty,a))$, $-u'' - cu' = u(1-u)$ and $\alpha u' - dv' = \alpha u'$. Therefore u' is continuous and, using $-u'' - cu' = u(1-u)$, u'' is also continuous and the previous differential equation is satisfied pointwise.

Now, we get by induction that u is $\mathcal{C}^\infty$ in $(-\infty, a)$. Since it does not explode on the left of a, it is the restriction of a solution on a strictly larger interval. Since u is regular, $u'(a) = 0$ and by the Cauchy–Lipschitz theorem, u is identically null. By the same reasoning, v is also identically null.

To prevent u and v from being both null on the whole real line, either one of the two normalization sequences $(u_k(0))_{k>1} = \left(\frac{1}{2}\right)$ and $(v_k(0)) = \left(\frac{1}{2}\right)$ combined with locally uniform convergence suffices. □

Remark. We already knew that $uv = 0$ everywhere. Thus the previous lemma ensures that, for any $a < b$, $u_{|[a,b]} = v_{|[a,b]} = 0$ is not possible; one of the two densities has to be positive whereas the other has to be null.

Lemma 1.9. *Neither u nor v can be positive everywhere.*

Proof. If $c \leq 0$, the normalization sequence is $(u_k(0)) = \left(\frac{1}{2}\right)$. It ensures that u is not null. We define $\xi_u = \sup\{\xi \in \mathbb{R} \mid u(\xi) > 0\} \in (-\infty, +\infty]$.

If $\xi_u = +\infty$ (that is, u positive everywhere), v is null.

In such a case, we have u decreasing, bounded between 0 and 1, with limits at infinity, non-constant by normalization, and $-u'' - cu' = u(1-u)$ everywhere with $u \in \mathcal{C}^\infty(\mathbb{R})$.

Standard elliptic estimates [80] yield then that $\lim_{-\infty} u = 1$ and $\lim_{+\infty} u = 0$.

Thus u is a traveling wave for the Fisher–KPP equation with speed $c \leq 0 < \sqrt{2}$, hence the contradiction [104].

If $c > 0$, we just apply this reasoning to v with normalization $(v_k(0)) = \left(\frac{1}{2}\right)$. □

Corollary 1.10. *The two quantities* $\sup\{\xi \in \mathbb{R} \mid u(\xi) > 0\}$ *and* $\inf\{\xi \in \mathbb{R} \mid v(\xi) > 0\}$ *are real and equal. Up to translation, we can assume it to be* 0*. By continuity of u and v, $u(0) = v(0) = 0$.*

Lemma 1.11. *We have:*

1. $u \in \mathcal{C}^\infty((-\infty, 0) \cup (0, +\infty))$,
2. $v \in \mathcal{C}^\infty((-\infty, 0) \cup (0, +\infty))$.

Moreover, we can extend u' and v' by continuity on the left and on the right respectively and obtain $u'(0) = \lim_{\xi\to 0, \xi<0} u'(\xi)$ and $v'(\xi_v) = \lim_{\xi\to 0,\xi>0} v'(\xi)$ which are finite and satisfy $-\alpha u'(0) = dv'(0) > 0$.

Proof. u is identically zero on $(0, \infty)$ so $u_{|(0,+\infty)}$ is trivially $\mathcal{C}^\infty$. In $(-\infty, 0)$, it is a weak, and then regular (same routine), solution of $u'' + cu' + u(1-u) = 0$.

Eventually, just recall that $\alpha u - dv \in \mathcal{C}^1(\mathbb{R})$. If its derivative at 0 is zero, by the same kind of Cauchy–Lipschitz reasoning, $u = v = 0$ everywhere. □

Remark. The relation $\alpha u'(0) + dv'(0) = 0$ is essentially the free boundary condition obtained by Nakashima and Wakasa in [120].

Lemma 1.12. $\lim_{-\infty} u = 1$ *and* $\lim_{+\infty} v = 1$.

Proof. Same as before. □

Lemma 1.13. $c \in \left(-2\sqrt{rd}, 2\right)$*, that is* $c \notin \left\{-2\sqrt{rd}, 2\right\}$.

Proof. Let us assume, for example, $c = -2\sqrt{rd}$. Let $\xi^\star > 0$ such that $v(\xi^\star) = \frac{1}{2}$.

We know from Fisher and KPP [104] that $c = -2\sqrt{rd}$ is the maximal speed for which there exists a traveling wave v_{KPP} positive, going from 0 at $-\infty$ to 1 at $+\infty$, which satisfies

$$-dv''_{KPP} - cv'_{KPP} = rv_{KPP}\left(1 - v_{KPP}\right).$$

We normalize by fixing $v_{KPP}(\xi^\star) = \frac{1}{2}$. Let $f = v_{KPP} - v$.

First, we can easily check that f is in $\mathcal{C}(\mathbb{R}) \cap \mathcal{C}^\infty((-\infty, 0) \cup (0, +\infty))$ and satisfies

$$-df'' - cf' = rf(1-f) - 2rvf$$

in $(0, +\infty)$.

For any $\xi > \xi^\star$, $1 - f(\xi) - 2v(\xi) = 1 - v_{KPP}(\xi) - v(\xi) < 0$, with $f(\xi^\star) = 0$. We can therefore apply the maximum principle to the elliptic operator

$$d\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} + c\frac{\mathrm{d}}{\mathrm{d}\xi} + r(1 - f - 2v)$$

in any interval $(\xi^\star, b)$, $b > \xi^\star$. Since $\lim_{+\infty} f = 0$, it gives us that $f(\xi) \leq 0$ for any $\xi \in (\xi^\star, +\infty)$. But we can also apply the minimum principle to the same operator, and we eventually get that f is identically zero in $(\xi^\star, +\infty)$. This way, $f'(\xi^\star) = 0$, hence f is identically zero in $(0, +\infty)$, which is impossible since $f(0) > 0$ and f is continuous in $\mathbb{R}$. □

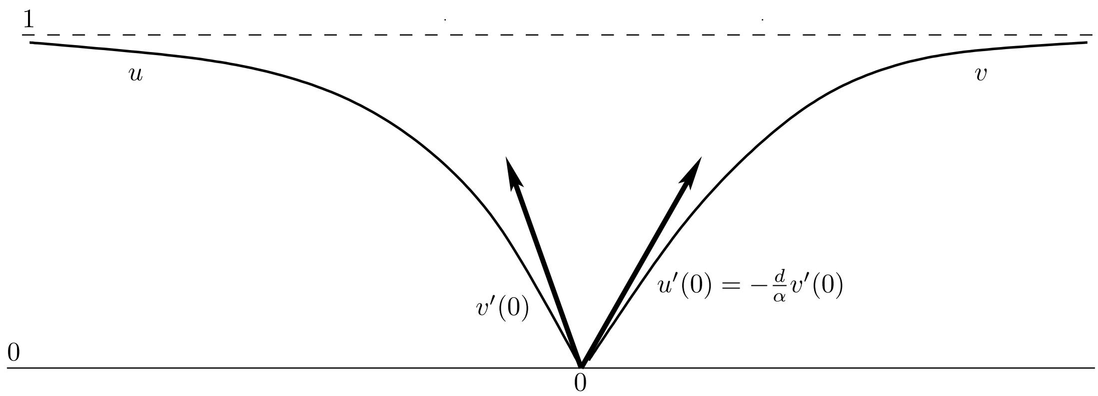
1.3.3 Limit problem

Let us sum up all these results in the following theorem.

Theorem 1.14. *There exist locally uniform limits u and v of (u_k) and (v_k) respectively. They satisfy:*

1. $u, v \in \mathcal{C}(\mathbb{R}) \cap \mathcal{C}^\infty((-\infty, 0) \cup (0, +\infty))$;
2. $\lim_{\xi \to -\infty} u(\xi) = 1$;
3. $\lim_{\xi \to +\infty} v(\xi) = 1$;
4. $u_{|\mathbb{R}_+} = 0$;
5. $v_{|\mathbb{R}_-} = 0$;
6. $u' \leq 0$ *in* $\mathbb{R}_-$ *with* $u'(0)$ *defined by left-continuity;*
7. $v' \geq 0$ *in* $\mathbb{R}_+$ *with* $v'(0)$ *defined by right-continuity;*
8. $-u'' - cu' = u(1 - u)$ *in* $(-\infty, 0)$;
9. $-dv'' - cv' = rv(1 - v)$ *in* $(0, +\infty)$;
10. $\alpha u'(0) = -dv'(0)$.

The behavior of these limits is illustrated with the following figure.



1.3.4 Uniqueness of the limit points

The following theorem is due to Du and Lin [59, 60].

Theorem 1.15. *For any $c > -2$, the problem*

$$\begin{cases} -y_c'' - cy_c' = y_c(1 - y_c) & in \quad (0, +\infty) \\ y_c(0) = 0 \end{cases}$$

admits a unique positive solution.

It satisfies $y_c' > 0$ in $\mathbb{R}_+$ and $\lim_{\xi \to +\infty} y(\xi) = 1$.

Furthermore, the function $\gamma : c \mapsto y_c'(0)$ is increasing and continuous.

Remark. We need to change u and v before pursuing this direction. Let us consider $\tilde{u} : \xi \mapsto u(-\xi)$ and $\tilde{v} : \xi \mapsto v\left(\sqrt{\frac{d}{r}}\xi\right)$. Then $\tilde{u}$ is a solution of the problem

$$\begin{cases} -\tilde{u}'' + c\tilde{u}' = \tilde{u}(1 - \tilde{u}) & \text{in} \quad (0, +\infty) \\ \tilde{u}(0) = 0 \end{cases}$$

and $\tilde{v}$ is a solution of the problem

$$\begin{cases} -\tilde{v}'' - \frac{c}{\sqrt{rd}}\tilde{v}' = \tilde{v}(1 - \tilde{v}) & \text{in} \quad (0, +\infty) \\ \tilde{v}(0) = 0. \end{cases}$$

Also, $c \in \left(-2\sqrt{rd}, 2\right)$ so that $-c > -2$ and $\frac{c}{\sqrt{rd}} > -2$. Therefore we can apply the theorem.

Corollary 1.16. *For any $d > 0$, there exists a unique (u, v, c) satisfying the limit problem.*

Proof. The equality $-\alpha u'(0) = dv'(0)$ can be written as $\alpha\gamma(-c) = \sqrt{rd}\gamma\left(\frac{c}{\sqrt{rd}}\right)$. Now we consider the two functions $x \mapsto \alpha\gamma(-x)$ and $x \mapsto \sqrt{rd}\gamma\left(\frac{x}{\sqrt{rd}}\right)$. They necessarily have an intersection point since c exists. But as they are respectively decreasing and increasing, this intersection point is unique.

The uniqueness of c implies by the previous theorem the uniqueness of u and v. □

The triplet (u, v, c) of the above corollary is hereafter denoted $(u_{\infty,d}, v_{\infty,d}, c_{\infty,d})$.

Corollary 1.17. *The sequences (c_k), (u_k) and (v_k) each have a unique limit point. Hence the pointwise convergence of (c_k) and locally uniform convergence of (u_k) and (v_k) are fully proved and there is no need to consider extracted subsequences anymore.*

Proof. Recall that, in any metric space, a sequence whose image is relatively compact and which has a unique limit point converges to this limit point. □

Remark. It is now clear that the sum up theorem of the previous section gives sufficient but far from necessary conditions for uniqueness. For any c, u and v are unique if and only if they are positive and satisfy points 4, 5, 8 and 9 and then the uniqueness of c is just a consequence of point 10.

Proposition 1.18. *The convergence of $(d \mapsto c_{k,d})_{k>1}$ to $d \mapsto c_{\infty,d}$ is locally uniform.*

Proof. Actually, one can see easily that the whole proof of pointwise convergence of $(d \mapsto c_{k,d})_{k>1}$ holds if we do not fix a priori d. It suffices to have $d \in [D_1, D_2]$, with $D_2 > D_1 > 0$ fixed, so that we can replace bounds like $-2\sqrt{rd}$ by $-2\sqrt{rD_2}$. □

1.4 Dependency of the invasion speed on the diffusion rates

This last section is where we derive from the limit problem the result: how does the invasion speed c depend on the diffusion rate d? Thanks to the convergence of (c_k) to c, we will then be able to extend it to c_k (for k large enough).

Theorem 1.19. *The function $d \mapsto c_{\infty,d}$ has exactly the sign of $\alpha^2 - rd$.*

Proof. The sign of $c_{\infty,d}$ is actually a simple consequence of the relation $\alpha\gamma\left(-c\right) = \sqrt{rd}\gamma\left(\frac{c}{\sqrt{rd}}\right)$. Indeed, let us prove that $rd < \alpha^2$ implies $c_{\infty,d} > 0$. Indeed, if $rd < \alpha^2$, then $\frac{\sqrt{rd}}{\alpha} < 1$ and as $\gamma\left(\frac{c}{\sqrt{rd}}\right) > 0$, we get $\frac{\sqrt{rd}}{\alpha}\gamma\left(\frac{c}{\sqrt{rd}}\right) < \gamma\left(\frac{c}{\sqrt{rd}}\right)$. Since γ is increasing, $\frac{c}{\sqrt{rd}} > -c$, which clearly implies that $c > 0$. The case $rd > \alpha^2$ is similar.

If $rd = \alpha^2$, the relation becomes $\gamma\left(-c\right) = \gamma\left(\frac{c}{\sqrt{rd}}\right)$. An obvious zero of $s \mapsto \gamma\left(-s\right) - \gamma\left(\frac{s}{\sqrt{rd}}\right)$ is 0, and by monotonicity it is unique, hence $c = 0$. ☐

Proposition 1.20. *The function $d \mapsto c_{\infty,d}$ is continuous in $(0, +\infty)$.*

Proof. This could follow from the continuity of each $d \mapsto c_{k,d}$ and the locally uniform convergence, but the continuity of $d \mapsto c_{k,d}$ is actually a more difficult problem (and is not solved by Kan-On [100]). Therefore, we prove the continuity of $d \mapsto c_{\infty,d}$ directly. Our proof being basically a repetition of the whole previous section of this article, we give only a sketch of it.

First, let $0 < D_1 < D_2$. We have:

$$\begin{aligned}
\left\{c_{\infty,d} \mid d \in [D_1, D_2]\right\} &\subset \left\{c_{\infty,d} \mid d \in [D_1, D_2] \cap \left(\frac{\alpha^2}{r}, +\infty\right)\right\} \cup \{0\} \cup \left\{c_{\infty,d} \mid d \in \cap \left(0, \frac{\alpha^2}{r}\right)\right\} \\
&\subset \left(\bigcup_{d \in [D_1, D_2] \cap \left(\frac{\alpha^2}{r}, +\infty\right)} \left[-2\sqrt{rd}, 0\right]\right) \cup [0, 2] \\
&\subset \left[-2\sqrt{rD_2}, 2\right].
\end{aligned}$$

Thus, $\{c_{\infty,d} \mid d \in [D_1, D_2]\}$ is a relatively compact subset of $\mathbb{R}$.

Now, let $\delta \in [D_1, D_2]$ and $(\delta_n)_{n\in\mathbb{N}} \in [D_1, D_2]^{\mathbb{N}}$ a positive sequence which converges to δ. Up to extraction, (c_{∞,δ_n}) converges to a limit point C.

If $C \leq 0$, we translate each couple $(u_{\infty,\delta_n}, v_{\infty,\delta_n})$ so that $(u_{\infty,\delta_n}(0)) = \left(\frac{1}{2}\right)$. If $C > 0$, we translate each couple $(u_{\infty,\delta_n}, v_{\infty,\delta_n})$ so that $(v_{\infty,\delta_n}(0)) = \left(\frac{1}{2}\right)$. In either case, $\{u_{\infty,d} \mid d \in [D_1, D_2]\}$ and $\{v_{\infty,d} \mid d \in [D_1, D_2]\}$ are relatively compact in each $C\left(K_i\right)$ by the Ascoli–Arzela theorem, and, up to extraction, (u_{∞,δ_n}) and (v_{∞,δ_n}) converge locally uniformly. Let U and V be their limits.

1. We have $-\alpha U'' + \delta V'' - \alpha C U' + C V' = \alpha U\left(1 - U\right) - rV\left(1 - V\right)$ in $\mathcal{D}'\left(\mathbb{R}\right)$.
2. U and V are continuous, $\alpha U - \delta V$ is C^1.
3. U and V are positive and have finite limits at infinity.
4. $UV = 0$.
5. If $C \leq 0$, U is not identically null by normalization and V cannot be identically null since if it was, U would be a traveling wave for the Fisher–KPP equation with a speed smaller than 2. The same reasoning applies for $C > 0$ and finally, neither U nor V can be identically null.

6. U and V cannot be both null on a compact subset by continuity of $(\alpha U - \delta V)'$ and a Cauchy–Lipschitz argument.

Now we translate back so that

$$\sup \{\xi \in \mathbb{R} \mid U(\xi) > 0\} = \inf \{\xi \in \mathbb{R} \mid V(\xi) > 0\} = 0.$$

This yields $U_{|\mathbb{R}_+} = 0$, $V_{|\mathbb{R}_-} = 0$, $-U'' - CU' = U(1-U)$ in $(-\infty, 0)$, $-\delta V'' - CV' = rV(1-V)$ in $(0, +\infty)$ and $\alpha U'(0) = -\delta V'(0)$. Basically, C, U and V satisfy the exact same problem as $c_{\infty,\delta}$, $u_{\infty,\delta}$ and $v_{\infty,\delta}$. By uniqueness, $C = c_{\infty,\delta}$, that is $c_{\infty,\delta}$ is the unique limit point of (c_{∞,δ_n}) and eventually $c_{\infty,\delta_n} \to c_{\infty,\delta}$. Therefore $d \mapsto c_{\infty,d}$ is indeed continuous. □

1.5 Conclusion

We have proved our "Unity is not strength" theorem. Some remaining questions concern the shape of the asymptotic speed: What are the limits when $d \to 0$ or $d \to +\infty$? Are there optimal diffusion rates so that the invasion of one species or the other is the fastest? And eventually, how fast is the convergence to this asymptotic limit and, for example, is it monotone?

These could be addressed with the knowledge of the derivatives of the speed as a function of k or d. These might be determined analytically thanks to Kan-On formulas [100]. However, we did not manage to compute the sign of these derivatives, that is, the monotonicity of the speed with respect to k or d. We leave it as an open problem.

Chapitre 2

Compétition en milieu périodique : I – Existence d'ondes pulsatoires

Résumé

Ce chapitre étudie l'existence d'ondes pulsatoires en milieu spatialement périodique pour un système de compétition – diffusion de Lotka – Volterra à deux espèces bistable. En se restreignant aux systèmes fortement compétitifs, une simple condition suffisante de type « haute fréquence ou faibles amplitudes » est mise en avant. Cette condition est de fait suffisante pour garantir que tout état de coexistence périodique converge vers l'état de co-extinction, et ainsi se déstabilise et devienne envahissable par les états de semi-extinction, quand la compétition devient suffisamment forte.

Ce chapitre a fait l'objet d'une publication sous le titre *Competition in periodic media : I – Existence of pulsating fronts* dans *Discrete and Continuous Dynamical Systems – Series B* [Gir17].

2.1 Introduction

This is the first part of a sequel to our previous article with Grégoire Nadin [GN15]. In this prequel, we studied the sign of the speed of bistable traveling wave solutions of the following competition–diffusion problem:

$$\begin{cases} \partial_t u_1 - \partial_{xx} u_1 = u_1 (1-u_1) - k u_1 u_2 & \text{in } (0,+\infty)\times\mathbb{R} \\ \partial_t u_2 - d\partial_{xx} u_2 = r u_2 (1-u_2) - \alpha k u_1 u_2 & \text{in } (0,+\infty)\times\mathbb{R}. \end{cases}$$

We proved that, as $k \to +\infty$, the speed of the traveling wave connecting $(1,0)$ to $(0,1)$ converges to a limit which has exactly the sign of $\alpha^2 - rd$. In particular, if $\alpha = r = 1$ and if k is large enough, the more motile species is the invader: this is what we called the "unity is not strength" result.

In view of this result, it would seem natural to try to generalize it in heterogeneous spaces, that is to systems with non-constant coefficients. Is the more motile species still the invading one?

The first obstacle toward this generalization is that of the existence of traveling fronts –or of some suitable generalization of these– for such a problem. Indeed, while past work had already established the existence of competitive bistable traveling waves in the case of homogeneous spaces (recall for instance Gardner [77] and Kan-On [100]), to the best of our knowledge, there is at this time no such pre-established result in the case of fully heterogeneous spaces (see the recent review of Guo and Wu [84]).

One of the main difficulties regarding this existence problem is of course the combination of unboundedness and heterogeneity. This yields additional difficulties (for instance, there are multiple non-equivalent definitions of the principal eigenvalue [22] and convenient integration-wise boundary conditions are lacking). Therefore, it is likely easier to first treat a simple case. With this in mind, we focus in this article on a simple, yet relevant application-wise heterogeneity: the periodic one. We hope to pave the way for a possible future generalization.

Periodic spaces are likely the type of unbounded heterogeneous spaces we know best how to handle mathematically and thus a literature about scalar equations in periodic spaces has been developed during the past few years. Concerning scalar reaction–diffusion in periodic spaces and with "KPP"-type non-linearities, important results have been established recently by Berestycki and his collaborators [14, 16, 17] (see also Nadin [117, 118] in space-time periodic media). We will rely a lot on these scalar results. Regarding bistable non-linearities, we refer to the work of Ding, Hamel and Zhao [57] and Zlatos [148].

For the sake of simplicity, we will assume that diffusion and interspecific competition rates are constant. We expect our main ideas to be generalizable to systems with periodic diffusion and interspecific competition rates, but we also expect a lot of technical details to get messy and there might very well be some major issues. As a counterpart to this loss in generality, we will be able to treat a much larger class of growth–saturation terms since the explicit form of these will not be prescribed a priori. We will only require some reasonable "KPP non-linearities" assumptions.

Since our final goal is to study the limits of these pulsating fronts as the competition becomes infinite, we will only consider systems in which competition is the main underlying mechanism, that is for large values of the interspecific competition rate. A first consequence of this approach is that our system will always be bistable. A second consequence is that segregation phenomena will be involved quite frequently. Competition-induced segregation in homogeneous spaces have been a main center of interest of Dancer, Terracini and others since the nineties ([38, 41, 42, 45, 46, 47, 49, 52] among others). They basically confirmed the intuitive idea that competitors tend to live in different ecological niches.

To investigate the existence of bistable pulsating fronts connecting two extinction states, we have at our disposal recent abstract results about monotone semiflows stated by Weinberger [141] (monostable case) and Fang and Zhao [69] (bistable case). Even though both articles were mostly concerned by scalar equations, they were careful enough to include monotone systems, such as two-species competitive ones, in their framework. Notice that Yu and Zhao [146] used a similar framework to prove, in the weak competition case, the existence of monostable pulsating fronts connecting two extinction states despite the presence of an intermediate coextinction state (Weinberger's framework requires no intermediate stationary state) (see also Fang–Yu–Zhao [68] for a similar work in space-time periodic media).

The core idea of Fang and Zhao's theorem is as follows: provided a bistable monotone problem, if all intermediate stationary states are unstable and if they are invaded by the stable states, then bistable traveling waves do exist. While these hypotheses might be easily verified for some problems (say, scalar or space-homogeneous), in the case exposed here, real issues arise from the segregation phenomenon. Indeed, stable intermediate segregated periodic coexistence states might a priori exist. Therefore it is natural to wonder whether periodicity might induce some simple, yet relevant, sufficient condition to enforce the non-existence of segregated periodic coexistence states. We will indeed state one such condition and will show that this condition is moreover sufficient to guarantee that all remaining periodic stationary states are unstable and invaded by the stable ones.

The following pages will be organized as follows: in the first section, the core hypotheses and framework will be precisely formulated and the main results stated. The second section will be dedicated to the proof of the existence of pulsating front solutions; in particular, we will perform a quite thorough study of the stability of periodic coexistence states.

The study of the limit as $k \to +\infty$ of these pulsating fronts will be the object of the second part [GN18].

2.2 Preliminaries and main results

Let $d, k, \alpha, L > 0$, $C = (0, L) \subset \mathbb{R}$ and $(f_1, f_2) : [0, +\infty) \times \mathbb{R} \to \mathbb{R}^2$ L-periodic with respect to its second variable. For any $u : \mathbb{R}^2 \to [0, +\infty)$ and $i \in \{1, 2\}$, we refer to $(t, x) \mapsto f_i\left(u\left(t, x\right), x\right)$ as $f_i\left[u\right]$. Our interest lies in the following competition–diffusion problem:

$$\begin{cases} \partial_t u_1 = \partial_{xx} u_1 + u_1 f_1\left[u_1\right] - k u_1 u_2 \\ \partial_t u_2 = d \partial_{xx} u_2 + u_2 f_2\left[u_2\right] - \alpha k u_1 u_2 \end{cases} \quad (\mathcal{P}_k)$$

2.2.1 Preliminaries

2.2.1.1 Redaction conventions.

- Mirroring the definition of $f_1\left[u\right]$ and $f_2\left[u\right]$, for any function of two real variables f and any real-valued function u of two real variables, $f\left[u\right]$ will refer to $(t, x) \mapsto f\left(u\left(t, x\right), x\right)$. For any real-valued function u of one real variable, $f\left[u\right]$ will refer to $x \mapsto f\left(u\left(x\right), x\right)$. For any function f of one real variable and any real-valued function u of one or two real variables, $f\left[u\right]$ will simply refer to $f \circ u$.
- For the sake of brevity, although we could index everything ($(\mathcal{P})$, u_1, $u_2\dots$) on k and d, the dependencies on k or d will mostly be implicit and will only be made explicit when it definitely facilitates the reading.
- Since we consider the limit of this system when $k \to +\infty$, many (but finitely many) results will only be true when "k is large enough". Hence, we define by induction the positive

number $k^\star$, whose value is initially 1 and is updated each time a statement is only true when "k is large enough" in the following way: if the statement is true for any $k \geq k^\star$, the value of $k^\star$ is unchanged; if, conversely, there exists $K > k^\star$ such that the statement is true for any $k \geq K$ but false for any $k \in [k^\star, K)$, the value of $k^\star$ becomes that of K. In the text, we will indifferently write "for k large enough" or "provided $k^\star$ is large enough". Moreover, when k indexes appear, they a priori indicate that we are considering families indexed on (equivalently, functions defined on) $[k^\star, +\infty)$, but for the sake of brevity, when sequential arguments imply extractions of sequences and subsequences indexed themselves on increasing elements of $[k^\star, +\infty)^{\mathbb{N}}$, we will not explicitly define these sequences of indexes and will simply stick with the indexes k, reindexing along the course of the proof the considered objects. In such a situation, the statement "as $k \to +\infty$" should be understood unambiguously.

— Periodicity will always implicitly mean L-periodicity (unless explicitly stated otherwise). For any functional space X on $\mathbb{R}$, X_{per} denotes the subset of L-periodic elements of X.

— We will use the classical partial order on the space of functions from any $\Omega \subset \mathbb{R}^N$ to $\mathbb{R}$: $g \leq h$ if and only if, for any $x \in \Omega$, $g(x) \leq h(x)$, and $g < h$ if and only if $g \leq h$ and $g \neq h$. We recall that when $g < h$, there might still exists $x \in \Omega$ such that $g(x) = h(x)$. If, for any $x \in \Omega$, $g(x) < h(x)$, we use the notation $g \ll h$. In particular, if $g \geq 0$, we say that g is non-negative, if $g > 0$, we say that g is non-negative non-zero, and if $g \gg 0$, we say that g is positive (and we define similarly non-positive, non-positive non-zero and negative functions). Eventually, if $g_1 \leq h \leq g_2$, we write $h \in [g_1, g_2]$, if $g_1 < h < g_2$, we write $h \in (g_1, g_2)$, and if $g_1 \ll h \ll g_2$, we write $h \in \langle g_1, g_2 \rangle$.

— We will also use the partial order on the space of vector functions $\Omega \to \mathbb{R}^{N'}$ naturally derived from the preceding partial order. It will involve similar notations.

— The periodic principal eigenvalue of a second order elliptic operator $\mathcal{L}$ with periodic coefficients will be generically referred to as $\lambda_{1,per}(-\mathcal{L})$. Recall (from Berestycki–Hamel–Roques [16] for instance) that the periodic principal eigenvalue of $\mathcal{L}$ is the unique real number λ such that there exists a periodic function $\varphi \gg 0$ satisfying:

$$\begin{cases} -\mathcal{L}\varphi = \lambda\varphi \text{ in } \mathbb{R} \\ \|\varphi\|_{L^\infty(C)} = 1 \end{cases}$$

The Dirichlet principal eigenvalue of an elliptic operator $\mathcal{L}$ in a sufficiently smooth domain Ω will be referred to as $\lambda_{1,Dir}(-\mathcal{L}, \Omega)$. Since our framework is spatially one-dimensional, such elliptic operators will involve first and second derivatives with respect to the spatial variable x.

2.2.1.2 Hypotheses on the reaction.

For any $i \in \{1, 2\}$, we have in mind functions f_i such that the reaction term $uf_i[u]$ is of logistic type (also known as KPP type). At least, we want to cover the largest possible class of $(u, x) \mapsto \mu(x) - \nu(x)u$. This is made precise by the following assumptions.

($\mathcal{H}_1$) f_i is $\mathcal{C}^1$ with respect to its first variable up to 0 and Hölder-continuous with respect to its second variable with a Hölder exponent larger than or equal to $\frac{1}{2}$.

($\mathcal{H}_2$) There exists a constant $m_i > 0$ such that $f_i[0] \geq m_i$.

($\mathcal{H}_3$) f_i is decreasing with respect to its first variable and there exists $a_i > 0$ such that, if $u > a_i$, then for any $x \in \mathbb{R}$ $f_i(u, x) < 0$.

Remark. If f_i is in the class of all $(u,x) \mapsto \mu(x) - \nu(x) u$, then $\mu, \nu \in \mathcal{C}^{0,1/2}_{per}(\mathbb{R})$, $\mu \gg 0$, $\nu \gg 0$. More generally, from $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ and the periodicity of $f_i[0]$, it follows immediately that there exists a constant $M_i > m_i$ such that $f_i[0] \leq M_i$. Without loss of generality, we assume that m_i and M_i are optimal, that is $m_i = \min_{\overline{C}} f_i[0]$ and $M_i = \max_{\overline{C}} f_i[0]$.

We refer to $\max(M_1, M_2)$ (resp. $\min(m_1, m_2)$) as M (resp. m).

Furthermore, we need a coupled hypothesis on the pair (f_1, f_2).

$(\mathcal{H}_{freq})$ The constants d, M_1 and M_2 satisfy $L < \pi\left(\frac{1}{\sqrt{M_1}} + \sqrt{\frac{d}{M_2}}\right)$.

Remark. Even if this might not be clear right now, this is the key hypothesis. $(\mathcal{H}_{freq})$ means that, given a fixed amplitude, we consider high frequencies, or equivalently, given a fixed frequency, we consider low amplitudes. This sufficient condition for existence might be a bit relaxed but the best condition we can give is very verbose and only slightly better. See the proof of Proposition 2.13, which is where $(\mathcal{H}_{freq})$ plays its role.

2.2.2 Two main results and a conjecture

Using known results about scalar equations and periodic principal eigenvalues [16], the following lemma is quite straightforward (as will show Subsection 2.2.3.3).

Lemma 2.1. *Assume that f_1 and f_2 satisfy $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$.*

The set of all periodic stationary states of the problem $(\mathcal{P})$ contains $(0,0)$, which is unstable, and a pair $\{(\tilde{u}_1, 0), (0, \tilde{u}_2)\}$ with $(\tilde{u}_1, \tilde{u}_2) \in \mathcal{C}^2_{per}\left(\mathbb{R}, (0,+\infty)^2\right)$.

As usual in the literature concerning competitive systems, hereafter, the stationary states with exactly one null component are referred to as *extinction states* whereas the stationary states with no null component are referred to as *coexistence states*. The extinction states of $(\mathcal{P})$ are periodic and some of its coexistence states may be periodic as well.

Our contribution to the study of the stationary states is the following theorem.

Theorem 2.2. *Assume that f_1 and f_2 satisfy $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$ and that (f_1, f_2) satisfies $(\mathcal{H}_{freq})$.*

Then there exists $k^\star > 0$ such that, for any $k > k^\star$, each extinction state is locally asymptotically stable and any periodic coexistence state is unstable.

Furthermore, let $(u_{1,k}, u_{2,k})_{k>k^\star}$ be a family of $\mathcal{C}^2_{per}(\mathbb{R}, \mathbb{R}^2)$ such that, for any $k > k^\star$, $(u_{1,k}, u_{2,k})$ is an unstable periodic stationary state of $(\mathcal{P}_k)$. Then $(u_{1,k}, u_{2,k})$ converges in $\mathcal{C}_{per}(\mathbb{R}, \mathbb{R}^2)$ to $(0,0)$ as $k \to +\infty$.

Remark. We stress that we did not investigate the existence nor the countability of the subset of periodic coexistence states. We stress as well that we did not investigate at all aperiodic coexistence states. We believe that a sharper description of the set of stationary states of $(\mathcal{P})$ could follow from bifurcation arguments (see Hutson–Lou–Mischaikow [96] or Furter–López-Gómez [76]). Since it was not our point at all (instability of periodic coexistence states was only a required step toward existence of pulsating fronts), we chose to leave this subject as an open question.

Thanks to the previous theorem, it is then possible to prove the following existence theorem.

Theorem 2.3. *Assume that f_1 and f_2 satisfy $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$ and that (f_1, f_2) satisfies $(\mathcal{H}_{freq})$.*

Then there exists $k^\star > 0$ such that, for any $k > k^\star$, the problem $(\mathcal{P})$ admits a bistable pulsating front solution connecting the two extinction states.

To end this subsection, let us present an important conjecture about the existence problem and about the sharpness of $(\mathcal{H}_{freq})$. We did not address this question but hopefully others will.

Conjecture. *Neither $(\mathcal{H}_{freq})$ nor the nonexistence of a stable periodic coexistence state are necessary conditions for the existence of a bistable pulsating front solution connecting the two extinction states.*

Furthermore, there exists a non-empty set of parameters $(L, d, \alpha, k, f_1, f_2)$ such that no such pulsating front exists.

We point out that, according to the present work, any of the following two conditions enforces that either $(\mathcal{H}_{freq})$ is not satisfied or $k \leq k^\star$:

— the existence of a stable periodic coexistence state;

— the nonexistence of a bistable pulsating front solution.

Moreover, our work will show that, if $k > k^\star$, any stable periodic coexistence state has the "close to segregation" form (which will be rigorously defined later on; roughly speaking, "close to segregation" periodic coexistence states converge as $k \to +\infty$ to a non-trivial periodic coexistence state satisfying $u_1 u_2 = 0$). This important property might be the starting point of a future work on the preceding conjecture.

2.2.3 A few more preliminaries

2.2.3.1 Compact embeddings of Hölder spaces

We recall a well-known result of functional analysis.

Proposition 2.4. *Let $(a, a') \in (0, +\infty)^2$ and n, n', β, β' such that $(a, a') = (n + \beta, n' + \beta')$, n and n' are non-negative integers and β and β' are in $(0, 1]$.*

If $a \leq a'$, then the canonical embedding $i : \mathcal{C}^{n',\beta'}(C) \hookrightarrow \mathcal{C}^{n,\beta}(C)$ is continuous and compact.

It will be clear later on that this problem naturally involves uniform bounds in $\mathcal{C}^{0,1/2}$ and in $\mathcal{C}^{2,1/2}$. Therefore, we fix once and for all $\beta \in \left(0, \frac{1}{2}\right)$ and we will use systematically the compact embeddings $\mathcal{C}^{n,1/2} \hookrightarrow \mathcal{C}^{n,\beta}$, meaning that uniform bounds in $\mathcal{C}^{n,1/2}$ yield relative compactness in $\mathcal{C}^{n,\beta}$.

2.2.3.2 Existence and uniqueness for the evolution system

Proposition 2.5. *Let $k > 0$. Equipped with an initial non-negative condition $(u_{1,0}, u_{2,0}) \in \mathcal{C}^{0,1/2}\left(\mathbb{R}, \mathbb{R}^2\right)$, the problem $(\mathcal{P})$ is well-posed: there exists a unique non-negative entire solution $(u_1, u_2) \in \mathcal{C}^{1,1/4}\left([0, +\infty), \mathcal{C}^{2,1/2}\left(\mathbb{R}, \mathbb{R}^2\right)\right)$.*

Furthermore, if $(u_{1,0}, u_{2,0}) > 0$, then $(u_1, u_2) \gg 0$, and if $(u_{1,0}, u_{2,0}) \in \mathcal{C}_{per}\left(\mathbb{R}, \mathbb{R}^2\right)$, then $(u_1, u_2) \in \mathcal{C}^1\left([0, +\infty), \mathcal{C}^2_{per}\left(\mathbb{R}, \mathbb{R}^2\right)\right)$.

Remark. We do not give a fully detailed proof of this statement. Ideas similar to those given in Berestycki–Hamel–Roques [16, Remark 2.7] suffice. The existence of solutions for the truncated system in $(-n, n)$ with Dirichlet boundary conditions can be proved with Pao's super- and sub-solutions theorem for competitive systems [124].

2.2.3.3 Extinction states

Lemma 2.6. *The periodic principal eigenvalues of* $-\frac{d^2}{dx^2} - f_1[0]$ *and* $-d\frac{d^2}{dx^2} - f_2[0]$ *are negative.*

Proof. This follows from $(\mathcal{H}_2)$ and the monotonicity of the periodic principal eigenvalue with respect to the zeroth order term of the elliptic operator. Indeed, for instance:

$$\lambda_{1,per}\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} - f_1[0]\right) \leq \lambda_{1,per}\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} - m_1\right) = -m_1 < 0.$$

□

From this lemma and hypotheses $(\mathcal{H}_1)$ and $(\mathcal{H}_3)$, a fundamental result from Berestycki–Hamel–Roques [16] can be applied.

Theorem 2.7. *For any* $\delta > 0$ *and any* $i \in \{1,2\}$*, the equation:*

$$-\delta z'' = z f_i[z]$$

admits a unique positive solution in $\mathcal{C}^2_{per}(\mathbb{R})$*.*

Hereafter, $\tilde{u}_1$ and $\tilde{u}_2$ are the respective unique positive periodic solutions of:

$$-z'' = z f_1[z],$$

$$-dz'' = z f_2[z].$$

$(\tilde{u}_1, 0)$ and $(0, \tilde{u}_2)$ are indeed the extinction states of any $(\mathcal{P}_k)$.

2.2.3.4 Monotone evolution system

One of the most important specificities of two-species competitive systems is that, up to a slight transformation, they are monotone systems. It is the key behind the results of Fang–Zhao [69] and Weinberger [141]. Let us recall this transformation.

Lemma 2.8. *Let* $J : z \mapsto \tilde{u}_2 - z$*, for any* $z \in \mathcal{C}^2_{per}(\mathbb{R})$ *or* $z \in \mathcal{C}^1\left([0,+\infty), \mathcal{C}^2_{per}(\mathbb{R})\right)$ *(with a slight abuse of notation). Let* $k > k^\star$ *and let* (u_1, u_2) *be a solution of* $(\mathcal{P})$ *and* $v_2 = J(u_2)$*.*

Then (u_1, v_2) *satisfies the following cooperative problem with periodicity conditions:*

$$\begin{cases} \partial_t u_1 - \partial_{xx} u_1 = u_1 f_1[u_1] + k u_1(-\tilde{u}_2 + v_2) \\ \partial_t v_2 - d\partial_{xx} v_2 = \tilde{u}_2 f_2[\tilde{u}_2] - (\tilde{u}_2 - v_2) f_2[\tilde{u}_2 - v_2] + \alpha k u_1(\tilde{u}_2 - v_2). \end{cases} \quad (\mathcal{M}_k)$$

Corollary 2.9. *Any solution* (u_1, u_2) *of* $(\mathcal{P})$ *with initial condition* $(0,0) < (u_{1,0}, u_{2,0}) < (\tilde{u}_1, \tilde{u}_2)$ *satisfies* $(0,0) \ll (u_1, u_2) \ll (\tilde{u}_1, \tilde{u}_2)$*.*

2.2.3.5 Segregated reaction terms

As $k \to +\infty$, the following functions will naturally appear:

$$\eta : (z,x) \mapsto f_1\left(\frac{z}{\alpha}, x\right) z^+ - \frac{1}{d} f_2\left(-\frac{z}{d}, x\right) z^-,$$

$$\gamma : (z,x) \mapsto f_1(0,x) z^+ - \frac{1}{d} f_2(0,x) z^-,$$

where $z^+ = \max(z,0)$ and $z^- = -\min(z,0)$ so that $z = z^+ - z^-$.

2.2.3.6 Derivatives of the reaction terms

We will denote g_i the partial derivative of $(u,x) \mapsto uf_i(u,x)$ with respect to u:

$$g_i : (u,x) \mapsto f_i(u,x) + u\partial_1 f_i(u,x) \text{ for all } i \in \{1,2\}.$$

2.3 Existence of pulsating fronts

2.3.1 Aim: Fang–Zhao's theorem

We recall that, for any $k > k^\star$ and any $t > 0$, the Poincaré's map Q_t associated with $(\mathcal{M})$ is defined as the operator:

$$Q_t : \mathcal{C}\left(\mathbb{R},\mathbb{R}^2\right) \cap [(0,0),(\tilde{u}_1,\tilde{u}_2)] \to \mathcal{C}\left(\mathbb{R},\mathbb{R}^2\right) \cap [(0,0),(\tilde{u}_1,\tilde{u}_2)]$$

which associates with some initial condition $(u_{1,0}, v_{2,0})$ the solution (u_1, v_2) of $(\mathcal{M})$evaluated at time $t > 0$.

From Fang and Zhao [69], we know that $(\mathcal{M})$ admits a pulsating front solution connecting $(\tilde{u}_1,\tilde{u}_2)$ to $(0,0)$ if:

1. $(0,0)$ and $(\tilde{u}_1,\tilde{u}_2) \gg (0,0)$ are locally asymptotically stable periodic stationary states of $(\mathcal{M})$ and all intermediate periodic stationary states of $(\mathcal{M})$ are unstable;
2. for any intermediate periodic stationary state (u_1,v_2), the sum of the spreading speeds associated with front-like initial data connecting respectively $(\tilde{u}_1,\tilde{u}_2)$ to (u_1,v_2) and (u_1,v_2) to $(0,0)$ is positive (notice that these sub-problems are of monostable type);
3. and if, for any $t > 0$, Q_t satisfies the following hypotheses:
   a) Q_t is spatially periodic;
   b) Q_t is continuous with respect to the topology of the locally uniform convergence;
   c) Q_t is strongly monotone, in the sense that if $(u_1,v_2) > \left(u^1,v^2\right)$, then:
   $$Q_t((u_1,v_2)) \gg Q_t\left(\left(u^1,v^2\right)\right);$$
   d) Q_t is compact with respect to the topology of the locally uniform convergence;

It is quite standard to check that the last four hypotheses are indeed satisfied. The verification of the first two, on the contrary, is the object of the remaining of this paper.

2.3.2 Stability of all extinction states

Proposition 2.10. *Provided $k^\star$ is large enough, $(\tilde{u}_1,0)$ and $(0,\tilde{u}_2)$ are locally asymptotically stable.*

Remark. For the case $k=1$, the proof of the local asymptotic stability of the extinction states was done by Dockery and his coauthors [58] with the help of Mora's theorem [113]. It works here too with a very slight adaptation; we give the proof for the sake of completeness.

Proof. Thanks to Mora's theorem [113], we know that $(\tilde{u}_1,0)$ is asymptotically stable if the periodic principal eigenvalue of the elliptic part of the monotone problem $(\mathcal{M})$ linearized at $(\tilde{u}_1,\tilde{u}_2) = (u,J(0))$ is positive. Therefore we consider the differential operator $\mathcal{A}_{(\tilde{u}_1,0)} : \mathcal{C}^2_{per}(\mathbb{R}) \to \mathcal{C}_{per}(\mathbb{R})$ defined as:

$$\mathcal{A}_{(\tilde{u}_1,0)} = \begin{pmatrix} \frac{\mathrm{d}^2}{\mathrm{d}x^2} + g_1[\tilde{u}_1] & k\tilde{u}_1 \\ 0 & d\frac{\mathrm{d}^2}{\mathrm{d}x^2} + f_2[0] - \alpha k\tilde{u}_1 \end{pmatrix}$$

From the special "triangular" form of $\mathcal{A}_{(\tilde{u}_1,0)}$, it is clear that:

$$\min\left(\mathrm{sp}\left(-\mathcal{A}_{(\tilde{u}_1,0)}\right)\right) = \min\left(\lambda_{1,per}\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} - g_1\left[\tilde{u}_1\right]\right), \lambda_{1,per}\left(-d\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \left(f_2\left[0\right] - \alpha k \tilde{u}_1\right)\right)\right).$$

By monotonicity of the periodic principal eigenvalue and $(\mathcal{H}_3)$, we obtain:

$$\lambda_{1,per}\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} - g_1\left[\tilde{u}_1\right]\right) > \lambda_{1,per}\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} - f_1\left[\tilde{u}_1\right]\right).$$

For any k large enough, $f_2\left[0\right] - \alpha k \tilde{u}_1 < f_2\left[\tilde{u}_2\right]$ holds, so that:

$$\lambda_{1,per}\left(-d\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \left(f_2\left[0\right] - \alpha k \tilde{u}_1\right)\right) > \lambda_{1,per}\left(-d\frac{\mathrm{d}^2}{\mathrm{d}x^2} - f_2\left[\tilde{u}_2\right]\right).$$

Moreover, from the equation solved by $\tilde{u}_1$, $\tilde{u}_1$ is actually an eigenfunction for the following eigenvalue:

$$\lambda_{1,per}\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} - f_1\left[\tilde{u}_1\right]\right) = 0.$$

Similarly,

$$\lambda_{1,per}\left(-d\frac{\mathrm{d}^2}{\mathrm{d}x^2} - f_2\left[\tilde{u}_2\right]\right) = 0.$$

Thus:

$$\lambda_{1,per}\left(-\mathcal{A}_{(\tilde{u}_1,0)}\right) > 0.$$

The same proof holds for $(0, \tilde{u}_2)$. □

2.3.3 Instability of all periodic coexistence states

In this subsection, we prove that $(\mathcal{M})$ admits no stable periodic stationary states in $\langle (0,0), (\tilde{u}_1, \tilde{u}_2)\rangle$.
For any $k > k^\star$, let:

$$S_k \subset \mathcal{C}^2_{per}\left(\mathbb{R}, \mathbb{R}^2\right)$$

be the set of periodic solutions of the following problem:

$$\begin{cases} -u_1'' = u_1 f_1\left[u_1\right] - k u_1 u_2 \\ -d u_2'' = u_2 f_2\left[u_2\right] - \alpha k u_1 u_2 \\ u_1 \in \langle 0, \tilde{u}_1\rangle \\ u_2 \in \langle 0, \tilde{u}_2\rangle. \end{cases}$$

Any $(u_1, u_2) \in S$ is a periodic coexistence state.

2.3.3.1 Basic properties of periodic coexistence states

Lemma 2.11. *Let $k > k^\star$. Any $(u_1, u_2) \in S$ satisfies:*

$$\begin{cases} k \min u_2 \leq \max f_1\left[\max u_1\right] \\ \alpha k \min u_1 \leq \max f_2\left[\max u_2\right] \\ \min f_1\left[\min u_1\right] \leq k \max u_2 \\ \min f_2\left[\min u_2\right] \leq \alpha k \max u_1, \end{cases}$$

each extrema being implicitly over $\overline{C}$.

Proof. We only prove the first inequality, the three others being proved similarly.

Let $\overline{x} \in \overline{C}$ such that $u_1(\overline{x}) = \max u_1$. Since $u_1 \in \mathcal{C}^2(\mathbb{R})$, $u_1''(\overline{x}) \leq 0$, that is:

$$\max u_1 f_1 [\max u_1] \geq \max u_1 k u_2 (\overline{x}).$$

Since $u_1 > 0$, we can divide by $\max u_1$. The claimed result easily follows. □

Remark. This lemma will be used together with $m > 0$ to prove that ku_1 and ku_2 stay non-zero as $k \to +\infty$. Thus, for the forthcoming study, it is not sufficient to merely assume that $\lambda_{1,per}\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} - f_1[0]\right)$ and $\lambda_{1,per}\left(-d\frac{\mathrm{d}^2}{\mathrm{d}x^2} - f_2[0]\right)$ are negative (as was done for instance by Dockery and his collaborators [58]).

Proposition 2.12. *As $k \to +\infty$, the family $(S_k)_{k>k^\star}$ is relatively compact in $\mathcal{C}^{0,\beta}_{per}(\mathbb{R},\mathbb{R}^2)$. $(0,0)$ is one of its limit points. Any other limit point $(u_{1,seg}, u_{2,seg}) \in \mathcal{C}^{0,\beta}_{per}(\mathbb{R},\mathbb{R}^2)$ is called a periodic segregated state and is such that $\alpha u_{1,seg} - du_{2,seg}$ is a non-zero sign-changing solution in $\mathcal{C}^{2,\beta}_{per}(\mathbb{R})$ of the following elliptic equation:*

$$-z'' = \eta[z].$$

Proof. Let $k > k^\star$.

Multiplying by $u_{1,k}$ the first equation of the stationary system and integrating over C yields easily:

$$\begin{aligned} \|u'_{1,k}\|_{L^2(C)} &\leq M_1 \|u_{1,k}\|_{L^2(C)} \\ &\leq M_1 \|\tilde{u}_1\|_{L^2(C)}, \end{aligned}$$

whence, for all $(x,y) \in C^2$:

$$|u_{1,k}(x) - u_{1,k}(y)| \leq M_1 \|\tilde{u}_1\|_{L^2(C)} |x-y|^{1/2}.$$

Moreover, $\|u_{1,k}\|_{L^\infty(C)} \leq \|\tilde{u}_1\|_{L^\infty(C)}$, and therefore $(u_{1,k})_{k>k^\star}$ is uniformly bounded in $\mathcal{C}^{0,1/2}(C)$ and relatively compact in $\mathcal{C}^{0,\beta}(C)$. The same proof holds for $(u_2)_{k>k^\star}$.

Let $(u_{1,\infty}, u_{2,\infty}) \in \mathcal{C}^{0,\beta}_{per}(\mathbb{R},\mathbb{R}^2)$ be a limit point of $(S_k)_{k>k^\star}$. There exists a sequence of periodic coexistence states $((u_{1,k}, u_{2,k}))_{k>k^\star}$ whose limit in $\mathcal{C}^{0,\beta}_{per}(\mathbb{R},\mathbb{R}^2)$ is $(u_{1,\infty}, u_{2,\infty})$. By elliptic regularity and thanks to the following equation:

$$-\alpha u''_{1,k} + dv''_{2,k} = \alpha u_{1,k} f_1[u_{1,k}] - u_{2,k} f_2[u_{2,k}],$$

which holds for any $k > k^\star$ and is obtained by linear combination of the equations of the stationary system, $(\alpha u_{1,k} - du_{2,k})$ converge in $\mathcal{C}^{2,\beta}_{per}(\mathbb{R})$ to $v = \alpha u_{1,\infty} - du_{2,\infty} \in \mathcal{C}^{2,\beta}_{per}(\mathbb{R})$.

Multiplying by a test function $\varphi \in \mathcal{D}(\mathbb{R})$ the equation defining $u_{1,k}$, integrating and dividing by k, we obtain easily that $(u_{1,k}u_{2,k})$ converges as $k \to +\infty$ in $\mathcal{D}'(\mathbb{R})$ to 0. Hence $u_{1,\infty}u_{2,\infty} = 0$ and then $\alpha u_{1,\infty} = v^+$ and $du_{2,\infty} = v^-$. In particular, v satisfies as claimed:

$$-v'' = \eta[v]$$

Let:

$$\begin{aligned} C_1 &= \{x \in C \mid v(x) > 0\}, \\ C_2 &= \{x \in C \mid v(x) < 0\}, \\ \Gamma &= \{x \in C \mid v(x) = 0\}, \end{aligned}$$

so that:

$$C \subset C_1 \cup C_2 \cup \Gamma \subset \overline{C}.$$

Exactly four cases are a priori possible:

1. $C_1 = C$: then by continuity $v = \alpha u_{1,\infty}$ in $\overline{C}$ whereas $u_{2,\infty} = 0$ in $\overline{C}$, hence $u_{1,\infty} \in \mathcal{C}^{2,\beta}_{per}(\mathbb{R})$ is a non-negative non-zero solution of
$$-u''_{1,\infty} = u_{1,\infty} f_1 [u_{1,\infty}]$$
in $\mathbb{R}$, and eventually by the elliptic strong minimum principle $u_{1,\infty} \gg 0$, meaning that $u_{1,\infty} = \tilde{u}_1$, and $C_2 = \Gamma = \emptyset$;
2. $C_2 = C$: then similarly $C_1 = \Gamma = \emptyset$, $u_{1,\infty} = 0$ and $u_{2,\infty} = \tilde{u}_2$;
3. $C_1 \neq \emptyset$ and $C_2 \neq \emptyset$.
4. $C_1 = \emptyset$ and $C_2 = \emptyset$: $\Gamma = C$, $u_{1,\infty}$ and $v_{2,\infty}$ are uniformly 0;

It is easily seen that Lemma 2.11 excludes the cases 1 (use the second inequality) and 2 (use the first inequality). ☐

Proposition 2.13. *The following set equalities hold:*
$$\left\{z \in \mathcal{C}^2_{per}(\mathbb{R}) \mid -z'' = \gamma[z]\right\} = \{0\},$$
$$\left\{z \in \mathcal{C}^2_{per}(\mathbb{R}) \mid -z'' = \eta[z]\right\} = \{-d\tilde{u}_2, 0, \alpha\tilde{u}_1\}.$$

Proof. In the γ case, solutions of constant sign are excluded by:
$$\lambda_{1,per}\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} - f_1[0]\right) < 0,$$
$$\lambda_{1,per}\left(-d\frac{\mathrm{d}^2}{\mathrm{d}x^2} - f_2[0]\right) < 0.$$

In the η case, solutions of constant sign are unique (see Berestycki–Hamel–Roques [16]) and are exactly $\alpha\tilde{u}_1$ and $-d\tilde{u}_2$. It only remains to prove that non-zero sign-changing solutions are excluded, and up to a shift of C it suffices to prove that non-zero sign-changing solutions which are equal to 0 at 0 and L are excluded.

For any $x \in \mathbb{R}$, any $f \in \mathcal{C}^0_{per}(\mathbb{R}, [m, M])$ and any $\delta \in \{1, d\}$, let $R(x, f, \delta) > 0$ such that:
$$\lambda_{1,Dir}\left(-\delta\frac{\mathrm{d}^2}{\mathrm{d}x^2} - f, B(x, R(x, f, \delta))\right) = 0.$$

Since the following function:
$$R \mapsto \lambda_{1,Dir}\left(-\delta\frac{\mathrm{d}^2}{\mathrm{d}x^2} - f, B(x, R)\right)$$
is continuous, decreasing and has positive and negative values (its limits as $R \to 0$ or $R \to +\infty$ are respectively $+\infty$ and $\lambda_{1,per}\left(-\delta\frac{\mathrm{d}^2}{\mathrm{d}x^2} - f\right) < 0$, as proved in [16]), $R(x, f, \delta)$ is uniquely defined. Since $\lambda_{1,Dir}\left(-\delta\frac{\mathrm{d}^2}{\mathrm{d}x^2} - f, B(x, R)\right)$ is non-increasing with respect to f and decreasing with respect to R, it is easy to check that $f \mapsto R(x, f, \delta)$ is non-increasing.

Remark that $R(x, f, \delta)$ and $\lambda_{1,Dir}\left(-\delta\frac{\mathrm{d}^2}{\mathrm{d}x^2} - f, B(x, R(x, f, \delta))\right)$ do not depend on x if f does not depend on x. Remark that, in such a case, $R(0, f, \delta)$ can be easily determined analytically and is equal to $\frac{\pi}{2}\sqrt{\frac{\delta}{f}}$.

With these notations, $(\mathcal{H}_{freq})$ means:

$$L < 2\left(R\left(0, M_1, 1\right) + R\left(0, M_2, d\right)\right).$$

Let z be a solution of $-z'' = \gamma[z]$ or a solution of $-z'' = \eta[z]$. Let:

$$C_+ = z^{-1}\left((0,+\infty)\right) \cap C,$$
$$C_- = z^{-1}\left((-\infty,0)\right) \cap C.$$

Assume by contradiction that both are non-empty. Let n be the number of zeros of z in C. Then:

- in virtue of the Hopf lemma, of:

  $$\min\left(\min_{x\in\overline{C}} R\left(x, f_1[0], 1\right), \min_{x\in\overline{C}} R\left(x, f_2[0], d\right)\right) > 0$$

  and of the continuity of z, n is finite and odd, say $n = 2p+1$ with p a non-negative integer, and C_+ and C_- both have precisely $p+1$ connected components, each of them being a one-dimensional ball (that is an interval); let $\left(x_i^+\right)_{1\leq i\leq p+1}$ (resp. $\left(x_i^-\right)_{1\leq i\leq p+1}$) be the ordered centers of the connected components of C_+ (resp. C_-);
- in the γ case:

  $$\begin{aligned} |C_+| &= 2\sum_{i=1}^{p+1} R\left(x_i^+, f_1[0], 1\right) \\ &\geq 2\sum_{i=1}^{p+1} R\left(x_i^+, M_1, 1\right) \\ &\geq 2\left(p+1\right) R\left(0, M_1, 1\right) \\ &\geq 2R\left(0, M_1, 1\right), \end{aligned}$$

  and similarly:

  $$\begin{aligned} |C_-| &= 2\sum_{i=1}^{p+1} R\left(x_i^-, f_2[0], d\right) \\ &\geq 2R\left(0, M_2, d\right), \end{aligned}$$

  whence we get the contradiction;
- in the η case:

  $$\begin{aligned} |C_+| &= 2\sum_{i=1}^{p+1} R\left(x_i^+, f_1\left[\frac{z}{\alpha}\right], 1\right) \\ &\geq 2\sum_{i=1}^{p+1} R\left(x_i^+, f_1[0], 1\right), \end{aligned}$$

  $$\begin{aligned} |C_-| &= 2\sum_{i=1}^{p+1} R\left(x_i^-, f_2\left[-\frac{z}{d}\right], d\right) \\ &\geq 2\sum_{i=1}^{p+1} R\left(x_i^-, f_2[0], d\right) \end{aligned}$$

  yield a similar contradiction.

□

Corollary 2.14. *Any family* $(u_{1,k}, u_{2,k})_{k>k^\star}$ *of periodic coexistence states converges in* $\mathcal{C}^{0,\beta}_{per}\left(\mathbb{R},\mathbb{R}^2\right)$ *as* $k \to +\infty$ *to* $(0,0)$.

Remark. This result has a very natural interpretation from an ecological point of view: if the wavelength of the distribution of resources is small enough, or if the resources are rare enough even in the most favorable areas, the species are not able to settle periodically in a favorable habitat smaller than the wavelength. Either one of them is strong enough to overcome unfavorable areas while eliminating the competitor and then it settles in the whole habitat, either both go extinct. Basically, at a given average intrinsic growth rate, the more fragmented the habitat is, the higher the chances of extinction are.

Lemma 2.15. *There exists* $R_1 \in (0,+\infty)$ *and* $R_2 \in (R_1,+\infty)$ *such that, provided* $k^\star$ *is large enough, for any* $k > k^\star$ *and any* $(u_{1,k}, u_{2,k}) \in S_k$*:*

$$R_1 \leq \frac{\|u_{2,k}\|_{L^\infty(C)}}{\alpha\|u_{1,k}\|_{L^\infty(C)}} \leq R_2.$$

Remark. Proof inspired by Dancer–Du [45, Lemma 2.1].

Proof. By contradiction, assume that there exists a sequence of periodic coexistence states $((u_{1,k}, u_{2,k}))_{k>k^\star}$ such that $\left(\frac{\|u_{2,k}\|_{L^\infty(C)}}{\alpha\|u_{1,k}\|_{L^\infty(C)}}\right)_{k>k^\star}$ is neither bounded from above nor from below by a positive constant. By symmetry, we can assume without loss of generality that it is not bounded from below by a positive constant. Up to extraction, $\frac{\|u_{2,k}\|_{L^\infty(C)}}{\alpha\|u_{1,k}\|_{L^\infty(C)}} \to 0$ as $k \to +\infty$.

Suppose first that $\left(\alpha k\|u_{1,k}\|_{L^\infty(C)}\right)_{k>k^\star}$ is bounded. Necessarily, $k\|u_{2,k}\|_{L^\infty(C)} \to 0$ as $k \to +\infty$.

For any non-negative $f \in \mathcal{C}(\mathbb{R},\mathbb{R})$, the following problem:

$$-z'' = zf_1[z] - zf$$

with periodicity conditions has a unique positive periodic solution z_f if and only if:

$$\lambda_{1,per}\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} - (f_1 - f)\right) < 0$$

(see Berestycki–Hamel–Roques [16]). Moreover, z_f depends continuously on f as a map from $\mathcal{C}_{per}(C)$ into itself (see Berestycki–Rossi [22]). Hence $u_{1,k} = z_{ku_{2,k}} \to z_0$ as $k \to +\infty$, where z_0 solves:

$$-z_0'' = z_0 f_1[z_0]$$

with periodicity conditions (that is $u[0] = \tilde{u}_1$). Since $k\|\tilde{u}_1\|_{L^\infty(C)} \to +\infty$, we get a contradiction.

Hence $\left(\alpha k\|u_{1,k}\|_{L^\infty(C)}\right)_{k>k^\star}$ is unbounded. Up to extraction, we can assume that $k\|u_{1,k}\|_{L^\infty(C)} \to +\infty$.

For any $k > k^\star$, let $\hat{u}_{1,k} = \frac{u_{1,k}}{\|u_{1,k}\|_{L^\infty(C)}}$, $\hat{u}_{2,k} = \frac{u_{2,k}}{\|u_{2,k}\|_{L^\infty(C)}}$. Clearly, $(\hat{u}_{1,k}, \hat{u}_{2,k})$ satisfies:

$$\begin{cases} -\hat{u}_{1,k}'' = \hat{u}_{1,k} f_1\left[\|u_{1,k}\|_{L^\infty(C)}\hat{u}_{1,k}\right] - k\|u_{2,k}\|_{L^\infty(C)}\hat{u}_{1,k}\hat{u}_{2,k} \\ -d\hat{u}_{2,k}'' = \hat{u}_{2,k} f_2\left[\|u_{2,k}\|_{L^\infty(C)}\hat{u}_{2,k}\right] - \alpha k\|u_{1,k}\|_{L^\infty(C)}\hat{u}_{1,k}\hat{u}_{2,k}. \end{cases}$$

From there, it follows with the same estimates as in the proof of Proposition 2.12 that $\hat{u}_{1,k}$ and $\hat{u}_{2,k}$ converge up to extraction in $\mathcal{C}^{0,\beta}_{per}(\mathbb{R})$. Let $\hat{u}_{1,\infty}$ and $\hat{u}_{2,\infty}$ be their limits; for any $i \in \{1,2\}$ $\|\hat{u}_{i,\infty}\|_{L^\infty(C)} = 1$, hence $u_{i,\infty} \neq 0$.

Then, we consider the system above in $\mathcal{D}'(C)$. Let $\varphi \in \mathcal{D}(C)$ and use it as a test function. On the second line, we see that, since:

$$\int \left(d\hat{u}''_{2,k} + \hat{u}_{2,k} f_2 \left[\|u_{2,k}\|_{L^\infty(C)} \hat{u}_{2,k}\right]\right) \varphi$$

is k-uniformly bounded, the same is true of:

$$\int \alpha k \|u_{1,k}\|_{L^\infty(C)} \hat{u}_{1,k} \hat{u}_{2,k} \varphi.$$

Thus:

$$\int k \|u_{2,k}\|_{L^\infty(C)} \hat{u}_{1,k} \hat{u}_{2,k} \varphi = \frac{\|u_{2,k}\|_{L^\infty(C)}}{\alpha \|u_{1,k}\|_{L^\infty(C)}} \int \left(\alpha k \|u_{1,k}\|_{L^\infty(C)} \hat{u}_{1,k} \hat{u}_{2,k} \varphi\right) \to 0$$

Therefore, considering the first line, we see that, by dominated convergence, the limit satisfies in the distributional sense:

$$-\hat{u}''_{1,\infty} = \hat{u}_{1,\infty} f_1 \left[\|u_{1,\infty}\|_{L^\infty(C)} \hat{u}_{1,\infty}\right].$$

Since $\hat{u}_{1,\infty}$ is in $\mathcal{C}^{0,\beta}_{per}(\mathbb{R})$, it is actually a solution in $\mathcal{C}^{2,\beta}_{per}(\mathbb{R})$ by classical elliptic regularity. In virtue of the elliptic strong minimum principle, $\hat{u}_{1,\infty} \gg 0$. But it is also true, using the same arguments as before, that $\hat{u}_{1,\infty} \hat{u}_{2,\infty} = 0$, hence $\hat{u}_{2,\infty} = 0$, which is indeed a contradiction. □

Lemma 2.16. *Let $\left((u_{1,k}, u_{2,k})\right)_{k>k^\star}$ be a sequence of periodic coexistence states. Then $\left((ku_{1,k}, ku_{2,k})\right)_{k>k^\star}$ is k-uniformly bounded in $L^\infty(C)$.*

Proof. From Lemma 2.15, it suffices to assume that there exists a sequence $\left((u_1, u_2)\right)_{k>k^\star}$ such that $k\|u_{1,k}\|_{L^\infty(C)} \to +\infty$ as $k \to +\infty$ and to get a contradiction.

With the same notations as in the proof of Lemma 2.15, up to extraction we can assume that $\hat{u}_{1,k} \to \hat{u}_{1,\infty}$ and $\hat{u}_{2,k} \to \hat{u}_{2,\infty}$ in $\mathcal{C}^{0,\beta}_{per}(\mathbb{R})$. We have for any $i \in \{1,2\}$ $\|\hat{u}_{i,\infty}\|_{L^\infty(C)} = 1$, hence $u_{i,\infty} \neq 0$. Considering the limit of the equation satisfied by $\hat{u}_{2,k}$ in $\mathcal{D}'(C)$ shows that $\hat{u}_{1,\infty} \hat{u}_{2,\infty} = 0$. Thanks to Lemma 2.15, up to extraction, we can assume that there exists $l > 0$ such that $\frac{\alpha \|u_{1,k}\|_{L^\infty(C)}}{\|u_{2,k}\|_{L^\infty(C)}} \to l$. Moreover, considering the equation satisfied by $\hat{u}_{1,k}$ in $\mathcal{D}'(C)$ shows that, for any $\varphi \in \mathcal{D}(C)$:

$$\int k \|u_{2,k}\|_{L^\infty(C)} \hat{u}_{1,k} \hat{u}_{2,k} \varphi$$

is k-uniformly bounded.

Multiplying the equation defining $\hat{u}_{1,k}$ by l and subtracting from it the equation defining $\hat{u}_{2,k}$ yields:

$$\begin{aligned} -l\hat{u}''_{1,k} + d\hat{u}''_{2,k} &= l\hat{u}_{1,k} f_1 \left[\|u_{1,k}\|_{L^\infty(C)} \hat{u}_{1,k}\right] - \hat{u}_{2,k} f_2 \left[\|u_{2,k}\|_{L^\infty(C)} \hat{u}_{2,k}\right] \\ &\quad + \left(\frac{\alpha \|u_{1,k}\|_{L^\infty(C)}}{\|u_{2,k}\|_{L^\infty(C)}} - l\right) k \|u_{2,k}\|_{L^\infty(C)} \hat{u}_{1,k} \hat{u}_{2,k}. \end{aligned}$$

Considering it in $\mathcal{D}'(C)$, passing to the limit (with, in virtue of Corollary 2.14, $\|u_{i,k}\|_{L^\infty(C)} \to 0$) and defining $v = l\hat{u}_{1,\infty} - d\hat{u}_{2,\infty}$, it becomes:

$$-v'' = \gamma[v].$$

By classical elliptic regularity, v is actually a solution in $\mathcal{C}^{2,\beta}_{per}(\mathbb{R})$. Then Proposition 2.13 implies $l\hat{u}_{1,\infty} = d\hat{u}_{2,\infty}$, but together with $\hat{u}_{1,\infty} \hat{u}_{2,\infty} = 0$ and the fact that the pair $(u_{1,\infty}, u_{2,\infty})$ is non-zero, this is a contradiction. □

Lemma 2.17. *Provided $k^\star$ is large enough, the following lower bound holds:*

$$\inf_{k>k^\star}\inf_{(u_1,u_2)\in S_k}\min\left\{\min_{\overline{C}}(ku_1),\min_{\overline{C}}(ku_2)\right\}>0$$

Proof. Let $((u_{1,k},u_{2,k}))_{k>k^\star}$. For any $i\in\{1,2\}$ and any $k>k^\star$, let $U_{i,k}=ku_{i,k}$. $(U_{1,k},U_{2,k})$ satisfies the following system:

$$\begin{cases} -U_{1,k}''=U_{1,k}f_1\left[\frac{U_{1,k}}{k}\right]-U_{1,k}U_{2,k}\\ -dU_{2,k}''=U_{2,k}f_2\left[\frac{U_{2,k}}{k}\right]-\alpha U_{1,k}U_{2,k}.\end{cases}$$

Since $U_{1,k}$ and $U_{2,k}$ are k-uniformly bounded in $L^\infty(C)$ in virtue of Lemma 2.16, we can prove with the same arguments as before that, for any $i\in\{1,2\}$ and up to extraction, $U_{i,k}$ converges in $\mathcal{C}^{0,\beta}_{per}(\mathbb{R})$ to some $U_{i,\infty}\geq 0$, and by Lemma 2.11 (third and fourth inequalities), $U_{i,\infty}\neq 0$. The limits satisfy the remarkable following system:

$$\begin{cases} -U_{1,\infty}''=U_{1,\infty}f_1[0]-U_{1,\infty}U_{2,\infty}\\ -dU_{2,\infty}''=U_{2,\infty}f_2[0]-\alpha U_{1,\infty}U_{2,\infty}.\end{cases}$$

At first this system is to be understood in the distributional sense, but once more thanks to classical elliptic regularity $U_{1,\infty}$ and $U_{2,\infty}$ are actually in $\mathcal{C}^{2,\beta}_{per}(\mathbb{R})$. Thanks to the elliptic strong minimum principle, for any $i\in\{1,2\}$, $U_{i,\infty}\gg 0$.

In C, $-\frac{U_{1,\infty}''}{U_{1,\infty}}=f_1[0]-U_{2,\infty}\leq M_1$. Integration over C yields:

$$\int_C f_1[0]=-\int_C\left|\frac{U_{1,\infty}'}{U_{1,\infty}}\right|^2+\int_C U_{2,\infty}\leq\int_C U_{2,\infty}.$$

Similarly,

$$\int_C f_2[0]\leq\int_C U_{1,\infty}.$$

Then $(\mathcal{H}_2)$ shows that $(U_{1,\infty},U_{2,\infty})$ is at positive distance of the origin in $L^1(C)$, and then in $L^\infty(C)$ by classical embeddings. Harnack's inequality yields eventually that $\min\left(\min_{\overline{C}}(U_{1,\infty}),\min_{\overline{C}}(U_{2,\infty})\right)$ is bounded from below by a real number $\epsilon>0$. By uniform convergence and provided $k^\star$ is large enough, the infimum of the sequence $\left(\min\left\{\min_{\overline{C}}(ku_{1,k}),\min_{\overline{C}}(ku_{2,k})\right\}\right)_{k>k^\star}$ is greater than, say, $\frac{3\epsilon}{4}$. This ϵ depends on m, C, but neither on the limit point $(U_{1,\infty},U_{2,\infty})$ nor on the choice of a convergent subsequence of $((u_1,u_2))_{k>k^\star}$, whence the bound holds for any convergent subsequence of $((u_1,u_2))_{k>k^\star}$. Furthermore, the bound does not depend on the choice of the sequence $((u_1,u_2))_{k>k^\star}$ itself, whence it holds for any convergent subsequence of any sequence.

The conclusion on the whole set is a standard compactness argument. □

2.3.3.2 Instability of periodic coexistence states close to $(0,0)$

Lemma 2.18. *Provided $k^\star$ is large enough, for any $(u_1,u_2)\in S$, the differential operator $\mathcal{A}_{(u_1,u_2)}:\mathcal{C}^2_{per}(\mathbb{R})\to\mathcal{C}_{per}(\mathbb{R})$ defined as:*

$$\mathcal{A}_{(u_1,u_2)}=\begin{pmatrix}\frac{d^2}{dx^2}+g_1[u_1]-ku_2 & ku_1\\ \alpha ku_2 & d\frac{d^2}{dx^2}+g_2[u_2]-\alpha ku_1\end{pmatrix}$$

is strongly positive.

Proof. It is well-known that $\mathcal{A}_{(u_1,u_2)}$ is strongly positive (i.e. satisfies the strong minimum principle) if there exists a pair of positive functions whose image by $-\mathcal{A}_{(u_1,u_2)}$ is itself non-negative (see for instance Figueiredo–Mitidieri [54]). From $(\mathcal{H}_1)$, if k is large enough, there exists a constant $R > 0$ which depends only on $x \mapsto \partial_1 f_1(0,x)$ and $x \mapsto \partial_1 f_2(0,x)$ such that:

$$\begin{cases} \partial_1 f_1 [u_1] \in [-R, 0] \\ \partial_1 f_2 [u_2] \in [-R, 0] . \end{cases}$$

From here, it is easy to check that, up to extraction and using the notations of the proof of Lemma 2.17,

$$-\mathcal{A}_{(u_{1,k},u_{2,k})} \begin{pmatrix} U_{1,\infty} \\ U_{2,\infty} \end{pmatrix} \to \begin{pmatrix} U_{1,\infty}U_{2,\infty} \\ \alpha U_{1,\infty}U_{2,\infty} \end{pmatrix}$$

uniformly in C as $k \to +\infty$.

This limit being positive, thanks to standard compactness arguments, we get indeed the claimed statement. □

Proposition 2.19. *For any $k > k^\star$, any $(u_1, u_2) \in S$ is unstable.*

Proof. Thanks to Mora's theorem [113], we know that (u_1, u_2) is unstable if the principal eigenvalue of the elliptic part of the monotone problem $(\mathcal{M})$ linearized at $(u_1, J(u_2))$ is negative. It is easy to verify that the linearized operator is in fact:

$$\mathcal{A}_{(u_1,u_2)} = \begin{pmatrix} \frac{\mathrm{d}^2}{\mathrm{d}x^2} + g_1[u_1] - ku_2 & ku_1 \\ \alpha k u_2 & d\frac{\mathrm{d}^2}{\mathrm{d}x^2} + g_2[u_2] - \alpha k u_1 \end{pmatrix}$$

$\mathcal{A}_{(u_1,u_2)}$ being strongly positive (see Lemma 2.18), it is injective and, up to a restriction of its codomain, it is invertible. Krein–Rutman's theorem and a well-known routine involving the compact canonical embedding $\mathcal{C}^{2,\beta}(C) \hookrightarrow \mathcal{C}^{0,\beta}_{loc}(C)$ prove the existence of the periodic principal eigenvalue $\lambda_{1,per}\left(-\mathcal{A}_{(u_1,u_2)}\right)$.

Now, we have to prove that $\lambda_{1,per}\left(-\mathcal{A}_{(u_1,u_2)}\right) < 0$. Recall the following characterization from Krein–Rutman's theorem:

$$\lambda_{1,per}\left(-\mathcal{A}_{(u_1,u_2)}\right) = \inf\left\{\lambda \in \mathbb{R} \mid \exists \varphi \in \mathcal{C}^2_{per}\left(\mathbb{R}, (0,+\infty)^2\right) \ \left(-\mathcal{A}_{(u_1,u_2)} - \lambda\right)\varphi \leq 0 \text{ in } \mathbb{R}\right\}.$$

Therefore, we only need to find some $\lambda < 0$ and some $\varphi \in \mathcal{C}^2_{per}\left(\mathbb{R}, (0,+\infty)^2\right)$ satisfying:

$$\left(-\mathcal{A}_{(u_1,u_2)} - \lambda\right)\varphi \leq 0.$$

Using $(\mathcal{H}_1)$, it is easy to check that there exists a constant $R > 0$ which depends only on $x \mapsto \partial_1 f_1(0,x)$ and $x \mapsto \partial_1 f_2(0,x)$ such that:

$$\begin{aligned} \left(-\mathcal{A}_{(u_1,u_2)}\right)\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} &= \begin{pmatrix} -u_1^2 \partial_1 f_1[u_1] - ku_1u_2 \\ -u_2^2 \partial_1 f_2[u_2] - \alpha k u_1 u_2 \end{pmatrix} \\ &\leq \begin{pmatrix} (Ru_1 - ku_2)\,u_1 \\ (Ru_2 - \alpha k u_1)\,u_2 \end{pmatrix} \\ &\leq -\min\left\{\min_{\overline{C}}(ku_2 - Ru_1), \min_{\overline{C}}(\alpha k u_1 - Ru_2)\right\}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}. \end{aligned}$$

In virtue of Lemma 2.17, provided $k^\star$ is large enough, for any $K > k^\star$ and any $(u_{1,K}, u_{2,K}) \in S_K$:

$$\min\left\{\min_{\overline{C}}\left(Ku_{2,K} - Ru_{1,K}\right), \min_{\overline{C}}\left(\alpha Ku_{1,K} - Ru_{2,K}\right)\right\} > 0.$$

Consequently it holds for k and (u_1, u_2).

Now, if we define λ as $-\min\left\{\min_{\overline{C}}\left(ku_2 - Ru_1\right), \min_{\overline{C}}\left(\alpha ku_1 - Ru_2\right)\right\}$ and φ as (u_1, u_2), it is obvious that $\left(-\mathcal{A}_{(u_1,u_2)} - \lambda\right)\varphi \leq 0$. Therefore, (u_1, u_2) is unstable. □

2.3.4 Counter-propagation

In this subsection, we prove the so-called counter-propagation hypothesis. Let us recall from Fang–Zhao [69] that, since every intermediate periodic stationary state is unstable (Proposition 2.19), their set is totally unordered.

Proposition 2.20. *Let $k > k^\star$ and $(u_1, u_2) \in S$.*

Let $c^\star_+\left(\left(u_1, \tilde{u}_2 - u_2\right), \left(\tilde{u}_1, \tilde{u}_2\right)\right) \in \mathbb{R}$ and $c^\star_-\left(\left(u_1, \tilde{u}_2 - u_2\right), (0,0)\right) \in \mathbb{R}$ be the spreading speeds associated with front-like initial data connecting respectively $(\tilde{u}_1, \tilde{u}_2)$ to $(u_1, \tilde{u}_2 - u_2)$ and $(u_1, \tilde{u}_2 - u_2)$ to $(0,0)$.

Then:

$$c^\star_+\left(\left(u_1, \tilde{u}_2 - u_2\right), \left(\tilde{u}_1, \tilde{u}_2\right)\right) + c^\star_-\left(\left(u_1, \tilde{u}_2 - u_2\right), (0,0)\right) > 0.$$

Remark. At least formally, since (u_1, u_2) vanishes as $k \to +\infty$, we have:

$$c^\star_+\left(\left(u_1, \tilde{u}_2 - u_2\right), \left(\tilde{u}_1, \tilde{u}_2\right)\right) \to c^\star_+\left(\left(0, \tilde{u}_2\right), \left(\tilde{u}_1, \tilde{u}_2\right)\right),$$

$$c^\star_-\left(\left(u_1, \tilde{u}_2 - u_2\right), (0,0)\right) \to c^\star_-\left(\left(0, \tilde{u}_2\right), (0,0)\right).$$

It is easily seen that the first limit is in fact the spreading speed of the scalar KPP pulsating front connecting $\tilde{u}_1$ to 0 for the equation $\partial_t u_1 - \partial_{xx} u_1 = u_1 f_1[u_1]$ whereas the second one is in fact the spreading speed of the scalar KPP pulsating front connecting $\tilde{u}_2$ to 0 for the equation $\partial_t u_2 - d\partial_{xx} u_2 = u_2 f_2[u_2]$. These limiting speeds are both positive. Hence, heuristically, we expect that both $c^\star_+\left(\left(u_1, \tilde{u}_2 - u_2\right), \left(\tilde{u}_1, \tilde{u}_2\right)\right)$ and $c^\star_-\left(\left(u_1, \tilde{u}_2 - u_2\right), (0,0)\right)$ are positive whenever k is large enough, and this is indeed what we will prove.

Proof. Let $k > k^\star$, $(u_1, u_2) \in S$, $\mathcal{A}_{(u_1,u_2)}$ be the associated linear elliptic operator defined as in Lemma 2.18, $t > 0$, Q_t be the semiflow associated with $(\mathcal{M})$ and $Q_t^{u,lin}$ be the linear semiflow associated with $\partial_t - \mathcal{A}_{(u_1,u_2)}$. We intend to use Weinberger's theory [141, Theorem 2.4] in order to establish that:

$$c^\star_+\left(\left(u_1, \tilde{u}_2 - u_2\right), \left(\tilde{u}_1, \tilde{u}_2\right)\right) \geq \inf_{\mu > 0} \frac{-\lambda_{1,per}\left(-\mu^2 \operatorname{diag}(1,d) - \mathcal{A}_{(u_1,u_2)}\right)}{\mu}.$$

(The exponential relation between the periodic principal eigenvalue of the elliptic operator $\mathcal{A}_{(u_1,u_2)}$ and that of the semiflow $Q_t^{u,lin}$ is classical and not detailed here.)

On one hand, to apply [141, Theorem 2.4], we have to find $\delta \in (0,1)$ and $\eta_+ > 0$ such that, for all $(v_1, v_2) \in [(0,0), (\eta_+, \eta_+)]$:

$$Q_t\left[(v_1, v_2) + \left(u_1, \tilde{u}_2 - u_2\right)\right] - \left(u_1, \tilde{u}_2 - u_2\right) \geq (1-\delta)\, Q_t^{u,lin}\left[(v_1, v_2)\right],$$

that is such that:

$$\delta Q_t^{u,lin}\left[(v_1, v_2)\right] \geq Q_t^{u,lin}\left[(v_1, v_2)\right] + \left(u_1, \tilde{u}_2 - u_2\right) - Q_t\left[(v_1, v_2) + \left(u_1, \tilde{u}_2 - u_2\right)\right].$$

On the other hand, by definition of $Q^{u,lin}$, for all $\varepsilon > 0$, we have the existence of $\eta_\varepsilon > 0$ such that, if $(v_1, v_2) \in [(0,0), (\eta_\varepsilon, \eta_\varepsilon)]$:

$$\left| Q_t^{u,lin} [(v_1, v_2)] + (u_1, \tilde{u}_2 - u_2) - Q_t [(v_1, v_2) + (u_1, \tilde{u}_2 - u_2)] \right| \leq \varepsilon \max \left(\max_{\overline{C}} v_1, \max_{\overline{C}} v_2 \right).$$

Hence it would be sufficient to show, for all $(v_1, v_2) \in [(0,0), (\eta_\varepsilon, \eta_\varepsilon)]$, the following inequality:

$$\varepsilon \max \left(\max_{\overline{C}} v_1, \max_{\overline{C}} v_2 \right) \leq \delta \min \left(\min_{\overline{C}} Q_t^{u,lin} [(v_1, v_2)]_1, \min_{\overline{C}} Q_t^{u,lin} [(v_1, v_2)]_2 \right),$$

which is a straightforward consequence of the positivity of $\mathcal{A}_{(u_1,u_2)}$ and of the instability of (u_1, u_2) (fixing for instance $\delta = \frac{1}{2}$ and then choosing ε small enough). Finally we define $\eta_+ = \eta_\varepsilon$.

Applying the same sketch of proof and being careful with the signs, we prove the existence of $\eta_- > 0$ such that, for all $(v_1, v_2) \in [(0,0), (\eta_-, \eta_-)]$:

$$-Q_t [-(v_1, v_2) + (u_1, \tilde{u}_2 - u_2)] + (u_1, \tilde{u}_2 - u_2) \geq \frac{1}{2} Q_t^{u,lin} [(v_1, v_2)],$$

whence a second inequality is established:

$$c_-^\star ((u_1, \tilde{u}_2 - u_2), (0,0)) \geq \inf_{\mu > 0} \frac{-\lambda_{1,per} \left(-\mu^2 \text{diag}(1, d) - \mathcal{A}_{(u_1,u_2)} \right)}{\mu}.$$

It is worthy to point out that both spreading speeds are estimated from below by the same quantity.

To conclude, we just have to notice the following inequality, true for all $\mu > 0$:

$$\lambda_{1,per} \left(-\mu^2 \text{diag}(1, d) - \mathcal{A}_{(u_1,u_2)} \right) \leq -\mu^2 \min(1, d) + \lambda_{1,per} \left(-\mathcal{A}_{(u_1,u_2)} \right) < 0.$$

In particular, from:

$$\frac{-\lambda_{1,per} \left(-\mu^2 \text{diag}(1, d) - \mathcal{A}_{(u_1,u_2)} \right)}{\mu} \geq \inf_{\mu > 0} \left(\mu \min(1, d) - \frac{\lambda_{1,per} \left(-\mathcal{A}_{(u_1,u_2)} \right)}{\mu} \right),$$

we deduce the following estimate:

$$\inf_{\mu > 0} \frac{-\lambda_{1,per} \left(-\mu^2 \text{diag}(1, d) - \mathcal{A}_{(u_1,u_2)} \right)}{\mu} \geq 2 \sqrt{\min(1, d) \left| \lambda_{1,per} \left(-\mathcal{A}_{(u_1,u_2)} \right) \right|} > 0.$$

□

2.3.5 Existence of pulsating fronts connecting both extinction states

We are now able to state rigorously the existence of pulsating fronts thanks to Fang–Zhao [69].

Theorem 2.21. *For any $k > k^\star$, there exists $c \in \mathbb{R}$ and $(\varphi_1, \varphi_2) \in \mathcal{C}\left(\mathbb{R}^2, \mathbb{R}^2\right)$ such that the following properties hold.*

1. *φ_1 and φ_2 are respectively non-increasing and non-decreasing with respect to their first variable, generically noted ξ.*
2. *φ_1 and φ_2 are periodic with respect to their second variable, generically noted x.*

3. *As* $\xi \to +\infty$,

$$\max_{x \in [0,L]} |(\varphi_1, \varphi_2)(-\xi, x) - (\tilde{u}_1, 0)(x)| + \max_{x \in [0,L]} |(\varphi_1, \varphi_2)(\xi, x) - (0, \tilde{u}_2)(x)| \to 0.$$

4. $(u_1, u_2) : (t, x) \mapsto (\varphi_1, \varphi_2)(x - ct, x)$ *is a classical solution of* $(\mathcal{P})$.

Remark. For any $\xi_0 \in \mathbb{R}$, $(\xi, x) \mapsto (\varphi_1, \varphi_2)(\xi + \xi_0, x)$ is a pulsating front solution of $(\mathcal{P})$ as well.

Regarding the regularity of (φ_1, φ_2), we recall that, even if Fang–Zhao [69] (as well as Weinberger [141]) worked in the framework of continuous functions, by classical parabolic regularity, a continuous solution of $(\mathcal{P})$ is in $\mathcal{C}^1_{loc}\left(\mathbb{R}, \mathcal{C}^2_{loc}\left(\mathbb{R}, \mathbb{R}^2\right)\right)$. Hence (φ_1, φ_2) is a fortiori in $\mathcal{C}^1_{loc}\left(\mathbb{R}^2, \mathbb{R}^2\right)$. This can be improved provided f_1 and f_2 are $\mathcal{C}^1$ with respect to x. Indeed, differentiating $(\mathcal{P})$ with respect to t and x shows similarly that $\partial_t (u_1, u_2) \in \mathcal{C}^1_{loc}\left(\mathbb{R}, \mathcal{C}^2_{loc}\left(\mathbb{R}, \mathbb{R}^2\right)\right)$ and $\partial_x (u_1, u_2) \in \mathcal{C}^1_{loc}\left(\mathbb{R}, \mathcal{C}^2_{loc}\left(\mathbb{R}, \mathbb{R}^2\right)\right)$. In such a case, (φ_1, φ_2) is at least in $\mathcal{C}^2\left(\mathbb{R}^2, \mathbb{R}^2\right)$.

Thanks

The author would like to thank Grégoire Nadin for the attention he paid to this work. The author is especially grateful to Jian Fang who explained in person some technicalities with great patience and clarity. The author would also like to thank the anonymous referee for valuable input which considerably simplified the verification of the counter-propagation hypothesis.

Chapitre 3

Compétition en milieu périodique : II – Limite ségrégative d'ondes pulsatoires et résultat de type « L'union ne fait pas la force »

Résumé

Ce chapitre s'intéresse à la limite, quand le taux de compétition interspécifique tend vers l'infini, d'ondes pulsatoires solutions d'un système de compétition – diffusion de Lotka – Volterra bistable en milieu spatialement périodique. On distingue deux cas importants : vitesse limite nulle et non-nulle. Dans le premier cas, on montre l'existence d'équilibres stationnaires ségrégés. Dans le second cas, on est capable d'établir l'unicité de l'onde pulsatoire ségrégée, et ainsi de prouver la convergence. L'onde pulsatoire ségrégée est solution d'un problème de frontière libre intéressant. On étudie également le signe de la vitesse limite, vue comme une fonction des paramètres du système. On est en mesure de déterminer complètement ce signe, avec des conditions explicites dépendant uniquement de ces paramètres. En particulier, si l'une des deux espèces est suffisamment plus mobile ou compétitive que l'autre, alors il s'agit de l'envahisseur. Ce résultat est donc de type « L'union ne fait pas la force ».

Ce chapitre, co-écrit avec Grégoire Nadin, a fait l'objet d'une publication sous le titre *Competition in periodic media : II – Segregative limit of pulsating fronts and "Unity is not strength"-type result* dans *Journal of Differential Equations* [GN18].

3.1 Introduction

This is the second part of a sequel to our previous article [GN15]. In the prequel, we studied the sign of the speed of bistable traveling wave solutions of the following competition–diffusion problem:

$$\begin{cases} \partial_t u_1 - \partial_{xx} u_1 = u_1 (1 - u_1) - k u_1 u_2 & \text{in } (0, +\infty) \times \mathbb{R} \\ \partial_t u_2 - d\partial_{xx} u_2 = r u_2 (1 - u_2) - \alpha k u_1 u_2 & \text{in } (0, +\infty) \times \mathbb{R}. \end{cases}$$

We proved that, as $k \to +\infty$, the speed of the traveling wave connecting $(1, 0)$ to $(0, 1)$ converges to a limit which has exactly the sign of $\alpha^2 - rd$. In particular, if $\alpha = r = 1$ and if k is large enough, the more motile species is the invader: this is what we called the "Unity is not strength" result.

In view of this result, it would seem natural to try to generalize it in heterogeneous spaces, that is to systems with non-constant coefficients. Is the more motile species still the invading one?

Competition–diffusion problems in bounded heterogeneous spaces with various boundary conditions have been widely studied during the past decades. Dockery, Hutson, Mischaikow and Pernarowski [58] showed (in particular) that for the heterogeneous system:

$$\begin{cases} \partial_t u_1 - d_1 \Delta_x u_1 = a_1 (x) u_1 - u_1^2 - u_1 u_2 & \text{in } (0, +\infty) \times \Omega \\ \partial_t u_2 - d_2 \Delta_x u_2 = a_2 (x) u_2 - u_2^2 - u_1 u_2 & \text{in } (0, +\infty) \times \Omega \end{cases}$$

with a_1 and a_2 non-constant functions, d_1 and d_2 constant, Ω a bounded open subset of some Euclidean space and homogeneous Neumann boundary conditions, the persistent species is actually the less motile one. The interspecific competition rate of this system is equal to 1 and the system is therefore monostable. On the contrary, as soon as the competition rate is large enough, the system is bistable. We wonder whether this qualitative change might be sufficient to reverse their conclusion. If we are able to extend in some satisfying way our space-homogeneous result, then the conclusion will be reversed indeed.

In the first part [Gir17] of this sequel, the first author studied the existence of bistable pulsating front solutions for the following problem:

$$\begin{cases} \partial_t u_1 = \partial_{xx} u_1 + u_1 f_1 (u_1, x) - k u_1 u_2 & \text{in } (0, +\infty) \times \mathbb{R} \\ \partial_t u_2 = d\partial_{xx} u_2 + u_2 f_2 (u_2, x) - \alpha k u_1 u_2 & \text{in } (0, +\infty) \times \mathbb{R}. \end{cases}$$

Here, the non-linearities $(u, x) \mapsto u f_i (u, x)$, $i \in \{1, 2\}$, are of "KPP"-type and, most importantly, are spatially periodic. Thanks to Fang–Zhao's theorem [69], it was showed that, provided k is large enough and (f_1, f_2) satisfies a high-frequency algebraic hypothesis (we highlight that the condition was algebraic and not asymptotic), there exists indeed such a pulsating front.

While the forthcoming main ideas might be generalizable to systems with periodic diffusion and interspecific competition rates, an existence result is lacking. Therefore we naturally stick with the aforementioned system. Let us recall moreover that the fully heterogeneous problem (non-periodic non-constant coefficients) is, as far as we know, still completely open at this time.

Let us recall as well that several important results about scalar reaction–diffusion equations in periodic media have been established recently (about "KPP"-type, see [16, 17, 117, 118, 121]; about "ignition"-type and monostable non-linearities, see [14]; about bistable non-linearities, see [57, 56, 144]). The first author used extensively the results about "KPP"-type equations in [Gir17]. In the forthcoming work, we will use the whole collection of results. Especially, we will use several times, in slightly different contexts, the sliding method of Berestycki–Hamel [14].

Integration over a bounded domain with Neumann boundary conditions and over a periodicity cell are somehow similar operations and thus Neumann and periodic boundary conditions yield in general analogous results. The periodic extension of the persistence result by Dockery and his collaborators seems in fact quite straightforward and, conversely, it should be possible to adapt the forthcoming ideas to determine the persistent species in a bistable space-heterogeneous Neumann problem with large competition rate. The comparison is therefore even more meaningful.

The competition-induced segregation phenomenon highlighted by Dancer, Terracini and others (see for instance [38, 45, 47, 49, 52]) has been one of our main tools in the preceding pair of articles [Gir17, GN15] and will still be a cornerstone here. In particular, segregation in two or more dimensions generically yields free boundary problems and this will be a major difference between the space-homogeneous case and this study: here, we will need to dedicate a few pages to the natural free boundary problem induced by the segregation of pulsating fronts. Thanks to the specific setting of pulsating fronts (monotonicity in time, spatial periodicity of the profile, limiting conditions, etc.), we will be able to prove that the free boundary is the graph of a strictly monotonic, bijective and continuous function without resorting to blow-up arguments or monotonicity formulas. We believe that our approach of the free boundary has interest of its own and that the ideas presented here might fond applications in other frameworks.

The following pages will be organized as follows: in the first section, the core hypotheses and framework will be precisely formulated and the main results stated. The second section will focus on the so-called “segregative limit” and will finally lead us to the third section and the statement of the periodic extension of the “Unity is not strength” theorem.

3.2 Preliminaries and main results

Remark. Subsections 3.2.1 and 3.2.3 are mostly a repetition of the preliminaries of the first author’s article [Gir17] where the existence of competitive pulsating fronts was investigated. A reader well aware of this article may safely skip these. On the contrary, Subsections 3.2.2 and 3.2.4 respectively state the main results of this article and highlight the differences between the present set of technical hypotheses and that of the first author’s article [Gir17].

Let $d, k, \alpha, L > 0$, $C = (0, L) \subset \mathbb{R}$ and $(f_1, f_2) : [0, +\infty) \times \mathbb{R} \to \mathbb{R}^2$ L-periodic with respect to its second variable. For any $u : \mathbb{R}^2 \to [0, +\infty)$ and $i \in \{1, 2\}$, we refer to $(t, x) \mapsto f_i\left(u\left(t, x\right), x\right)$ as $f_i\left[u\right]$. Our interest lies in the following competition–diffusion problem:

$$\begin{cases} \partial_t u_1 = \partial_{xx} u_1 + u_1 f_1\left[u_1\right] - k u_1 u_2, \\ \partial_t u_2 = d \partial_{xx} u_2 + u_2 f_2\left[u_2\right] - \alpha k u_1 u_2. \end{cases} \qquad (\mathcal{P}_k)$$

3.2.1 Preliminaries

3.2.1.1 Redaction conventions.

- Mirroring the definition of $f_1\left[u\right]$ and $f_2\left[u\right]$, for any function of two real variables f and any real-valued function u of two real variables, $f\left[u\right]$ will refer to $(t, x) \mapsto f\left(u\left(t, x\right), x\right)$. For any real-valued function u of one real variable, $f\left[u\right]$ will refer to $x \mapsto f\left(u\left(x\right), x\right)$. For any function f of one real variable and any real-valued function u of one or two real variables, $f\left[u\right]$ will simply refer to $f \circ u$.
- For the sake of brevity, although we could index everything $((\mathcal{P}), u_1, u_2 \ldots)$ on k and d, the dependencies on k or d will mostly be implicit and will only be made explicit when it definitely facilitates the reading.
- Since we consider the limit of this system when $k \to +\infty$, many (but finitely many) results will only be true when “k is large enough”. Hence, we define by induction the positive

number $k^\star$, whose value is initially 1 and is updated each time a statement is only true when "k is large enough" in the following way: if the statement is true for any $k \geq k^\star$, the value of $k^\star$ is unchanged; if, conversely, there exists $K > k^\star$ such that the statement is true for any $k \geq K$ but false for any $k \in [k^\star, K)$, the value of $k^\star$ becomes that of K. In the text, we will indifferently write "for k large enough" or "provided $k^\star$ is large enough". Moreover, when k indexes appear, they *a priori* indicate that we are considering families indexed on $[k^\star, +\infty)$, but for the sake of brevity, when sequential arguments involve sequences indexed themselves on increasing elements of $[k^\star, +\infty)^{\mathbb{N}}$, we will not explicitly define these sequences of indexes and will simply stick with the indexes k, reindexing along the course of the proof the considered objects. In such a situation, the statement "as $k \to +\infty$" should be understood unambiguously.

— Periodicity will always implicitly mean L-periodicity (unless explicitly stated otherwise). For any functional space X on $\mathbb{R}$, X_{per} denotes the subset of L-periodic elements of X.

— We will use the classical partial order on the space of functions from any $\Omega \subset \mathbb{R}^N$ to $\mathbb{R}$: $g \leq h$ if for any $x \in \Omega$ $g(x) \leq h(x)$ and $g < h$ if $g \leq h$ and $g \neq h$. We recall that when $g < h$, there might still exists $x \in \Omega$ such that $g(x) = h(x)$. If, for any $x \in \Omega$, $g(x) < h(x)$, we use the notation $g \ll h$. In particular, if $g \geq 0$, we say that g is non-negative, if $g > 0$, we say that g is non-negative non-zero, and if $g \gg 0$, we say that g is positive. Finally, if $g_1 \leq h \leq g_2$, we write $h \in [g_1, g_2]$, if $g_1 < h < g_2$, we write $h \in (g_1, g_2)$, and if $g_1 \ll h \ll g_2$, we write $h \in \langle g_1, g_2 \rangle$.

— We will also use the partial order on the space of vector functions $\Omega \to \mathbb{R}^{N'}$ naturally derived from the preceding partial order. It will involve similar notations.

— Functions f of two or more real variables will sometimes be identified with the maps $t \mapsto (x \mapsto f(t,x))$. This is quite standard in parabolic theory but we stress that the variable of the map will always be the first variable of f, even if this variable is not called t: we will use indeed functions of the pair of variables $(\xi, x) \in \mathbb{R}^2$ and then the maps will be $\xi \mapsto (x \mapsto f(\xi, x))$. So for instance if we say that a function f of (ξ, x) is an element of a functional space $X(\mathbb{R}, Y)$, the latter should be understood unambiguously.

3.2.1.2 Hypotheses on the reaction.

For any $i \in \{1,2\}$, we have in mind functions f_i such that the reaction term $uf_i[u]$ is of logistic type (also known as "KPP"-type). At least, we want to cover the largest possible class of $(u,x) \mapsto \mu(x)(a-u)$. This is made precise by the following assumptions.

($\mathcal{H}_1$) f_i is in $\mathcal{C}^1([0,+\infty) \times \mathbb{R})$.

($\mathcal{H}_2$) There exists a constant $m_i > 0$ such that $f_i[0] \geq m_i$.

($\mathcal{H}_3$) f_i is decreasing with respect to its first variable and there exists $a_i > 0$ such that, for any $x \in \mathbb{R}$, $f_i(a_i, x) = 0$.

Remark. If f_i is in the class of all $(u,x) \mapsto \mu(x)(a-u)$, then $\mu \in \mathcal{C}^1_{per}(\mathbb{R})$, $\mu \gg 0$ and $a > 0$. More generally, from ($\mathcal{H}_1$), ($\mathcal{H}_2$) and the periodicity of $f_i[0]$, it follows immediately that there exists a constant $M_i > m_i$ such that $f_i[0] \leq M_i$. Without loss of generality, we assume that m_i and M_i are optimal, that is $m_i = \min_{\overline{C}} f_i[0]$ and $M_i = \max_{\overline{C}} f_i[0]$.

3.2.1.3 Extinction states

The periodic principal eigenvalues of $\frac{\mathrm{d}^2}{\mathrm{d}x^2} + f_1[0]$ and $d\frac{\mathrm{d}^2}{\mathrm{d}x^2} + f_2[0]$ are negative (as proved by the first author in [Gir17]). Recall (from Berestycki–Hamel–Roques [16] for instance) that the periodic principal eigenvalue of $\mathcal{L}$ is the unique real number λ such that there exists a periodic function $\varphi \gg 0$ satisfying:

$$\begin{cases} -\mathcal{L}\varphi = \lambda\varphi \text{ in } \mathbb{R} \\ \|\varphi\|_{L^\infty(C)} = 1 \end{cases}$$

From this observation, it follows from Berestycki–Hamel–Roques [16] that a_1 (respectively a_2) is the unique periodic non-negative non-zero solution of $-z'' = zf_1[z]$ (resp. $-dz'' = zf_2[z]$).

The states $(a_1, 0)$ and $(0, a_2)$ are clearly periodic stationary states of $(\mathcal{P}_k)$ (for any $k > k^\star$) and are referred to as the *extinction states* of $(\mathcal{P}_k)$ (remark that they are the unique periodic stationary states with one null component and the other one positive, so that it makes sense to call them "the" extinction states). Provided $k^\star$ is large enough, they are moreover locally asymptotically stable (again, as proved in [Gir17]).

We recall also that, for any $k > k^\star$, by virtue of the scalar parabolic comparison principle, any solution (u_1, u_2) of $(\mathcal{P}_k)$ with initial condition $(0,0) < (u_{1,0}, u_{2,0}) < (a_1, a_2)$ satisfies $(0,0) \ll (u_1, u_2) \ll (a_1, a_2)$.

3.2.1.4 Pulsating front solutions of $(\mathcal{P})$

Let us add a necessary existence hypothesis.

$(\mathcal{H}_{exis})$ There exists $k^\star > 0$ such that, for any $k > k^\star$, there exists $c_k \in \mathbb{R}$ and $(\varphi_{1,k}, \varphi_{2,k}) \in \mathcal{C}^2\left(\mathbb{R}^2\right)^2$ such that the following properties hold.

- $(u_{1,k}, u_{2,k}) : (t, x) \mapsto (\varphi_{1,k}, \varphi_{2,k})(x - c_k t, x)$ is a classical solution of $(\mathcal{P}_k)$.
- $\varphi_{1,k}$ and $\varphi_{2,k}$ are respectively non-increasing and non-decreasing with respect to their first variable, generically noted ξ.
- $\varphi_{1,k}$ and $\varphi_{2,k}$ are periodic with respect to their second variable, generically noted x.
- As $\xi \to -\infty$,
$$\max_{x\in[0,L]} |(\varphi_{1,k}, \varphi_{2,k})(\xi, x) - (a_1, 0)| \to 0.$$
- As $\xi \to +\infty$,
$$\max_{x\in[0,L]} |(\varphi_{1,k}, \varphi_{2,k})(\xi, x) - (0, a_2)| \to 0.$$

The pair $(u_{1,k}, u_{2,k})$ is referred to as a *pulsating front solution* of $(\mathcal{P}_k)$ with *speed* c_k and *profile* $(\varphi_{1,k}, \varphi_{2,k})$.

Before going any further, it is natural to wonder if such a solution is unique.

Conjecture. *Let $k > k^\star$. Let $(\hat\varphi_1, \hat\varphi_2)$ and $\hat c$ be respectively the profile and the speed of a pulsating front solution $(\hat u_1, \hat u_2)$ of $(\mathcal{P})$. Then $\hat c = c_k$ and there exists $\hat\xi \in \mathbb{R}$ such that $(\hat\varphi_1, \hat\varphi_2)$ coincides with:*

$$(\xi, x) \mapsto (\varphi_{1,k}, \varphi_{2,k})\left(\xi - \hat\xi, x\right).$$

This conjecture is due to the following observation: in most (if not all) problems concerned with bistable traveling or pulsating fronts, the front is unique (in the same sense as above: two fronts have the same speed and have the same profile up to translation).

We refer to Gardner [77], Kan-On [100], Berestycki–Hamel [14] or Ding–Hamel–Zhao [57] for proofs of this type of result in slightly different settings.

Because the proof of such a result:

- would involve precise estimates of the exponential decay of the profiles as $\xi \to \pm\infty$ that cannot be obtained briefly (in the scalar case, see Hamel [88]) and have no additional interest in the forthcoming work,

— would be strongly analogous to the proofs of the preceding collection of references,

we choose to leave this as an open question here for the sake of brevity. We might address this question in a future sequel.

Still, it is useful to have this uniqueness in mind because it clearly motivates our study of $\lim_{k\to+\infty} c_k$.

3.2.2 "Unity is not strength" theorem for periodic media

In the forthcoming theorem, the parameters d, α, f_1 and f_2 may vary (in some sense which is made precise), but immediately after that they are fixed again (at least up to Section 3.4).

Theorem 3.1. *["Unity is not strength", periodic case] Assume that there exists an open connected set $\mathfrak{P}$ of parameters:*

$$(d,\alpha,f_1,f_2) \in (0,+\infty)^2 \cap \mathcal{C}\left([0,+\infty),\mathcal{C}_{per}\left(\mathbb{R}\right)\right)^2$$

in which $(\mathcal{H}_1)$, $(\mathcal{H}_2)$, $(\mathcal{H}_3)$ and $(\mathcal{H}_{exis})$ are satisfied.

The sequence $\left((d,\alpha,f_1,f_2)\in\mathfrak{P}\mapsto c_k\right)_{k>k^\star}$ converges pointwise as $k\to+\infty$ to some continuous function $(d,\alpha,f_1,f_2)\in\mathfrak{P}\mapsto c_\infty$. If the function $(d,\alpha,f_1,f_2)\in\mathfrak{P}\mapsto k^\star$ is locally bounded, then this convergence is in fact locally uniform in $\mathfrak{P}$.

Furthermore, for any $(d,\alpha,f_1,f_2)\in\mathfrak{P}$, there exist $\overline{r}>0$, $\underline{r}\in(0,\overline{r}]$ (both dependent on (f_1,f_2) only) and a non-empty closed interval $\mathcal{R}^0\subset[\underline{r},\overline{r}]$ (dependent on (d,f_1,f_2) only) such that the sign of c_∞ satisfies the following properties.

1. *$c_\infty>0$ if and only if $\frac{\alpha^2}{d}>\max\mathcal{R}^0$.*
2. *$c_\infty<0$ if and only if $\frac{\alpha^2}{d}<\min\mathcal{R}^0$.*
3. *If, for any $i\in\{1,2\}$, f_i has the particular form $(u,x)\mapsto\mu_i(x)(1-u)$, then:*
   a) *c_∞ is null or has the sign of:*
   $$\alpha^2-d\frac{\|\mu_2\|_{L^1(C)}}{\|\mu_1\|_{L^1(C)}};$$
   b) *$(\underline{r},\overline{r})$ satisfies:*
   $$\frac{\min_{\overline{C}}(\mu_2)}{\max_{\overline{C}}(\mu_1)}\leq\underline{r}\leq\overline{r}\leq\frac{\max_{\overline{C}}(\mu_2)}{\min_{\overline{C}}(\mu_1)}.$$

The objects $\overline{r}$, $\underline{r}$ and $\mathcal{R}^0$ are respectively defined by formulas $(\mathfrak{F}_{\overline{r}})$, $\left(\mathfrak{F}_{\underline{r}}\right)$ and $(\mathfrak{F}_{\mathcal{R}^0})$ (see page 130).

Remark. We emphasize the interest of $\underline{r}$ and $\overline{r}$, which are upper and lower bounds for $\mathcal{R}^0$ which are uniform with respect to d.

We will explain in Section 3.4 that if $(\mathcal{H}_{exis})$ is derived from the existence result of the first author [Gir17], then a set $\mathfrak{P}$ exists: the main assumption of our theorem makes sense indeed.

The strategy of the proof is as follows.

We will begin with some compactness estimates uniform with respect to k so that a limiting speed and an associated limiting solution, possibly non-unique at this point, can be extracted. This will require a crucial distinction between two cases: limiting speed null or not.

Regarding the first case, we will give some regularity properties of the corresponding solution, that will be called a *segregated stationary equilibrium*. It is unclear whether the segregated stationary equilibrium is unique but this is not surprising: the null speed case is known to be quite degenerate (see for instance Ding–Hamel–Zhao [57]).

On the contrary, the second case will be fully characterized: the corresponding solution, the *segregated pulsating front*, is actually unique (up to translation). Such a uniqueness result will require several intermediary results and in particular a (possibly not complete but already quite thorough) study of its intrinsic free boundary problem.

Subsequently, the uniqueness of the segregated pulsating front will follow from a sliding argument which will also provide us with an exclusion result: there exists a segregated stationary equilibrium for a particular choice of parameters (d, α, f_1, f_2) if and only if there does not exist a segregated pulsating front. Thanks to this result, the uniqueness of the limiting speed will be deduced even though the null case is still degenerate.

We will then obtain a necessary and sufficient condition on (d, α, f_1, f_2) for the existence of a segregated stationary equilibrium thanks to its regularity at the interface (which is, in some sense, the counterpart to the free boundary problem leading to the uniqueness of the segregated pulsating front) and finally, thanks to a classical integration by parts, obtain the sign of the speed provided it is already known to be non-zero.

3.2.3 A few more preliminaries

3.2.3.1 Compact embeddings of Hölder spaces

Proposition 3.2. *Let $(a, a') \in (0, +\infty)^2$ and n, n', β, β' such that $(a, a') = (n + \beta, n' + \beta')$, n and n' are non-negative integers and β and β' are in $(0, 1]$.*

If $a \leq a'$, then the canonical embedding $i : \mathcal{C}^{n',\beta'}(C) \hookrightarrow \mathcal{C}^{n,\beta}(C)$ is continuous and compact.

It will be clear later on that this problem naturally involves uniform bounds in $\mathcal{C}^{0,1/2}$. Therefore, we fix once and for all $\beta \in \left(0, \frac{1}{2}\right)$ and we will use systematically the compact embeddings $\mathcal{C}^{n,1/2} \hookrightarrow \mathcal{C}^{n,\beta}$, meaning that uniform bounds in $\mathcal{C}^{n,1/2}$ yield relative compactness in $\mathcal{C}^{n,\beta}$.

3.2.3.2 Additional notations regarding the pulsating fronts

Let $E = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. For any $k > k^\star$, $(c_k, \varphi_{1,k}, \varphi_{2,k})$ satisfies the following system:

$$\begin{cases} -\mathrm{div}\left(E\nabla\varphi_{1,k}\right) - c_k\partial_\xi\varphi_{1,k} = \varphi_{1,k} f_1\left[\varphi_{1,k}\right] - k\varphi_{1,k}\varphi_{2,k} \\ -d\mathrm{div}\left(E\nabla\varphi_{2,k}\right) - c_k\partial_\xi\varphi_{2,k} = \varphi_{2,k} f_2\left[\varphi_{2,k}\right] - \alpha k\varphi_{1,k}\varphi_{2,k}. \end{cases} \qquad (\mathcal{PF}_{sys,k})$$

Remark. Be aware that, since $\mathrm{sp}E = \{0, 2\}$, the differential operator:

$$\mathrm{div}\left(E\nabla\right) = \partial_{\xi\xi} + \partial_{xx} + 2\partial_{\xi x}$$

is only degenerate elliptic. This will trigger difficulties unknown in the space-homogeneous case. Most regularity results will come from the parabolic system $(\mathcal{P})$ and we will need to go back and forth a lot between the so-called "parabolic coordinates" (t, x) and the so-called "traveling coordinates" (ξ, x). This will be possible if and only if the propagation speed is non-zero, whence a necessary distinction of cases.

For any $k > k^\star$, let:

$$\psi_{d,k} = \alpha\varphi_{1,k} - d\varphi_{2,k},$$

$$\psi_{1,k} = \alpha\varphi_{1,k} - \varphi_{2,k},$$

$$v_{d,k} = \alpha u_{1,k} - du_{2,k},$$

$$v_{1,k} = \alpha u_{1,k} - u_{2,k}.$$

A linear combination of the equations of $(\mathcal{PF}_{sys,k})$ yields:

$$-\mathrm{div}\left(E\nabla\psi_{d,k}\right) - c_k\partial_\xi\psi_{1,k} = \alpha\varphi_{1,k}f_1\left[\varphi_{1,k}\right] - \varphi_{2,k}f_2\left[\varphi_{2,k}\right] \quad (\mathcal{PF}_k).$$

$(\mathcal{PF}_k)$ does not depend explicitly on k.
$(u_{1,k}, u_{2,k}, v_{d,k}, v_{1,k})$ is isomorphic to $(\varphi_{1,k}, \varphi_{2,k}, \psi_{d,k}, \psi_{1,k})$ if and only if $c_k \neq 0$. In parabolic coordinates, $(\mathcal{PF}_k)$ becomes:

$$\partial_t v_{1,k} - \partial_{xx} v_{d,k} = \alpha u_{1,k} f_1\left[u_{1,k}\right] - u_{2,k} f_2\left[u_{2,k}\right].$$

As $k \to +\infty$, the following function will naturally appear:

$$\eta : (z,x) \mapsto f_1\left(\frac{z}{\alpha}, x\right) z^+ - \frac{1}{d} f_2\left(-\frac{z}{d}, x\right) z^-,$$

where $z^+ = \max(z,0)$ and $z^- = -\min(z,0)$ so that $z = z^+ - z^-$.
We will also denote g_i the partial derivative of $(u,x) \mapsto u f_i(u,x)$ with respect to u:

$$g_i : (u,x) \mapsto f_i(u,x) + u\partial_1 f_i(u,x) \text{ for all } i \in \{1,2\}.$$

3.2.4 Comparison between the first and the second part

In addition to the new notations introduced in the preceding subsection ($(\mathcal{PF}_{sys})$, $(\mathcal{PF})$, “parabolic coordinates”, “traveling coordinates”, ψ_d, ψ_1, v_d, v_1), the following differences are pointed out.

- In the first part [Gir17], f_1 and f_2 were only assumed to be Hölder-continuous with respect to x, whereas here we need them to be at least continuously differentiable. Thanks to this technical hypothesis, it is then possible to differentiate with respect to x the various equations and systems involved. In particular, continuous pulsating front solutions of $(\mathcal{P})$ are in fact in $\mathcal{C}^2_{loc}\left(\mathbb{R}^2\right)$. This will similarly yield a stronger regularity at the limit. Nevertheless, we think that Hölder-continuity might actually suffice to obtain most of the forthcoming results.
- The positive zero of $u \mapsto f_i(u,x)$ cannot depend on x anymore. Consequently, while, in the first part [Gir17], the unique positive solution of $-z'' = zf_1[z]$, $\tilde{u}_1$, and the unique positive solution of $-dz'' = zf_2[z]$, $\tilde{u}_2$, were periodic functions of x, here they are the constants a_1 and a_2. This restriction is standard in bistable pulsating front problems (see for instance [57, 56, **?**]) and is especially related to the method generically used to determine the sign of the speed of the pulsating fronts. Still, most of the forthcoming pages is easily generalized (actually, many results need no adaptation at all). We will highlight where this hypothesis is truly needed and will give some indications regarding the non-constant case. In the end, it should be clear why we conjecture that “Unity is not strength” holds true even in the non-constant case.
- A trade-off to these more restrictive assumptions is that here we do not assume *a priori* the high-frequency hypothesis:

$$L < \pi\left(\frac{1}{\sqrt{M_1}} + \sqrt{\frac{d}{M_2}}\right). \quad (\mathcal{H}_{freq})$$

  We merely assume existence of pulsating fronts, this hypothesis being referred to as $(\mathcal{H}_{exis})$. It was proved in the first part that if $(\mathcal{H}_{freq})$ is satisfied, then so is $(\mathcal{H}_{exis})$.

3.3 Asymptotic behavior: the infinite competition limit

3.3.1 Existence of a limiting speed

In order to prove that $(c_k)_{k>k^\star}$ has at least one limit point, we recall an important result from the Fisher–KPP scalar case (see Berestycki–Hamel–Roques [17]).

Theorem 3.3. *For any $\delta \in \{1, d\}$ and $i \in \{1, 2\}$, there exists $c^\star[\delta, i] > 0$ such that, for any $s \in \mathbb{R}$, there exists in $\mathcal{C}^2\left(\mathbb{R}^2\right)$ a pulsating front solution of:*

$$\partial_t z - \delta \partial_{xx} z = z f_i[z]$$

connecting a_i to 0 at speed s if and only if $s \geq c^\star[\delta, i]$.

Lemma 3.4. *Provided $k^\star$ is large enough, for any $k > k^\star$ and any pulsating front solution of $(\mathcal{P}_k)$, its speed c satisfies:*

$$-c^\star[d, 2] < c < c^\star[1, 1].$$

In particular, the family $(c_k)_{k>k^\star}$ is uniformly bounded with respect to k.

Remark. Here, the assumption that k is large enough might in fact be redundant with the underlying assumption of bistability. Indeed, this proof does not use any limiting behavior but only requires that:

$$k > \max\left\{\frac{1}{a_2}\max_{\overline{C}}\left(f_1[0]\right), \frac{1}{\alpha a_1}\max_{\overline{C}}\left(f_2[0]\right)\right\}.$$

In the space-homogeneous logistic case, this condition reduces to $k > \max\left\{1, \alpha^{-1}\right\}$, that is precisely the necessary and sufficient condition for the system to be bistable. In the space-periodic case, according to the proof of [Gir17, Proposition 2.1], both a_i are stable if the condition above is satisfied. Yet an optimal threshold should involve periodic principal eigenvalues instead of these maxima. Furthermore, the instability of any other periodic steady state has only been established for (really) large k (see [Gir17, Theorem 1.2]) and when $(\mathcal{H}_{freq})$ holds true. Even for arbitrarily large k, it is unclear whether stable coexistence periodic steady states might exist when $(\mathcal{H}_{freq})$ does not hold.

We point out that the following proof provides us with an instance of a detailed proof using the sliding method [14] that will be referred to later on.

Proof. Assume by contradiction that there exists $k > 0$ such that there exists a pulsating front solution (z_1, z_2) of $(\mathcal{P}_k)$ with a speed $c \notin (-c^\star[d, 2], c^\star[1, 1])$ and a profile (φ_1, φ_2). For instance, assume $c \geq c^\star[1, 1]$ (the other case being obviously symmetric), and let $\underline{c} = c^\star[1, 1] \leq c$. By virtue of Theorem 3.3, $\underline{c} > 0$ and there exists a pulsating front solution z of :

$$\partial_t z - \partial_{xx} z = z f_1[z]$$

with speed $\underline{c}$ and profile φ.

Now we are in position to use the sliding method to compare z and z_1. This will finally lead to a contradiction.

Step 1: existence of a translation of the profile associated with the higher speed such that it is locally below the other profile.

Fix $\zeta \in \mathbb{R}$. Then let $\zeta_1 \in \mathbb{R}$ such that:

$$\max_{x \in \overline{C}} \varphi_1(\zeta_1, x) < \min_{x \in \overline{C}} \varphi(\zeta, x).$$

Let:

$$\tau = \zeta - \zeta_1,$$
$$\varphi_1^\tau : (\xi, x) \mapsto \varphi_1 (\xi - \tau, x),$$
$$\Phi^\tau = \varphi - \varphi_1^\tau,$$

so that:

$$\min_{x \in \overline{C}} \Phi^\tau (\zeta, x) = \min_{x \in \overline{C}} (\varphi (\zeta, x) - \varphi_1 (\zeta_1, x)) > 0.$$

Step 2: up to some extra term, this ordering is global on the left.
Let $\mathcal{U} = (-\infty, \zeta) \times \overline{C}$. Since $\varphi \gg 0$ in $\mathcal{U}$ and $\varphi_1^\tau \in L^\infty (\mathcal{U})$, there exists $\kappa > 0$ such that:

$$\kappa\varphi - \varphi_1^\tau \geq 0 \text{ in } \mathcal{U}.$$

Notice that, since $\Phi^\tau (\xi, x) \to 0$ as $\xi \to \pm\infty$ (uniformly with respect to x), any such κ is larger than or equal to 1.

Step 3: this extra term is actually unnecessary, thanks to the maximum principle.
Let:

$$\kappa^\star = \inf \left\{ \kappa > 1 \mid \inf_{\mathcal{U}} (\kappa\varphi - \varphi_1^\tau) > 0 \right\}$$

and let us prove that $\kappa^\star = 1$. We assume by contradiction that $\kappa^\star > 1$ and we take a sequence $(\kappa_n)_{n\in\mathbb{N}} \in (1, \kappa^\star)^{\mathbb{N}}$ which converges to $\kappa^\star$ from below.

There exists a sequence $((\xi_n, x_n)) \in \mathcal{U}^{\mathbb{N}}$ such that for any $n \in \mathbb{N}$,

$$\kappa_n \varphi (\xi_n, x_n) < \varphi_1^\tau (\xi_n, x_n).$$

Since $\kappa_n > 1$, the limits when $\xi \to -\infty$ prove that (ξ_n) is bounded from below, and since it is also bounded from above by ζ, we can extract a convergent subsequence with limit $\xi^\star \in (-\infty, \zeta]$. Similarly, we can extract a convergent subsequence of $(x_n) \in \overline{C}^{\mathbb{N}}$ with limit $x^\star \in \overline{C}$. By continuity, $(\kappa^\star\varphi - \varphi_1^\tau)(\xi^\star, x^\star) = 0$ and, necessarily, $\xi^\star < \zeta$.

Back to parabolic variables, recall that $\underline{c} > 0$ and let:

$$t^\star = \frac{x^\star - \xi^\star}{\underline{c}},$$
$$\hat{z}_i^\tau : (t, x) \mapsto \varphi_i (x - \underline{c}t - \tau, x) \text{ for any } i \in \{1, 2\},$$
$$v^\star = \kappa^\star z - \hat{z}_1^\tau,$$
$$f : (t, x) \mapsto -(c - \underline{c}) (\partial_\xi \varphi_1^\tau) (x - \underline{c}t, x)$$
$$E = \left\{ (t, x) \in \mathbb{R}^2 \mid x - \underline{c}t < \zeta \right\}.$$

By virtue of $(\mathcal{H}_3)$ and $\kappa^\star > 1$:

$$\kappa^\star z f_1 [z] > \kappa^\star z f_1 [\kappa^\star z] \text{ in } E,$$

and moreover:

$$\partial_t v^\star - \partial_{xx} v^\star = \kappa^\star z f_1 [z] - \hat{z}_1^\tau f_1 [\hat{z}_1^\tau] + k \hat{z}_1^\tau \hat{z}_2^\tau + f \text{ in } E,$$
$$f \geq 0 \text{ in } E.$$

Now, from the Lipschitz-continuity of f_1 with respect to its first variable, it follows that of $(u, x) \mapsto u f_1 (u, x)$, whence there exists $q \in L^\infty (E)$ such that:

$$\partial_t v^\star - \partial_{xx} v^\star \geq q v^\star \text{ in } E.$$

In the end, $v^\star$ is a non-negative super-solution which vanishes at some interior point: by virtue of the parabolic strong minimum principle, it is identically null in $((-\infty, t^\star] \times \mathbb{R}) \cap E$.

But in such an unbounded set, it is always possible to construct an element of $\{\zeta\} \times \overline{C}$, which contradicts:

$$\min_{x \in \overline{C}} (\kappa^\star \varphi - \varphi_1^\tau)(\zeta, x) > 0.$$

Therefore $\kappa^\star = 1$,

$$\kappa^\star \varphi - \varphi_1^\tau = \Phi^\tau \geq 0 \text{ in } \mathcal{U}$$

and then by periodicity and, once more, by virtue of the parabolic strong minimum principle:

$$\Phi^\tau \gg 0 \text{ in } (-\infty, \zeta) \times \mathbb{R}.$$

Step 4: up to some (possibly different) extra term, this ordering is global on the right.

Near $+\infty$ (in $(\zeta, +\infty) \times \mathbb{R}$), on the contrary, multiplying φ by some $\kappa \gg 1$ is not going to yield a clear ordering anymore since we are interested in the behavior as $\varphi \sim 0$ and $\varphi_1 \sim 0$ (and replacing φ and φ_1^τ by respectively $a_1 - \varphi$ and $a_1 - \varphi_1^\tau$ will not suffice since the monostability has no underlying symmetry).

But it is natural, for instance, to replace this multiplication by the addition of some $\varepsilon \geq 0$ and to prove in the next step that $\varepsilon^\star = 0$. This is actually what was done originally by Berestycki–Hamel [14].

Step 5: this (possibly different) extra term is also unnecessary.

We define $\varepsilon^\star$ as the following quantity:

$$\varepsilon^\star = \inf \left\{ \varepsilon > 0 \mid \inf_{(\zeta, +\infty) \times \overline{C}} (\varphi - \varphi_1^\tau + \varepsilon) > 0 \right\}.$$

We assume by contradiction that $\varepsilon^\star > 0$ and this yields as before a contact point $(\xi^\star, x^\star) \in (\zeta, +\infty) \times \overline{C}$.

Now the main difficulty is that $u \mapsto u f_1[u]$ is increasing near 0, so that we really cannot hope to have:

$$z f_1[z] \geq (z + \varepsilon) f_1[z + \varepsilon].$$

Still, it is possible to assume without loss of generality that, during the construction of τ, ζ_1 has also been chosen so that:

$$\frac{a_2}{2} \leq \varphi_2(\xi, x) \leq a_2 \text{ for any } (\xi, x) \in [\zeta_1, +\infty) \times \overline{C}.$$

It follows that:

$$\varphi_1^\tau (f_1[\varphi_1^\tau] - k\varphi_2^\tau) \leq \varphi_1^\tau \left(f_1[\varphi_1^\tau] - k\frac{a_2}{2} \right) \text{ in } [\zeta, +\infty) \times \overline{C}.$$

By virtue of the hypotheses $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$, provided $k^\star$ is large enough, for any $K > k^\star$, the following non-linearity:

$$u \mapsto u \left(f_1[u] - K\frac{a_2}{2} \right)$$

is decreasing in a neighborhood of 0 (in fact, it is decreasing in $[0, +\infty)$). Then, in addition to this monotonicity, it suffices to use:

$$\varphi f_1[\varphi] \geq \varphi f_1[\varphi] - k\frac{a_2}{2}\varphi$$

and the Lipschitz-continuity of f_1 to conclude this step.

Step 6: thanks to the maximum principle again, the speeds are equal and the profiles are equal up to some translation.

Thus in fact:

$$\Phi^\tau \gg 0 \text{ in } \mathbb{R}^2.$$

Now, let:

$$\tau^\star = \sup\left\{\tau \in \mathbb{R} \mid \Phi^\tau \geq 0 \text{ in } \mathbb{R}^2\right\}.$$

The limits as $\xi \to \pm\infty$ of φ and φ_1 ensure that $\tau^\star < +\infty$. By continuity,

$$\Phi^{\tau^\star} \geq 0.$$

Let us verify quickly that, by virtue of the maximum principle, either $\Phi^{\tau^\star} = 0$ and $c = \underline{c}$, either $\Phi^{\tau^\star} \gg 0$. For instance, assume that $\left(\Phi^{\tau^\star}\right)^{-1}(\{0\})$ is non-empty, so that $\Phi^{\tau^\star} \gg 0$ does not hold. Then there exists $(\xi^\star, x^\star) \in \mathbb{R}^2$ such that $\Phi^{\tau^\star}(\xi^\star, x^\star) = 0$. Once more, we introduce:

$$t^\star = \frac{x^\star - \xi^\star}{\underline{c}},$$

$$v^{\tau^\star}(t,x) = \Phi^{\tau^\star}(x - \underline{c}t, x),$$

and using the parabolic linear inequality satisfied by $v^{\tau^\star}$, we verify that $v^{\tau^\star}$ is a non-negative super-solution which vanishes at $(t^\star, x^\star)$. Then, by the strong parabolic maximum principle and periodicity with respect to x of $\Phi^{\tau^\star}$, it is actually deduced that $\Phi^{\tau^\star} = 0$, which in turn implies (reinserting $v^{\tau^\star} = 0$ into the original non-linear equation satisfied by $v^{\tau^\star}$ and considering the function f which has been defined earlier) that $c = \underline{c}$.

Finally, assume by contradiction that $\Phi^{\tau^\star} \gg (0,0)$, i.e. assume that for any $B > 0$,

$$\min_{[-B,B]\times\overline{C}} \Phi^{\tau^\star} > (0,0).$$

Fix $B > 0$. By continuity, there exists $\epsilon > 0$ such that:

$$\min_{[-B,B]\times\overline{C}} \Phi^{\tau} > (0,0) \text{ for any } \tau \in [\tau^\star, \tau^\star + \epsilon).$$

We can now repeat Steps 2, 3, 4, 5 to show that, for any such τ:

$$\Phi^\tau \gg 0 \text{ in } (\mathbb{R}\setminus(-B,B)) \times \mathbb{R}.$$

The maximality of $\tau^\star$ being contradicted, this ends this step.

Step 7: the contradiction.

If $c = \underline{c}$ and $z = z_1$, then thanks to the equations satisfied by z and z_1, $z_2 = 0$ in $\mathbb{R}^2$. This contradicts the limit of φ_2 as $\xi \to +\infty$. □

Corollary 3.5. *$(c_k)_{k>k^\star}$ has a limit point $c_\infty \in [-c^\star[d,2], c^\star[1,1]]$.*

Remark. Similarly, we do expect that $c_\infty \notin \{-c^\star[d,2], c^\star[1,1]\}$ but will not address this question for the sake of brevity.

3.3.2 Existence of a limiting density provided the speed converges

In this subsection, we fix a sequence $(c_k)_{k>k^\star}$ such that it converges to c_∞.

Then we prove the relative compactness of the associated sequence of pulsating front solutions $((u_{1,k}, u_{2,k}))_{k>k^\star}$, which will follow from classical parabolic estimates similar to those used by Dancer and his collaborators (see for instance [47]) supplemented by some estimates specific to the pulsating front setting. This supplement will lead indeed to a stronger compactness result than the one presented in the aforementioned work.

If $c_\infty \neq 0$, we will see that $((u_{1,k}, u_{2,k}))_{k>k^\star}$ is relatively compact if and only if $((\varphi_{1,k}, \varphi_{2,k}))_{k>k^\star}$ is relatively compact. Moreover, we will show that the compactness result can be improved further thanks to additional pulsating front estimates.

3.3.2.1 Normalization

Before going any further, we point out that, at this point, for any $k > k^\star$, (φ_1, φ_2) is fixed completely arbitrarily among the one-dimensional family of translated profiles. By monotonicity of the profiles with respect to ξ, this choice can in fact be normalized. In the space-homogeneous problem [GN15], the normalization was used to guarantee that the extracted limit point had no null component. It should be clear that this part of the proof will be strongly analogous. Therefore we choose now normalizations reminiscent to the space-homogeneous ones.

— On one hand, if $c_\infty \leq 0$, we fix without loss of generality for any $k > k^\star$ the normalization:

$$0 = \inf \left\{ \xi \in \mathbb{R} \mid \exists x \in \overline{C} \quad \varphi_{1,k}(\xi, x) < \frac{a_1}{2} \right\}.$$

— On the other hand, if $c_\infty > 0$, we fix without loss of generality for any $k > k^\star$ the normalization:

$$0 = \sup \left\{ \xi \in \mathbb{R} \mid \exists x \in \overline{C} \quad \varphi_{2,k}(\xi, x) < \frac{a_2}{2} \right\}.$$

Remark also that $((u_{1,k}, u_{2,k}))_{k>k^\star}$ is normalized (in the sense that its value at some arbitrary initial time is entirely prescribed) if and only if $((\varphi_{1,k}, \varphi_{2,k}))_{k>k^\star}$ is normalized.

3.3.2.2 Compactness results

Proposition 3.6. *The following collection of properties holds independently of the sign of c_∞.*

1. *[Segregation] $(\varphi_{1,k}\varphi_{2,k})_{k>k^\star}$ converges to 0 in $L^1_{loc}(\mathbb{R} \times C)$.*
2. *[Persistence] $(0,0)$ is not a limit point of $((\varphi_{1,k}, \varphi_{2,k}))_{k>k^\star}$ in $L^1_{loc}(\mathbb{R}^2, \mathbb{R}^2)$.*
3. *[Uniform bound in the diagonal direction] For any $n \in \mathbb{N}$, $((\partial_x + \partial_\xi)(\varphi_{1,k}, \varphi_{2,k}))_{k>k^\star}$ is uniformly bounded with respect to k in $L^2((-n,n) \times C, \mathbb{R}^2)$.*
4. *[Uniform bound in the ξ direction] For any $k > k^\star$ and any $x \in \overline{C}$,*

$$\int_{\mathbb{R}} \partial_\xi \varphi_{1,k}(\zeta, x)\, d\zeta = -\int_{\mathbb{R}} |\partial_\xi \varphi_{1,k}(\zeta, x)|\, d\zeta = -a_1$$

and

$$\int_{\mathbb{R}} \partial_\xi \varphi_{2,k}(\zeta, x)\, d\zeta = \int_{\mathbb{R}} |\partial_\xi \varphi_{2,k}(\zeta, x)|\, d\zeta = a_2.$$

5. *[Uniform bound in the x direction] For any $T > 0$, $((u_{1,k}, u_{2,k}))_{k>k^\star}$ is uniformly bounded with respect to k in $L^2((-T,T), H^1(C, \mathbb{R}^2))$.*

6. *[Uniform bound in the t direction] For any $T > 0$, $(\partial_t v_{1,k})_{k>k^\star}$ is uniformly bounded with respect to k in $L^2\left((-T,T), \left(H^1(C)\right)'\right)$.*
7. *[Compactness in traveling coordinates] $((\varphi_{1,k}, \varphi_{2,k}))_{k>k^\star}$ is relatively compact in the topology of $L^1_{loc}(\mathbb{R}^2, \mathbb{R}^2)$.*
8. *[Compactness in parabolic coordinates] There exists:*

$$(u_{1,\infty}, u_{2,\infty}) \in \left(L^\infty(\mathbb{R}^2) \cap L^2\left((-T,T), H^1((0,L))\right)\right)^2$$

such that:

a) $\partial_t v_{1,\infty} \in L^2\left((-T,T), \left(H^1((0,L))\right)'\right)$;

b) $(u_{1,\infty}, u_{2,\infty})$ *is a limit point of* $((u_{1,k}, u_{2,k}))_{k>k^\star}$ *in the topology of* $L^1_{loc}(\mathbb{R}^2, \mathbb{R}^2)$;

c) $u_{1,\infty}$ *and* $u_{2,\infty}$ *are in* $\mathcal{C}^{0,\beta}_{loc}(\mathbb{R}^2)$;

d) $u_{1,\infty} = \alpha^{-1} v^+_{d,\infty} = \alpha^{-1} v^+_{1,\infty}$ *and* $u_{2,\infty} = d^{-1} v^-_{d,\infty} = v^-_{1,\infty}$.

Proof. The segregation property comes directly from an integration of, say, the first equation of $(\mathcal{PF}_{sys,k})$ over some $(-n,n) \times C$. The persistence of at least one component is a consequence of the choice of normalization: for instance, if $c_\infty \leq 0$, necessarily $(\varphi_{1,k})_{k>k^\star}$ does not vanish.

To get the uniform bound in the diagonal direction, we introduce a cut-off function. For any $n \in \mathbb{N}$, there exists a non-negative non-zero function $\chi \in \mathcal{D}(\mathbb{R}^2)$ such that, for any $x \in \overline{C}$, $\chi(\xi, x) = 0$ if $\xi \notin [-n-1, n+1]$ and $\chi(\xi, x) = 1$ if $\xi \in [-n, n]$.

Let $k > k^\star$. Multiplying the first equation of $(\mathcal{PF}_{sys})$ by $\varphi_{1,k}\chi$ and integrating by parts in $\mathbb{R} \times C$, we obtain:

$$\int (\partial_\xi \varphi_{1,k})^2 \chi - \frac{1}{2} \int \varphi_{1,k}^2 \partial_{\xi\xi} \chi + \int \chi (\partial_x \varphi_{1,k})^2 + 2 \int \chi \partial_\xi \varphi_{1,k} \partial_x \varphi_{1,k} \leq \int M_1 \varphi_{1,k}^2 \chi - c \int \frac{\varphi_{1,k}^2}{2} \partial_\xi \chi.$$

(The integrals being implicitly over $\mathbb{R} \times C$.)

Using $\chi \geq \mathbf{1}_{[-n,n]}$, the k-uniform L^∞-bound for $(\varphi_{1,k})_{k>k^\star}$ and:

$$|c| \leq \max\{c^\star[d,2], c^\star[1,1]\},$$

we deduce the existence of a constant R_n independent on k such that:

$$\int_{[-n,n]\times C} |\partial_\xi \varphi_{1,k} + \partial_x \varphi_{1,k}|^2 \leq R_n.$$

The same proof holds for $\varphi_{2,k}$. Finally, the same computation in parabolic coordinates gives immediately the uniform bound in the x direction.

The uniform bound in the ξ direction is a straightforward result. Provided the uniform bound in the x direction, the uniform bound in the t direction comes from an integration over $(0,T) \times C$ of $(\mathcal{PF})$ multiplied by some test function in $L^2\left((0,T), H^1(C)\right)$.

The relative compactness in both systems of coordinates follows from the embedding:

$$L^2_{loc}(\mathbb{R}^2) \hookrightarrow L^1_{loc}(\mathbb{R}^2)$$

and the compact embedding:

$$W^{1,1}_{loc}(\mathbb{R}^2, \mathbb{R}^2) \hookrightarrow L^1_{loc}(\mathbb{R}^2).$$

To obtain the continuity of $u_{1,\infty}$ and $u_{2,\infty}$, we consider a convergent subsequence. Since the convergence occurs a.e. up to extraction, the limit point is actually in $L^\infty\left(\mathbb{R}^2,\mathbb{R}^2\right)$, whence:

$$v_{1,\infty}\in L^\infty\left(\mathbb{R}\times(0,L)\right)\cap L^2\left(\left(-T,T\right),H^1\left((0,L)\right)\right),$$

$$\partial_t v_{1,\infty}\in L^2\left(\left(-T,T\right),\left(H^1\left((0,L)\right)\right)'\right).$$

It follows from a standard regularity result that $v_{1,\infty}\in\mathcal{C}\left(\left[-T,T\right],L^2\left((0,L)\right)\right)$ (see for instance Evans [67, 5.9.2]).

Then, we pass the parabolic version of $(\mathcal{PF})$ to the limit in $\mathcal{D}'\left(\mathbb{R}\right)$ and we can apply DiBenedetto's theory [55]: $v_{1,\infty}$ is a locally bounded weak solution of the following parabolic equation:

$$\partial_t z-\partial_x\left(\left(\mathbf{1}_{z>0}+d\mathbf{1}_{z<0}\right)\partial_x z\right)=f_1\left[\frac{z}{\alpha}\right]z^+-f_2\left[-z\right]z^-.$$

In a large class of degenerate parabolic equations which contains in particular this equation, locally bounded weak solutions are, for any $\delta\in(0,1)$, spatially $\mathcal{C}^{0,\delta}_{loc}$ and temporally $\mathcal{C}^{0,\delta/2}_{loc}$, whence *a fortiori* $v_{1,\infty}\in\mathcal{C}^{0,\beta}_{loc}\left(\mathbb{R}^2\right)$ (with $\delta=2\beta\in(0,1)$).

Finally, by virtue of the segregation property:

$$u_{1,\infty}=\alpha^{-1}v_{1,\infty}^+\text{ a.e.,}$$

$$u_{2,\infty}=v_{1,\infty}^-\text{ a.e..}$$

From this, it follows that $v_{1,\infty}$ is in $\mathcal{C}^{0,\beta}_{loc}\left(\mathbb{R}^2\right)$ if and only if $u_{1,\infty}$ and $u_{2,\infty}$ are themselves in $\mathcal{C}^{0,\beta}_{loc}\left(\mathbb{R}^2\right)$, whence

$$\left(u_{1,\infty},u_{2,\infty}\right)\in\mathcal{C}^{0,\beta}_{loc}\left(\mathbb{R}^2,\mathbb{R}^2\right).$$

□

Remark. At this point, we do not know if the limit points in parabolic coordinates and in traveling coordinates are related. Yet, when $c_\infty\neq 0$, we can improve the preceding results and relate the limit points indeed.

Proposition 3.7. *Assume $c_\infty\neq 0$. The following additional collection of properties holds.*

1. *[Improved uniform bound in the ξ direction] Provided $k^\star$ is large enough, $\left(\partial_\xi\left(\varphi_{1,k},\varphi_{2,k}\right)\right)_{k>k^\star}$ is uniformly bounded with respect to k in $L^2\left(\mathbb{R}\times C,\mathbb{R}^2\right)$.*
2. *[Improved compactness] There exists $\left(\varphi_{1,seg},\varphi_{2,seg}\right)\in L^\infty\left(\mathbb{R}^2,\mathbb{R}^2\right)\cap H^1_{loc}\left(\mathbb{R}^2,\mathbb{R}^2\right)$ such that, up to extraction:*
   a) *$\left(\left(\varphi_{1,k},\varphi_{2,k}\right)\right)_{k>k^\star}$ converges to $\left(\varphi_{1,seg},\varphi_{2,seg}\right)$ strongly in $L^2_{loc}\left(\mathbb{R}^2,\mathbb{R}^2\right)$ and a.e.;*
   b) *$\left(\left(\nabla\varphi_{1,k},\nabla\varphi_{2,k}\right)\right)_{k>k^\star}$ converges to $\left(\nabla\varphi_{1,seg},\nabla\varphi_{2,seg}\right)$ weakly in $L^2_{loc}\left(\mathbb{R}^2,\mathbb{R}^4\right)$;*
   c) *$\left(\left(u_{1,k},u_{2,k}\right)\right)_{k>k^\star}$ converges to:*

   $$\left(u_{1,seg},u_{2,seg}\right):(t,x)\mapsto\left(\varphi_{1,seg},\varphi_{2,seg}\right)\left(x-c_\infty t,x\right)$$

   strongly in $L^2_{loc}\left(\mathbb{R}^2,\mathbb{R}^2\right)$, a.e., and $\left(\left(\nabla u_{1,k},\nabla u_{2,k}\right)\right)_{k>k^\star}$ converges weakly in $L^2_{loc}\left(\mathbb{R}^2,\mathbb{R}^4\right)$.

Proof. Since $c_\infty\neq 0$, we assume without loss of generality that $k^\star$ is sufficiently large to ensure that $c_k\neq 0$ for any $k>k^\star$.

We start by showing that the uniform boundedness in $L^2\left(\mathbb{R}\times C\right)$ of $\left(\partial_\xi\varphi_{1,k}\right)_{k>k^\star}$ is equivalent to that of $\left(\partial_\xi\varphi_{2,k}\right)_{k>k^\star}$ and to that of $\left(\partial_\xi\psi_{d,k}\right)_{k>k^\star}$. □

— First step of the equivalence: assume that $\left(\|\partial_\xi \varphi_{1,k}\|_{L^2(\mathbb{R}\times C)}\right)_{k>k^\star}$ is uniformly bounded. Let $k > k^\star$. Multiply $(\mathcal{PF}_k)$ by $\partial_\xi \psi_{d,k}$, remark that:

$$\partial_\xi \psi_{1,k} = \frac{1}{d}\left(\alpha\left(d-1\right)\partial_\xi \varphi_{1,k} + \partial_\xi \psi_{d,k}\right)$$

and integrate by parts over $(-n,n)\times C$ with some $n\in\mathbb{N}$. By classical parabolic estimates, the terms involving E vanish as $n\to+\infty$. By change of variable, for any $i\in\{1,2\}$,

$$\int_C \int_{-n}^{n} \varphi_{i,k} f_i\left[\varphi_{i,k}\right]\partial_\xi \varphi_{i,k} = \int_C \int_{\varphi_i(-n,x)}^{\varphi_i(+n,x)} z f_i\left(z,x\right)\mathrm{d}z\mathrm{d}x,$$

whence as $n\to+\infty$:

$$\int_C \int_{-n}^{n} \varphi_{1,k} f_1\left[\varphi_{1,k}\right]\partial_\xi \varphi_{1,k} \to -\int_C \int_0^{a_1} z f_1\left(z,x\right)\mathrm{d}z\mathrm{d}x,$$

$$\int_C \int_{-n}^{n} \varphi_{2,k} f_2\left[\varphi_{2,k}\right]\partial_\xi \varphi_{2,k} \to \int_C \int_0^{a_2} z f_1\left(z,x\right)\mathrm{d}z\mathrm{d}x.$$

It follows that:

$$\begin{aligned}\left(-\frac{c_k}{d}\right)\int_{\mathbb{R}\times C}\left(\alpha\left(d-1\right)\partial_\xi\varphi_{1,k}+\partial_\xi\psi_{d,k}\right)\partial_\xi\psi_{d,k} \quad = \quad & -\alpha\int_C\int_0^{a_1} z f_1\left(z,x\right)\mathrm{d}z\mathrm{d}x \\ & +\alpha\int_{\mathbb{R}\times C}\varphi_{1,k}f_1\left[\varphi_{1,k}\right]\left(-d\partial_\xi\varphi_{2,k}\right) \\ & +\int_{\mathbb{R}\times C}\left(-\varphi_{2,k}\right)f_2\left[\varphi_{2,k}\right]\left(\alpha\partial_\xi\varphi_{1,k}\right) \\ & +d\int_C\int_0^{a_2} z f_1\left(z,x\right)\mathrm{d}z\mathrm{d}x.\end{aligned}$$

Dividing by $-\frac{c_k}{d}$ which stays away from 0, the result reduces to:

$$\begin{aligned}\alpha\left(d-1\right)\int\partial_\xi\varphi_{1,k}\partial_\xi\psi_{d,k}+\int\left|\partial_\xi\psi_d\right|^2 \quad = \quad & \frac{\alpha d}{c_k}\int d\varphi_{1,k}f_1\left[\varphi_{1,k}\right]\partial_\xi\varphi_{2,k} \\ & +\frac{\alpha d}{c_k}\int\varphi_{2,k}f_2\left[\varphi_{2,k}\right]\partial_\xi\varphi_{1,k} \\ & +\frac{\alpha d}{c_k}\int_C\int_0^{a_1} z f_1\left(z,x\right)\mathrm{d}z\mathrm{d}x \\ & -\frac{d^2}{c_k}\int_C\int_0^{a_2} z f_1\left(z,x\right)\mathrm{d}z\mathrm{d}x.\end{aligned}$$

Using the boundedness in L^∞ of $\varphi_{i,k} f_i\left[\varphi_{i,k}\right]$ and the relations:

$$\int\left|\partial_\xi\varphi_{1,k}\right| = La_1,$$

$$\int\left|\partial_\xi\varphi_{2,k}\right| = La_2,$$

we obtain that the right-hand side is uniformly bounded. Since $\partial_\xi \varphi_{1,k}$ and $\partial_\xi \psi_{d,k}$ are both non-positive non-zero, if $d \geq 1$, the uniform boundedness of $\left(\int |\partial_\xi \psi_{d,k}|^2\right)_{k>k^\star}$ follows. Otherwise, there exists $R > 0$ such that:

$$\begin{aligned}\int |\partial_\xi \psi_{d,k}|^2 &\leq R + |\alpha (d-1)| \int \partial_\xi \varphi_{1,k} \partial_\xi \psi_{d,k} \\ &\leq R + |\alpha (d-1)| \left(\int |\partial_\xi \varphi_{1,k}|^2\right)^{1/2} \left(\int |\partial_\xi \psi_{d,k}|^2\right)^{1/2}.\end{aligned}$$

This shows that $\left(\int |\partial_\xi \psi_{d,k}|^2\right)^{1/2}$, which is positive, is also smaller than or equal to the largest zero of the following polynomial:

$$X^2 - |\alpha (d-1)| \, \|\partial_\xi \varphi_{1,k}\|_{L^2(\mathbb{R}\times C)} X - R$$

(which is itself positive and uniformly bounded).

- Second step of the equivalence: assume that $\left(\|\partial_\xi \varphi_{2,k}\|_{L^2(\mathbb{R}\times C)}\right)_{k>k^\star}$ is uniformly bounded. A slight adaptation of the first step (using $\partial_\xi \psi_1 = \partial_\xi \psi_d + (d-1)\, \partial_\xi \varphi_2$) shows that the third statement is implied indeed.
- Third step of the equivalence: assume that $\left(\|\partial_\xi \psi_{d,k}\|_{L^2(\mathbb{R}\times C)}\right)_{k>k^\star}$ is uniformly bounded. Since, for any $k > k^\star$:

$$\|\partial_\xi \psi_d\|^2_{L^2} = \alpha^2 \|\partial_\xi \varphi_1\|^2_{L^2} + d^2 \|\partial_\xi \varphi_2\|^2_{L^2} - 2\alpha d \left\langle \partial_\xi \varphi_1, \partial_\xi \varphi_2 \right\rangle_{L^2},$$

  with a positive third term, the first and the second statements are immediately implied.

Proof. Now that the equivalence is established, we simply show that if $c_\infty > 0$, $\left(\|\partial_\xi \varphi_{1,k}\|_{L^2(\mathbb{R}\times C)}\right)_{k>k^\star}$ is uniformly bounded, and conversely if $c_\infty < 0$, $\left(\|\partial_\xi \varphi_{2,k}\|_{L^2(\mathbb{R}\times C)}\right)_{k>k^\star}$ is uniformly bounded. Multiplying the first equation of $(\mathcal{PF}_{sys})$ by $\partial_\xi \varphi_1$, integrating over $\mathbb{R}\times C$, and using the sign of $\partial_\xi \varphi_1$ and classical parabolic estimates at $\pm\infty$, the result reduces to:

$$\begin{aligned} c \int_{\mathbb{R}\times(0,L)} |\partial_\xi \varphi_1|^2 &= k \int_{\mathbb{R}\times(0,L)} \varphi_1 \varphi_2 \partial_\xi \varphi_1 + \int_0^L \int_0^{a_1} z f_1(z,x)\, \mathrm{d}z \mathrm{d}x \\ &\leq \int_0^L \int_0^{a_1} z f_1(z,x)\, \mathrm{d}z \mathrm{d}x.\end{aligned}$$

Similarly, we obtain:

$$\begin{aligned} c \int_{\mathbb{R}\times(0,L)} |\partial_\xi \varphi_2|^2 &= \alpha k \int_{\mathbb{R}\times(0,L)} \varphi_1 \varphi_2 \partial_\xi \varphi_2 - \int_0^L \int_0^{a_2} z f_2(z,x)\, \mathrm{d}z \mathrm{d}x \\ &\geq - \int_0^L \int_0^{a_2} z f_2(z,x)\, \mathrm{d}z \mathrm{d}x.\end{aligned}$$

The improved uniform bound in the ξ direction immediately follows.

The improved relative compactness of $((\varphi_{1,k}, \varphi_{2,k}))_{k>k^\star}$ is a straightforward consequence of the previous lemmas, of Sobolev's embeddings and of Banach–Alaoglu's theorem. For the relative compactness of $((u_{1,k}, u_{2,k}))_{k>k^\star}$, let $[s] : (t,x) \mapsto (x - st, x)$, so that for any $k > k^\star$ $(u_1, u_2) = (\varphi_1, \varphi_2) \circ [c]$. For any $i \in \{1,2\}$:

$$\|u_i - u_{i,seg}\|_{L^2_{loc}} \leq \|\varphi_i \circ [c] - \varphi_i \circ [c_\infty]\|_{L^2_{loc}} + \|\varphi_i \circ [c_\infty] - \varphi_{i,seg} \circ [c_\infty]\|_{L^2_{loc}}.$$

Then, by virtue of Fréchet–Kolmogorov's theorem, the right-hand side vanishes as $k \to +\infty$. The same argument holds for the weak convergence of the derivatives. □

Remark. We point out that the preceding result is specific to the case of constant a_1 and a_2 (without this assumption, one term due to E does not vanish after the integration by parts). In the general case, we do not know if the bounds of Proposition 3.6 can be improved.

Corollary 3.8. *If $c_\infty \neq 0$, the parabolic limit point $(u_{1,seg}, u_{2,seg})$ obtained with the improved compactness result from Proposition 3.7 is also a limit point $(u_{1,\infty}, u_{2,\infty})$ in the sense of Proposition 3.6. In particular, $(u_{1,seg}, u_{2,seg}) \in \mathcal{C}^{0,\beta}_{loc}\left(\mathbb{R}^2, \mathbb{R}^2\right)$, whence $(\varphi_{1,seg}, \varphi_{2,seg}) \in \mathcal{C}^{0,\beta}_{loc}\left(\mathbb{R}^2, \mathbb{R}^2\right)$ as well.*

Remark. The case $c_\infty = 0$ is somehow degenerate and does not really correspond to what intuition calls a "pulsating" front. Moreover, we will need quite different techniques to handle the two cases and, even in the very end, there will be no clear common framework. Therefore, hereafter, we call the case $c_\infty = 0$ "segregated stationary equilibrium" whereas the case $c_\infty \neq 0$ is referred to as "segregated pulsating front". These terms will be precisely defined in a moment.

3.3.3 Characterization of the segregated stationary equilibrium

In this subsection, we assume $c_\infty = 0$ and we use Proposition 3.6 to get an extracted convergent subsequence of pulsating fronts, still denoted $\left((u_{1,k}, u_{2,k})\right)_{k>k^\star}$, with limit $(u_{1,\infty}, u_{2,\infty})$. Up to an additional extraction, we assume a.e. convergence of $(u_{1,k}, u_{2,k}, u_{1,k}u_{2,k})$ to $(u_{1,\infty}, u_{2,\infty}, 0)$.

Obviously, since $c_\infty = 0$, we expect that $(u_{1,\infty}, u_{2,\infty})$ does not depend on t. This will be true indeed, so that it makes sense to refer to this case as "stationary equilibrium". To stress this particularity, we fix t_{cv} such that $\left((u_1, u_2)_{|\{t_{cv}\}\times\mathbb{R}}\right)_{k>k^\star}$ converges a.e. and we define $e = (v_{d,\infty})_{|\{t_{cv}\}\times\mathbb{R}}$, so that if $(u_{1,\infty}, u_{2,\infty})$ is constant with respect to t, $(\alpha u_{1,\infty}, d u_{2,\infty})(t,x) = (e^+, e^-)(x)$ for any $(t,x) \in \mathbb{R}^2$.

We start with an important particular case.

Lemma 3.9. *Assume that, provided $k^\star$ is large enough, $(c_k)_{k>k^\star} = 0$. Then:*

- *for any $k > k^\star$, (u_1, u_2) reduces to:*
$$(t,x) \mapsto (\varphi_1, \varphi_2)(x,x),$$
- *for any $(t,x) \in \mathbb{R}^2$:*
$$(\alpha u_{1,\infty}, d u_{2,\infty})(t,x) = \left(e^+, e^-\right)(x),$$
- *the convergence of $\left((\alpha u_1, d u_2)_{|\{t_{cv}\}\times\mathbb{R}}\right)_{k>k^\star}$ to (e^+, e^-) actually occurs in $\mathcal{C}^{0,\beta}_{loc}(\mathbb{R})$,*
- *the convergence of $\left((v_d)_{|\{t_{cv}\}\times\mathbb{R}}\right)_{k>k^\star}$ to e actually occurs in $\mathcal{C}^{2,\beta}_{loc}(\mathbb{R})$,*
- *e satisfies:*
$$-e'' = \eta\,[e]\,.$$

Proof. The system $(\mathcal{P})$ reduces to an elliptic system. It is then easy to deduce the locally uniform convergence, the time-independence and the limiting equation. We refer, for instance, to [Gir17] for details. □

Some of the preceding results can be extended.

Lemma 3.10. *The properties:*

— *for any* $(t,x) \in \mathbb{R}^2$*,* $(\alpha u_{1,\infty}, du_{2,\infty})(t,x) = (e^+, e^-)(x)$ *;*
— $e \in \mathcal{C}^2(\mathbb{R})$ *and* $-e'' = \eta[e]$*;*

hold true regardless of any sign assumption on the sequence $(c_k)_{k>k^\star}$*.*

Proof. The two statements are actually quite easy to verify. Let $(t,t',x) \in \mathbb{R}^3$ such that, for any $i \in \{1,2\}$ and any $\tau \in \{t,t'\}$, $u_{i,k}(\tau,x) \to u_{i,\infty}(\tau,x)$ as $k \to +\infty$. Recalling that:

$$\int_{\mathbb{R}} \partial_t u_{i,k} = -c_k \int_{\mathbb{R}} \partial_\xi \varphi_{i,k} \to 0$$

as $k \to +\infty$ is sufficient to show that in the following inequality:

$$\begin{aligned} |u_{i,\infty}(t,x) - u_{i,\infty}(t',x)| \leq\ & |u_{i,\infty}(t,x) - u_{i,k}(t,x)| \\ & + \left| \int_t^{t'} \partial_t u_{i,k}(\tau,x)\,\mathrm{d}\tau \right| \\ & + |u_{i,k}(t',x) - u_{i,\infty}(t',x)| \end{aligned}$$

the right-hand side converges to 0 as $k \to +\infty$. Therefore the left-hand side is 0, whence $u_{i,\infty}$ is constant with respect to the time variable in a dense subset of $\mathbb{R}^2$, and then by continuity, it holds *a fortiori* everywhere in $\mathbb{R}^2$.

As for the regularity and limiting equation, the equation is satisfied *a priori* in the distributional sense, then in the classical sense by elliptic regularity. □

Lemma 3.11. *For any* $x \in \mathbb{R}$*, the sequence* $(e(x+nL))_{n\in\mathbb{N}}$ *is non-increasing.*

Proof. By monotonicity with respect to ξ and periodicity with respect to x, for any $(t,x) \in \mathbb{R}^2$ and any $k > k^\star$:

$$\begin{aligned} v_d(t,x+L) - v_d(t,x) &\leq \psi_d(x-ct+L,x+L) - \psi_d(x-ct,x) \\ &\leq \psi_d(x-ct+L,x) - \psi_d(x-ct,x) \\ &\leq 0. \end{aligned}$$

In particular, for any $(t,x) \in \mathbb{R}^2$ and any $k > k^\star$, the sequence $(v_{d,k}(t,x+nL))_{n\in\mathbb{N}}$ is non-increasing, and then, passing to the limit as $k \to +\infty$, the sequence $(e(x+nL))_{n\in\mathbb{N}}$ is non-increasing. This holds for any x in a dense subset of $\mathbb{R}$ and then for any $x \in \mathbb{R}$ by continuity of e. □

Lemma 3.12. e *is non-zero and sign-changing. Moreover:*

$$\inf e^{-1}((-\infty,0)) > -\infty.$$

Proof. The normalization:

$$0 = \inf\left\{\xi \in \mathbb{R} \mid \exists x \in \overline{C} \quad \varphi_{1,k}(\xi,x) < \frac{a_1}{2}\right\},$$

implies that $u_{1,\infty} \neq 0$, whence $e \neq 0$. It shows also that the set:

$$\left\{n \in \mathbb{Z} \mid \exists x \in \overline{C} \quad \varphi_{1,k}(x+nL,x+nL) < \frac{a_1}{2}\right\}$$

is uniformly bounded with respect to k from below. In particular, it has a minimum $\underline{n}_k \in \mathbb{Z}$. Then let:

$$x_k = \inf\left\{x \in \overline{C} \mid \varphi_{1,k}(x+\underline{n}_k L, x+\underline{n}_k L) < \frac{a_1}{2}\right\},$$

so that:

$$\varphi_{1,k}(x,x) > \frac{a_1}{2} \text{ for any } x < x_k + \underline{n}_k L.$$

By monotonicity, we deduce:

$$\varphi_{1,k}(\xi,x) > \frac{a_1}{2} \text{ for any } \xi < x < \underline{n}_k L.$$

If (up to extraction) $\underline{n}_k \to +\infty$ as $k \to +\infty$, then the definition of the normalization is contradicted by the preceding inequality evaluated at $\xi = 0$ and $x \in [L, 2L]$, whence $(\underline{n}_k)_{k>k^\star}$ is uniformly bounded from above as well. In particular, up to extraction, $(\underline{n}_k)_{k>k^\star}$ converges to a finite limit. The finiteness of $\inf\{x \in \mathbb{R} \mid e(x) < 0\}$ follows immediately.

By uniqueness, if $e > 0$, $e = \alpha a_1$. This is discarded by the finiteness of $\lim\limits_{k\to+\infty} \underline{n}_k$, whence e is sign-changing. □

Remark. If, instead of the normalization sequence:

$$0 = \inf\left\{\xi \in \mathbb{R} \mid \exists x \in \overline{C} \quad \varphi_1(\xi,x) < \frac{a_1}{2}\right\} \text{ for any } k > k^\star,$$

we choose:

$$0 = \sup\left\{\xi \in \mathbb{R} \mid \exists x \in \overline{C} \quad \varphi_2(\xi,x) < \frac{a_2}{2}\right\} \text{ for any } k > k^\star,$$

and if we consider once again the case $c_\infty = 0$, the preceding results hold apart from $\inf e^{-1}((-\infty,0)) > -\infty$, which is naturally replaced by:

$$\sup e^{-1}((0,+\infty)) < +\infty.$$

In view of these results, we state the following definition.

Definition 3.13. A function $z \in \mathcal{C}^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ is called a *segregated stationary equilibrium* if:

1. $-z'' = \eta[z]$;
2. for any $x \in \overline{C}$, $(z(x+nL))_{n\in\mathbb{N}}$ is non-increasing;
3. z is non-zero and sign-changing;
4. $\inf z^{-1}((-\infty,0)) > -\infty$ or $\sup z^{-1}((0,+\infty)) < +\infty$.

Corollary 3.14. *e is a segregated stationary equilibrium.*

Let us derive some properties necessarily satisfied by any segregated stationary equilibrium. The first one is obvious but will be useful.

Proposition 3.15. *If z is a segregated stationary equilibrium, then for any $n \in \mathbb{Z}$, $x \mapsto z(x+nL)$ is a segregated stationary equilibrium as well.*

The following one is easily derived from the second order necessary conditions satisfied at a local extremum.

Proposition 3.16. *Let z be a segregated stationary equilibrium. Then $-da_2 < z < \alpha a_1$.*

The following one highlights some difficulties which are intrinsic to the null speed limit.

Proposition 3.17. *Let z be a segregated stationary equilibrium and*

$$\mathcal{Z}(z) = z^{-1}(\{0\}).$$

The set $\mathcal{Z}(z)$ is a discrete set. If it is a finite set, its cardinal is odd. Moreover, it has a minimum or a maximum.

Proof. The fact that $\mathcal{Z}(z)$ is a discrete set follows easily from Hopf's lemma and the regularity of z. Provided finiteness of the set, the monotonicity of $(z(x+nL))_{n\in\mathbb{N}}$ for any $x \in C$ yields the parity of $\#\mathcal{Z}(z)$. Finally, the existence of an extremum comes from the definition of the segregated stationary equilibrium. □

Remark. Under the more restrictive assumption $(\mathcal{H}_{freq})$ presented by the first author in [Gir17], it is possible to prove that every segregated stationary equilibrium has a unique zero. It is basically deduced from the fact that, when there are multiple zeros, the segregated stationary equilibrium restricted to any interval delimited by two consecutive zeros is the unique solution of a semi-linear Dirichlet problem. The monotonicity of $(e(x+nL))_{n\in\mathbb{N}}$ ensures that the distance between these consecutive zeros is smaller than L and then, considering the next zero and using $(\mathcal{H}_{freq})$, a contradiction arises. We do not detail this proof here.

Proposition 3.18. *Let z be a stationary segregated equilibrium.*
If $z^{-1}(\{0\})$ has a minimum, as $n \to +\infty$,

$$\|z - \alpha a_1\|_{\mathcal{C}^2([-(n+1)L,-nL])} \to 0.$$

If $z^{-1}(\{0\})$ has a maximum, as $n \to +\infty$,

$$\|z - d a_2\|_{\mathcal{C}^2([nL,(n+1)L])} \to 0.$$

Proof. We assume that $z^{-1}(\{0\})$ has a minimum, the other case being similar. Since, for any $x \in [0,L)$, $(z(x-nL))_{n\in\mathbb{N}}$ is bounded and non-decreasing, it converges to a limit $z_{-\infty}(x)$. Using Lipschitz-continuity of z, we are able to prove that $z_{-\infty}$ is Lipschitz-continuous in $\overline{C}$. Using elliptic regularity, the distributional equation:

$$-z''_{-\infty} = z_{-\infty} f_1\left[\frac{z_{-\infty}}{\alpha}\right]$$

and Arzela–Ascoli's theorem, we are able to prove in fact that $z_{-\infty} \in \mathcal{C}^{2,\beta}\left(\overline{C}\right)$ and that the convergence occurs in $\mathcal{C}^{2,\beta}\left(\overline{C}\right)$. This proves that $z_{-\infty}$ also satisfies in the classical sense the equation. Moreover,

$$|z(x-(n+1)L) - z(x-nL)| \to 0$$

as $n \to +\infty$ and, this proves that $z_{-\infty}$ is periodic. Since it is also positive, by uniqueness, $z_{-\infty} = \alpha a_1$. □

3.3.4 Characterization of the segregated pulsating fronts

In this subsection, we assume $c_\infty \neq 0$ and we use Proposition 3.7 to get an extracted convergent subsequence of profiles, still denoted $((\varphi_{1,k},\varphi_{2,k}))_{k>k^\star}$, with limit $(\varphi_{1,seg},\varphi_{2,seg})$. Up to an additional extraction, we assume a.e. convergence of $(\varphi_{1,k},\varphi_{2,k},\varphi_{1,k}\varphi_{2,k})$ to $(\varphi_{1,seg},\varphi_{2,seg},0)$. We define $\phi = \alpha\varphi_{1,seg} - d\varphi_{2,seg}$ and $w = \alpha u_{1,seg} - d u_{2,seg}$ (that is, (ϕ, w) is the limit of $((\psi_{d,k}, v_{d,k}))_{k>k^\star}$).

Here, parabolic limit points and traveling limit points are naturally related by the isomorphism $(t,x) \mapsto (x - c_\infty t, x)$. Therefore we can freely use the more convenient system of variables.

3.3.4.1 Definitions and asymptotics

Hereafter,

$$\sigma : z \mapsto \mathbf{1}_{z>0} + \frac{1}{d}\mathbf{1}_{z<0},$$
$$\hat{\sigma} : z \mapsto \mathbf{1}_{z>0} + d\mathbf{1}_{z<0}.$$

Remark. Clearly, for any $z \in \mathcal{C}\left(\mathbb{R}^2\right)$:

- $\sigma[z]$ and $\hat{\sigma}[z]$ are in $L^\infty\left(\mathbb{R}^2\right)$;
- $\sigma[z]$ and $\hat{\sigma}[z]$ vanish if and only if z vanish;
- $\sigma[z]\hat{\sigma}[z] = 1$ in $\mathbb{R}^2$ apart from the zero set of z;
- $\sigma[z]z$ and $\hat{\sigma}[z]z$ are in $\mathcal{C}\left(\mathbb{R}^2\right)$; furthermore, if $z \in W^{1,\infty}\left(\mathbb{R}^2\right)$, then they are Lipschitz-continuous.

Lemma 3.19. *The equalities:*

$$\sigma[w](t,x) = \sigma[\phi](x - c_\infty t, x),$$

$$\hat{\sigma}[w](t,x) = \hat{\sigma}[\phi](x - c_\infty t, x),$$

hold for all $(t,x) \in \mathbb{R}^2$.

Furthermore, the following equalities hold in $L^2_{loc}\left(\mathbb{R}^2\right)$:

$$\partial_t(\sigma[w]w) = \sigma[w]\partial_t w,$$

$$\partial_x w = \hat{\sigma}[w]\partial_x(\sigma[w]w),$$
$$\partial_\xi(\sigma[\phi]\phi) = \sigma[\phi]\partial_\xi\phi,$$
$$\partial_x\phi = \hat{\sigma}[\phi]\partial_x(\sigma[\phi]\phi).$$

Proof. The equalities between the weak derivatives are derived easily from the weak formulation of $(\mathcal{PF})$ (recall the proof of Proposition 3.6). When passing to the limit $k \to +\infty$, it is possible to obtain equivalently all these equations (we restrict ourselves here to parabolic coordinates, the equalities in traveling coordinates being obtained analogously):

$$\sigma[w]\partial_t w - \partial_{xx}w = \eta[w],$$

$$\partial_t(\sigma[w]w) - \partial_{xx}w = \eta[w],$$
$$\partial_t(\sigma[w]w) - \partial_x(\hat{\sigma}[w]\partial_x(\sigma[w]w)) = \eta[w].$$

□

Definition 3.20. Let $s \in \mathbb{R}\backslash\{0\}$ and $\mathcal{C}^1_0\left(\mathbb{R}^2\right)$ be the subset of compactly supported elements of $\mathcal{C}^1\left(\mathbb{R}^2\right)$.

We say that $\varphi \in \mathcal{C}\left(\mathbb{R}^2\right) \cap H^1_{loc}\left(\mathbb{R}^2\right)$ is a weak solution of:

$$-\mathrm{div}(E\nabla\varphi) - s\partial_\xi(\sigma[\varphi]\varphi) = \eta[\varphi] \quad (\mathcal{SPF}[s])$$

if, for any test function $\zeta \in \mathcal{C}^1_0\left(\mathbb{R}^2\right)$:

$$\int E\nabla\varphi.\nabla\zeta + s\int\sigma[\varphi]\varphi\partial_\xi\zeta = \int\eta[\varphi]\zeta.$$

Lemma 3.21. *ϕ is a weak solution of $(\mathcal{SPF}[c_\infty])$.*

Proof. This is merely the traveling formulation of the limiting equation obtained *a priori* in $\mathcal{D}'\left(\mathbb{R}^2\right)$ and *a fortiori* holding in the weak sense. $\square$

Remark. Since $c_\infty \sigma[\phi]\,\partial_\xi \phi$ and $\eta[\phi]$ are in $L^2_{loc}\left(\mathbb{R}^2\right)$, $-\mathrm{div}\,(E\nabla\phi)$ is actually in $L^2_{loc}\left(\mathbb{R}^2\right)$ as well and we can also consider test functions in $L^2_{loc}\left(\mathbb{R}^2\right)$, but then we cannot integrate by parts as in the equality above.

Proposition 3.22. *Let $s \in \mathbb{R}\backslash\{0\}$. If φ is a weak solution of $(\mathcal{SPF}[s])$, then $z:(t,x)\mapsto \varphi(x-st,x)$ is a weak solution of:*

$$\partial_t(\sigma[z]\,z) - \partial_{xx}z = \eta[z],$$

in the sense that for any $\zeta \in \mathcal{C}^1_0\left(\mathbb{R}^2\right)$, the following holds:

$$\int (\sigma[z]\,z\partial_t\zeta - \partial_x z\partial_x\zeta + \eta[z]\,\zeta) = 0.$$

Remark. Similarly, we can restrict ourselves regarding this weak parabolic equation to test functions $\zeta \in L^2_{loc}\left(\mathbb{R}^2\right)$ but then we cannot integrate by parts.

Lemma 3.23. *ϕ is periodic with respect to x and non-increasing with respect to ξ.*

Proof. Thanks to the a.e. convergence, periodicity with respect to x and monotonicity with respect to ξ are preserved a.e., that is at least in a dense subset of $\mathbb{R}^2$. Continuity extends these behaviors everywhere. $\square$

Lemma 3.24. *ϕ is non-zero and sign-changing.*

Remark. This statement holds if and only if both $\varphi_{1,seg}$ and $\varphi_{2,seg}$ are non-zero (or equivalently non-negative non-zero).

Proof. Assume for example $c_\infty < 0$. The normalization gives immediately $\varphi_{1,seg} \neq 0$. If $\varphi_{2,seg} = 0$, $u_{1,seg}$ is a non-negative solution in $\mathbb{R}^2$ of:

$$\partial_t z - \partial_{xx} z = z f_1[z].$$

By the parabolic strong minimum principle, $u_{1,seg} \gg 0$, and by parabolic regularity, $u_{1,seg}$ is regular. By classical parabolic estimates, as $\xi \to -\infty$, $\varphi_{1,seg}$ converges uniformly in x to a positive periodic solution of:

$$-\partial_{xx} z = z f_1[z],$$

that is to a_1. Similarly, $\varphi_{1,seg}$ converges to 0 as $\xi \to +\infty$.

Thus $\varphi_{1,seg}$ is a pulsating front connecting a_1 to 0 at speed $c_\infty < 0$. This is a contradiction (see Theorem 3.3).

A symmetric proof discards the case $c_\infty > 0$. $\square$

In view of these results, we state the following definition.

Definition 3.25. Let:

$$s \in \mathbb{R}\backslash\{0\},$$
$$z \in \mathcal{C}^{0,\beta}_{loc}\left(\mathbb{R}^2\right) \cap H^1_{loc}\left(\mathbb{R}^2\right) \cap L^\infty\left(\mathbb{R}^2\right),$$
$$\varphi : (\xi,x) \mapsto z\left(\frac{x-\xi}{s}, x\right).$$

z is called a *segregated pulsating front with speed s and profile φ* if:

1. φ is a weak solution of $(\mathcal{SPF}[s])$;
2. φ is non-increasing with respect to ξ;
3. φ is periodic with respect to x;
4. φ is non-zero and sign-changing.

Corollary 3.26. *w is a segregated pulsating front with speed c_∞ and profile ϕ.*

Proposition 3.27. *Let z be a segregated pulsating front with profile φ. As $\xi \to +\infty$,*

$$\max_{x\in\overline{C}} |\varphi(-\xi, x) - \alpha a_1| + \max_{x\in\overline{C}} |\varphi(\xi, x) + d a_2| \to 0.$$

Proof. It follows from classical parabolic estimates and the monotonicity of φ with respect to ξ. □

3.3.4.2 The intrinsic free boundary problem

We intend to conclude the characterization of the segregated pulsating front with a uniqueness result. Our proof will use a sliding argument and the continuity of $\partial_x z$. Obviously, in $\mathbb{R}^2 \backslash z^{-1}(\{0\})$, classical parabolic regularity applies and the regularity of a segregated pulsating front is only limited by that of η. On the contrary, the regularity of z at the free boundary $z^{-1}(\{0\})$ is a tough problem and, as usual in free boundary problems, requires a detailed study of the regularity of the free boundary itself. This study is the object of the following pages.

Let us stress here that our interest does not lie in the most general study of the free boundaries of the solutions of $(\mathcal{SPF}[s])$. To show that $\partial_x z$ is continuous, Lipschitz-continuity of the free boundary is sufficient, and we are able to prove such a regularity only using the monotonicity properties of the segregated pulsating fronts as well as the parabolic maximum principle. We believe that this proof has interest of its own. Yet, at the end of this subsection, we will explain why we expect the free boundary to actually be $\mathcal{C}^1$ and $\partial_t z$ to be continuous without any additional assumption.

Up to the next subsection, let z be a segregated pulsating front with speed $s \neq 0$ and profile φ and let:

$$\Gamma = \left\{(t,x) \in \mathbb{R}^2 \mid z(t,x) = 0\right\},$$
$$\Omega_+ = \left\{(t,x) \in \mathbb{R}^2 \mid z(t,x) > 0\right\},$$
$$\Omega_- = \left\{(t,x) \in \mathbb{R}^2 \mid z(t,x) < 0\right\}.$$

Before going any further, let us state precisely the results of this subsection in the following proposition.

Theorem 3.28. *There exists a continuous bijection $\Xi : \mathbb{R} \to \mathbb{R}$ such that Γ is the graph of Ξ and such that:*

$$\begin{cases} \Omega_+ = \left\{(t,x) \in \mathbb{R}^2 \mid x < \Xi(t)\right\} \\ \Omega_- = \left\{(t,x) \in \mathbb{R}^2 \mid x > \Xi(t)\right\}. \end{cases}$$

Moreover, $\partial_x z \in \mathcal{C}^{0,\beta}\left(\mathbb{R}^2\right)$ and $(\partial_x z)_{|\Gamma} \ll 0$.

Remark. Of course, this type of result is strongly reminiscent of the celebrated paper by Angenent [4] about the number of zeros of a solution of a parabolic equation. We stress that this result cannot be applied here because of the non-linearity due to $\sigma[z]$. It will be clearly established during the proof that this lack of regularity is compensated here by the monotonicity of z.

The proof of Theorem 3.28 begins with a couple of lemmas leading to the existence of Ξ.

Lemma 3.29. *The quantities:*

$$\Xi_{+}(t) = \sup\{x \in \mathbb{R} \mid z(t,x) > 0\},$$

$$\Xi_{-}(t) = \inf\{x \in \mathbb{R} \mid z(t,x) < 0\},$$

are well-defined and finite.

Proof. By Proposition 3.27, for any $(t,x) \in \mathbb{R}^2$:

$$\lim_{n\to+\infty} \max_{x\in\overline{C}} |\varphi(x+nL-st,x)+da_2| = 0,$$

$$\lim_{n\to+\infty} \max_{x\in\overline{C}} |\varphi(x-nL-st,x)-\alpha a_1| = 0.$$

By periodicity with respect to x:

$$\begin{aligned}\varphi(x\pm nL-st,x) &= \varphi(x\pm nL-st,x\pm nL)\\ &= z(t,x\pm nL)\end{aligned}$$

and thus $x\mapsto z(t,x)$ is negative at $+\infty$, positive at $-\infty$, whence $\Xi_{+}(t)$ and $\Xi_{-}(t)$ are well-defined and finite. □

Lemma 3.30. *Let $(t,x) \in \mathbb{R}^2$.*

1. *If $s > 0$ and $z(t,x) \leq 0$, then for any $y > x$, $z(t,y) < 0$.*
2. *If $s < 0$ and $z(t,x) \geq 0$, then for any $y < x$, $z(t,y) > 0$.*

Proof. Let us show for instance the first statement, the other one being symmetric.

By Lemma 3.29, there exists $X > x$ such that $z(t,X) < 0$. Since φ is non-increasing with respect to ξ, z is non-decreasing with respect to t, whence for any $t' < t$, $z(t',x) \leq 0$ and $z(t',X) < 0$. Moreover, by Proposition 3.27, there exists $T > 0$ such that:

$$z(t-T,y) < 0 \text{ for any } y \in [x,X].$$

By continuity of z, there exists $\tau > 0$ such that:

$$z \ll 0 \text{ in } [t-T,t-T+\tau]\times[x,X].$$

Let:

$$\tau^\star = \sup\{\tau\in(0,T) \mid z\ll 0 \text{ in } [t-T,t-T+\tau]\times(x,X)\}$$

and let us check that $\tau^\star = T$.

If $\tau^\star < T$, then there exists $y \in (x,X)$ such that $z(t-T+\tau^\star,y) = 0$. But in the parabolic cylinder $[t-T,t-T+\tau^\star]\times[x,X]$, $z<0$ satisfies a regular parabolic equation and satisfies also the strong parabolic maximum principle, which immediately contradicts the strict sign of z at $t-T$.

Thus $\tau^\star = T$ and then, if there exists $y \in (x,X)$ such that $z(t,y) = 0$, applying once more the strong parabolic maximum principle gives the same contradiction.

The proof is ended by passing to the limit $X \to +\infty$. □

Corollary 3.31. *For any $t \in \mathbb{R}$, the zero of $x \mapsto z(t,x)$ is unique, or equivalently, $\Xi_{+}(t) = \Xi_{-}(t)$.*

Lemma 3.32. *For any $t \in \mathbb{R}$, let $\Xi(t)$ be the unique zero of $x \mapsto z(t,x)$.*

Then $\Xi : \mathbb{R} \to \mathbb{R}$ is unbounded, non-decreasing if $s > 0$ and non-increasing if $s < 0$, and continuous.

Furthermore, Γ is exactly the graph of Ξ,

$$\Omega_- = \left\{(t,x) \in \mathbb{R}^2 \mid x > \Xi(t)\right\},$$

$$\Omega_+ = \left\{(t,x) \in \mathbb{R}^2 \mid x < \Xi(t)\right\}.$$

Proof. Assume for instance and up to the end of the proof $s > 0$ (the case $s < 0$ is similar).

Since φ is non-increasing with respect to ξ, z is non-decreasing with respect to t. Assume by contradiction that there exists $t, t' \in \mathbb{R}$ such that $t' < t$ and $\Xi(t) < \Xi(t')$. By Lemma 3.30, for any $x > \Xi(t)$, $z(t,x) < 0$, whence in particular $z(t, \Xi(t')) < 0$, whence by monotonicity of z, $z(t', \Xi(t')) < 0$, which contradicts the definition of $\Xi(t')$. Thus Ξ is non-decreasing.

The unboundedness is straightforward: considering the limiting signs of $t \mapsto z(t,x)$ shows by continuity that this function has at least one zero for any $x \in \mathbb{R}$. But if Ξ was bounded, thanks to Lemma 3.30 once again, it would be possible to build a counter-example.

Finally, continuity is also straightforward, since it is well-known that a monotonic function admits left-sided and right-sided limits at every point and that every discontinuity it has is a jump discontinuity. The existence of such a discontinuity, that is of a segment $\{t^\star\} \times [x^\star, x^\star + X]$ included in the free boundary, would immediately contradict Lemma 3.30. ☐

Corollary 3.33. *Both Ω_+ and Ω_- have a Lipschitz boundary.*

Proof. It suffices to recall that every point of the graph of a monotone function satisfies an interior cone condition and that such a condition characterizes Lipschitz boundaries. ☐

In view of this regularity of $\Omega_\pm$ and by means of easy integration by parts, we are now able to generalize to any segregated pulsating front a property that was immediately satisfied by w (Lemma 3.19).

Corollary 3.34. *The following equalities hold in $L^2_{loc}\left(\mathbb{R}^2\right)$:*

$$\partial_t \left(\sigma[z]\, z\right) = \sigma[z]\, \partial_t z,$$

$$\partial_x z = \hat{\sigma}[z]\, \partial_x \left(\sigma[z]\, z\right),$$

$$\partial_\xi \left(\sigma[\varphi]\, \varphi\right) = \sigma[\varphi]\, \partial_\xi \varphi,$$

$$\partial_x \varphi = \hat{\sigma}[\varphi]\, \partial_x \left(\sigma[\varphi]\, \varphi\right).$$

Proof. Let us show for instance the first one. Let $(\zeta_n)_{n\in\mathbb{N}} \in \left(\mathcal{D}\left(\mathbb{R}^2\right)\right)^{\mathbb{N}}$ such that (ζ_n) converges in L^2_{loc} to some test function $\zeta \in L^2_{loc}$. For any $n \in \mathbb{N}$, we have:

$$\begin{aligned} \int \partial_t \left(\sigma[z]\, z\right) \zeta_n &= -\int \sigma[z]\, z \partial_t \zeta_n \\ &= -\int_{\Omega_+} z \partial_t \zeta_n - \int_{\Omega_-} \frac{1}{d} z \partial_t \zeta_n. \end{aligned}$$

Since $\Omega_\pm$ have a Lipschitz boundary, we can integrate by parts once again (recalling that, by definition, $z_{|\Gamma} = 0$):

$$\begin{aligned} \int \partial_t \left(\sigma[z]\, z\right) \zeta_n &= \int_{\Omega_+} \partial_t z \zeta_n + \int_{\Omega_-} \frac{1}{d} \partial_t z \zeta_n \\ &= \int \sigma[z]\, \partial_t z \zeta_n. \end{aligned}$$

Passing to the limit $n \to +\infty$ ends the proof. □

More interestingly, we are now closer to an explicit free boundary condition. The following three lemmas are dedicated to this question.

Lemma 3.35. *Let Ξ be defined as in Lemma 3.32.*

Then the traces $(\partial_x z^+)_{|\partial\Omega_+}$ and $(\partial_x z^-)_{|\partial\Omega_-}$ are well-defined in $L^2_{loc}(\partial\Omega_+)$ and $L^2_{loc}(\partial\Omega_-)$ respectively.

Proof. Since $\partial\Omega_+$ (respectively $\partial\Omega_-$) is a Lipschitz boundary, let us prove that $(\partial_x(z^+))_{|\Omega_+}$ (resp.$(\partial_x(z^-))_{|\Omega_-}$) is in $H^1_{loc}(\Omega_+)$ (resp. $H^1_{loc}(\Omega_-)$). It is already established that it is in $L^2_{loc}(\mathbb{R}^2)$. Considering the equation satisfied by z then shows immediately that $(\partial_{xx}(z^+))_{|\Omega_+}$ (resp. $(\partial_{xx}(z^-))_{|\Omega_-}$) is in $L^2_{loc}(\mathbb{R}^2)$ as well. To conclude, it remains to prove that $(\partial_{tx}(z^+))_{|\Omega_+}$ (resp. $(\partial_{tx}(z^-))_{|\Omega_-}$) is in $L^2_{loc}(\Omega_+)$ (resp. $L^2_{loc}(\Omega_-)$).

Let $t_1, t_2, x_1, x_2 \in \mathbb{R}$ such that $t_1 < t_2$, $x_1 < x_2$ and $[t_1, t_2] \times [x_1, x_2] \subset \Omega_+$. Let $\chi \in \mathcal{D}(\mathbb{R}^2)$ be a non-negative non-zero function identically equal to 1 in $[t_1, t_2] \times [x_1, x_2]$. From the following equation, satisfied in the classical sense in Ω_+:

$$\partial_t(\partial_t z) - \partial_{xx}(\partial_t z) = g_1\left[\frac{z}{\alpha}\right]\partial_t z,$$

multiplied by $\partial_t z\chi$ and integrated over $\mathbb{R}^2$, we deduce:

$$-\int \frac{1}{2}|\partial_t z|^2\partial_t\chi + \int |\partial_{xt}z|^2\chi - \frac{1}{2}\int |\partial_t z|^2\partial_{xx}\chi = \int g_1\left[\frac{z}{\alpha}\right]|\partial_t z|^2\chi.$$

It follows that there exists a constant $R > 0$ such that :

$$\|\partial_{xt}z\|^2_{L^2([t_1,t_2]\times[x_1,x_2])} \leq R\|\partial_t z\|_{L^2([t_1,t_2]\times[x_1,x_2])}\|\chi\|_{H^2(\mathbb{R}^2)},$$

whence $\partial_{tx}z^+ \in L^2_{loc}(\Omega_+)$ indeed.

Similarly, $\partial_{tx}(z^-) \in L^2_{loc}(\Omega_-)$.

In the end, Ω_+ and Ω_- are Lipschitz domains, $(\partial_x(z^+))_{|\Omega_+} \in H^1_{loc}(\Omega_+)$ and $(\partial_x(z^-))_{|\Omega_-} \in H^1_{loc}(\Omega_-)$, whence their traces can be rigorously defined in $L^2_{loc}(\partial\Omega_+)$ and $L^2_{loc}(\partial\Omega_-)$ respectively. □

Lemma 3.36. *Let Ξ be defined as in Lemma 3.32.*

For any non-negative test function with compact support $\zeta \in \mathcal{C}^1_0(\mathbb{R}^2)$, the following equalities hold:

$$\int_{\Omega_+}(\sigma[z]z\partial_t\zeta - \partial_x z\partial_x\zeta + \eta[z]\zeta) = \int_{\partial\Omega_+}\partial_x z\zeta,$$

$$\int_{\Omega_-}(\sigma[z]z\partial_t\zeta - \partial_x z\partial_x\zeta + \eta[z]\zeta) = \int_{\partial\Omega_-}\partial_x z\zeta.$$

Proof. We prove the equality concerning Ω_+, the other one being similar.

First, it is straightforward that:

$$(\sigma[z])_{|\Omega_+} = 1.$$

Let $\varepsilon > 0$ and:

$$\Omega^\varepsilon_+ = \left\{(t,x) \in \mathbb{R}^2 \mid \Xi(t) - \varepsilon \leq x < \Xi(t)\right\}.$$

Then:

$$\int_{\Omega_+} \left(\sigma\left[z\right] z\partial_t\zeta - \partial_x z\partial_x\zeta\right) = \int_{\Omega_+^\varepsilon} \left(z\partial_t\zeta - \partial_x z\partial_x\zeta\right) + \int_{\Omega_+\backslash\Omega_+^\varepsilon} \left(z\partial_t\zeta - \partial_x z\partial_x\zeta\right)$$

Let

$$\tau_\varepsilon : x \mapsto \inf\left\{t\in\mathbb{R} \mid \Xi\left(t\right) = x+\varepsilon\right\}.$$

This function is increasing, piecewise-continuous, measurable and satisfies the following equality:

$$\mathbf{1}_{\Omega_+\backslash\Omega_+^\varepsilon} = \mathbf{1}_{\{(t,x)\in\mathbb{R}^2 \ \mid \ \tau_\varepsilon(x)\leq t\}}.$$

By integration by parts and using the equation satisfied by z in $\Omega_+\backslash\Omega_+^\varepsilon$:

$$\begin{aligned}\int_{\Omega_+\backslash\Omega_+^\varepsilon} \left(z\partial_t\zeta - \partial_x z\partial_x\zeta\right) &= -\int_{\Omega_+\backslash\Omega_+^\varepsilon} \eta\left[z\right]\zeta \\ &\quad -\int_{\mathbb{R}} \partial_x z\left(t,\Xi\left(t\right)-\varepsilon\right)\zeta\left(t,\Xi\left(t\right)-\varepsilon\right)\mathrm{d}t \\ &\quad -\int_{\mathbb{R}} z\left(\tau_\varepsilon\left(x\right),x\right)\zeta\left(\tau_\varepsilon\left(x\right),x\right)\mathrm{d}x.\end{aligned}$$

By the Cauchy–Schwarz inequality and dominated convergence, as $\varepsilon\to 0$:

$$\int_{\Omega_+^\varepsilon} \left(z\partial_t\zeta - \partial_x z\partial_x\zeta\right) \to 0,$$

$$\int_{\Omega_+\backslash\Omega_+^\varepsilon} \eta\left[z\right]\zeta \to \int_{\Omega_+} \eta\left[z\right]\zeta,$$

$$\int_{\mathbb{R}} z\left(\tau_\varepsilon\left(x\right),x\right)\zeta\left(\tau_\varepsilon\left(x\right),x\right)\mathrm{d}x \to 0.$$

Therefore, the following convergence holds as $\varepsilon\to 0$:

$$-\int_{\mathbb{R}} \partial_x z\left(t,\Xi\left(t\right)-\varepsilon\right)\zeta\left(t,\Xi\left(t\right)-\varepsilon\right)\mathrm{d}t \to \int_{\Omega_+} \left(\sigma\left[z\right] z\partial_t\zeta - \partial_x z\partial_x\zeta\right) + \int_{\Omega_-} \eta\left[z\right]\zeta.$$

Lemma 3.35 indicates that the trace of $\partial_x z\zeta$ at $\partial\Omega_+$ is well-defined in L^2. Therefore, it remains to show that:

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}} \partial_x z\left(t,\Xi\left(t\right)-\varepsilon\right)\zeta\left(t,\Xi\left(t\right)-\varepsilon\right)\mathrm{d}t = -\int_{\partial\Omega_+} \partial_x z\zeta$$

Define, for any $\varepsilon>0$:

$$z_\varepsilon : \left(t,x\right) \mapsto z\left(t,x-\varepsilon\right),$$

$$\zeta_\varepsilon : \left(t,x\right) \mapsto \zeta\left(t,x-\varepsilon\right).$$

It is clear that the trace of $\partial_x z_\varepsilon\zeta_\varepsilon$ is well-defined in L^2 as well and satisfies:

$$\int_{\mathbb{R}} \partial_x z\left(t,\Xi\left(t\right)-\varepsilon\right)\zeta\left(t,\Xi\left(t\right)-\varepsilon\right)\mathrm{d}t = \int_{\partial\Omega_+} \left(-1\right)\partial_x z_\varepsilon\zeta_\varepsilon.$$

Now, by virtue of the trace's theorem, there exists a constant $R>0$ such that :

$$\|\partial_x z_\varepsilon\zeta_\varepsilon - \partial_x z\zeta\|_{L^2(\partial\Omega_+)} \leq R\|\partial_x z_\varepsilon\zeta_\varepsilon - \partial_x z\zeta\|_{H^1(\Omega_+)}.$$

Integrating by parts and using the continuity of z and $\partial_x\zeta$, it is easily deduced that the right-hand side converges to 0 as $\varepsilon\to 0$. Hence the claimed result follows. ☐

We can now prove that Ξ is bijective and that a free boundary condition is satisfied in a weak sense.

Lemma 3.37. *Let Ξ be defined as in Lemma 3.32.*
Then Ξ is bijective and the functions:

$$z_{x,-}: t \mapsto (\partial_x z)_{|\partial\Omega_-}(t, \Xi(t)),$$

$$z_{x,+}: t \mapsto (\partial_x z)_{|\partial\Omega_+}(t, \Xi(t)),$$

where $(\partial_x z)_{|\partial\Omega_\pm}$ are the traces of $\partial_x z$ at each side of Γ, are in $L^2_{loc}(\mathbb{R})$ and are equal a.e..
Furthermore, if $s > 0$, $z_{x,-} \ll 0$, and if $s < 0$, $z_{x,+} \ll 0$.

Proof. Assume for instance $s > 0$, the other case being similar.
First, we prove the a.e. equality of $z_{x,+}$ and $z_{x,-}$, as well as the sign of $z_{x,-}$.
Let $\zeta \in \mathcal{C}^1_0(\mathbb{R}^2)$ be any non-negative test function and let $\zeta_\Gamma : t \mapsto \zeta(t, \Xi(t))$. By Lemma 3.36:

$$\int_{\partial\Omega_+} \partial_x z \zeta + \int_{\partial\Omega_-} \partial_x z \zeta = 0$$

where the unit vector normal to $\partial\Omega_+$ is the opposite of the one normal to $\partial\Omega_-$, whence we obtain:

$$\int_{\mathbb{R}} z_{x,+}\zeta_\Gamma = \int_{\mathbb{R}} z_{x,-}\zeta_\Gamma.$$

That is, for a.e. t, $z_{x,+}(t) = z_{x,-}(t)$, or, in other words, for a.e. $t \in \mathbb{R}$, $x \mapsto \partial_x z(t,x)$ is continuous. The sign of $z_{x,-}(t)$ follows directly from Hopf's lemma applied at the vertex $(t, \Xi(t))$ of the smooth parabolic cylinder $(t-1, t) \times (\Xi(t), \Xi(t)+1)$.

Then, it is clear that a continuous unbounded real-valued function is necessarily surjective, whence Ξ is bijective if and only if it is injective (or equivalently if and only if it is strictly monotonic). We are going to prove directly that Ξ is injective.

Differentiating (firstly in the distributional sense) the equation satisfied by z with respect to t in $\mathbb{R}^2 \backslash \Gamma$ yields the following regular and linear parabolic equations:

$$\begin{cases} \partial_t(\partial_t z) - \partial_{xx}(\partial_t z) - \alpha g_1\left[\frac{z}{\alpha}\right]\partial_t z = 0 & \text{in } \Omega_+ \\ \partial_t(\partial_t z) - d\partial_{xx}(\partial_t z) + dg_2\left[-\frac{z}{d}\right]\partial_t z = 0 & \text{in } \Omega_-. \end{cases}$$

Let $x \in \mathbb{R}$. Assume that $\Xi^{-1}(\{x\})$ is not a singleton. By (large) monotonicity, it is then a segment, say $[t_1, t_2]$. Applying classical parabolic regularity on this system of equations in $(t_1, t_2) \times (x, x+1)$ shows that $\partial_t z$ is $\mathcal{C}^1$ with respect to t and $\mathcal{C}^2$ with respect to x up to $(t_1, t_2) \times \{x\}$. Moreover, $\partial_t z = 0$ along $(t_1, t_2) \times \{x\}$. By classical parabolic regularity and Hopf's lemma, for any $t \in (t_1, t_2)$, the right-sided and the left-sided limit of $\partial_x \partial_t z(t, y)$ as $y \to x$ exists and have opposite sign.

Remark that, away from Γ, the equations satisfied by z, $\partial_t z$ and $\partial_x z$ suffice to show that $z \in \mathcal{C}^2(\Omega_+) \cap \mathcal{C}^2(\Omega_-)$. Therefore Schwarz' theorem can be applied away from Γ.

Thus, for any $t, t' \in (t_1, t_2)$ and some $\varepsilon > 0$ small enough, we get:

$$\begin{aligned} \partial_x z(t, x \pm \varepsilon) - \partial_x z(t', x \pm \varepsilon) &= \int_{t'}^{t} \partial_t \partial_x z(\tau, x \pm \varepsilon)\,\mathrm{d}\tau \\ &= \int_{t'}^{t} \partial_x \partial_t z(\tau, x \pm \varepsilon)\,\mathrm{d}\tau. \end{aligned}$$

These two integrals have an opposite strict sign: with respect to t, $\partial_x z$ is decreasing on one side of $(t_1, t_2) \times \{x\}$ and increasing on the other. This contradicts the fact that, for a.e. $t \in \mathbb{R}$, $x \mapsto \partial_x z\,(t,x)$ is continuous (see the first step of the proof). Therefore for any $x \in \mathbb{R}$, $\mathbb{R} \times \{x\} \cap \Gamma$ is a singleton, whence Ξ is bijective. □

Corollary 3.38. *The function $x \mapsto x - s\Xi^{-1}(x)$ is continuous and periodic. Furthermore,*

$$\left\{\left(x - s\Xi^{-1}(x), x\right) \in \mathbb{R}^2 \mid x \in \mathbb{R}\right\} = \varphi^{-1}(\{0\}).$$

Proof. The periodicity comes from the periodicity with respect to x of φ. □

Remark. This corollary confirms that, roughly speaking, the free boundary is located near the straight line of equation $x = st + \Xi(0)$. In other words, Ξ can be represented as the sum of $t \mapsto st$ and a $\frac{L}{s}$-periodic function Ξ_{per}.

Corollary 3.39. *The monotonicity of z with respect to t is strict. Equivalently, φ is decreasing with respect to ξ.*

Proof. Just apply the strong maximum principle to the equations satisfied by $\partial_t z$ in each component of $\mathbb{R}^2\backslash\Gamma$ to get that, in $\mathbb{R}^2\backslash\Gamma$, $\partial_t z \gg 0$ if $s > 0$ and $\partial_t z \ll 0$ if $s < 0$, which is sufficient to obtain strict monotonicity since the measure of Γ (as a measurable subset of $\mathbb{R}^2$) is zero. □

Now, thanks to a technique developed by Aronson for the porous media equation [7], we are able to prove the continuity of $\partial_x z$.

Lemma 3.40. *Let Ξ be defined as in Lemma 3.32 and $z_{x,+}$ and $z_{x,-}$ be defined as in Lemma 3.37.*

If $s > 0$ (respectively $s < 0$), $z_{x,+}(t)$ (resp. $z_{x,-}(t)$) is actually defined for any $t \in \mathbb{R}$. Moreover, the function $z_{x,+}$ (resp. $z_{x,-}$) is non-positive and locally uniformly bounded from below.

Proof. We only prove the result in the case $s > 0$, the other one being symmetric.

Let $t \in \mathbb{R}$ and $x, x' \in \mathbb{R}$ such that $x < x' < \Xi(t)$. For any $\tilde{x} \in (x, x')$,

$$\partial_{xx} z\,(t,\tilde{x}) = \partial_t z\,(t,\tilde{x}) - z\,(t,\tilde{x})\, f_1\left(z\,(t,\tilde{x}),\tilde{x}\right).$$

On one hand, the term $z\,(t,\tilde{x})\, f_1\left(z\,(t,\tilde{x}),\tilde{x}\right)$ is bounded from below by 0 and from above by a constant R independent on $\tilde{x}$. On the other hand, $\partial_t z\,(t,\tilde{x}) > 0$. Thus:

$$\partial_{xx} z\,(t,\tilde{x}) \geq -R.$$

Integrating this inequality, we obtain:

$$\partial_x z\,(t,x') \geq \partial_x z\,(t,x) - R\,(x' - x).$$

It follows that:

$$\liminf_{x' \to \Xi(t)} \partial_x z\,(t,x') \geq \partial_x z\,(t,x) - R\,(\Xi(t) - x),$$

and then:

$$\liminf_{x' \to \Xi(t)} \partial_x z\,(t,x') \geq \limsup_{x \to \Xi(t)} \partial_x z\,(t,x).$$

Hence:

$$\lim_{x \to 0, x > 0} \partial_x z\,(t, \Xi(t) - x)$$

exists. From the sign of z in Ω_+, it is clear that it is non-positive. Using once more the inequality:

$$\liminf_{x'\to\Xi(t)} \partial_x z\left(t,x'\right) \geq \partial_x z\left(t,x\right) - R\left(\Xi\left(t\right)-x\right)$$

together with the local boundedness of $\partial_x z$ in Ω_+, it follows that the limit is locally uniformly bounded from below. Finally, it necessarily coincides with $z_{x,+}\left(t\right)$. □

Corollary 3.41. $\partial_x z \in L^\infty\left(\mathbb{R}^2\right)$.

Lemma 3.42. *We have* $\partial_x z \in \mathcal{C}^{0,\beta}_{loc}\left(\mathbb{R}^2\right)$.

Proof. Let $\zeta \in \mathcal{C}^2_0\left(\mathbb{R}^2\right)$. Choosing as test functions in the weak formulation in L^2_{loc} of:

$$\sigma\left[z\right]\partial_t z - \partial_{xx} z = \eta\left[z\right]$$

a sequence of smooth functions converging in $L^2_{loc}\left(\mathbb{R}^2\right)$ to $\hat{\sigma}\left[z\right]\partial_x\zeta$, we obtain:

$$\int \partial_t z \partial_x \zeta - \int \hat{\sigma}\left[z\right]\partial_{xx} z \partial_x \zeta = \int \hat{\sigma}\left[z\right]\eta\left[z\right]\partial_x\zeta.$$

Remarking the following equalities:

$$\begin{aligned}\int \partial_t z \partial_x \zeta &= -\int z \partial_t\left(\partial_x\zeta\right) \\ &= -\int z\partial_x\left(\partial_t\zeta\right) \\ &= \int \partial_x z \partial_t \zeta,\end{aligned}$$

$$\begin{aligned}\int \hat{\sigma}\left[z\right]\eta\left[z\right]\partial_x\zeta &= -\int \partial_x\left(\hat{\sigma}\left[z\right]\eta\left[z\right]\right)\partial_x\zeta \\ &= -\int \hat{\sigma}\left[z\right]\partial_x\left(\eta\left[z\right]\right)\partial_x\zeta,\end{aligned}$$

(where, by virtue of $(\mathcal{H}_1)$, $\partial_x\left(\eta\left[z\right]\right)$ is piecewise-continuous and *a fortiori* is in $L^\infty\left(\mathbb{R}^2\right)$), we deduce:

$$-\int \partial_x z \partial_t \zeta + \int \hat{\sigma}\left[z\right]\partial_{xx} z \partial_x\zeta = \int \hat{\sigma}\left[z\right]\partial_x\left(\eta\left[z\right]\right)\zeta.$$

Hence we can once more apply DiBenedetto's theory [55]: $\partial_x z$, which is both in $L^\infty\left(\mathbb{R}^2\right)$ and in $\mathcal{C}_{loc}\left(\mathbb{R}, L^2_{loc}\left(\mathbb{R}\right)\right)$ (by classical parabolic estimates similar to those detailed previously in the proof of Proposition 3.6), is a locally bounded weak solution of:

$$\partial_t Z - \partial_x\left(\hat{\sigma}\left[z\right]\partial_x Z\right) = \hat{\sigma}\left[z\right]\partial_x\left(\eta\left[z\right]\right)$$

and therefore is locally Hölder-continuous indeed. □

Remark. Let us explain here why $\partial_t z$ is very likely to be continuous as well (equivalently, Ξ is very likely to be continuously differentiable). There are in fact some articles related to this free boundary problem and although none of them is exactly what we need here, they strongly lead to this conjecture (let us cite for instance Evans [66], Cannon–Yin [33] and Jensen [99]).

Roughly speaking, the idea would be to regularize ($\mathcal{SPF}[s]$), to show the uniqueness of the weak solution of the problem written in divergence form, to prove thanks to the maximum principle that the regularization of $\| (\partial_t z)(\partial_x z)^{-1} \|_{L^\infty}$ is bounded uniformly with respect to the regularization, to obtain consequently that Ξ is Lipschitz-continuous, and then to deduce from Caffarelli's classical results about one–phase Stefan problems [32] that $\Xi \in \mathcal{C}^1(\mathbb{R})$, whence finally $\partial_t z \in \mathcal{C}(\mathbb{R}^2)$.

Since we do not need such results to conclude this study about pulsating fronts, we choose not to investigate further in this direction. Nevertheless, the rigorous proof of the continuity of $\partial_t z$ in the more general framework of weak solutions of ($\mathcal{SPF}[s]$) might be the object of a future follow-up to this article.

Let us conclude this subsection with the following corollary, which takes into account the previous remark and gives an interesting formula.

Corollary 3.43. *If $d = 1$, then $\partial_t z, \partial_{xx} z \in \mathcal{C}^{0,\beta}_{loc}(\mathbb{R}^2)$ and $\Xi \in \mathcal{C}^1(\mathbb{R})$.*

If $d \neq 1$ and if $\partial_t z \in L^\infty(\mathbb{R}^2)$, then $\Xi \in \mathcal{C}^1(\mathbb{R})$, $\partial_t z \in \mathcal{C}(\mathbb{R}^2)$, $\hat{\sigma}[z]\partial_{xx} z \in \mathcal{C}(\mathbb{R}^2)$ and the following equality holds for any $t \in \mathbb{R}$:

$$\Xi'(t) = \frac{d}{1-d} \frac{\lim_{\varepsilon \to 0, \varepsilon > 0} \left(\partial_{xx} z(t, \Xi(t) - \varepsilon) - \partial_{xx} z(t, \Xi(t) + \varepsilon)\right)}{\partial_x z(t, \Xi(t))}.$$

Proof. Regularity in the symmetrical case $d = 1$ follows from classical parabolic regularity.

Provided $d \neq 1$ and global boundedness of $\partial_t z$, let $\varepsilon > 0$ small enough so that the implicit function theorem can be applied at the level set $z^{-1}(\{\pm\varepsilon\})$. There exists $\Xi_{\pm\varepsilon} \in \mathcal{C}^1(\mathbb{R})$ such that $\Xi_{+\varepsilon} \ll \Xi \ll \Xi_{-\varepsilon}$ and such that:

$$\Xi'_{\pm\varepsilon}(t) = -\frac{\partial_t z(t, \Xi_{\pm\varepsilon}(t))}{\partial_x z(t, \Xi_{\pm\varepsilon}(t))}.$$

Passing to the limit $\varepsilon \to 0$, we deduce that Ξ is Lipschitz-continuous. Then, by Caffarelli [32], $\partial_t z, \partial_{xx} z \in \mathcal{C}(\overline{\Omega_+}) \cap \mathcal{C}(\overline{\Omega_-})$ and $\Xi \in \mathcal{C}^1(\mathbb{R})$. Thus $\Xi_{\pm\varepsilon} \to \Xi$ in $\mathcal{C}^1_{loc}(\mathbb{R})$ as $\varepsilon \to 0$, whence $\partial_t z$ is moreover continuous at Γ. Then, since $\hat{\sigma}[z]\partial_{xx} z = \partial_t z - \hat{\sigma}[z]\eta[z]$, $\hat{\sigma}[z]\partial_{xx} z$ is continuous in $\mathbb{R}^2$ as well. Finally, the formula relating Ξ' to the jump discontinuity of $\partial_{xx} z$ is easily obtained:

$$\begin{aligned}
\lim_{\varepsilon \to 0, \varepsilon > 0} \left(\partial_{xx} z(t, \Xi(t) - \varepsilon) - \partial_{xx} z(t, \Xi(t) + \varepsilon)\right) &= \lim_{\varepsilon \to 0, \varepsilon > 0} \left(\partial_t z(t, \Xi(t) - \varepsilon) - \frac{1}{d}\partial_t z(t, \Xi(t) + \varepsilon)\right) \\
&\quad + \lim_{\varepsilon \to 0, \varepsilon > 0} \left(\eta[z](t, \Xi(t) - \varepsilon) - \eta[z](t, \Xi(t) + \varepsilon)\right) \\
&= \left(1 - \frac{1}{d}\right)\partial_t z(t, \Xi(t)) \\
&= -\left(1 - \frac{1}{d}\right)\Xi'(t)\,\partial_x z(t, \Xi(t)).
\end{aligned}$$

□

3.3.4.3 Uniqueness

We are now able to end our characterization.

Theorem 3.44. *Let z_1 and z_2 be segregated pulsating fronts with respective speeds $s_1 \neq 0$ and $s_2 \neq 0$ and respective profiles φ_1 and φ_2.*

Then $s_1 = s_2$ and there exists $\tau \in \mathbb{R}$ such that $\varphi_1^\tau = \varphi_2$, where $\varphi_1^\tau : (\xi, x) \mapsto \varphi_1(\xi - \tau, x)$.

In other words, the speed is unique and the profile is unique up to translation with respect to ξ.

Proof. We are going to use once more the sliding method. Remark that, up to the free boundary, this is the most simple case: bistable scalar equation. Therefore we refer to the proof of Lemma 3.4 for the details and only point out here some technical differences due to the presence of the free boundary.

Step 1: existence of a translation of the profile associated with the highest speed such that it is locally below the other profile.

Here it is useful to additionally require that, at ζ, the upper profile is positive (uniformly with respect to x) whereas the lower profile is negative (uniformly as well). This will simplify some arguments in Steps 2, 3, 4 and 5 since it is now clear that the contact points $(\xi^\star, x^\star)$ are necessarily located away from the free boundary, whence the arguments of the usual sliding method for regular pulsating fronts (Berestycki–Hamel [14]) apply straightforwardly.

Step 2: up to some extra term, this ordering is global on the left.

No new idea here: multiply the upper profile by some $\kappa \geq 1$.

Step 3: this extra term is actually unnecessary, thanks to the maximum principle.

Similarly, there is no new idea here as well and it follows easily that $\kappa^\star = 1$.

Step 4: up to some (possibly different) extra term, this ordering is global on the right.

Thanks to the underlying symmetry due to the bistable structure, the proof of this step is much simpler here: just change every profile into its opposite and repeat straightforwardly Step 2.

Step 5: this (possibly different) extra term is also unnecessary.

Similarly, repeat Step 3 to prove that $\kappa^\star = 1$.

Step 6: thanks to the maximum principle again, the speeds are equal and the profiles are equal up to some translation.

This is the step which requires additional care because of the free boundary. To this end, let us introduce some notations.

We assume that $s_1 \leq s_2$. Let:

$$v_2 : (t,x) \mapsto \varphi_2 \left(x - s_1 t, x\right),$$

$$v_1^{\tau^\star} : (t,x) \mapsto \varphi_1 \left(x - s_1 t - \tau^\star, x\right),$$

$$v = v_2 - v_1^{\tau^\star},$$

where $\tau^\star$ is defined as in Lemma 3.4.

At this step of the proof, it is established that $v \geq 0$. Let $\mathcal{Z} = v^{-1}\left(\{0\}\right)$. With the same argument as in Lemma 3.4, we can discard the possibility $\mathcal{Z} = \emptyset$. Now there are basically three cases.

1. There exists $(t^\star, x^\star) \in \mathcal{Z}$ such that $v_2\left(t^\star, x^\star\right) > 0$. Then by virtue of the usual parabolic strong maximum principle, $\left(v_1^{\tau^\star}\right)^+ = \left(v_2\right)^+$ in some parabolic cylinder whose final time is $t^\star$ and whose spatial center is $x^\star$. Thus v is identically null in this cylinder, whence by strict monotonicity (see Corollary 3.39) of φ_2 with respect to ξ, $s_1 = s_2$, $v_2 = z_2$ in this cylinder, and then by periodicity of $\varphi_1 - \varphi_2$ with respect to x, $\left(v_1^{\tau^\star}\right)^+ = v_2^+$ in $\mathbb{R}^2$ and their free boundaries (i.e. zero sets) coincide. Thus there exists a unique bijection Ξ such that this free boundary is the graph of Ξ. By continuity of $\partial_x v_1^{\tau^\star}$ and $\partial_x v_2$ (see Proposition 3.28), $\partial_x v = 0$ on the other side of the free boundary, whence by virtue of Hopf's lemma the equality $v_1^{\tau^\star} = v_2$ extends everywhere.
2. There exists $(t^\star, x^\star) \in \mathcal{Z}$ such that $v_2\left(t^\star, x^\star\right) < 0$. Then, by the exact same argument (this is once more due to the underlying symmetry), $v_1^{\tau^\star} = v_2$ in $\mathbb{R}^2$.

3. Every $(t^\star, x^\star) \in \mathcal{Z}$ is such that $v_1^{\tau^\star}(t^\star, x^\star) = v_2(t^\star, x^\star) = 0$. Thanks to Hopf's lemma again, this case is actually contradictory. On one hand, since $\partial_x v \in \mathcal{C}(\mathbb{R}^2)$ and v is non-negative non-zero in $\mathbb{R}^2$, for any $(t^\star, x^\star) \in \mathcal{Z}$, $\partial_x v(t^\star, x^\star) = 0$. On the other hand, although the free boundaries of $v_1^{\tau^\star}$ and v_2 are here *a priori* distinct, we can still apply Hopf's lemma at $(t^\star, x^\star)$ in a suitable parabolic cylinder and get a strict sign for $\partial_x v(t^\star, x^\star)$.

□

Remark. At this point, it would be tempting to notice that this kind of proof can be easily generalized if one of the two speeds is zero (in this case, the argument is usually referred to as a "quenching" or "blocking" argument) and then to use it to show that a segregated stationary equilibrium cannot coexist with a segregated pulsating front. Unfortunately, this is not possible. A segregated stationary equilibrium is *a priori* a much more general notion than what could be defined as a "segregated pulsating front with null speed" (the basic reason being that, when $c_\infty = 0$, the change of variables $(t,x) \mapsto (x - c_\infty t, x)$ is not an isomorphism anymore).

Nevertheless, it is still possible to use some kind of more elaborated quenching argument, as shows the following theorem.

Theorem 3.45. *If there exists a segregated pulsating front, there does not exist a segregated stationary equilibrium.*

Proof. Assume that there exist both a segregated pulsating front z with speed $s \neq 0$ and profile φ and a segregated stationary equilibrium e.

Assume for instance that $s > 0$ and that e has a smallest zero:

$$x_1 = \min e^{-1}(\{0\}) \in \mathbb{R}.$$

As in the usual sliding method, we construct (and do not detail these constructions) $\tau \in \mathbb{R}$ and $\kappa > 1$ such that:

$$(\xi, x) \mapsto \kappa e(\xi) - \varphi(\xi - \tau, x)$$

is positive everywhere in $(-\infty, x_1) \times \mathbb{R}$, with a fixed gap at $\{x_1\} \times \mathbb{R}$ (constructing for instance τ such that $\max_{x \in \overline{C}} \varphi(x_1 - \tau, x) = -\frac{da_2}{2}$). Then we define $\kappa^\star$ as the infimum of these κ, we assume by contradiction that $\kappa^\star > 1$ and we construct consequently the first contact point $(\xi^\star, x^\star)$ with $\xi^\star < x_1$. By virtue of Proposition 3.16, $\xi^\star > -\infty$. Let $t^\star = \frac{x^\star - \xi^\star}{s}$.

Notice that there exists a neighborhood of $(\xi^\star, x^\star)$ such that $\varphi \gg 0$ in this neighborhood. Consequently, there exists $\varepsilon > 0$ such that both functions:

$$x \mapsto \varphi(x - st^\star - \tau, x),$$

$$v_{\tau,\kappa^\star} : x \mapsto \kappa^\star e(x + \xi^\star - x^\star) - \varphi(x - st^\star - \tau, x),$$

are non-negative non-zero everywhere in $[x^\star - \varepsilon, x^\star + \varepsilon]$. Moreover, $v_{\tau,\kappa^\star}(x^\star) = 0$. Thanks to the inequality:

$$\kappa\eta[e] \geq \kappa\eta[\kappa e] \text{ in } (x^\star - \varepsilon, x^\star + \varepsilon),$$

we get:

$$-\kappa e''(x + \xi^\star - x^\star) \geq \kappa\eta(\kappa e(x + \xi^\star - x^\star), x + \xi^\star - x^\star) \text{ for any } x \in (x^\star - \varepsilon, x^\star + \varepsilon),$$

whence, since $\partial_t z > 0$, $v_{\tau,\kappa^\star}$ satisfies:

$$-v''_{\tau,\kappa^\star}(x) > q_{\kappa^\star}(x) v_{\tau,\kappa^\star}(x) \text{ for any } x \in (x^\star - \varepsilon, x^\star + \varepsilon),$$

where $q_{\kappa^\star} \in L^\infty(\mathbb{R})$ is defined as:

$$q_{\kappa^\star} : x \mapsto \begin{cases} \frac{\eta(\kappa^\star e(x+\xi^\star - x^\star), x+\xi^\star-x^\star) - \eta(\varphi(x-st^\star-\tau,x),x)}{v_{\tau,\kappa^\star}} & \text{if } v_{\tau,\kappa^\star}(x) \neq 0 \\ 1 & \text{if } v_{\tau,\kappa^\star}(x) = 0. \end{cases}$$

The function $v_{\tau,\kappa^\star}$ is a non-negative non-zero super-solution of some elliptic problem. Since the elliptic strong maximum principle contradicts the existence of $\xi^\star$, $\kappa^\star = 1$ indeed.

Repeating the argument near $\xi = +\infty$ with some $\kappa \leq 1$ then proves that (up to some increase of τ) $e(\xi) - \varphi(\xi - \tau, x) \gg 0$ actually holds in $\mathbb{R}^2$. Note that in this case, the proof is simpler, since the negativity of φ in $(\xi^\star, +\infty) \times \mathbb{R}$ follows from its normalization and monotonicity. We point out that, *a priori*, there are two cases, depending on the existence of $\max e^{-1}(\{0\})$. But in fact these two cases do not require different arguments.

Now, just as usual, we can define:

$$\tau^\star = \sup\left\{\tau \in \mathbb{R} \mid e(\xi) - \varphi(\xi - \tau, x) \geq 0 \text{ for any } (\xi, x) \in \mathbb{R}^2\right\}.$$

Assume by contradiction that:

$$\min_{[-B,B]\times\mathbb{R}} (e(\xi) - \varphi(\xi - \tau^\star, x)) > 0$$

for any $B > 0$ such that:

$$e(B) < 0,$$
$$\min_{x\in\mathbb{R}} \varphi(-B - \tau^\star, x) > 0.$$

By continuity, we then obtain for $\tau > \tau^*$ close enough,

$$\min_{[-B,B]\times\mathbb{R}} (e(\xi) - \varphi(\xi - \tau, x)) > 0,$$

$$\min_{x\in\mathbb{R}} \varphi(-B - \tau, x) > 0.$$

It follows from the same type of arguments as those presented at the beginning of this proof that:

$$e(\xi) - \varphi(\xi - \tau, x) \gg 0 \text{ in } (\mathbb{R}\backslash(-B, B)) \times \mathbb{R},$$

thus contradicting the maximality of τ^*.

Hence, there exists $B > 0$ such that:

$$\min_{[-B,B]\times\mathbb{R}} (e(\xi) - \varphi(\xi - \tau^\star, x)) = 0,$$

i.e. there exists $(\xi^\star, x^\star) \in [-B, B] \times \mathbb{R}$ such that:

$$e(\xi^\star) - \varphi(\xi^\star - \tau^\star, x^\star) = 0.$$

Let:

$$t^\star = \frac{x^\star - \xi^\star}{s},$$
$$v : (t, x) \mapsto e(x + \xi^\star - x^\star) - \varphi(x - st - \tau^\star, x)$$

and notice that:

$$v(t, x) > 0 \text{ for any } (t, x) \in [t^\star - 1, t^\star) \times \mathbb{R},$$
$$v(t^\star, x^\star) = 0.$$

Now, we need to distinguish two cases, as in the proof of Theorem 3.44:

— if $\xi^\star \notin e^{-1}(\{0\})$, using the continuity of v and the strong parabolic maximum principle in some parabolic cylinder $[t^\star - \varepsilon, t^\star] \times [x^\star - \varepsilon, x^\star + \varepsilon]$ (with a small enough ε so that the signs of $e(x + \xi^\star - x^\star)$ and of $\varphi(x - st - \tau^\star, x)$ do not change in this cylinder), we get a contradiction;

— if $x^\star \in e^{-1}(\{0\})$, using the continuity of e' and $\partial_x z$ and Hopf's lemma at the vertex $(t^\star, x^\star)$ of the parabolic cylinder $[t^\star - 1, t^\star] \times [x^\star, x^\star + 1]$, we get a contradiction as well.

The pair (z, e) cannot exist.

If $s < 0$, we change $v_{\tau,\kappa^\star}$ into $-v_{\tau,\kappa^\star}$ so that $\partial_t z < 0$ yields a negative sub-solution and we deduce similarly $e(\xi) - \varphi(\xi - \tau, x) \gg 0$. The end of the proof is carried on similarly.

If $\min e^{-1}(\{0\})$ does not exist, then $\max e^{-1}(\{0\})$ does: it suffices to change the roles of e and φ, in the sense that now we have to show that $\varphi(\xi - \tau, x) - e(\xi) \gg 0$. Near $\xi = -\infty$, the studied quantity is $\kappa e - \varphi$ with $\kappa \leq 1$, and near $\xi = +\infty$, the studied quantity is $\kappa e - \varphi$ with $\kappa \geq 1$. Once $\kappa^\star = 1$ is established, the end of the proof is exactly the same. □

Remark. The preceding proof only works in the case of constant a_1 and a_2. In the case of non-constant extinction states, this type of quenching argument does not hold anymore because Proposition 3.16 is not true anymore and therefore we cannot prove that $\xi^\star < -\infty$ when trying to prove that $\kappa^\star = 1$. We do not know how to prove the theorem in such a case and we stress that this is really unsatisfying. Still, we think it is natural to make the following conjecture.

Conjecture 3.46. *Theorem 3.45 still holds true in the non-constant case.*

3.3.5 Uniqueness of the asymptotic speed

From now on, $(c_k)_{k>k^\star}$ refers to the general family indexed on $(k^\star, +\infty)$ instead of an *a priori* extracted convergent sequence. In the following, we will prove that $(c_k)_{k>k^\star}$ converges indeed to c_∞ as $k \to +\infty$.

Definition 3.47. We say that $s \in \mathbb{R}$ *satisfies Property* $(\mathcal{E}(d, \alpha, f_1, f_2))$ if one of the following holds:

— $s = 0$ and there exists a segregated stationary equilibrium;

— $s \neq 0$ and there exists a segregated pulsating front with speed s.

The set of all $s \in \mathbb{R}$ satisfying Property $(\mathcal{E}(d, \alpha, f_1, f_2))$ is referred to as $\Sigma_{(d,\alpha,f_1,f_2)}$.

Remark. This set does not depend at all on $k^\star$.

Following Theorems 3.44 and 3.45, we deduce the following uniqueness result.

Corollary 3.48. *There is at most one $s \in \mathbb{R}$ satisfying Property $(\mathcal{E}(d, \alpha, f_1, f_2))$.*

To conclude about the convergence of the speeds, it suffices to recall that c_∞ satisfies of course Property $(\mathcal{E}(d, \alpha, f_1, f_2))$.

Proposition 3.49. *The limit at $+\infty$ of the function $k \mapsto c_k$ is well-defined.*

Remark. If a_1 and a_2 are non-constant, as explained before, the quenching argument cannot be used and we do not have the uniqueness in $\mathbb{R}$ of the elements satisfying Property $(\mathcal{E}(d, \alpha, f_1, f_2))$. Still, we have the uniqueness in $\mathbb{R} \backslash \{0\}$, whence in particular the countability of the limit points of $k \mapsto c_k$ as $k \to +\infty$. Therefore, using the intermediate value theorem, we can still prove that the limit of the continuous function $k \mapsto c_k$ as $k \to +\infty$ is well-defined. In other words, the convergence of (c_k) can be proved even without proving Conjecture 3.46.

3.3.6 Conclusion of this section

The function $k \mapsto c_k$ converges at $+\infty$.

If its limit c_∞ is non-zero, then both families $((u_{1,k}, u_{2,k}))_{k>k^\star}$ and $((\varphi_{1,k}, \varphi_{2,k}))_{k>k^\star}$ have a unique limit point (which are respectively the segregated pulsating front w traveling with speed c_∞ and its profile ϕ), and therefore the functions $k \mapsto (\varphi_{1,k}, \varphi_{2,k})$ and $k \mapsto (u_{1,k}, u_{2,k})$ converge as well as $k \to +\infty$.

If $c_\infty = 0$, then $((u_{1,k}, u_{2,k}))_{k>k^\star}$ might have multiple limit points, each one of them being a segregated stationary equilibrium.

3.4 Sign of the asymptotic speed depending on the parameters

In this final section, we investigate the sign of c_∞ as a function of (d, α), which is consequently not considered as fixed anymore ($L > 0$ and (f_1, f_2) are still fixed nevertheless).

We assume the existence of $D_{exis} \geq 0$ such that, for any $d > D_{exis}$ and any $\alpha > 0$, $(\mathcal{H}_{exis})$ is satisfied.

Once $(d, \alpha) \in (D_{exis}, +\infty) \times (0, +\infty)$ is given, c_∞ is naturally defined. If $c_\infty \neq 0$, ϕ and w are well-defined as well.

Remark. These assumptions are natural in view of the existence result under the hypothesis $(\mathcal{H}_{freq})$ exhibited by the first author [Gir17]. Indeed, if $(\mathcal{H}_{freq})$ is assumed, then it implies $(\mathcal{H}_{exis})$ and the existence of an explicit D_{exis}:

$$D_{exis} = \begin{cases} M_2 \left(\frac{L}{\pi} - \frac{1}{\sqrt{M_1}} \right)^2 & \text{if } L\sqrt{M_1} > \pi \\ 0 & \text{if } L\sqrt{M_1} \leq \pi. \end{cases}$$

3.4.1 Necessary and sufficient conditions on the parameters for the asymptotic speed to be zero

Here the idea is to follow what we did in the space-homogeneous case [GN15] to deduce a free boundary condition satisfied by any segregated stationary equilibrium. To this end, we need the following result, which shares some similarities with Proposition 4.1 of Du–Lin [59, 60] but is, on one hand, restricted to the null speeds and, on the other hand, extended to the space-periodic non-linearities.

Proposition 3.50. *Let $x_0 \in \mathbb{R}$ and $f : [0, +\infty) \times \mathbb{R} \to \mathbb{R}$, periodic with respect to x and satisfying $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$. The following problem:*

$$\begin{cases} -z'' = zf[z] & \text{in } (x_0, +\infty) \\ z(x_0) = 0 \end{cases}$$

admits a unique non-negative non-zero solution $z_{x_0,f} \in \mathcal{C}^2([x_0, +\infty))$.

Furthermore, the function

$$\Theta : (x_0, f) \mapsto z'_{x_0,f}(x_0)$$

(that is the right-sided derivative of $z_{x_0,f}$ at x_0) satisfies:

1. *$\Theta \gg 0$;*
2. *Θ is continuous with respect to the canonical topology of $\mathbb{R} \times \mathcal{C}^1(\mathbb{R}^2, \mathbb{R})$;*

3. *Θ is periodic with respect to its first variable;*
4. *for any $\kappa > 0$,*

$$\Theta\left(x_0, (z,x) \mapsto f\left(\frac{z}{\kappa}, x\right)\right) = \kappa\Theta\left(x_0, f\right).$$

Proof. Firstly, let us point out that Du–Lin's proposition [59, 60] is readily extended to generic "KPP"-type non-linearities which do not depend on the spatial variable. We do not detail this extension here.

Thus, let $\overline{f} : z \mapsto \max_{y\in\overline{C}} f(z,y)$. It can be checked that $z \mapsto z\overline{f}[z]$ is indeed a KPP-type non-linearity (mostly, it reduces to the proof of the fact that $\overline{f}$ is decreasing and negative after some fixed value). Then, let $\overline{z}$ be the solution given by (the aforementioned extension of) Du–Lin's proposition of:

$$\begin{cases} -z''(x) = z\overline{f}[z] & \text{in } (x_0, +\infty) \\ z(x_0) = 0. \end{cases}$$

Similarly, let $\underline{f} : z \mapsto \min_{y\in\overline{C}} f(z,y)$ and $\underline{z}$ be the solution of:

$$\begin{cases} -z''(x) = z\underline{f}[z] & \text{in } (x_0, +\infty) \\ z(x_0) = 0. \end{cases}$$

We intend to prove that $\overline{z}$ and $\underline{z}$ form an ordered pair of super- and sub-solution for the problem at hand.

Let a be the positive constant given by $(\mathcal{H}_3)$ such that $f(a,x) = 0$ for all $x \in \overline{C}$. By standard elliptic estimates,

$$\lim_{+\infty} \overline{z} = \lim_{+\infty} \underline{z} = a.$$

By Du–Lin's proposition, we know that $\overline{z}'(x_0)$ and $\underline{z}'(x_0)$ (understood as right-sided derivatives) are finite, whence there exists $\kappa > 0$ such that:

$$\kappa\overline{z} - \underline{z} \geq 0 \text{ in } (x_0, +\infty).$$

Let:

$$\kappa^\star = \inf\left\{\kappa > 0 \mid \kappa\overline{z} - \underline{z} \gg 0 \text{ in } (x_0, +\infty)\right\}$$

and assume by contradiction that $\kappa^\star > 1$. We can fix a sequence $(\kappa_n)_{n\in\mathbb{N}} \in (1, \kappa^\star)^{\mathbb{N}}$ which converges to $\kappa^\star$ from below. There exists a sequence $(x_n)_{n\in\mathbb{N}} \in (x_0, +\infty)^{\mathbb{N}}$ such that:

$$(\kappa_n\overline{z} - \underline{z})(x_n) < 0.$$

Since $\lim_{+\infty}(\kappa_n\overline{z} - \underline{z}) = (\kappa_n - 1)a > 0$, the sequence $(x_n)_{n\in\mathbb{N}}$ is bounded and then convergent up to extraction.

If x_∞ is the limit of (x_n), then by continuity:

$$\kappa^\star\overline{z}(x_\infty) = \underline{z}(x_\infty).$$

Now, remarking that:

$$\kappa^\star\overline{z}\overline{f}[\overline{z}] \geq \kappa^\star\overline{z}\overline{f}[\kappa^\star\overline{z}]$$

by monotonicity of $\overline{f}$, it follows by Lipschitz-continuity of $\overline{f}$ that $\kappa^\star\overline{z} - \underline{z}$ is a positive super-solution of some linear elliptic problem which vanishes at x_∞. Provided $x_\infty \neq x_0$, this contradicts the elliptic strong minimum principle and the strict ordering at $+\infty$.

But if $x_\infty = x_0$, then Hopf's lemma implies that:

$$(\kappa^\star \overline{z} - \underline{z})'(x_0) > 0.$$

From this inequality, the optimality of $\kappa^\star$ is easily contradicted.

Hence $\kappa^\star = 1$, that is $\overline{z}$ and $\underline{z}$ are indeed a pair of ordered super- and sub-solution of the problem. Since f depends on x (the special case of f constant with respect to x, that is Du–Lin's case, can be discarded here without loss of generality), they are not solutions themselves, whence their ordering is strict:

$$\underline{z} \ll \overline{z} \text{ in } (x_0, +\infty).$$

Finally, by virtue of classical existence–comparison results for semi-linear elliptic problems, there exists a solution of the problem $z_{x_0,f}$ satisfying furthermore:

$$\underline{z} \ll z_{x_0,f} \ll \overline{z}.$$

The uniqueness of $z_{x_0,f}$ follows from similar arguments.

The positivity of Θ easily follows from $z_{x_0,f} \gg \underline{z}$. Its continuity comes from the uniqueness of $z_{x_0,f}$ and classical compactness arguments. Its periodicity with respect to x comes from the uniqueness of $z_{x_0,f}$ and the periodicity of f with respect to x. The last property comes from the following easy fact. Let $\kappa > 0$ and $Z = \kappa z_{x_0,f}$. It is easily verified that:

$$-Z'' = Zf\left[\frac{Z}{\kappa}\right] \text{ in } (x_0, +\infty)$$

and then by uniqueness $Z = z_{x_0,f_\kappa}$ where $f_\kappa : (z,x) \mapsto f\left(\frac{z}{\kappa}, x\right)$. ◻

Before going any further, we recall that it suffices to choose different normalization sequences to deduce that, if $c_\infty = 0$, there exists at least one segregated stationary equilibrium e_1 satisfying:

$$\inf e_1^{-1}\left((-\infty, 0)\right) > -\infty$$

and at least one segregated stationary equilibrium e_2 satisfying:

$$\sup e_2^{-1}\left((0, +\infty)\right) < +\infty.$$

If $c_\infty = 0$, we define consequently $x_1 = \min e_1^{-1}(\{0\})$ and $x_2 = \max e_2^{-1}(\{0\})$. Recall that, without loss of generality, we can assume that $(x_1, x_2) \in [0, L)^2$.

Lemma 3.51. *Let $(d, \alpha) \in (D_{exis}, +\infty) \times (0, +\infty)$, $f_{1,x_1} : (z, x) \mapsto f_1(z, 2x_1 - x)$ and Θ be defined as in Proposition 3.50. Assume $c_\infty = 0$.*

Then:

$$\alpha\Theta(x_1, f_{1,x_1}) \geq d\Theta\left(x_1, \frac{1}{d} f_2\right),$$

$$\alpha\Theta(x_2, f_{1,x_2}) \leq d\Theta\left(x_2, \frac{1}{d} f_2\right).$$

Proof. We prove the first inequality, the second one being proved similarly (using e_2 instead of e_1).

First, if:

$$e_1^{-1}(\{0\}) \setminus \{x_1\} = \emptyset,$$

then e_1 has a unique zero. Now, consider the problems satisfied by the functions:

$$z_1 : x \mapsto e_1^+ (2x_1 - x),$$

$$z_2 : x \mapsto e_1^- (x).$$

It is clear that:

$$(z_1, z_2) = \left(z_{x_1,(z,x)\mapsto f_1\left(\frac{z}{\alpha}, 2x_1 - x\right)}, z_{x_1,(z,x)\mapsto \frac{1}{d} f_2\left(\frac{z}{d}, x\right)} \right).$$

Since $e_1 \in \mathcal{C}^2(\mathbb{R})$, $z_1'\left(x_1^+\right) = z_2'\left(x_1^+\right)$ is necessary. From the relations:

$$\Theta\left(x_1, (z,x) \mapsto f_1\left(\frac{z}{\alpha}, 2x_1 - x\right)\right) = \alpha\Theta\left(x_1, f_{1,x_1}\right),$$

$$\Theta\left(x_1, (z,x) \mapsto \frac{1}{d} f_2\left(\frac{z}{d}, x\right)\right) = d\Theta\left(x_1, \frac{1}{d} f_2\right),$$

we see that we are in the case of equality.

Next, if:

$$e_1^{-1}(\{0\}) \setminus \{x_1\} \neq \emptyset,$$

then let:

$$y_1 = \min e_1^{-1}(\{0\}) \setminus \{x_1\}.$$

Clearly, $z_3 = \left(e_1^-\right)_{|(x_1,y_1)}$ is the unique non-negative non-zero solution of:

$$\begin{cases} -dz'' = zf_2\left[\frac{z}{d}\right] & \text{in } (x_1, y_1) \\ z(x_1) = z(y_1) = 0. \end{cases}$$

Now it can be easily verified that z_3 is a sub-solution for the problem satisfied by $z_{x_1,(z,x)\mapsto \frac{1}{d} f_2\left(\frac{z}{d}, x\right)}$. The inequality follows. □

Remark. We explained previously that, if $(\mathcal{H}_{freq})$ [Gir17] is assumed, each segregated stationary equilibrium has a unique zero x_e. In such a case, we have equality:

$$\alpha\Theta\left(x_e, f_{1,x_e}\right) = d\Theta\left(x_e, \frac{1}{d} f_2\right).$$

Let $(d, \alpha) \in (0, +\infty)^2$. With the same notations as before, we define the following sets:

$$X_{(d,\alpha)}^+ = \left\{ x \in [0, L) \mid \alpha\Theta\left(x, f_{1,x}\right) \geq d\Theta\left(x, \frac{1}{d} f_2\right) \right\},$$

$$X_{(d,\alpha)}^- = \left\{ x \in [0, L) \mid \alpha\Theta\left(x, f_{1,x}\right) \leq d\Theta\left(x, \frac{1}{d} f_2\right) \right\}.$$

Clearly, from the preceding corollary, if $c_\infty = 0$,

$$\begin{cases} X_{(d,\alpha)}^+ \neq \emptyset \\ X_{(d,\alpha)}^- \neq \emptyset. \end{cases}$$

Proposition 3.52. *Let $(d,\alpha) \in (0,+\infty)^2$, $f_{1,x} : (z,y) \mapsto f_1(z, 2x-y)$, Θ be defined as in Proposition 3.50 and :*

$$A_d : x \mapsto \frac{d\Theta\left(x, \frac{1}{d}f_2\right)}{\Theta\left(x, f_{1,x}\right)}.$$

The function A_d is continuous, positive and periodic, does not depend on α and satisfies the following properties.

— *If there exists $x \in X^+_{(d,\alpha)}$, then $\alpha \geq A_d(x)$.*

— *If there exists $x \in X^-_{(d,\alpha)}$, then $\alpha \leq A_d(x)$.*

— *It has a global minimum and a global maximum .*

Consequently, provided $d > D_{exis}$, $\alpha \in [\min A_d, \max A_d]$ if and only if $c_\infty = 0$.

Proof. Everything is straightforward apart maybe the following implication: if $\alpha \in [\min A_d, \max A_d]$, then $c_\infty = 0$. In fact, if there exists $x_e \in [0,L)$ such that $\alpha = A(x_e)$, then the following function:

$$z : y \mapsto \begin{cases} z_{x_e,(z,x)\mapsto f_1\left(\frac{z}{\alpha}, 2x_e - x\right)}(2x_e - y) & \text{if } y < x_e, \\ -z_{x_e,(z,x)\mapsto \frac{1}{d}f_2\left(\frac{z}{d}, x\right)}(y) & \text{if } y \geq x_e, \end{cases}$$

is a segregated stationary equilibrium, which implies by uniqueness (see Theorem 3.45) that $c_\infty = 0$. □

Remark. The preceding proposition characterizes sharply $\{\alpha > 0 \mid c_\infty = 0\}$. Moreover, it also gives an implicit characterization of the diffusion rates such that $c_\infty = 0$. With this in mind, understanding whether A_d is constant or not would be of great interest.

Let us recall that if a_1 and a_2 are not constant, we do not know how to prove Theorem 3.45. Therefore in such a case the preceding sharpness is lost and we might still have a non-zero c_∞ for some $\alpha \in [\min A_d, \max A_d]$. This pathological situation seems highly unlikely (recall Conjecture 3.46).

From this result, we can also deduce an explicit estimate for the range of parameters (d,α), as indicated by the following statement.

Proposition 3.53. *Let $\Lambda \subset \mathbb{R}^2$ be the following set:*

$$\left\{(d,\alpha) \in (0,+\infty)^2 \mid X^+_{(d,\alpha)} \neq \emptyset \text{ and } X^-_{(d,\alpha)} \neq \emptyset\right\}.$$

There exists $\underline{r} > 0$ and $\overline{r} \geq \underline{r}$, defined by formulas $\left(\mathfrak{F}_{\underline{r}}\right)$ and $\left(\mathfrak{F}_{\overline{r}}\right)$ which only depend on (f_1, f_2), such that, for any $(d,\alpha) \in \Lambda$,

$$\underline{r} \leq \frac{\alpha^2}{d} \leq \overline{r}.$$

Remark. Although these estimates do not depend on d, they are also less precise than the previous statement. Indeed, we will see in the course of the proof that, for any $d > 0$:

$$\sqrt{\underline{r}d} \leq \min A_d,$$

$$\max A_d \leq \sqrt{\overline{r}d},$$

and furthermore it should be expected that these inequalities are actually strict. Thus the interest of this proposition lies mostly in the fact that $\underline{r}$ and $\overline{r}$ do not depend on d.

Proof. Recalling from Proposition 3.50 the definition of $z_{x_0,f}$, we define for any $d>0$ and any $y\in\overline{C}$ the following functions:

$$z_{1,y}=z_{y,f_{1,y}},$$
$$z_{2,y}:x\mapsto z_{y,\frac{1}{d}f_2}\left(\sqrt{d}x+y\right).$$

Most importantly, $z_{2,y}$ satisfies:

$$\begin{cases}-z''_{2,y}(x)=z_{2,y}(x)\,f_2\left(z_{2,y}(x),\sqrt{d}x+y\right) & \text{for any } x\in(0,+\infty),\\ z_{2,y}(0)=0.\end{cases}$$

Let $\overline{f_2}:z\mapsto\max_{x\in\overline{C}}f_2(z,x)$ and $\overline{z}$ be the solution of:

$$\begin{cases}-z''=z\overline{f_2}[z] & \text{in } (0,+\infty)\\ z(0)=0.\end{cases}$$

Similarly, let $\underline{f_2}:z\mapsto\min_{x\in\overline{C}}f_2(z,x)$ and $\underline{z}$ be the solution of:

$$\begin{cases}-z''=z\underline{f_2}[z] & \text{in } (0,+\infty)\\ z(0)=0.\end{cases}$$

It can easily be checked (see the proof of Proposition 3.50) that the solutions $\underline{z}$ and $\overline{z}$ form a pair of sub-solution and super-solution for the problem satisfied by $z_{2,y}$. By uniqueness, $\underline{z}\leq z_{2,y}\leq\overline{z}$. Since $\sqrt{d}\Theta\left(y,\frac{1}{d}f_2\right)=z'_{2,y}(0)$, consequently:

$$\underline{z}'(0)\leq\sqrt{d}\Theta\left(y,\frac{1}{d}f_2\right)\leq\overline{z}'(0).$$

Then, for any $(d,\alpha)\in\Lambda$, we deduce from the preceding estimate and from the definitions of $X^+_{(d,\alpha)}$ and $X^-_{(d,\alpha)}$ that there exists $(x_1,x_2)\in[0,L]^2$ such that:

$$\alpha\Theta\left(x_1,f_{1,x_1}\right)\geq\sqrt{d}\underline{z}'(0),$$
$$\alpha\Theta\left(x_2,f_{1,x_2}\right)\leq\sqrt{d}\overline{z}'(0).$$

The conclusion follows from the following definitions:

$$\underline{r}=\left(\frac{\underline{z}'(0)}{\max_{x\in\overline{C}}\Theta\left(x,f_{1,x}\right)}\right)^2, \quad (\mathfrak{F}_{\underline{r}})$$

$$\overline{r}=\left(\frac{\overline{z}'(0)}{\min_{x\in\overline{C}}\Theta\left(x,f_{1,x}\right)}\right)^2. \quad (\mathfrak{F}_{\overline{r}})$$

□

Corollary 3.54. *Assume that, for any $i\in\{1,2\}$, f_i has the particular form $(u,x)\mapsto\mu_i(x)(1-u)$ with $\mu_i\in\mathcal{C}^1_{per}(\mathbb{R})$, $\mu_i\gg0$.*

Then:

$$\frac{\min_{\overline{C}}(\mu_2)}{\max_{\overline{C}}(\mu_1)}\leq\underline{r}\leq\overline{r}\leq\frac{\max_{\overline{C}}(\mu_2)}{\min_{\overline{C}}(\mu_1)}.$$

Proof. In such a case, the functions $\overline{f_2}$ and $\underline{f_2}$ defined in the proof of Proposition 3.53 reduce to:

$$\overline{f_2} : z \mapsto \max_{\overline{C}} (\mu_2)(1-z),$$

$$\underline{f_2} : z \mapsto \min_{\overline{C}} (\mu_2)(1-z).$$

Define analogously:

$$\overline{f_1} : z \mapsto \max_{\overline{C}} (\mu_1)(1-z),$$

$$\underline{f_1} : z \mapsto \min_{\overline{C}} (\mu_1)(1-z).$$

Denoting the functions $\overline{z}$ and $\underline{z}$ defined in the proof of Proposition 3.53 as $\overline{z}_2$ and $\underline{z}_2$, the definitions of $\underline{r}$ and $\overline{r}$ read:

$$\underline{r} = \left(\frac{\underline{z}_2'(0)}{\max_{x\in\overline{C}} \Theta(x, f_{1,x})}\right)^2,$$

$$\overline{r} = \left(\frac{\overline{z}_2'(0)}{\min_{x\in\overline{C}} \Theta(x, f_{1,x})}\right)^2.$$

Defining analogously the functions $\overline{z}_1$ and $\underline{z}_1$, we obtain by a super- and sub-solution argument similar to that of Proposition 3.53 the following estimates:

$$\underline{z}_1'(0) \leq \min_{x\in\overline{C}} \Theta(x, f_{1,x}) \leq \max_{x\in\overline{C}} \Theta(x, f_{1,x}) \leq \overline{z}_1'(0),$$

which lead subsequently to:

$$\underline{r} \geq \left(\frac{\underline{z}_2'(0)}{\overline{z}_1'(0)}\right)^2,$$

$$\overline{r} \leq \left(\frac{\overline{z}_2'(0)}{\underline{z}_1'(0)}\right)^2.$$

Now let us determine $\Theta(0, z \mapsto r(1-z))$ for any constant $r > 0$. Multiplying the equality satisfied by $z = z_{0,z\mapsto r(1-z)}$ by z', we find:

$$-\left(\frac{(z')^2}{2}\right)' = r\left(\frac{z^2}{2}\right)' - r\left(\frac{z^3}{3}\right)'.$$

Integrating between 0 and $+\infty$, it follows $(z'(0))^2 = \frac{r}{6}$, that is:

$$\Theta(0, z \mapsto r(1-z)) = \sqrt{\frac{r}{6}}.$$

Applying this equality with $r = \max_{\overline{C}} (\mu_2)$, $r = \min_{\overline{C}} (\mu_2)$, $r = \max_{\overline{C}} (\mu_1)$ and $r = \min_{\overline{C}} (\mu_1)$, the claimed estimates for $\underline{r}$ and $\overline{r}$ follow directly. □

Thanks to the existence of $\underline{r}$ and $\overline{r}$, we now know that the quantity $\frac{\alpha^2}{d}$ plays a particular role (and this is obviously reminiscent of the space-homogeneous case [GN15]). Therefore, we also state the following (immediate) proposition.

Proposition 3.55. *For any $d \in (0, +\infty)$, let:*

$$\mathcal{R}_d^0 = \left[\frac{(\min A_d)^2}{d}, \frac{(\max A_d)^2}{d} \right]. \quad (\mathfrak{F}_{\mathcal{R}^0})$$

The set $\mathcal{R}_d^0$ is a non-empty, closed, subinterval of $[\underline{r}, \overline{r}]$.
Assume moreover that $d > D_{exis}$. Then $c_\infty = 0$ if and only if $\frac{\alpha^2}{d} \in \mathcal{R}_d^0$.

Remark. Once more, in the case of non-constant a_1 and a_2, one implication is lacking, but proving Conjecture 3.46 would be sufficient to recover it.

The length of $\mathcal{R}_d^0$ is a very interesting open question (which is obviously equivalent to that of the constancy of A_d). Recall that in the space-homogeneous case [GN15], $\mathcal{R}_d^0 = \left\{ \frac{f_2[0]}{f_1[0]} \right\}$ is a singleton which does not depend on d.

3.4.2 Sign of a non-zero asymptotic speed

Proposition 3.56. *Let $(d, \alpha) \in (0, +\infty)^2$. Let z be a segregated pulsating front with speed $s \neq 0$ and profile φ.*

Then s has the sign of:

$$\int_0^L \int_{-da_2}^{\alpha a_1} \eta(z, x)\, dz dx = \int_0^L \left(\alpha^2 \int_0^{a_1} z f_1(z, x)\, dz - d \int_0^{a_2} z f_2(z, x)\, dz \right) dx.$$

Remark. In view of well-known results about bistable scalar traveling waves, and more recently pulsating fronts (see for instance Ding–Hamel–Zhao [57]), such a result was to be expected.

It could be tempting to try to get rid of the *a priori* condition $s \neq 0$ and to show that the existence of a segregated stationary equilibrium implies:

$$\int_C \int_{-da_2}^{\alpha a_1} \eta(z, x)\, \mathrm{d}z\mathrm{d}x = 0.$$

But Zlatos [148] showed on the contrary that it is possible to build counter-examples of pure bistable non-linearities F of positive integral such that:

$$\partial_t z - \partial_{xx} z = F[z]$$

does not admit any transition front with non-zero speed. Therefore we do not investigate further in this direction.

Proof. We have justified previously that in the equation $(\mathcal{SPF}[s])$, every term $(\operatorname{div}(E\nabla\varphi)$, $\partial_\xi \varphi$ and $\eta[\varphi])$ is well-defined in $L^2_{loc}(\mathbb{R}^2)$. Thus we consider the test function $\partial_\xi \varphi \mathbf{1}_{[-B,B]\times\overline{C}} \in L^2_{loc}(\mathbb{R}^2)$ for some large enough $B > 0$. By large, we mean here that we assume the following:

$$\min_{x \in \overline{C}} \varphi(\xi, x) > 0 \text{ for any } \xi < -B,$$

$$\max_{x \in \overline{C}} \varphi(\xi, x) < 0 \text{ for any } \xi > B.$$

Hence the subset of the free boundary $\left\{ (\xi, x) \in \mathbb{R} \times \overline{C} \mid \varphi(\xi, x) = 0 \right\}$ is included in $(-B, B) \times \overline{C}$.

Multiplying ($\mathcal{SPF}[s]$) by $\partial_\xi \varphi$ and integrating over $(-B,B)\times\overline{C}$ yield:

$$\int_{-B}^{B}\int_0^L \operatorname{div}\left(E\nabla\varphi\right)\partial_\xi\varphi + s\int_{-B}^{B}\int_0^L \sigma\left[\varphi\right]\left(\partial_\xi\varphi\right)^2 = -\int_{-B}^{B}\int_0^L \eta\left[\varphi\right]\partial_\xi\varphi.$$

First, by change of variable, Lipschitz-continuity of the free boundary (see Proposition 3.28) and definition of η:

$$\begin{aligned} -\int_0^L\int_{-B}^{B}\eta\left[\varphi\right]\partial_\xi\varphi &= \int_0^L\int_{\varphi(B,x)}^{\varphi(-B,x)}\eta\left(z,x\right)\mathrm{d}z\mathrm{d}x \\ &= \int_0^L\left(\alpha^2\int_0^{\varphi(-B,x)/\alpha} zf_1\left(z,x\right)\mathrm{d}z - d\int_0^{-\varphi(B,x)/d} zf_2\left(z,x\right)\mathrm{d}z\right)\mathrm{d}x. \end{aligned}$$

Then, since we do not know that $\partial_\xi\varphi$ is continuous, the term $\int_{-B}^{B}\int_0^L \operatorname{div}\left(E\nabla\varphi\right)\partial_\xi\varphi$ is dealt with a standard mollification procedure. There exists a sequence of non-negative non-zero mollifiers $\left(\theta_n\right)_{n\in\mathbb{N}}\in\mathcal{D}\left(\mathbb{R}\right)$. For any $n\in\mathbb{N}$, let:

$$\varphi_n : (\xi,x)\mapsto\int\varphi\left(\xi-\zeta,x\right)\theta_n\left(\zeta\right)\mathrm{d}\zeta.$$

On one hand, for any $n\in\mathbb{N}$, it is clear that all the terms $\partial_{\xi\xi}\varphi_n$, $\partial_{xx}\varphi_n$, $\partial_{\xi x}\varphi_n$ are classically defined. By periodicity and integration by parts, we easily obtain:

$$\int_{-B}^{B}\int_0^L \operatorname{div}\left(E\nabla\varphi_n\right)\partial_\xi\varphi_n = \frac{1}{2}\int_0^L\left(\left[\left(\partial_\xi\varphi_n\right)^2(\xi,x)\right]_{-B}^{B} - \left[\left(\partial_x\varphi_n\right)^2(\xi,x)\right]_{-B}^{B}\right)\mathrm{d}x.$$

It can be easily verified that if both sets:

$$\pm B + 2\mathrm{supp}\theta_1 = \pm B + 2\bigcup_{n\in\mathbb{N}}\mathrm{supp}\theta_n$$

do not intersect the free boundary, that is if B is large enough indeed, then as $n\to+\infty$:

$$\max_{x\in\overline{C}}\left|\partial_\xi\varphi_n\left(\pm B,x\right)-\partial_\xi\varphi\left(\pm B,x\right)\right| + \max_{x\in\overline{C}}\left|\partial_x\varphi_n\left(\pm B,x\right)-\partial_x\varphi\left(\pm B,x\right)\right|\to 0.$$

It follows that:

$$\frac{1}{2}\int_0^L\left(\left[\left(\partial_\xi\varphi_n\right)^2(\xi,x)\right]_{-B}^{B} - \left[\left(\partial_x\varphi_n\right)^2(\xi,x)\right]_{-B}^{B}\right)\mathrm{d}x \to \frac{1}{2}\int_0^L\left(\left[\left(\partial_\xi\varphi\right)^2(\xi,x)\right]_{-B}^{B} - \left[\left(\partial_x\varphi\right)^2(\xi,x)\right]_{-B}^{B}\right)\mathrm{d}x.$$

On the other hand:

$$\int_{-B}^{B}\int_0^L \operatorname{div}\left(E\nabla\varphi\right)\partial_\xi\left(\varphi-\varphi_n\right) \leq \|\operatorname{div}\left(E\nabla\varphi\right)\|_{L^2((-B,B)\times C)}\|\partial_\xi\left(\varphi-\varphi_n\right)\|_{L^2((-B,B)\times C)},$$

$$\int_{-B}^{B}\int_0^L \operatorname{div}\left(E\nabla\left(\varphi-\varphi_n\right)\right)\partial_\xi\varphi_n \leq \|\operatorname{div}\left(E\nabla\left(\varphi-\varphi_n\right)\right)\|_{L^2((-B,B)\times C)}\sup_{n\in\mathbb{N}}\|\partial_\xi\varphi_n\|_{L^2((-B,B)\times C)},$$

and, once more by standard mollification theory, $\|\partial_\xi\left(\varphi-\varphi_n\right)\|_{L^2((-B,B)\times C)}$ and $\|\operatorname{div}\left(E\nabla\left(\varphi-\varphi_n\right)\right)\|_{L^2((-B,B)\times C)}$ converge to 0 as $n\to+\infty$.

Therefore, passing to the limit $n \to +\infty$, we obtain the expected equality:

$$\int_{-B}^{B}\int_{0}^{L} \operatorname{div}\left(E\nabla\varphi\right)\partial_\xi\varphi = \frac{1}{2}\int_0^L\left(\left[\left(\partial_\xi\varphi\right)^2\left(\xi,x\right)\right]_{-B}^{B} - \left[\left(\partial_x\varphi\right)^2\left(\xi,x\right)\right]_{-B}^{B}\right)\mathrm{d}x.$$

Finally, using these computations to pass to the limit $B \to +\infty$ in the equality:

$$\int_{-B}^{B}\int_{0}^{L} \operatorname{div}\left(E\nabla\varphi\right)\partial_\xi\varphi + s\int_{-B}^{B}\int_{0}^{L}\sigma\left[\varphi\right]\left(\partial_\xi\varphi\right)^2 = -\int_{-B}^{B}\int_{0}^{L}\eta\left[\varphi\right]\partial_\xi\varphi,$$

it follows:

$$s\int_{\mathbb{R}\times C}\sigma\left[\varphi\right]\left(\partial_\xi\varphi\right)^2 = \int_0^L\left(\alpha^2\int_0^{a_1} zf_1\left(z,x\right)\mathrm{d}z - d\int_0^{a_2} zf_2\left(z,x\right)\mathrm{d}z\right)\mathrm{d}x,$$

and since:

$$0 < \min\left\{1,\frac{1}{d}\right\}\left\|\partial_\xi\varphi\right\|^2_{L^2(\mathbb{R}\times C)} \leq \int_{\mathbb{R}\times C}\sigma\left[\varphi\right]\left(\partial_\xi\varphi\right)^2,$$

the claimed relationship between s and $\int_0^L\int_{-da_2}^{\alpha a_1}\eta\left(z,x\right)\mathrm{d}z\mathrm{d}x$ follows. □

Corollary 3.57. *Let $(d,\alpha)\in(D_{exis},+\infty)\times(0,+\infty)$. Then:*

1. *if $\frac{\alpha^2}{d} > \max\mathcal{R}_d^0$, $c_\infty > 0$;*
2. *if $\frac{\alpha^2}{d} < \min\mathcal{R}_d^0$, $c_\infty < 0$.*

Proof. It suffices to remark that, for any $i\in\{1,2\}$, $\int_0^L\int_0^{a_i} zf_i\left(z,x\right)\mathrm{d}z\mathrm{d}x > 0$. □

Remark. We recall that, in the proof of Proposition 3.56, the fact that a_1 and a_2 are constant is crucial. This issue has already been encountered (see the remark following Proposition 3.7). Therefore, in the general setting, it is not possible to obtain such an explicit formula for the sign of c_∞. Nevertheless, let us point out that the results of Corollary 3.57 should still hold in this case:

- there still exists $\overline{r}\geq\underline{r}>0$ such that $0\notin\Sigma_{(d,\alpha,f_1,f_2)}$ if (d,α) does not satisfy $\underline{r}\leq\frac{\alpha^2}{d}\leq\overline{r}$, since the whole subsection 3.4.1 can be easily generalized (even though:
  - we cannot prove that $c_\infty = 0$ if $\alpha\in[\min A_d,\max A_d]$, i.e. if $\frac{\alpha^2}{d}\in\mathcal{R}_d^0$ (but recall Conjecture 3.46);
  - additional care is needed since a non-constant a_2 would *a priori* depend on d);
- we will prove in the next section that $(d,\alpha)\mapsto c_\infty$ is continuous at least in

$$\left\{(d,\alpha)\in(D_{exis},+\infty)\times(0,+\infty)\ \mid\ \frac{\alpha^2}{d}\notin\mathcal{R}_d^0\right\};$$

- the study of the limit of the segregated pulsating front as $\alpha\to 0$ or $\alpha\to+\infty$ (which can be rigorously done since D_{exis} does not depend on α) should easily yield the sign of the speed at such limits:
  - formally, as $\alpha\to 0$, the positive part of w vanishes and we are left with a Fisher–KPP pulsating front connecting 0 to $-da_2$, consequently with a negative speed;
  - formally, as $\alpha\to+\infty$, the negative part of $\frac{w}{\alpha}$ vanishes and we are left with a Fisher–KPP pulsating front connecting a_1 to 0, consequently with a positive speed;

— hence, by connectedness and continuity, Corollary 3.57 would be recovered indeed.

To conclude, let us highlight an important particular case.

Corollary 3.58. *Assume that, for any $i \in \{1,2\}$, f_i has the particular form $(u,x) \mapsto \mu_i(x)(1-u)$ with $\mu_i \in \mathcal{C}^1_{per}(\mathbb{R})$, $\mu_i \gg 0$.*

Let:

$$r = \frac{\|\mu_2\|_{L^1(C)}}{\|\mu_1\|_{L^1(C)}}.$$

If $c_\infty \neq 0$, then it has the sign of $\alpha^2 r - d$.

Proof. In such a case, for any $i \in \{1,2\}$, $a_i = 1$ and:

$$\int_0^L \int_0^1 z f_i(z,x)\,\mathrm{d}z\mathrm{d}x = \frac{1}{6}\int_0^L \mu_i(x)\,\mathrm{d}x.$$

□

3.4.3 Continuity of the asymptotic speed with respect to the parameters

In this final subsection, we even allow (f_1, f_2) to vary in the set $\mathcal{F}$ of all L-periodic $f : [0,+\infty) \times \mathbb{R} \to \mathbb{R}$ satisfying $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$, equipped with the canonical topology of $\mathcal{C}^1(\mathbb{R}^2, \mathbb{R})$.

Proposition 3.59. *Assume that for any $(f_1,f_2) \in \mathcal{F}^2$, there exists a non-negative $D_{exis} = D^{f_1,f_2}_{exis}$ as defined before.*

Let:

$$\mathfrak{P} = \left\{(d,\alpha,f_1,f_2) \in (0,+\infty)^2 \times \mathcal{F}^2 \mid d > D^{f_1,f_2}_{exis}\right\}.$$

The function:

$$\begin{array}{ccc} \mathfrak{P} & \to & \mathbb{R} \\ (d,\alpha,f_1,f_2) & \mapsto & c_\infty \end{array}$$

is well-defined and continuous.

Assume moreover that the function $(d,\alpha,f_1,f_2) \in \mathfrak{P} \mapsto k^\star$ is locally bounded. Then the convergence of $((d,\alpha,f_1,f_2) \in \mathfrak{P} \mapsto c_k)_{k>k^\star}$ to $(d,\alpha,f_1,f_2) \in \mathfrak{P} \mapsto c_\infty$ is locally uniform.

Remark. If $(\mathcal{H}_{exis})$ follows from $(\mathcal{H}_{freq})$ [Gir17] and if:

$$D_{exis} = \begin{cases} M_2\left(\frac{L}{\pi} - \frac{1}{\sqrt{M_1}}\right)^2 & \text{if } L\sqrt{M_1} > \pi, \\ 0 & \text{if } L\sqrt{M_1} \leq \pi, \end{cases}$$

then $(f_1,f_2) \mapsto D^{f_1,f_2}_{exis}$ is indeed well-defined (and actually continuous) in $\mathcal{F}^2$.

Proof. Just verify (with the same integrations by parts than those used in the course of the proofs of Propositions 3.6 and 3.7) that:

- all families of segregated pulsating fronts satisfy some locally uniform estimates (with respect to (d,α,f_1,f_2)) in $\mathcal{C}_{loc}\left(\mathbb{R}, L^2_{loc}(\mathbb{R})\right) \cap L^2_{loc}\left(\mathbb{R}, H^1_{loc}(\mathbb{R})\right)$ and therefore, by virtue of DiBenedetto's theory [55], in $\mathcal{C}^{0,\beta}_{loc}\left(\mathbb{R}^2\right)$;
- all families of segregated stationary equilibrium satisfy some locally uniform estimates (with respect to (d,α,f_1,f_2)) in $\mathcal{C}^{2,\beta}_{loc}(\mathbb{R})$.

The continuity of c_∞ is then a classical consequence of Theorems 3.44 and 3.45 and of compactness arguments.

The locally uniform convergence is proved with similar compactness arguments, this time using the fact that the compactness estimates of Propositions 3.6 and 3.7 are locally uniform. ☐

Remark. We recall that in the case of non-constant a_1 and a_2, we cannot prove Theorem 3.45. Therefore it is not possible to prove complete continuity of c_∞. In the whole subset:

$$\left\{(d,\alpha,f_1,f_2)\in\mathfrak{P}\mid\frac{\alpha^2}{d}\in\mathcal{R}^0_{d,f_1,f_2}\right\},$$

c_∞ might not be continuous and jump between 0 and some non-zero values. Still, it is not possible to jump directly from a positive value to a negative one, whence the zero set is in any case non-empty. Moreover, we recall that these issues are completely subordinated to Conjecture 3.46.

3.4.3.1 As a conclusion: what about monotonicity?

Regarding the monotonicity of $\alpha\mapsto c_\infty$:

it should be easily established, via super- and sub-solutions, that $\alpha\mapsto c_\infty$ is non-decreasing (a proof that we do not detail here for the sake of brevity). Recall moreover that we already suggested in the previous subsection that $c_\infty\to -c^\star[d,2]$ as $\alpha\to 0$ and $c_\infty\to c^\star[1,1]$ as $\alpha\to+\infty$, whence $\alpha\mapsto c_\infty$ would in fact be from $(0,+\infty)$ onto $(-c^\star[d,2],c^\star[1,1])$.

Regarding the monotonicity of $d\mapsto c_\infty$:

on the contrary, such a result should in general not be expected. We recall that:

— the dependency of the speed of a bistable front on its diffusion coefficient is in general unclear;

— even for the Fisher–KPP equation, as long as heterogeneity is introduced, the monotonicity of the minimal speed as a function of the diffusion coefficient is in general lost (for instance, in space-time periodic media, a counter example has been exhibited by the second author [118]).

Chapitre 4

Compétition en milieu périodique : III – Existence & stabilité d'états de coexistence périodiques ségrégés

Résumé

Ce chapitre s'intéresse au système de compétition – diffusion de Lotka – Volterra à deux espèces avec fort taux de compétition ainsi qu'à deux équations de réaction – diffusion scalaires liées. On montre que dans certains milieux périodiques avec large période, il existe des états stationnaires de coexistence périodiques, non-constants et stables. Les résultats sont comparés à des résultats déjà connus sur l'existence et la non-existence de telles solutions. Enfin, une interprétation écologique est proposée.

Ce chapitre, co-écrit avec Alessandro Zilio, a fait l'objet d'une soumission sous le titre *Competition in periodic media : III – Existence & stability of segregated periodic coexistence states* dans *Mathematische Annalen* [GZ18].

4.1 Introduction

We construct stable periodic sign-changing steady states in one-dimensional spatially periodic media for the equation

$$\partial_t z - \partial_{xx} z = f(z, x) \tag{4.1.1}$$

and its quasi-linear counterpart

$$\partial_t (\sigma(z)z) - \partial_{xx} z = f(z, x), \tag{4.1.2}$$

where

$$f : (z, x) \mapsto \mu_1(x)\left(a_1 - \frac{1}{\alpha}z\right)z^+ - \frac{1}{d}\mu_2(x)\left(a_2 + \frac{1}{d}z\right)z^-$$

and the positive function σ is

$$\sigma : z \mapsto \mathbf{1}_{z>0} + \frac{1}{d}\mathbf{1}_{z<0}.$$

Here L, a_1, a_2, α and d are positive constants, $\mu_1, \mu_2 \in L^\infty(\mathbb{R}, (0, +\infty))$ are positive L-periodic functions, $z^+ = \max(z, 0)$ and $z^- = -\min(z, 0)$ (so that $z = z^+ - z^-$).

We also construct stable periodic coexistence steady states for the following competition–diffusion system:

$$\begin{cases} \partial_t u_1 - \partial_{xx} u_1 = \mu_1(x)(a_1 - u_1)u_1 - k\omega(x)u_1u_2 \\ \partial_t u_2 - d\partial_{xx} u_2 = \mu_2(x)(a_2 - u_2)u_2 - \alpha k\omega(x)u_1u_2 \end{cases} \tag{4.1.3}$$

where $\omega \in L^\infty(\mathbb{R}, (0, +\infty))$ is positive and L-periodic (with a normalized mean value, say).

System (4.1.3) belongs to the wider class of elliptic or parabolic systems of Lotka–Volterra type in the presence of strong competition, and (4.1.1) and (4.1.2) are related to its singular *strong competition limit* $k \to +\infty$. To our knowledge, the study of the strong competition limit appeared first in [45] as a way to model biological species that are fiercely competing for the same resource. The literature on this subject is very vast, varying from existence and uniqueness results [43], multiplicity results in presence of strong competition [45] and the rigorous proof of Gause's competitive exclusion [47, 103] stating that in the homogeneous case, non-constant solutions are necessarily unstable (in convex domains). We refer the interested reader to these contributions and the references therein.

More recently, the strong competition limit in periodic media was the object of investigation of two papers [Gir17, GN18] by the first author and Nadin. According to [GN18], (4.1.2) is the equation satisfied, in the strong competition limit, by the quantity $\alpha u_1 - du_2$ with (u_1, u_2) solution of (4.1.3). Notice that, by normalizing (u_1, u_2), we can assume without loss of generality $a_1 = a_2 = 1$. This is assumed indeed from now on. Notice also that, although all results of [Gir17, GN18] are stated for $\omega = 1$, they are readily extended to the case of non-constant ω.

Steady states of (4.1.1) and of (4.1.2) satisfy the same elliptic semilinear equation:

$$-z''(x) = \mu_1(x)\left(1 - \frac{1}{\alpha}z(x)\right)z^+(x) - \frac{1}{d}\mu_2(x)\left(1 + \frac{1}{d}z(x)\right)z^-(x). \tag{4.1.4}$$

However, due to the different time dependencies, (4.1.1) and (4.1.2) involve in general different notions of stability and therefore different eigenproblems. Before going any further, let us precise this important point.

4.1.1 Notions of stability

For any functional space X, $X_{L\text{-per}}$ denotes the set of L-periodic functions whose restriction to any interval of length L are elements of X. Accordingly, for any second order monotone elliptic operator $\mathscr{L}$, $\lambda_{1,L\text{-per}}(-\mathscr{L})$ denotes the periodic principal eigenvalue of $\mathscr{L}$ given by the Krein–Rutman theorem. Recall that if (u_1, u_2) is a solution of (4.1.3), then the system satisfied by $(u_1, 1-u_2)$ is a monotone system, whence its linearization admits indeed a periodic principal eigenvalue (details can be found in [Gir17]).

Hereafter, a solution $z \in H^2_{L\text{-per}}(\mathbb{R})$ of (4.1.4) such that the L-periodic function

$$f_1[z] : x \mapsto \partial_1 f(z(x), x),$$

is well-defined (at least weakly) is referred to as *linearly stable in the sense of (4.1.1)* if

$$\lambda_{1,L\text{-per}}\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} - f_1[z]\right) > 0$$

and as *linearly stable in the sense of (4.1.2)* if

$$\lambda_{1,L\text{-per}}\left(-\hat{\sigma}(z)\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \hat{\sigma}(z) f_1[z]\right) > 0,$$

with

$$\hat{\sigma} : z \mapsto \mathbf{1}_{z\geq 0} + d\mathbf{1}_{z<0}.$$

The constant solutions of (4.1.4) are α, $-d$ and 0. It is easily verified that α and $-d$ are linearly stable in both senses whereas 0 is linearly unstable (namely, not linearly stable) in both senses.

The definition of linear stability in the sense of (4.1.2) can be formally understood by plugging perturbations of the form $\mathrm{e}^{-\lambda t}\varphi(x)$, with φ L-periodic, into the equation (4.1.2) linearized at an almost everywhere nonzero steady state z. Indeed, such a perturbation solves the linear equation if and only if

$$-\lambda\sigma(z)\varphi - \varphi'' = f_1[z]\varphi,$$

that is, due to the almost everywhere equality $\sigma(z(x))\,\hat{\sigma}(z(x)) = 1$, if and only if

$$-\hat{\sigma}(z)\varphi'' - \hat{\sigma}(z) f_1[z]\varphi = \lambda\varphi.$$

Similarly, a steady state solution (u_1, u_2) of (4.1.3) is a solution of

$$\begin{cases} -u_1''(x) = \mu_1(x)(1-u_1(x))\,u_1(x) - k\omega(x)u_1(x)u_2(x) \\ -du_2''(x) = \mu_2(x)(1-u_2(x))\,u_2(x) - \alpha k\omega(x)u_1(x)u_2(x) \end{cases} \tag{4.1.5}$$

and is referred to as *linearly stable* if

$$\lambda_{1,L\text{-per}}\left(-\begin{pmatrix} \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \mu_1(1-2u_1) - k\omega u_2 & k\omega u_1 \\ \alpha k\omega u_2 & d\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \mu_2(1-2u_2) - \alpha k\omega u_1 \end{pmatrix}\right) > 0.$$

The steady states $(1,0)$ and $(0,1)$ are linearly stable whereas $(0,0)$ is linearly unstable.

By analogy with the spatially homogeneous setting and in view of the stability of the constant solutions, (4.1.1), (4.1.2) and (4.1.3) are sometimes referred to as *bistable*. However our main contribution is to prove that this terminology can be misleading: because of the spatial heterogeneity, a third stable state can very well exist.

Let us point out that the previous two parts of the series "*Competition in periodic media*"[Gir17, GN18] only used the notion of stability in the sense of the system (4.1.3). This explains why the two notions of stability for the segregated equation (4.1.4) are only introduced now.

4.1.2 Main results

Let $(r_0, r_1, r_2) \in (0,1)^3$ such that $2r_0 + 2r_1 + 2r_2 = 1$. Let $(M_1, M_2) \in (0, +\infty)^2$ and define two 1-periodic functions $\mu_1^\star$ and $\mu_2^\star$ by

$$(\mu_1^\star)_{|[0,1]} = M_1 \mathbf{1}_{[0,r_1]} + M_1 \mathbf{1}_{[r_1+2r_0+2r_2,1]}$$

$$(\mu_2^\star)_{|[0,1]} = M_2 \mathbf{1}_{[r_1+r_0,r_1+r_0+2r_2]}$$

and, for all $L > 0$,

$$\left(\mu_1^L, \mu_2^L\right) : x \mapsto (\mu_1^\star, \mu_2^\star)\left(\frac{x}{L}\right).$$

Our first main result is concerned with the equation (4.1.4).

Theorem 4.1. *There exists $\underline{L} > 0$ such that, for all $L > \underline{L}$, (4.1.4) with $(\mu_1, \mu_2) = \left(\mu_1^L, \mu_2^L\right)$ or with $(\mu_1, \mu_2) = \left(\mu_1^L + \mu_2^L, \mu_1^L + \mu_2^L\right)$ admits a linearly stable in both senses, sign-changing, L-periodic solution.*

Furthermore, for all $L > \underline{L}$, there exist a neighborhood U_L of $\left(\mu_1^L, \mu_2^L\right)$ in the topology of $\left(L^\infty_{L\text{-}per}\right)^2$ and a neighborhood V_L of $\mu_1^L + \mu_2^L$ in the topology of $\left(L^\infty_{L\text{-}per}\right)$ such that, for all $(\mu_1, \mu_2) \in U_L$ and all $\mu \in V_L$, (4.1.4) with (μ_1, μ_2) or (μ, μ) admits a linearly stable in both senses, sign-changing, L-periodic solution.

This first result will be proved by explicit construction of v and non-trivial application of the implicit function theorem.

In biological terms, the growth rate $\mu_1^L + \mu_2^L$ corresponds to a periodic environment where large favorable areas are separated by large neutral areas. A neutral area could be, say, in a woodland inhabited by herbivorous animals looking for glades, an area densely covered by trees where predators live and hide and where linear death rates roughly equal linear birth rates and no intraspecific competition occurs. The associated stable steady state describes the situation where one competitor settles in the evenly numbered favorable areas whereas the other settles in the oddly numbered ones. This particular form is illustrated by Figure 4.2.1.

Let us point out that well-known density results yield immediately the following corollary.

Corollary 4.2. *For all $L > \underline{L}$, there exists $(\mu_1, \mu_2) \in \left(\mathscr{C}^\infty_{L\text{-}per}(\mathbb{R}, (0, +\infty))\right)^2$ such that (4.1.4) admits a linearly stable in both senses, sign-changing, L-periodic solution.*

Our second main result is concerned with the system (4.1.5) and states that the existence of stable steady states for the segregated equation implies the existence of stable steady states for the strongly competitive system. It will be proved as a consequence of Theorem 4.1 and of degree theory.

Theorem 4.3. *For all $L > \underline{L}$, there exist $k^\star > 0$ and $(\mu_1, \mu_2) \in \left(\mathscr{C}^\infty_{L\text{-}per}(\mathbb{R}, (0, +\infty))\right)^2$ such that, for all $k > k^\star$, (4.1.5) admits a linearly stable, component-wise positive, L-periodic solution.*

4.1.3 Discussion and comparison with known results

Theorem 4.1 and Theorem 4.3 complement interestingly a result of the first author [Gir17, Theorem 1.2] stating that, provided L is sufficiently small, that is

$$L \in \left(0, \pi\left(\left(\max_{[0,L]} \mu_1\right)^{-\frac{1}{2}} + \sqrt{d}\left(\max_{[0,L]} \mu_2\right)^{-\frac{1}{2}}\right)\right),$$

and provided k is large enough, all L-periodic coexistence states are unstable and vanish as $k \to +\infty$.

Theorem 4.1 is also strictly related to a result due to Ding, Hamel and Zhao [57, Theorem 1.5] which shows in particular that the regular bistable equation

$$\partial_t z - \partial_{xx} z = g_L(x, z),$$

with $g_L : (z, x) \mapsto g\left(z, \frac{x}{L}\right)$, g 1-periodic with respect to x and independent of L, 0 and 1 linearly stable steady states (in the standard sense) and $\theta \in \mathscr{C}_{1-\text{per}}(\mathbb{R}, (0, 1))$ intermediate zero of g, admits bistable pulsating fronts connecting 0 and 1 provided L is large enough and the nonlinearity g satisfies

$$\min_{x \in [0,L]} \int_0^1 g(x, z) \mathrm{d}z > 0 \text{ and } \min_{x \in [0,L]} \frac{\partial g}{\partial z}(x, \theta(x)) > 0.$$

Their proof is based on a very important result by Fang and Zhao [69] stating in a general setting that bistable pulsating fronts exist if all intermediate periodic steady states are unstable and invadable. Therefore the proof of Ding–Hamel–Zhao basically shows that the above conditions imply the nonexistence of stable periodic steady states. Importantly,

— on one hand, the family of scaled functions $(f_L)_{L > \underline{L}}$ in Theorem 4.1 satisfies

$$\min_{x \in [0,L]} \int_{-d}^{\alpha} f_L(x, z) \mathrm{d}z = 0 \quad \text{for all } L > \underline{L}$$

(recalling that here the two constant stable states are $-d$ and α instead of 0 and 1);

— on the other hand, any family of regularized and positive functions obtained from Corollary 4.2 satisfies indeed the above two positivity conditions, but by the result of Ding–Hamel–Zhao cannot be of the prescribed scaled form as L varies (in other words, the neighborhoods U_L and V_L obtained with the implicit function theorem are not uniform with respect to L and shrink as $L \to +\infty$).

We point out that a recent paper by Zlatŏs [148] constructed an example of periodic bistable nonlinearity admitting no pulsating front. His result is very related to ours but remains qualitatively different: we focus on stable intermediate steady states whereas Zlatŏs focuses on nonexistence of transition fronts. Furthermore, our construction has a very simple ecological interpretation and is valid for all large periods, whereas the construction of Zlatŏs requires a very precise period. In this regard, our paper is an interesting complement.

Theorem 4.1 is also related to a family of results stating, loosely speaking, that the geometry of a homogeneous domain with boundary can block bistable propagation. See for instance Berestycki–Bouhours–Chapuisat [13] and references therein.

Ecologically speaking, Theorem 4.3 shows that *strong interspecific competition* and *heterogeneity of the habitat* can lead together to *spatial segregation* and therefore to *speciation* and *increased biodiversity*. Having this interpretation in mind, we notice that the strength of the competition is crucial: indeed, in the weak competition case, Dockery–Hutson–Mischaikow–Pernarowski [58] showed on the contrary that heterogeneity leads to extinction of all competitors but the one with the lowest diffusion rate. Ecologically, strong competition occurs for instance when resources are rare. Mathematically, it is known to lead indeed to spatial segregation, or in other words pattern formation, in homogeneous domains with appropriate boundary conditions or initial conditions (see for instance [38, 41, 47] and references therein). As such, our result can be seen as a contribution to the overarching research program on pattern formation in strongly competing systems and as one of the first results in spatially heterogeneous domains.

It is worthy to recall that by a result of Berestycki–Hamel–Rossi [18, Proposition 6.6], the periodic principal eigenvalue of a self-adjoint periodic scalar elliptic operator coincides with the decreasing limit as $R \to +\infty$ of its Dirichlet principal eigenvalue in the ball $(-R, R)$. Consequently, if the domain of a linearly stable in both senses, periodic, sign-changing steady state solution z of (4.1.4) is restricted to a periodicity cell $(y, y+L)$ with y chosen so that $z(y) = 0$, then we obtain a steady state for the corresponding Dirichlet problem which is linearly stable in the following senses:

$$\lambda_{1,\mathrm{Dir}} \left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} - f_1[z], (y, y+L) \right) > 0,$$

$$\lambda_{1,\mathrm{Dir}} \left(-\hat{\sigma}(z)\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \hat{\sigma}(z) f_1[z], (y, y+L) \right) > 0.$$

4.1.4 What about more general bistable equations?

The particular shape of function f in (4.1.4) is due to the underlying ecological model. With very few modifications, Theorem 4.1 can be extended more general bistable equations in periodic media, like for instance the familiar Allen–Cahn equation

$$\partial_t z - \partial_{xx} z = \mu_L(x)(1 - z^2)z.$$

4.1.5 Structure of the paper

In Section 2, we prove Theorem 4.1, focusing first on the construction of v and then using the implicit function theorem to obtain the open neighborhood U. In Section 3, we prove Theorem 4.3 thanks to Theorem 4.1 and topological arguments.

4.2 The segregated bistable equation

Our goal in this section is to prove that (4.1.4) admits sign-changing solutions that are also stable in the sense of (4.1.1) and (4.1.2).

Before going any further, we observe the following: replacing $\left(\frac{\mu_1}{\alpha}, \frac{\mu_2}{d^2}\right)$ by (μ_1, μ_2), (4.1.4) reads

$$-z'' = \mu_1 (\alpha - z) z^+ - \mu_2 (d + z) z^-. \tag{4.2.1}$$

Hence up to end of this section we have in mind the above more compact form. The piecewise-constant functions $\mu_1^\star$ and $\mu_2^\star$ defined in the introduction are accordingly modified, with $\left(\frac{M_1}{\alpha}, \frac{M_2}{d^2}\right)$ replaced by (M_1, M_2).

In order to construct a sing-changing, periodic and stable solution to (4.2.1), we need a preliminary result concerning its linearization.

4.2.1 Linearization near a non-constant stationary solution

Since the right hand side of (4.2.1) is only Lipschitz continuous at $z = 0$, we need some caution in order to properly introduce the linearization of the equation around a sign-changing steady state. Many authors have already addressed similar issues (see, for instance, [47, Section 4.1]). Since we could not find the precise statement that we needed, we decided to present a complete proof. We wish to point out that the result can be adapted to more general equations (for instance bounded domains with Neumann boundary conditions).

For all $(\mu_1,\mu_2,z) \in \left(L^\infty_{L\text{-per}}\right)^2 \times H^2_{L\text{-per}}$, we define

$$\mathscr{F} : \left(L^\infty_{L\text{-per}}\right)^2 \times H^2_{L\text{-per}} \to L^2_{L\text{-per}}$$

such that, for all test functions $\varphi \in H^2_{L\text{-per}}$,

$$\langle \mathscr{F}(\mu_1,\mu_2,z),\varphi\rangle = \int_0^L z'\varphi' - \int_0^L \left(\mu_1(\alpha - z)z^+ - \mu_2(d+z)z^-\right)\varphi. \tag{4.2.2}$$

We recall that, by Sobolev embedding, the inclusion $H^2_{L\text{-per}} \hookrightarrow \mathscr{C}^{1,\frac{1}{2}}_{L\text{-per}}$ holds true.

Lemma 4.4. *Let $O \subset H^2_{L\text{-per}}$ be an open set in the topology of $H^2_{L\text{-per}}$ such that for all $z \in O$, the closed set $z^{-1}(\{0\})$ has zero Lebesgue measure.*

Then $\mathscr{F} \in \mathscr{C}^1\left(\left(L^\infty_{L\text{-per}}\right)^2 \times O, L^2_{L\text{-per}}\right)$.

For any $(\mu_1,\mu_2,z) \in \left(L^\infty_{L\text{-per}}\right)^2 \times O$ and any $(\eta_1,\eta_2,w) \in \left(L^\infty_{L\text{-per}}\right)^2 \times H^2_{L\text{-per}}$, the differential $\mathrm{d}\mathscr{F}[\mu_1,\mu_2,z]$ evaluated at (η_1,η_2,w) is

$$\varphi \mapsto \int_0^L w'\varphi' - \int_0^L \left(\eta_1(\alpha - z)z^+ - \eta_2(d+z)z^-\right)\varphi \\ - \int_0^L \left(\mu_1(\alpha - 2z)\mathbf{1}_{z>0} + \mu_2(d+2z)\mathbf{1}_{z<0}\right)w\varphi.$$

Remark. Some assumptions on the open set O are necessary. In general, the Gâteaux differential of $\mathscr{F}$ at (μ_1,μ_2,z) in the direction (η_1,η_2,w) fails to be linear with respect to (η_1,η_2,w). More precisely, it is the sum of the linear functional above and of

$$\varphi \mapsto -\int_0^L \left(\mu_1\alpha w^+ - \mu_2 d w^-\right)\mathbf{1}_{z=0}\varphi,$$

which is non-linear with respect to w. We can prove this by partitioning $\mathbb{R} = \{z>0\}\cup\{z=0\}\cup\{z<0\}$.

Proof. The linear mapping appearing in the statement above is readily continuous. Thus we only need to show that it is indeed the Gâteaux differential.

Fix $(\mu_1,\mu_2,z) \in \left(L^\infty_{L\text{-per}}\right)^2 \times O$ and $(\eta_1,\eta_2,w) \in \left(L^\infty_{L\text{-per}}\right)^2 \times H^2_{L\text{-per}}$. For all $t>0$ and all $\varphi \in H^2_{L\text{-per}}$,

$$\frac{1}{t}\left(\mathscr{F}\left[(\mu_1,\mu_2,z)+t(\eta_1,\eta_2,w)\right] - \mathscr{F}\left[(\mu_1,\mu_2,z)\right]\right)(\varphi) = \\ \int_0^L w'\varphi' - \frac{1}{t}\int_0^L \left((\mu_1+t\eta_1)(\alpha-(z+tw))(z+tw)^+ - \mu_1(\alpha - z)v^+\right)\varphi \\ + \frac{1}{t}\int_0^L \left((\mu_2+t\eta_2)(d+(z+tw))(z+tw)^- - \mu_2(d-z)z^-\right)\varphi.$$

The first term in the right hand side does not depend on t. We only need to consider the second one, as the third one can be dealt with in a similar way. Rearranging the terms, we find

$$\begin{aligned}\frac{1}{t}\int_0^L \left((\mu_1+t\eta_1)\,(\alpha-(z+tw))\,(z+tw)^+ - \mu_1(\alpha-z)v^+\right)\varphi \\ = \int_0^L \eta_1\,(\alpha-(z+tw))\,(z+tw)^+\varphi \\ + \int_0^L \mu_1\frac{(\alpha-(z+tw))\,(z+tw)^+ - (\alpha-z)v^+}{t}\varphi.\end{aligned}$$

The dominated convergence theorem yields

$$\int_0^L \eta_1\,(\alpha-(z+tw))\,(z+tw)^+\varphi \to \int_0^L \eta_1\,(\alpha-z)\,v^+\varphi \qquad \text{as } t\to 0.$$

Rearranging the last term of the preceding equality, we find

$$\begin{aligned}\int_0^L \mu_1\left(\frac{(\alpha-z-tw)\,(z+tw)^+ - (\alpha-z)\,z^+}{t}\right)\varphi \\ = \int_0^L \mu_1\left(\frac{(z+tw)^+ - z^+}{t}\right)(\alpha-z)\,\varphi - \int_0^L \mu_1 w\alpha\,(z+tw)^+\,\varphi.\end{aligned}$$

By dominated convergence,

$$\lim_{t\to 0}\int_0^L \mu_1 w\alpha\,(z+tw)^+\,\varphi = \int_0^L \mu_1 w\alpha z^+\varphi.$$

Since by assumption $z^{-1}(\{0\})$ has zero Lebesgue measure and the map $\zeta\mapsto\zeta^+$ is smooth away from 0, the dominated convergence theorem yields once again

$$\lim_{t\to 0}\int_0^L \mu_1\left(\frac{(z+tw)^+ - z^+}{t}\right)(\alpha-z)\,\varphi = \int_0^L \mu_1 w\mathbf{1}_{z>0}\,(\alpha-z)\,\varphi.$$

This concludes the proof. □

4.2.2 Construction of the solution

We now proceed by constructing the solution of (4.2.1). To do so, we first consider the equation with piecewise-constant coefficients. In this case, solutions can be constructed by gluing together different profiles. The implicit function theorem then leads to an open neighborhood of valid coefficients near this piecewise-constant pair.

4.2.2.1 Piecewise-constant coefficients

In the following result we collect some properties of the solutions of the logistic equation with non-zero Dirichlet conditions. These properties are well known and straightforward consequences of the comparison principle. For this reason, we do not present here a fully detailed proof.

Lemma 4.5. *For all $A > 0$, $M > 0$, $\nu \in \left[\frac{1}{2}, 1\right)$ and $R > 0$ there exists a unique positive solution $w_{A,M,\nu,R} \in \mathscr{C}^2\left([-R, R]\right)$ of*

$$\begin{cases} -w'' = M\left(A - w\right)w & \text{in } (-R, R) \\ w\left(\pm R\right) = \nu A. \end{cases}$$

The function $w_{A,M,\nu,R}$ is even and satisfies

$$\nu A < w_{A,M,\nu,R}(x) < A \qquad \text{for all } x \in (-R, R).$$

Furthermore, let

$$\Phi : (A, M, \nu, R) \mapsto w'_{A,M,\nu,R}\left(-R\right).$$

The following properties hold true.

1. *Φ is positive and continuous;*
2. *it holds*
$$\lim_{R\to 0^+} \Phi(A, M, \nu, R) = 0;$$
3. *there exists $\gamma_{A,M,\nu} \in (0, +\infty)$ such that*
$$\gamma_{A,M,\nu} = \lim_{R\to+\infty} \Phi\left(A, M, \nu, R\right).$$
Moreover, $(A, M, \nu) \mapsto \gamma_{A,M,\nu}$ is continuous with respect to A, M and ν, increasing with respect to A and M and decreasing with respect to ν. In particular $0 = \lim_{\nu\to 1} \gamma_{A,M,\nu} < \gamma_{A,M,\nu} < \gamma_{A,M,\frac{1}{2}}$;
4. *the function $R \mapsto \Phi(A, M, \nu, R)$ is an increasing homeomorphism from $(0, +\infty)$ onto $(0, \gamma_{A,M,\nu})$;*
5. *the function $\nu \mapsto \Phi(A, M, \nu, R)$ is a decreasing homeomorphism from $\left[\frac{1}{2}, 1\right)$ onto $\left(0, \Phi(A, M, \frac{1}{2}, R)\right]$.*

We point out that the upper limit $\gamma_{A,M,\nu}$ can actually be determined explicitly.

Proof. We perform the following change of variables

$$w(x) = AW_{\rho,\nu}\left(\sqrt{AM}x\right) \qquad \text{and} \qquad \rho = \sqrt{AM}R.$$

Here the function $W_{\rho,\nu}$ is a solution to the scaled equation

$$\begin{cases} -W'' = \left(1 - W\right)W & \text{in } (-\rho, \rho) \\ W\left(\pm\rho\right) = \nu. \end{cases} \tag{4.2.3}$$

We can rephrase all the statements of the result in terms of the dependence of $W_{\rho,\nu}$ on ρ and ν. Here we consider only the dependence on ρ. The same arguments can be adapted to show the corresponding results in terms of ν.

For any value of $\rho > 0$ and $\nu \in [\frac{1}{2}, 1)$, the previous equation admits a unique, positive solution which is even and is such that $\nu < W(x) < 1$ for all $x \in (-\rho, \rho)$. This follows by standard arguments. We just observe that the functions $x \mapsto \nu\cos(\gamma x)/\cos(\gamma\rho)$ are sub-solutions of (4.2.3) for γ small enough, while the constant 1 is always a super-solution.

Notice that, for all $\kappa > 1$:

$$-\left(\kappa W_{\rho,\nu}\right)'' = \left(1 - W_{\rho,\nu}\right)\kappa W_{\rho,\nu} \geq \left(1 - \kappa W_{\rho,\nu}\right)\kappa W_{\rho,\nu} \text{ in } \left(-\rho, \rho\right).$$

For all $\rho' > \rho > 0$, the following quantity is well-defined:

$$\kappa^\star = \inf\left\{\kappa > 1 \mid \kappa W_{\rho',\nu} \geq W_{\rho,\nu} \text{ in } (-\rho,\rho)\right\}.$$

Assuming by contradiction that $\kappa^\star > 1$ and applying the strong maximum principle, we get a contradiction. Hence the family $\left(W_{\rho,\nu}\right)_{\rho>0}$ is non-decreasing, and once more by the strong maximum principle, it is in fact increasing.

It follows that the function $\rho \mapsto \max_{[-\rho,\rho]} W_{\rho,\nu}(x)$ is increasing with limit 1 as $\rho \to +\infty$. By classical elliptic estimates (see Gilbarg–Trudinger [80]) the family converges locally uniformly to a bounded and positive solution of (4.2.3) defined on the whole line $\mathbb{R}$. Hence, as $\rho \to +\infty$, we find that $W_{\rho,\nu} \to 1$ locally in $\mathscr{C}^2$.

We now consider the shifted family of functions

$$\overline{W}_{\rho,\nu}(x) = W_{\rho,\nu}\left(x - \rho\right) \qquad \text{for} \quad x \in [0, 2\rho].$$

The family $\rho \mapsto \overline{W}_{\rho,\nu}$ is increasing. In particular, by the Hopf lemma,

$$\rho \mapsto \overline{W}'_{\rho,\nu}(0)$$

is increasing as well. Once again, classical elliptic estimates show that, as $\rho \to +\infty$, the family $\overline{W}_{\rho,\nu}$ converges locally uniformly to the unique solution $\overline{W}$ of

$$\begin{cases} -\overline{W}'' = \left(1 - \overline{W}\right)\overline{W} & \text{in } (0,+\infty) \\ \overline{W}\left(0\right) = \nu \\ \nu < \overline{W} < 1 & \text{in } (0,+\infty) \end{cases} \tag{4.2.4}$$

(see Du–Lin [59, 60, Proposition 4.1]). Thus, the limit as $\rho \to +\infty$ of $\overline{W}'_{\rho,\nu}(-\rho)$ is finite and positive. We can figure out its value by testing (4.2.4) against $\overline{W}'$. This yields the identity

$$\lim_{\rho\to+\infty} \overline{W}'_{\rho,\nu}(-\rho) = \sqrt{\frac{1}{3} + \nu^2\left(\frac{2}{3}\nu - 1\right)}.$$

Observe that the limit is always positive and bounded.

We conclude by observing that the continuity of $\overline{W}'_{\rho,\nu}$ with respect to ρ is a classical consequence of the uniqueness of $\overline{W}_{\rho,\nu}$ and of compactness arguments. □

From the previous result we deduce a property which is crucial for our construction. For sake of brevity, from now on we will simply write

$$\Phi_1(\nu, L) = \Phi(\alpha, M_1, \nu, r_1 L),$$
$$\Phi_2(\nu, L) = \Phi(d, M_2, \nu, r_2 L),$$

(recalling that $M_1 > 0$, $M_2 > 0$, $r_1 > 0$ and $r_2 > 0$ were fixed in the introduction).

We can finally construct the periodic stable solutions of (4.2.1) with the piecewise-constant coefficients.

Proposition 4.6. *There exists $\underline{L} > 0$ such that, for any $L > \underline{L}$, (4.2.1) with either $(\mu_1, \mu_2) = \left(\mu_1^L, \mu_2^L\right)$ or with $(\mu_1, \mu_2) = \left(\mu_1^L + \mu_2^L, \mu_1^L + \mu_2^L\right)$ admits a nonzero sign-changing solution $v \in H^2_{L\text{-per}}$ satisfying, for all L-periodic test functions $\varphi \in H^1_{L\text{-per}}$,*

$$\int_0^L v'\varphi' = \int_0^L \left(\mu_1\left(\alpha - v\right)v^+ - \mu_2\left(d + v\right)v^-\right)\varphi.$$

Furthermore, v is linearly stable in the sense of (4.1.1) and (4.1.2).

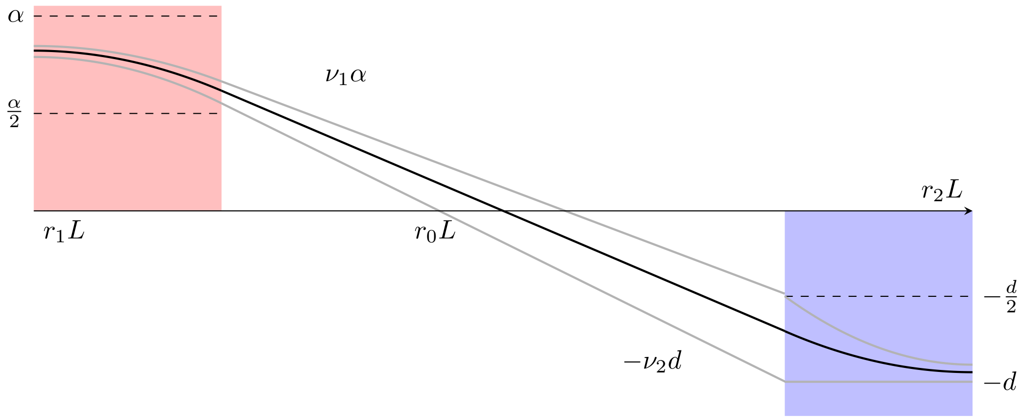


Figure 4.2.1 – Visual representation of the construction of v. In red, areas where $\mu_1^L = M_1$. In blue, areas where $\mu_2^L = M_2$. In gray, the bounds given by $\underline{\nu_L}$ and $\overline{\nu_L}$. In black, the solution v.

Proof. Let

$$\delta : (\nu, L) \mapsto -\Phi_1 (\nu, L) r_0 L + \alpha\nu.$$

The function $\nu \mapsto \delta(\nu, L)$ is, for all $L > 0$, an increasing homeomorphism from $\left[\frac{1}{2}, 1\right)$ onto

$$\left[-\Phi_1\left(\frac{1}{2}, L\right) r_0 L + \frac{\alpha}{2}, \alpha\right).$$

Since $L \mapsto -\Phi_1\left(\frac{1}{2}, L\right) r_0 L$ is decreasing and goes to $-\infty$ as $L \to +\infty$, we can define the unique $L_0 > 0$ satisfying

$$-\Phi_1\left(\frac{1}{2}, L_0\right) r_0 L_0 + \frac{\alpha}{2} = -d.$$

Then for all $L > L_0$, we can define the unique $\underline{\nu_L} \in \left(\frac{1}{2}, 1\right)$ and the unique $\overline{\nu_L} \in \left(\underline{\nu_L}, 1\right)$ satisfying respectively

$$\delta\left(\underline{\nu_L}, L\right) = -d \text{ and } \delta\left(\overline{\nu_L}, L\right) = -\frac{d}{2}.$$

Now let

$$\psi : (\nu, L) \mapsto \Phi_1 (\nu, L) - \Phi_2\left(-\frac{\delta(\nu, L)}{d}, L\right),$$

well-defined in $\left(\underline{\nu_L}, \overline{\nu_L}\right]$ for all $L > L_0$. For all $L > L_0$, $\nu \mapsto \psi(\nu, L)$ is a decreasing homeomorphism satisfying

$$\lim_{\nu \to \underline{\nu_L}} \psi(\nu, L) = \frac{\alpha\underline{\nu_L} + d}{r_0 L} > 0,$$

$$\psi\left(\overline{\nu_L}, L\right) = \frac{\alpha\overline{\nu_L} + \frac{d}{2}}{r_0 L} - \Phi_2\left(\frac{1}{2}, L\right).$$

Since $L \mapsto \psi\left(\overline{\nu_L}, L\right)$ goes to $-\gamma_{d, M_2, \frac{1}{2}} < 0$ as $L \to +\infty$, we can define $\underline{L} \geq L_0$ such that, for all $L > \underline{L}$,

$$\psi\left(\overline{\nu_L}, L\right) < 0$$

and deduce that for all $L > \underline{L}$, there exists a unique $\nu_L \in \left(\underline{\nu_L}, \overline{\nu_L}\right)$ satisfying $\psi\left(\nu_L, L\right) = 0$, that is

$$\Phi_1\left(\nu_L, L\right) = \Phi_2\left(-\frac{\delta\left(\nu_L, L\right)}{d}, L\right).$$

Next, we fix $L > \underline{L}$ and define $w_1 = w_{\alpha,M_1,\nu_L,r_1 L}$, $w_2 = w_{d,M_2,-d^{-1}\delta(\nu_L,L),r_2 L}$ as well as the nonzero, sign-changing, L-periodic function v by

$$v_{|[0,L)}(x) = \begin{cases} w_1\left(x\right) & \text{if } x \in [0, r_1 L) \\ -\Phi_1\left(\nu_L, L\right)\left(x - r_1 L\right) + \nu_L \alpha & \text{if } x \in [r_1 L, r_1 L + r_0 L) \\ w_2\left(x - r_1 L - r_0 L - r_2 L\right) & \text{if } x \in [r_1 L + r_0 L, r_1 L + r_0 L + 2 r_2 L) \\ \Phi_1\left(\nu_L, L\right)\left(x - L + r_1 L\right) + \nu_L \alpha & \text{if } x \in [r_1 L + r_0 L + 2 r_2 L, r_1 L + 2 r_0 L + 2 r_2 L) \\ w_1\left(x - L\right) & \text{if } x \in [r_1 L + 2 r_0 L + 2 r_1 L, L) \end{cases}$$

Since, by construction, v is a $\mathscr{C}^{1,1}_{L\text{-per}} \subset H^2_{L\text{-per}}$ juxtaposition of piecewise solutions of (4.2.1), we readily deduce that it is a solution of (4.2.1).

Regarding the stability of the solution v, from Lemma 4.4 we evince that the linearized elliptic operator at v, denoted $\mathscr{L} \in L\left(H^2_{L\text{-per}}, L^2_{L\text{-per}}\right)$, is

$$\mathscr{L} : \eta \mapsto \eta'' + \left[\mu_1\left(\alpha - 2v\right)\mathbf{1}_{v>0} + \mu_2\left(d + 2v\right)\mathbf{1}_{v<0}\right]\eta.$$

First we verify the stability in the sense of (4.1.1). Let λ be the corresponding periodic principal eigenvalue and $\psi \in H^2_{L\text{-per}}$ be the associated unique periodic positive eigenfunction, normalized in $L^2\left((0, L)\right)$. From the identity

$$\int_0^L \left(-\mathscr{L}\psi - \lambda\psi\right)\psi = 0$$

we deduce

$$\begin{aligned} \int_0^L \left(\psi'\right)^2 &= \int_0^L \left[\mu_1\left(\alpha - 2v\right)\mathbf{1}_{v>0} + \mu_2\left(d + 2v\right)\mathbf{1}_{v<0}\right]\psi^2 + \lambda \\ &= M_1 \int_{\{\mu_1>0\}\cap\{v>0\}} \left(\alpha - 2v\right)\psi^2 + M_2 \int_{\{\mu_2>0\}\cap\{v<0\}} \left(d + 2v\right)\psi^2 + \lambda. \end{aligned}$$

Since by construction

$$v \geq \nu_L \alpha > \frac{\alpha}{2} \text{ in } \{\mu_1 > 0\} \cap \{v > 0\}$$

and

$$v \leq -\left(-\frac{\delta\left(\nu_L, L\right)}{d}\right) d < -\frac{d}{2} \text{ in } \{\mu_2 > 0\} \cap \{v < 0\},$$

we deduce

$$\lambda > \int_0^L \left(\psi'\right)^2 > 0.$$

Similarly, we verify the stability of v in the sense of (4.1.2). The same computations as before lead us to the desired conclusion.

This conclude the proof of existence and stability of sign-changing solutions for piecewise-constant coefficients □

Remark. Going carefully through the proof, using $\overline{\nu_L} < 1$ and assuming that $\underline{L}$ is minimal, we obtain the estimate $\underline{L} < L^\star$, where $L^\star > 0$ is the unique solution of

$$\Phi_2\left(\frac{1}{2}, L^\star\right) L^\star = \frac{1}{r_0} \max\left(\alpha + \frac{d}{2}, \frac{\alpha}{2} + d\right).$$

Hence estimating $\underline{L}$ is only a matter of estimating $L \mapsto \Phi_2\left(\frac{1}{2}, L\right)$. Unfortunately, being unable to find any satisfying estimation of Φ_2, we do not pursue further.

4.2.2.2 With regular coefficients

The function v constructed in Proposition 4.6 is linear around $v = 0$. Thus there exists an open neighborhood $O \subset H^2_{L\text{-per}}$ satisfying the assumptions of Lemma 4.4.

Proposition 4.7. *Under the assumptions of Proposition 4.6, for any $L > \underline{L}$ there exists an open neighborhood $U \subset \left(L^\infty_{L\text{-per}}\right)^2$ of (μ_1, μ_2) such that for all $(\rho_1, \rho_2) \in U$, (4.2.1) with (ρ_1, ρ_2) admits a sign-changing, L-periodic, weak solution. The solution is also linearly stable in the sense of (4.1.1) and (4.1.2).*

Proof. Let $L > \underline{L}$ and let $(\mu_1, \mu_2, v) \in \left(L^\infty_{L\text{-per}}\right)^2 \times H^2_{L\text{-per}}$ be the solution constructed in Proposition 4.6.

The prerequisites of the implicit function theorem are readily satisfied for the functional $\mathscr{F}$ at (μ_1, μ_2, v). In particular, since the solution v is linearly stable in the sense of (4.1.1), the functional $\frac{\partial \mathscr{F}}{\partial z}[\mu_1, \mu_2, v]$ is invertible in the following sense: for all $f \in L^2_{L\text{-per}}$, there exists a unique weak solution $z_f \in H^2_{L\text{-per}}$ of

$$\frac{\partial \mathscr{F}}{\partial v}[\mu_1, \mu_2, z](z_f) = f.$$

This follows by standard regularity results.

By virtue of the implicit function theorem, there exists an open neighborhood $U \subset \left(L^\infty_{L\text{-per}}\right)^2$ of (μ_1, μ_2), an open neighborhood $V \subset O \subset H^2_{L\text{-per}}$ of v and a $\mathscr{C}^1$ diffeomorphism $\Psi : U \to V$ such that, for all $(\rho_1, \rho_2) \in U$,

$$\mathscr{F}[\rho_1, \rho_2, \Psi[\rho_1, \rho_2]] = 0.$$

Finally, since the map Ψ is $\mathscr{C}^1$, we find that the linear stability of the solution is preserved in a open neighborhood of (μ_1, μ_2). □

4.2.3 Uniqueness

We end this section with the following uniqueness result, which will be used later on. We emphasize that this is not a full uniqueness result.

Lemma 4.8. *Let $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, $L > \underline{L}$, $(\mu_1, \mu_2) \in U$ and v be the solution of (4.2.1) given by Proposition 4.7.*

Then any solution z of (4.2.1) satisfying

$$z^+ > \varepsilon_1 v^+ \text{ and } z^- > \varepsilon_2 v^-$$

coincides with v.

Proof. Let z be any such solution. By continuity of z and by the fact that z admits only one zero in each periodicity cell, the condition of the statement directly guarantees that $\{z > 0\} = \{v > 0\}$, $\{z < 0\} = \{v < 0\}$ and $\{z = 0\} = \{v = 0\}$.

Now we focus on one connected component of, say, $\{v > 0\}$. By translation, we can assume without loss of generality that this interval has the form $(0, R)$ with $R < L$. The functions $z_{|(0,R)}$ and $v_{|(0,R)}$ are then both solutions of the following Dirichlet problem:

$$\begin{cases} -z'' = \mu_1 (1 - z) z & \text{in } (0, R) \\ z(0) = z(R) = 0. \end{cases}$$

Since it is well-known that such a solution is unique (we refer for instance to Berestycki [12]), we deduce that z and v coincide in any connected component of $\{v > 0\}$.

Repeating subsequently the argument in $\{v < 0\}$, we obtain the claimed uniqueness. □

4.3 The strongly competitive competition–diffusion system

In the previous section we have considered the equation

$$-z'' = \frac{\mu_1}{\alpha} (\alpha - z) z^+ - \frac{\mu_2}{d^2} (d + z) z^-.$$

For this equation and particular choices of μ_1 and μ_2, we have constructed a sign-changing solution $v \in \mathscr{C}^{1,1}_{L\text{-per}}$ for periods L larger than a threshold $\underline{L}$. We have also shown that this solution is linearly stable in the sense of (4.1.1) and (4.1.2).

In this section, we aim at using this result to prove the existence of linearly stable solutions of (4.1.5). Specifically, fixing $L > \underline{L}$ and a positive L-periodic smooth function ω, our aim is to prove that for any $k > 0$ large enough there exists a positive and stable solution of (4.1.5) $(u_{1,k}, u_{2,k}) \in \mathscr{C}^{1,1}_{L\text{-per}}$ such that

$$(u_{1,k}, u_{2,k}) \to \left(\frac{v^+}{\alpha}, \frac{v^-}{d}\right) \qquad \text{as } k \to +\infty$$

in $H^1_{L\text{-per}}$ and $\mathscr{C}^{0,\gamma}_{L\text{-per}}$ for $\gamma \in (0, \frac{1}{2})$.

We will show the result in a series of steps: first, we give some *a priori* estimates of the solution of a more general class of systems. Then, by means of topological arguments, we deduce from these estimates the existence of solutions. Finally we establish the uniqueness and the linear stability of the solutions.

4.3.1 *A priori* estimate

We start by showing *a priori* estimates for the solutions of a family of systems that contains (4.1.5) as a special case. We are here interested in the L-periodic positive solutions of

$$\begin{cases} -u_1'' = \mu_1(x)(1 - u_1)u_1 - k\omega(x)u_1 \left[tu_2 + (1 - t)\frac{v^-}{d}\right] \\ -du_2'' = \mu_2(x)(1 - u_2)u_2 - \alpha k\omega(x)u_2 \left[tu_1 + (1 - t)\frac{v^+}{\alpha}\right] \end{cases} \tag{4.3.1}$$

where $k > 0$ and $t \in [0, 1]$. Observe that if we take $t = 1$, then (4.3.1) reduces to the original system (4.1.5). On the contrary, if $t = 0$, the equations in (4.3.1) are decoupled.

Lemma 4.9. *There exists a constant $C > 0$, independent of $k > 0$ and $t \in [0,1]$, such that if $(u_1, u_2) \in \mathscr{C}^{1,1}_{L\text{-}per}$ is a solution of (4.3.1) with*

$$u_1 > \frac{v^+}{2\alpha} \quad \text{and} \quad u_2 > \frac{v^-}{2d},$$

then

$$\|(u_1,u_2)\|_{H^1_{L\text{-}per}} + \|(u_1,u_2)\|_{\mathscr{C}^{0,\frac{1}{2}}_{L\text{-}per}} \le C.$$

Let $((u_{1,k}, u_{2,k}))_k$ be any sequence of solutions as before, with $k \to +\infty$ and $t = t_k \in [0,1]$. Then

$$(u_{1,k}, u_{2,k}) \to \left(\frac{v^+}{\alpha}, \frac{v^-}{d}\right) \qquad \text{as } k \to +\infty$$

in $H^1_{L\text{-}per}$ and in $\mathscr{C}^{0,\gamma}$ for any $\gamma \in (0, \frac{1}{2})$.

Proof. We first observe that if (u_1, u_2) is a non-negative solution of (4.3.1) then, by the maximum principle, $0 < u_1 < 1$ and $0 < u_2 < 1$.

We now consider the equation in u_1 in (4.3.1). By testing the equation against u_1 itself, we find

$$\int_0^L (u_1')^2 + \mu_1 u_1^3 + k\omega u_1^2 \left[t u_2 + (1-t)\frac{v^-}{d}\right] \le \int_0^L \mu_1 u_1^2 \le C$$

where the constant $C > 0$ can be chosen independently of t and k. Thus u_1 is uniformly bounded in $H^1_{L\text{-per}}$ and, by Sobolev's embeddings, u_1 is also uniformly bounded in $\mathscr{C}^{0,\frac{1}{2}}_{L\text{-per}}$. We can argue similarly for the component u_2.

Let us now consider a sequence of solutions $((u_{1,k}, u_{2,k}))_k$ as in the statement, with $k \to +\infty$. By testing the equation in $u_{1,k}$ against $\left(u_{1,k} - \frac{v^+}{\alpha}\right) \in H^1_{L\text{-per}}$, we obtain

$$\int_0^L u_{1,k}' \left(u_{1,k} - \frac{v^+}{\alpha}\right)' + k\omega u_{1,k}\left(u_{1,k} - \frac{v^+}{\alpha}\right)\left[t u_{2,k} + (1-t)\frac{v^-}{d}\right] = \int_0^L \mu_1 (1 - u_{1,k}) u_{1,k} \left(u_{1,k} - \frac{v^+}{\alpha}\right).$$

After some simple algebraic manipulations, this yields

$$\begin{aligned} &\int_0^L \left[\left(u_{1,k} - \frac{v^+}{\alpha}\right)'\right]^2 + k\omega \left(u_{1,k} - \frac{v^+}{\alpha}\right)^2 \frac{v^-}{2d} \\ &\qquad \le \int_0^L \left(\frac{v^+}{\alpha}\right)' \left(u_{1,k} - \frac{v^+}{\alpha}\right)' + \int_0^L \mu_1(1-u_{1,k})u_{1,k}\left(u_{1,k} - \frac{v^+}{\alpha}\right). \end{aligned} \tag{4.3.2}$$

By the uniform $H^1_{L\text{-per}}(\mathbb{R})$ estimates, we know that the right hand-side is bounded uniformly in k and t. Thus, if $\overline{u_1}$ is any limit of $(u_{1,k})_k$ (weak in $H^1_{L\text{-per}}$ and in $\mathscr{C}^{0,\gamma}_{L\text{-per}}$ for any $\gamma \in (0, \frac{1}{2})$), we find

$$\left(\overline{u_1} - \frac{v^+}{\alpha}\right)^2 v^- = 0 \qquad \text{a.e. in } \mathbb{R}$$

that is $\overline{u_1} = \frac{v^+}{\alpha} = 0$ where $v^- > 0$. Since by assumption $\overline{u_1} > \frac{v^+}{2\alpha}$, by Lemma 4.8, it must be $\overline{u_1} = \frac{v^+}{\alpha}$.

Going back to (4.3.2), the right hand side converges to 0 as $k \to +\infty$, which implies the strong convergence in $H^1_{L\text{-per}}$ of a subsequence of $(u_{1,k})_k$. We conclude the proof by pointing out that the same reasoning holds for any subsequence of $(u_{1,k})_k$. As a result we deduce the strong convergence of the whole original sequence of solutions. □

An interesting consequence of the previous result is that the solutions of (4.3.1), when k is large, are close to the segregated state $\left(\frac{v^+}{\alpha}, \frac{v^-}{d}\right)$, independently of the value of $t \in [0,1]$. More precisely, we have the following corollary.

Corollary 4.10. *For all $\gamma \in \left(0, \frac{1}{2}\right)$ and $\varepsilon > 0$, there exists $\bar{k} = \bar{k}(\gamma, \varepsilon) > 0$ such that, for all $t \in [0,1]$, $k > \bar{k}$ and $(u_1, u_2) \in \mathscr{C}^{1,1}_{L\text{-per}}$ solution of (4.3.1) such that*

$$u_1 > \frac{v^+}{2\alpha} \quad \textit{and} \quad u_2 > \frac{v^-}{2d},$$

we have

$$\left\| (u_1, u_2) - \left(\frac{v^+}{\alpha}, \frac{v^-}{d}\right) \right\|_{H^1_{L\text{-per}}} + \left\| (u_1, u_2) - \left(\frac{v^+}{\alpha}, \frac{v^-}{d}\right) \right\|_{\mathscr{C}^{0,\gamma}_{L\text{-per}}} \le \varepsilon.$$

4.3.2 Existence of solutions

We now show the existence of solution of (4.1.5) when k is large. We will prove this result in two steps, first proving the existence of solutions of an auxiliary problem, and then, making use of a homotopy argument, we will transfer this result to the original problem. Our argument is inspired by the method proposed in [45] to prove the existence of solutions of a related problem.

Lemma 4.11. *For any $k > 0$, there exists a unique positive solution $(u_1, u_2) \in \mathscr{C}^{1,1}_{L\text{-per}}$ of*

$$\begin{cases} -u_1'' = \mu_1(1-u_1)u_1 - k\omega u_1 \frac{v^-}{d} \\ -du_2'' = \mu_2(1-u_2)u_2 - \alpha k\omega u_2 \frac{v^+}{\alpha}. \end{cases} \tag{4.3.3}$$

Furthermore, the solution is linearly stable, $u_1 > \frac{v^+}{\alpha}$ and $u_2 > \frac{v^-}{d}$.

Proof. Since the equations in the system are actually decoupled, we can consider them one at a time. Thus, we show the proof only for the component u_1. We can apply the same reasoning to the equation in u_2.

We consider the equation

$$-u_1'' = \left[\left(\mu_1 - k\omega\frac{v^-}{d}\right) - \mu_1 u_1\right] u_1 \tag{4.3.4}$$

with periodicity conditions. We observe that $\frac{v^+}{\alpha}$ is a sub-solution whereas 1 is a super-solution, so that there exists indeed a solution u of (4.3.4) satisfying $\frac{v^+}{\alpha} < u_1 < 1$. Moreover, (4.3.4) falls in the general theory of periodic KPP reaction–diffusion equations developed by Berestycki–Hamel–Roques in [16]. In particular, it follows that the solution u_1 is unique, periodic and linearly stable [16, Theorem 2.4]. □

We now pass to the second step of the construction. For notation convenience, let $X = \mathscr{C}^{0,1/4}_{L\text{-per}}$ (any Hölder exponent $\gamma \in \left(0, \frac{1}{2}\right)$ would do) and let $L \in \mathcal{K}(X;X)$ be the linear compact operator defined as

$$z = Lf \iff \begin{cases} -z'' + z = f \\ \text{with } z, f \in X. \end{cases}$$

We consider the homotopy $H : X^2 \times [0,1] \to X^2$, defined by

$$H(u_1,u_2;t) = \begin{cases} u_1 - L\left(u_1 + \mu_1(1-u_1)u_1 - k\omega u_1\left[tu_2 + (1-t)\frac{v^-}{d}\right]\right) \\ u_2 - \frac{1}{d}L\left(du_2 + \mu_2(1-u_2)u_2 - \alpha k\omega u_2\left[tu_1 + (1-t)\frac{v^+}{\alpha}\right]\right). \end{cases}$$

Observe that the homotopy H is of the form $\mathrm{id} - K_t$ where $\mathrm{id} : X^2 \to X^2$ is the identity operator, and $K_t \in \mathcal{K}(X^2 \times [0,1]; X^2)$ is a compact operator for any $t \in [0,1]$ and is continuous in t by standard elliptic estimates. In this regard, we observe that k is fixed.

We have that $H(u_1,u_2;0) = 0$ if and only if (u_1,u_2) are solutions of (4.3.3), while $H(u_1,u_2;1) = 0$ if and only if (u_1,u_2) are solutions of (4.1.5). Our goal is to apply the theory of the Leray-Schauder degree in order to evince the existence of solutions of (4.1.5) from the existence of solutions of (4.3.3), Lemma 4.11.

Let $\mathcal{O}_\varepsilon \subset X^2$ be the connected open subset of X^2 defined as the set of all $(u_1,u_2) \in X^2$ such that

$$\frac{v^+}{2\alpha} < u_1 < 1, \ \frac{v^-}{2d} < u_2 < 1, \ \left\|(u_1,u_2) - \left(\frac{v^+}{\alpha}, \frac{v^-}{d}\right)\right\|_{X^2} < \varepsilon.$$

Lemma 4.12. *For any $\varepsilon > 0$ there exists $\bar{k} > 0$ such that the equation*

$$H(u_1,u_2;t) = 0$$

has no solutions for any $t \in [0,1]$ and $k \geq \bar{k}$ on $\partial\mathcal{O}_\varepsilon$.

This result follows directly from Corollary 4.10.

Lemma 4.13. *The equation*

$$H(u_1,u_2;0) = 0$$

has a unique solution in $\mathcal{O}_\varepsilon$. Moreover there exists $\bar{k} > 0$ such that if $k \geq \bar{k}$, then such solution has fixed point index 1, that is

$$ind_{X^2}(\mathcal{O}_\varepsilon;(u_1,u_2)) = 1.$$

This result follows from Lemma 4.11. We also recall that the fixed point index of isolated solution can be computed by linearization if the equation involves $\mathscr{C}^1$ operators, [3, Theorem 4.2.11].

We can thus conclude by virtue of the Leray–Schauder theorem (see [107] and [3, Theorem 4.3.4]).

Lemma 4.14. *For any $\varepsilon > 0$, there exists $\bar{k} > 0$ such that, for all $k > \bar{k}$, (4.1.5) has a solution $(u_{1,k},u_{2,k})$ in $\mathcal{O}_\varepsilon$. We have*

$$\lim_{k\to+\infty} \left\|(u_{1,k},u_{2,k}) - \left(\frac{v^+}{\alpha}, \frac{v^-}{d}\right)\right\|_{H^1_{L\text{-}per}} + \left\|(u_{1,k},u_{2,k}) - \left(\frac{v^+}{\alpha}, \frac{v^-}{d}\right)\right\|_{\mathscr{C}^{0,\gamma}} = 0$$

for any $\gamma \in (0,\frac{1}{2})$.

If needed, one can improve the convergence result, by stating that the solutions are uniformly bounded in the Lipschitz norm and converge in the $\mathscr{C}^{0,\gamma}$ norm for any $\gamma \in (0,1)$. See, on this subject, the results in [38].

4.3.3 Linear stability for k large

We now investigate the linear stability of the solutions obtained in Lemma 4.14. To this end, we consider the linearized system (4.1.5) at the solution (u_1, u_2) and introduce its periodic principal eigenvalue.

For all $k > \bar{k}$, let

$$\lambda_{1,k} = \lambda_{1,L\text{-per}} \left(- \begin{pmatrix} \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \mu_1 (1 - 2u_{1,k}) - k\omega u_{2,k} & k\omega u_{1,k} \\ \alpha k\omega u_{2,k} & d\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \mu_2 (1 - 2u_{2,k}) \psi - \alpha k\omega u_{1,k} \end{pmatrix} \right)$$

and assume that the associated periodic principal eigenfunction (φ_k, ψ_k) is normalized in such a way that

$$\max_{x\in[0,L]} (\alpha\varphi_k + d\psi_k)(x) = 1.$$

Observe that since both φ_k and ψ_k are positive, this automatically implies that the two functions are globally bounded.

We start by showing a priori estimates on the principal eigenvalue and the principal eigenfunctions.

Lemma 4.15. *The principal eigenvalues are uniformly bounded from below. There exists $C \in \mathbb{R}$ such that*

$$\lambda_{1,k} > -C \qquad \textit{for all } k > \bar{k}.$$

Proof. It suffices to take

$$C = \sup_{k>\bar{k}, x\in\mathbb{R}} (|\mu_1(1 - 2u_{1,k})| + |\mu_2(1 - 2u_{2,k})|).$$

Indeed, the solution $(u_{1,k}, u_{2,k}) \in \mathcal{O}_\varepsilon$ are uniformly bounded. Thus C is finite. We then consider the sum of the equation in $\alpha\varphi_k$ and in ψ_k. The conclusion follows from the fact that the equation

$$-(\alpha\varphi_k + d\psi_k)'' = \mu_1(1 - 2u_{1,k})\alpha\varphi_k + \mu_2(1 - 2u_{2,k})\psi_k + \lambda_{1,k}(\alpha\varphi_k + \psi_k),$$

where the right-hand side is smaller than or equal to $(C + \lambda_{1,k})(\alpha\varphi_k + \psi_k)$, has no positive L-periodic solution if $\lambda_{1,k} < -C$. □

Lemma 4.16. *For any $\varepsilon > 0$ and $\delta > 0$, there exists $\bar{k} > 0$ such that*

$$\sup_{\{v^- > \varepsilon\}} \varphi_k + \sup_{\{v^+ > \varepsilon\}} \psi_k \le \delta$$

for any $k \ge \bar{k}$.

Proof. We prove only the estimate in ψ_k, since the estimate in φ_k follows the same reasoning. Moreover we will implicitly prove the estimate in an interval of length L, and extend them by periodicity. By virtue of a scaling and a translation, we can also assume for simplicity that $\{v^+ > 0\} = (-1, 1)$.

By Lemma 4.14, we already know that

$$\omega(x)u_{1,k} > \omega(x)\frac{v^+}{2\alpha} \ge D(1 - x^2)^+$$

for k large enough. Here $D > 0$ is a small constant independent of k. Plugging this information in (4.1.3), we find that

$$\begin{cases} -du''_{2,k} + Dk(1-x^2)^+ u_{2,k} \leq \mu_2(1-u_{2,k})u_{2,k} \leq G & \text{in } (-1,1) \\ 0 < u_{2,k} < 1 \end{cases}$$

where

$$G = \sup_{k,x\in\mathbb{R}} \mu_2(1-u_{2,k})u_{2,k}$$

is by assumption a finite constant. We are then in position to apply the estimate of Lemma 4.20. This yields

$$Dk(1-x^2)^+ u_{2,k} \leq C\frac{1}{(1-x^2)^2} + 2G.$$

We now fix $\varepsilon > 0$ small. By the previous estimate we find that there exists $C_\varepsilon > 0$ such that

$$\sup_{x\in\left[-1+\frac{\varepsilon}{2},1-\frac{\varepsilon}{2}\right]} k\omega(x)u_{2,k} \leq C_\varepsilon.$$

From the equation in ψ_k, we deduce

$$\begin{cases} -d\psi''_k + Dk(1-x^2)^+\psi_k \leq C & \text{in } \left(-1+\frac{\varepsilon}{2}, 1-\frac{\varepsilon}{2}\right) \\ 0 < \psi_k < \frac{1}{d} \end{cases}$$

for another constant $C > 0$ that does not depend on k. We can conclude by applying again Lemma 4.20 to obtain the sought decay estimate of ψ_k in the interval $(-1+\varepsilon, 1-\varepsilon)$. □

With the uniform estimates of Lemma 4.15 and Lemma 4.16 we are now in position to show that the solution (u_1, u_2) constructed in the previous section is indeed linearly stable if k is sufficiently large.

Of course, if $\liminf_{k\to+\infty} \lambda_{1,k} = +\infty$, then the proof is done. Hence we assume from now on that $\liminf_{k\to+\infty} \lambda_{1,k} < +\infty$. Up to extraction of a subsequence, we also assume that $\lambda_{1,k} \to \liminf_{k\to+\infty} \lambda_{1,k}$ as $k \to +\infty$. In particular, $(\lambda_{1,k})_k$ is bounded.

Lemma 4.17. *For all $k > \bar{k}$, we define $Z_k \in \mathscr{C}^{1,1}_{L\text{-}per}$ as*

$$Z_k = \alpha\varphi_k + d\psi_k.$$

Then the sequence of positive functions $(Z_k)_k$ is uniformly bounded in $W^{2,p}_{L\text{-}per}$ and $\mathscr{C}^{1,\gamma}_{L\text{-}per}$ for any $p < \infty$ and $\gamma < 1$. Each Z_k solves

$$-Z''_k = \left[\mu_1\left(1-2\frac{v^+}{\alpha}\right) + \frac{1}{d}\mu_2\left(1+2\frac{v^-}{d}\right)\right] Z_k + \lambda_{1,k}\sigma(v)Z_k + o_k(1)$$

where $o_k(1)$ is a sequence of functions, bounded uniformly in L^∞ and such that $o_k(1) \to 0$ in $L^p_{L\text{-}per}$ for any $p < \infty$.

Proof. Once again, we take the sum of the equation in $\alpha\varphi_k$ and the equation in ψ_k. We thus find

$$-\left(\alpha\varphi_k + d\psi_k\right)'' = \mu_1\left(1-2u_{1,k}\right)\alpha\varphi_k + \mu_2\left(1-2u_{2,k}\right)\psi_k + \lambda_{1,k}\left(\alpha\varphi_k + \psi_k\right). \tag{4.3.5}$$

We observe that the terms in the right hand side of (4.3.5) are uniformly bounded. Thus there exists $Z \in (H^2 \cap \mathscr{C}^{1,\gamma})_{L\text{-per}}$ such that, up to subsequence, $Z_k \to Z \geq 0$. By uniform convergence we have $\max Z = 1$. As a consequence of Lemma 4.16, we also have that

$$(\alpha\varphi_k + \psi_k) \to \left(\mathbf{1}_{v>0} + \frac{1}{d}\mathbf{1}_{v<0}\right) Z = \sigma(v)Z$$

in L^p for any $p < \infty$.

We now rearrange the terms of (4.3.5) as follows:

$$\begin{aligned} -Z_k'' = &\left[\mu_1\left(1 - 2\frac{v^+}{\alpha}\right) + \frac{1}{d}\mu_2\left(1 + 2\frac{v^-}{d}\right)\right] Z_k + \lambda_{1,k}\sigma(v)Z_k \\ &+ \lambda_{1,k}\left[(\alpha\varphi_k + \psi_k) - \sigma(v)Z_k\right] \\ &+ \left[2\alpha\mu_1\left(\frac{v^+}{\alpha} - u_{1,k}\right)\varphi_k - 2\mu_2\left(\frac{v^-}{d} + u_{2,k}\right)\psi_k\right] \\ &- \left(\mu_1\left(1 - 2\frac{v^+}{\alpha}\right)d\psi_k + \frac{1}{d}\mu_2\left(1 + 2\frac{v^-}{d}\right)\alpha\varphi_k\right). \end{aligned}$$

In order to conclude, we need to show that the second, third and fourth lines in the previous equation are small contributions in the $L^p_{L\text{-per}}$ norm. Now, we just proved that the second line converges to zero in the L^p topology. The third line also converges to zero, since $(u_1, u_2)_k \to \left(\frac{v^+}{\alpha}, \frac{v^-}{d}\right)$ in $\mathscr{C}^{0,\gamma}$. Finally, by Lemma 4.16, the fourth line also converges to zero in $L^p_{L\text{-per}}$. □

We now recall that the solution v is, by construction, linearly stable in the sense of (4.1.2). This implies in particular that any eigenpair (λ, Z) satisfying

$$-Z'' - \left[\mu_1\left(1 - 2\frac{v^+}{\alpha}\right)\mathbf{1}_{v>0} + \frac{1}{d}\mu_2\left(1 + 2\frac{v^-}{d}\right)\mathbf{1}_{v<0}\right] Z = \lambda\sigma(v)Z \tag{4.3.6}$$

is such that λ has a positive real part. More precisely, using the uniqueness part of the Krein–Rutman theorem, we can establish the following convergence result.

Lemma 4.18. *There exists $\bar{k} > 0$ such that for any $k \geq \bar{k}$ the solution $(u_{1,k}, u_{2,k})$ is linearly stable.*

Furthermore, the sequence $((\lambda_{1,k}, Z_k))_k$ and the principal eigenpair (λ_1, Z) given by the notion of stability in the sense of (4.1.2) satisfy the following equalities:

$$\liminf_{k\to+\infty} \lambda_{1,k} = \lambda_1 > 0 \quad \text{and} \quad \lim_{k\to+\infty} Z_k = Z$$

in $W^{2,p}_{L\text{-per}}$ and $\mathscr{C}^{1,\gamma}_{L\text{-per}}$ for any $p < \infty$ and $\gamma < 1$.

Proof. In view of Lemma 4.17, $(Z_k)_k$ converges to some limit Z_∞ in $W^{2,p}_{L\text{-per}}$ and $\mathscr{C}^{1,\gamma}$ for any $p < \infty$ and $\gamma < 1$. This limit is obviously an eigenfunction associated with the eigenvalue $\liminf_{k\to+\infty} \lambda_{1,k}$ and, moreover, Z_∞ is L-periodic, $\max Z_\infty = 1$ and $Z_\infty > 0$. Hence, by uniqueness up to normalization of the positive eigenfunction, the result follows. □

4.3.4 Uniqueness of solutions

We conclude with the following observation.

Lemma 4.19. *The solution in Lemma 4.14 is unique in $\mathcal{O}_\varepsilon$.*

Proof. By homotopy, the Leray–Schauder degree of $H(u_1, u_2; t)$ at 0 is constant for $t \in [0, 1]$. For $t = 0$, we know that

$$\deg_{X^2}(H(\cdot, \cdot; t), \mathcal{O}_\varepsilon, 0) = \mathrm{ind}_{X^2}(\mathcal{O}_\varepsilon; (u_1, u_2)) = 1.$$

On the other hand, by Lemma 4.18, any solution of the equation $H(u_1, u_2; 1) = 0$ in $\mathcal{O}_\varepsilon$ is linearly stable, and thus also isolated. By conservation of the Leray–Schauder degree, it must be that the solution in $\mathcal{O}_\varepsilon$ is unique. □

Acknowledgments

The authors thank Grégoire Nadin for the attention he paid to this work and Cécile Carrère and Henri Berestycki for pointing out the interest of this problem.

4.A A technical estimate

Lemma 4.20. *There exists a universal constant $C > 0$ such that for all $A > 0$, $B > 0$ and $k > 0$, if u satisfies*

$$\begin{cases} -u'' + k(1 - x^2)u \leq B & \text{in } (-1, 1) \\ 0 < u < A \end{cases}$$

then the following estimate holds

$$k(1 - x^2)u(x) \leq C\frac{A}{(1 - x^2)^2} + 2B.$$

We observe that, for x close to 0 (the point of minimum of the right hand side) the estimate is sharp, at least with respect to the order. Indeed the solution of the equation with $u(0) = B/(2k)$ and $u'(0) = 0$ is not positive in the interval $(-1, 1)$.

Proof. For any $x \in (-1, 1)$, let $d(x) = 1 - |x|$ be the distance of x to $\{-1, 1\}$. We observe that $d(x) < (1 - x^2) < 2d(x)$. We have, for $y \in (-1, 1)$, that

$$\begin{cases} -u'' + kd(y)u \leq B & \text{in } B_{d(y)}(y) \\ 0 < u < A. \end{cases}$$

By [135, Lemma 2.2], we have that there exists a universal constant $C > 0$ such that

$$kd(y)u(y) \leq \frac{CA}{d(y)^2} + B$$

and we can easily reach the conclusion. □

Systèmes de Fisher – KPP non-monotones

« Ce n'est ni le plus fort de l'espèce
qui survit, ni le plus intelligent.
C'est celui qui sait le mieux
s'adapter au changement. »

(C. Darwin)

Chapitre 5

Systèmes de Fisher – KPP non-monotones : ondes progressives et comportement en temps long

Résumé

L'objet de ce chapitre est l'étude de systèmes de réaction – diffusion non-coopératifs dont la structure est similaire à celle de l'équation de Fisher – KPP. Cette similarité rend possible de prouver, entre autres, une dichotomie extinction – persistance et, en cas de persistance, l'existence d'un état stationnaire strictement positif, l'existence d'ondes progressives avec une demi-droite de vitesses admissibles et une vitesse minimale strictement positive, ainsi que l'égalité entre cette vitesse et la vitesse de propagation de solutions de certains problèmes de Cauchy. Les systèmes KPP non-coopératifs peuvent modéliser divers phénomènes impliquant les trois mécanismes suivants : diffusion spatiale locale, coopération linéaire et compétition surlinéaire.

Ce chapitre a fait l'objet d'une publication sous le titre *Non-cooperative Fisher–KPP systems : traveling waves and long-time behavior* dans *Nonlinearity* [Gir18b].

Dans une appendice, une question laissée ouverte dans l'article (extinction dans le cas critique) est résolue.

5.1 Introduction

In this paper, we study a large class of parabolic reaction–diffusion systems whose prototype is the so-called Lotka–Volterra mutation–competition–diffusion system:

$$\begin{cases} \partial_t u_1 - d_1 \partial_{xx} u_1 = r_1 u_1 - \left(\sum_{j=1}^{N} c_{1,j} u_j \right) u_1 - \mu u_1 + \mu u_2 \\ \partial_t u_2 - d_2 \partial_{xx} u_2 = r_2 u_2 - \left(\sum_{j=1}^{N} c_{2,j} u_j \right) u_2 - 2\mu u_2 + \mu u_1 + \mu u_3 \\ \vdots \\ \partial_t u_N - d_N \partial_{xx} u_N = r_N u_N - \left(\sum_{j=1}^{N} c_{N,j} u_j \right) u_N - \mu u_N + \mu u_{N-1} \end{cases}$$

where N is an integer larger than or equal to 2 and the coefficients d_i, r_i, $c_{i,j}$ (with $i, j \in \{1, \dots, N\}$) and μ are positive real numbers.

This system can be understood as an ecological model, where $(u_1, \dots, u_N)$ is a metapopulation density phenotypically structured, $\mu u_{i-1} - \mu u_i$ and $\mu u_{i+1} - \mu u_i$ are the step-wise mutations of the i-th phenotype with a mutation rate μ, d_i is its dispersal rate, r_i is its growth rate per capita in absence of mutation, $c_{i,j}$ is the rate of the competition exerted by the j-th phenotype on the i-th phenotype, $\frac{r_i}{c_{i,i}}$ is the carrying capacity of the i-th phenotype in absence of mutation and interphenotypic competition.

We are especially interested in spreading properties which describe the invasion of the population in an uninhabited environment and which are expected to involve so-called traveling wave solutions. Such solutions were first studied, independently and both in 1937, by Fisher [72] on one hand and by Kolmogorov, Petrovsky and Piskunov [104] on the other hand for the equation that is now well-known as the Fisher–KPP equation, Fisher equation or KPP equation:

$$\partial_t u - \partial_{xx} u = u(1-u).$$

While a lot of work has been accomplished about traveling waves and spreading properties for scalar reaction–diffusion equations, the picture is much less complete regarding coupled systems of reaction–diffusion equations. In particular, almost nothing is known about non-cooperative systems like the system above.

Before going any further, let us introduce more precisely the problem.

5.1.1 Notations

Let $(n, n') \in (\mathbb{N} \cap [1, +\infty))^2$. The set of the first n positive integers $[1, n] \cap \mathbb{N}$ is denoted $[n]$ (and $[0] = \emptyset$ by convention).

5.1.1.1 Typesetting conventions

In order to ease the reading, we reserve the italic typeface (x, f, X) for reals, real-valued functions or subsets of $\mathbb{R}$, the bold typeface ($\mathbf{v}$, $\mathbf{A}$) for euclidean vectors or vector-valued functions, in lower case for column vectors and in upper case for other matrices [1], the sans serif typeface in upper case (B, K) for subsets of euclidean spaces [2] and the calligraphic typeface in upper case ($\mathscr{C}$, $\mathscr{L}$) for functional spaces and operators.

1. This convention being superseded by the previous one when the dimension is specifically equal to 1.
2. Same exception.

5.1.1.2 Linear algebra notations

— The canonical basis of $\mathbb{R}^n$ is denoted $(\mathbf{e}_{n,i})_{i\in[n]}$. The euclidean norm of $\mathbb{R}^n$ is denoted $|\bullet|_n$. The open euclidean ball of center $\mathbf{v}\in\mathbb{R}^n$ and radius $r>0$ and its boundary are denoted $\mathsf{B}_n(\mathbf{v},r)$ and $\mathsf{S}_n(\mathbf{v},r)$ respectively.

— The space $\mathbb{R}^n$ is equipped with one partial order $\geq_n$ and two strict partial orders $>_n$ and $\gg_n$, defined as

$$\mathbf{v}\geq_n\hat{\mathbf{v}} \text{ if } v_i\geq\hat{v}_i \text{ for all } i\in[n],$$

$$\mathbf{v}>_n\hat{\mathbf{v}} \text{ if } \mathbf{v}\geq_n\hat{\mathbf{v}} \text{ and } \mathbf{v}\neq\hat{\mathbf{v}},$$

$$\mathbf{v}\gg_n\hat{\mathbf{v}} \text{ if } v_i>\hat{v}_i \text{ for all } i\in[n].$$

The strict orders $>_n$ and $\gg_n$ coincide if and only if $n=1$.
A vector $\mathbf{v}\in\mathbb{R}^n$ is *nonnegative* if $\mathbf{v}\geq_n\mathbf{0}$, *nonnegative nonzero* if $\mathbf{v}>_n\mathbf{0}$, *positive* if $\mathbf{v}\gg_n\mathbf{0}$. The sets of all nonnegative, nonnegative nonzero and positive vectors are respectively denoted K_n, K_n^+ and K_n^{++}.

— The sets $\mathsf{K}_n^+\cap\mathsf{S}_n(\mathbf{0},1)$ and $\mathsf{K}_n^{++}\cap\mathsf{S}_n(\mathbf{0},1)$ are respectively denoted $\mathsf{S}_n^+(\mathbf{0},1)$ and $\mathsf{S}_n^{++}(\mathbf{0},1)$.

— For any $X\subset\mathbb{R}$, the sets of X-valued matrices of dimension $n\times n'$ and $n\times n$ are respectively denoted $\mathsf{M}_{n,n'}(X)$ and $\mathsf{M}_n(X)$. If $X=\mathbb{R}$ and if the context is unambiguous, we simply write $\mathsf{M}_{n,n'}$ and M_n. As usual, the entry at the intersection of the i-th row and the j-th column of the matrix $\mathbf{A}\in\mathsf{M}_{n,n'}$ is denoted $a_{i,j}$ and the i-th component of the vector $\mathbf{v}\in\mathbb{R}^n$ is denoted v_i. For any vector $\mathbf{v}\in\mathbb{R}^n$, $\operatorname{diag}\mathbf{v}$ denotes the diagonal matrix whose i-th diagonal entry is v_i.

— Matrices are vectors and consistently we may apply the notations $\geq_{nn'}$, $>_{nn'}$ and $\gg_{nn'}$ as well as the vocabulary nonnegative, nonnegative nonzero and positive to matrices. We emphasize this convention because of the possible confusion with the notion of "positive definite square matrix".

— A matrix $\mathbf{A}\in\mathsf{M}_n$ is *essentially nonnegative*, *essentially nonnegative nonzero*, *essentially positive* if $\mathbf{A}-\min_{i\in[n]}(a_{i,i})\mathbf{I}_n$ is nonnegative, nonnegative nonzero, positive respectively.

— The identity of M_n and the element of $\mathsf{M}_{n,n'}$ whose every entry is equal to 1 are respectively denoted $\mathbf{I}_n$ and $\mathbf{1}_{n,n'}$ ($\mathbf{1}_n$ if $n=n'$) .

— We recall the definition of the Hadamard product of a pair of matrices $(\mathbf{A},\mathbf{B})^2\in(\mathsf{M}_{n,n'})^2$:

$$\mathbf{A}\circ\mathbf{B}=(a_{i,j}b_{i,j})_{(i,j)\in[n]\times[n']}.$$

The identity matrix under Hadamard multiplication is $\mathbf{1}_{n,n'}$.

— The spectral radius of any $\mathbf{A}\in\mathsf{M}_n$ is denoted $\rho(\mathbf{A})$. Recall from the Perron–Frobenius theorem that if $\mathbf{A}$ is nonnegative and irreducible, $\rho(\mathbf{A})$ is the dominant eigenvalue of $\mathbf{A}$, called the *Perron–Frobenius eigenvalue* $\lambda_{PF}(\mathbf{A})$, and is the unique eigenvalue associated with a positive eigenvector. Recall also that if $\mathbf{A}\in\mathsf{M}_n$ is essentially nonnegative and irreducible, the Perron–Frobenius theorem can still be applied. In such a case, the unique eigenvalue of $\mathbf{A}$ associated with a positive eigenvector is $\lambda_{PF}(\mathbf{A})=\rho\left(\mathbf{A}-\min_{i\in[n]}(a_{i,i})\mathbf{I}_n\right)+\min_{i\in[n]}(a_{i,i})$. Any eigenvector associated with $\lambda_{PF}(\mathbf{A})$ is referred to as a *Perron–Frobenius eigenvector* and the unit one is denoted $\mathbf{n}_{PF}(\mathbf{A})$.

5.1.1.3 Functional analysis notations

— We will consider a parabolic problem of two real variables, the "time" t and the "space" x. A (straight) *parabolic cylinder* in $\mathbb{R}^2$ is a subset of the form $(t_0, t_f) \times (a, b)$ with $(t_0, t_f, a, b) \in \overline{\mathbb{R}}^4$, $t_0 < t_f$ and $a < b$. The parabolic boundary $\partial_P \mathsf{Q}$ of a bounded parabolic cylinder Q is defined classically. A *classical solution* of some second-order parabolic problem of dimension n set in a parabolic cylinder $\mathsf{Q} = (t_0, t_f) \times (a, b)$ is a solution in

$$\mathscr{C}^1\left((t_0, t_f), \mathscr{C}^2((a,b), \mathbb{R}^n)\right) \cap \mathscr{C}(\mathsf{Q} \cup \partial\mathsf{Q}, \mathbb{R}^n).$$

Similarly, a classical solution of some second-order elliptic problem of dimension n set in an interval $(a, b) \subset \overline{\mathbb{R}}$ is a solution in

$$\mathscr{C}^2((a,b), \mathbb{R}^n) \cap \mathscr{C}((a,b) \cup \partial(a,b), \mathbb{R}^n).$$

— Consistently with $\mathbb{R}^n$, the set of functions $(\mathbb{R}^n)^{\left(\mathbb{R}^{n'}\right)}$ is equipped with

$$\mathbf{f} \geq_{\mathbb{R}^{n'},\mathbb{R}^n} \hat{\mathbf{f}} \text{ if } \mathbf{f}(\mathbf{v}) - \hat{\mathbf{f}}(\mathbf{v}) \in \mathsf{K}_n \text{ for all } \mathbf{v} \in \mathbb{R}^{n'},$$

$$\mathbf{f} >_{\mathbb{R}^{n'},\mathbb{R}^n} \hat{\mathbf{f}} \text{ if } \mathbf{f} \geq_{\mathbb{R}^{n'},\mathbb{R}^n} \hat{\mathbf{f}} \text{ and } \mathbf{f} \neq \hat{\mathbf{f}},$$

$$\mathbf{f} \gg_{\mathbb{R}^{n'},\mathbb{R}^n} \hat{\mathbf{f}} \text{ if } \mathbf{f}(\mathbf{v}) - \hat{\mathbf{f}}(\mathbf{v}) \in \mathsf{K}_n^{++} \text{ for all } \mathbf{v} \in \mathbb{R}^{n'}.$$

We define consistently nonnegative, nonnegative nonzero and positive functions [3].

— The composition of two compatible functions $\mathbf{f}$ and $\hat{\mathbf{f}}$ is denoted $\mathbf{f}\left[\hat{\mathbf{f}}\right]$, the usual $\circ$ being reserved for the Hadamard product.

— If the context is unambiguous, a functional space $\mathscr{F}(\mathsf{X}, \mathbb{R})$ is denoted $\mathscr{F}(\mathsf{X})$.

— For any smooth open bounded connected set $\Omega \subset \mathbb{R}^{n'}$ and any second order linear elliptic operator $\mathscr{L} : \mathscr{C}^2(\Omega, \mathbb{R}^n) \to \mathscr{C}(\Omega, \mathbb{R}^n)$ with coefficients in $\mathscr{C}_b(\Omega, \mathbb{R}^n)$, the *Dirichlet principal eigenvalue* of $\mathscr{L}$ in Ω, denoted $\lambda_{1,Dir}(-\mathscr{L}, \Omega)$, is well-defined if $\mathscr{L}$ is order-preserving in Ω. Recall from the Krein–Rutman theorem that $\lambda_{1,Dir}(-\mathscr{L}, \Omega)$ is the unique eigenvalue associated with a principal eigenfunction positive in Ω and null on $\partial\Omega$. Sufficient conditions for the order-preserving property are:
  — $n = 1$;
  — $n \geq 2$ and the system is *weakly coupled* (the coupling occurs only in the zeroth order term) and *fully coupled* (the zeroth order coefficient is an essentially nonnegative irreducible matrix). When $n \geq 2$, order-preserving operators are also referred to as *cooperative* operators.

5.1.2 Setting of the problem

From now on, an integer $N \in \mathbb{N} \cap [2, +\infty)$ is fixed. For the sake of brevity, the subscripts depending only on 1 or N in the various preceding notations will be omitted when the context is unambiguous.

We also fix $\mathbf{d} \in \mathsf{K}^{++}$, $\mathbf{D} = \operatorname{diag}\mathbf{d}$, $\mathbf{L} \in \mathsf{M}$ and $\mathbf{c} \in \mathscr{C}^1\left(\mathbb{R}^N, \mathbb{R}^N\right)$.

3. Regarding functions, some authors use $>$ to denote what is here denoted $\gg$. Thus the use of these two functional notations will be as sparse as possible and we will prefer the less ambiguous expressions "nonnegative nonzero" and "positive".

The semilinear parabolic evolution system under scrutiny is

$$\partial_t \mathbf{u} - \mathbf{D}\partial_{xx}\mathbf{u} = \mathbf{L}\mathbf{u} - \mathbf{c}\left[\mathbf{u}\right] \circ \mathbf{u}, \quad (E_{KPP})$$

the unknown being $\mathbf{u} : \mathbb{R}^2 \to \mathbb{R}^N$ (although (E_{KPP}) might occasionally be restricted to a parabolic cylinder).

The associated semilinear elliptic stationary system is

$$-\mathbf{D}\mathbf{u}'' = \mathbf{L}\mathbf{u} - \mathbf{c}\left[\mathbf{u}\right] \circ \mathbf{u}, \quad (S_{KPP})$$

the unknown being $\mathbf{u} : \mathbb{R} \to \mathbb{R}^N$ (although (S_{KPP}) might occasionally be restricted to an interval).

5.1.2.1 Restrictive assumptions

The main restrictive assumptions are the following ones.

(H_1) $\mathbf{L}$ is essentially nonnegative and irreducible.

(H_2) $\mathbf{c}(\mathsf{K}) \subset \mathsf{K}$.

(H_3) $\mathbf{c}(\mathbf{0}) = \mathbf{0}$.

(H_4) There exist

$$(\underline{\alpha}, \delta, \underline{\mathbf{c}}) \in [1, +\infty)^2 \times \mathsf{K}^{++}$$

such that

$$\sum_{j=1}^{N} l_{i,j} n_j \geq 0 \implies \alpha^{\delta} \underline{c}_i \leq c_i(\alpha \mathbf{n})$$

for all

$$(\mathbf{n}, \alpha, i) \in \mathsf{S}^{+}(\mathbf{0}, 1) \times [\underline{\alpha}, +\infty) \times [N].$$

A few immediate consequences of these assumptions deserve to be pointed out.

- (E_{KPP}) and (S_{KPP}) are not cooperative and do not satisfy a comparison principle.
- The Perron–Frobenius eigenvalue $\lambda_{PF}(\mathbf{L})$ is well-defined and the system $\mathbf{u}' = \mathbf{L}\mathbf{u}$ is cooperative.
- For all $\mathbf{v} \in \mathbb{R}^N$, the Jacobian matrix of $\mathbf{w} \mapsto \mathbf{c}(\mathbf{w}) \circ \mathbf{w}$ at $\mathbf{v}$ is

  $$\operatorname{diag}\mathbf{c}(\mathbf{v}) + (\mathbf{v}\mathbf{1}_{1,N}) \circ D\mathbf{c}(\mathbf{v}).$$

  In particular, at $\mathbf{v} = \mathbf{0}$, this Jacobian is null if and only if (H_3) is satisfied. Also, if $D\mathbf{c}(\mathbf{v}) \geq \mathbf{0}$ for all $\mathbf{v} \in \mathsf{K}$, then the system $\mathbf{u}' = -\mathbf{c}[\mathbf{u}] \circ \mathbf{u}$ is competitive.
- This framework contains both the Lotka–Volterra linear competition $\mathbf{c}(\mathbf{u}) = \mathbf{C}\mathbf{u}$ and the Gross–Pitaevskii quadratic competition $\mathbf{c}(\mathbf{u}) = \mathbf{C}(\mathbf{u} \circ \mathbf{u})$ (with, in both cases, $\mathbf{C} \gg \mathbf{0}$).

5.1.2.2 KPP property

The system (E_{KPP}) is, in some sense, a "multidimensional KPP equation". Let us recall the main features of scalar KPP nonlinearities:

1. $f'(0) > 0$ (instability of the null state),
2. $f'(0)v \geq f(v)$ for all $v \geq 0$ (no Allee effect),
3. there exists $K > 0$ such that $f(v) < 0$ if and only if $v > K$ (saturation).

Of course, our assumptions (H_1)–(H_4) aim to put forward a possible generalization of these features. A few comments are in order.

Regarding the saturation property, the growth at least linear of $\mathbf{c}$ (H_4) will imply an analogous statement. Ensuring uniform $\mathscr{L}^\infty$ estimates is really the main mathematical role of the competitive term.

Regarding the presence of an Allee effect, $\mathbf{c}(\mathsf{K}) \subset \mathsf{K}$ (H_2) and $\mathbf{c}(\mathbf{0}) = \mathbf{0}$ (H_3) clearly yield that $\partial_t \mathbf{u} - \mathbf{D}\partial_{xx}\mathbf{u} = \mathbf{L}\mathbf{u}$ is the linearization at $\mathbf{0}$ of (E_{KPP}) and moreover that $\mathbf{f} : \mathbf{v} \mapsto \mathbf{L}\mathbf{v} - \mathbf{c}(\mathbf{v}) \circ \mathbf{v}$ satisfies

$$D\mathbf{f}(\mathbf{0})\,\mathbf{v} \geq \mathbf{f}(\mathbf{v}) \text{ for all } \mathbf{v} \in \mathsf{K}.$$

Regarding the instability of the null state, we stress here that the notion of positivity of matrices is somewhat ambiguous and, consequently, finding a natural generalization of $f'(0) > 0$ is not completely straightforward.

In order to decide which positivity sense is the right one, we offer the following criterion. On one hand, a suitable multidimensional generalization of the KPP equation should enable generalizations of the striking results concerning its scalar counterpart. On the other hand, the most remarkable result about the KPP equation is that the answer to many natural questions (value of the spreading speed, persistence in bounded domains, etc.) only depends on $f'(0)$ (the importance of $f'(0)$ can already be seen in the features above). Thus, in our opinion, a KPP system should also be linearly determinate regarding these questions.

With this criterion in mind, let us explain for instance why positivity understood as positive definite matrices (i.e. positive spectrum) is not satisfying. In such a case, Lotka–Volterra competition–diffusion nonlinearities, whose linearization at $\mathbf{0}$ has the form $\operatorname{diag}\mathbf{r}$ with $\mathbf{r} \in \mathsf{K}^{++}$, would be KPP nonlinearities. Nevertheless, it is known that the spreading speed of a competition–diffusion system is not necessarily linearly determinate (for instance, see Lewis–Li–Weinberger [108]).

On the contrary, the main theorems of the present paper will show unambiguously that irreducibility and essential nonnegativity (H_1) supplemented with $\lambda_{PF}(\mathbf{L}) > 0$ is the right notion. This confirmation of the relevance of (H_1)–(H_4) will then lead us to a general definition of multidimensional KPP nonlinearity.

5.1.3 Main results

5.1.3.1 KPP-type theorems established under (H_1)–(H_4)

Theorem 5.1. *[Strong positivity] For all nonnegative classical solutions $\mathbf{u}$ of (E_{KPP}) set in $(0,+\infty) \times \mathbb{R}$, if $x \mapsto \mathbf{u}(0,x)$ is nonnegative nonzero, then $\mathbf{u}$ is positive in $(0,+\infty) \times \mathbb{R}$.*

Consequently, all nonnegative nonzero classical solutions of (S_{KPP}) are positive.

Theorem 5.2. *[Absorbing set and upper estimates] There exists a positive function $\mathbf{g} \in \mathscr{C}([0,+\infty), \mathsf{K}^{++})$, component-wise non-decreasing, such that all nonnegative classical solutions $\mathbf{u}$ of (E_{KPP}) set in $(0,+\infty) \times \mathbb{R}$ satisfy*

$$\mathbf{u}(t,x) \leq \left(g_i\left(\sup_{x\in\mathbb{R}} u_i(0,x)\right)\right)_{i\in[N]} \quad \text{for all } (t,x) \in [0,+\infty) \times \mathbb{R}$$

and furthermore, if $x \mapsto \mathbf{u}(0,x)$ is bounded, then

$$\left(\limsup_{t\to+\infty} \sup_{x\in\mathbb{R}} u_i(t,x)\right)_{i\in[N]} \leq \mathbf{g}(0).$$

Consequently, all bounded nonnegative classical solutions $\mathbf{u}$ *of* (S_{KPP}) *satisfy*

$$\mathbf{u} \leq \mathbf{g}(0).$$

Theorem 5.3. *[Extinction or persistence dichotomy]*

1. *Assume* $\lambda_{PF}(\mathbf{L}) < 0$. *Then all bounded nonnegative classical solutions of* (E_{KPP}) *set in* $(0,+\infty)\times\mathbb{R}$ *converge asymptotically in time, exponentially fast, and uniformly in space to* $\mathbf{0}$.
2. *Conversely, assume* $\lambda_{PF}(\mathbf{L}) > 0$. *Then there exists* $\nu > 0$ *such that all bounded positive classical solutions* $\mathbf{u}$ *of* (E_{KPP}) *set in* $(0,+\infty)\times\mathbb{R}$ *satisfy, for all bounded intervals* $I \subset \mathbb{R}$,

$$\left(\liminf_{t\to+\infty}\inf_{x\in I} u_i(t,x)\right)_{i\in[N]} \geq \nu \mathbf{1}_{N,1}.$$

Consequently, all bounded nonnegative nonzero classical solutions of (S_{KPP}) *are valued in*

$$\prod_{i=1}^{N}[\nu, g_i(0)].$$

Remark. The critical case $\lambda_{PF}(\mathbf{L}) = 0$, for which extinction still occurs, was unsolved at the time of writing of the initial article. It was solved later on thanks to an hint of Adrian Lam and its proof is presented in the appendix of the current chapter (see Section 5.A).

Although Theorem 5.3 proves that the attractor of the induced semiflow is reduced to $\{\mathbf{0}\}$ in the extinction case, in the persistence case the long-time behavior is unclear and might not be reduced to a locally uniform convergence toward a unique stable steady state. This direct consequence of the multidimensional structure of (E_{KPP}) is a major difference with the scalar KPP equation. Still, the following theorem provides some additional information about the steady states of (E_{KPP}) and confirms in some sense the preceding conjecture.

Theorem 5.4. *[Existence of steady states]*

1. *If* $\lambda_{PF}(\mathbf{L}) < 0$, *there exists no positive classical solution of* (S_{KPP}).
2. *If* $\lambda_{PF}(\mathbf{L}) = 0$ *and*

$$\mathit{span}(\mathbf{n}_{PF}(\mathbf{L})) \cap \mathsf{K} \cap \mathbf{c}^{-1}(\{\mathbf{0}\}) = \{\mathbf{0}\},$$

there exists no bounded positive classical solution of (S_{KPP}).

3. *If* $\lambda_{PF}(\mathbf{L}) > 0$, *there exists a constant positive classical solution of* (S_{KPP}).

Due to the unclear long-time behavior of (E_{KPP}) when $\lambda_{PF}(\mathbf{L}) > 0$, it seems inappropriate to consider only traveling wave solutions connecting $\mathbf{0}$ to some stable positive steady state (as is usually done in the monostable scalar setting). Hence we resort to the following more flexible definition.

Definition. A *traveling wave solution* of (E_{KPP}) is a pair

$$(\mathbf{p}, c) \in \mathscr{C}^2\left(\mathbb{R}, \mathbb{R}^N\right) \times [0,+\infty)$$

which satisfies:

1. $\mathbf{u} : (t,x) \mapsto \mathbf{p}(x - ct)$ is a bounded positive classical solution of (E_{KPP});
2. $\left(\liminf_{\xi\to-\infty} p_i(\xi)\right)_{i\in[N]} \in \mathsf{K}^+$;

3. $\lim_{\xi\to+\infty} \mathbf{p}(\xi) = \mathbf{0}$.

We refer to $\mathbf{p}$ as the *profile* of the traveling wave and to c as its *speed.* [4]

Theorem 5.5. *[Traveling waves] Assume $\lambda_{PF}(\mathbf{L}) > 0$.*

1. *There exists $c^\star > 0$ such that:*
   a) *there exists no traveling wave solution of (E_{KPP}) with speed c for all $c \in [0, c^\star)$;*
   b) *if, furthermore,*
   $$D\mathbf{c}(\mathbf{v}) \geq \mathbf{0} \text{ for all } \mathbf{v} \in \mathsf{K},$$
   then there exists a traveling wave solution of (E_{KPP}) with speed c for all $c \geq c^\star$.
2. *All profiles $\mathbf{p}$ satisfy*
   $$\mathbf{p} \leq \mathbf{g}(0).$$
3. *All profiles $\mathbf{p}$ satisfy*
   $$\left(\liminf_{\xi\to-\infty} p_i(\xi)\right)_{i\in[N]} \geq \nu \mathbf{1}_{N,1}.$$
4. *All profiles are component-wise decreasing in a neighborhood of $+\infty$.*

When traveling waves exist for all speeds $c \geq c^\star$, $c^\star$ is called the minimal wave speed.

Theorem 5.6. *[Spreading speed] Assume $\lambda_{PF}(\mathbf{L}) > 0$. For all $x_0 \in \mathbb{R}$ and all bounded nonnegative nonzero $\mathbf{v} \in \mathscr{C}\left(\mathbb{R}, \mathbb{R}^N\right)$, the classical solution $\mathbf{u}$ of (E_{KPP}) set in $(0, +\infty) \times \mathbb{R}$ with initial data $\mathbf{v}\mathbf{1}_{(-\infty,x_0)}$ satisfies*

$$\left(\lim_{t\to+\infty} \sup_{x\in(y,+\infty)} u_i(t, x+ct)\right)_{i\in[N]} = \mathbf{0} \text{ for all } c \in (c^\star, +\infty) \text{ and all } y \in \mathbb{R},$$

$$\left(\liminf_{t\to+\infty} \inf_{x\in[-R,R]} u_i(t, x+ct)\right)_{i\in[N]} \in \mathsf{K}^{++} \text{ for all } c \in [0, c^\star) \text{ and all } R > 0.$$

Of course, by well-posedness of (E_{KPP}), the solution with initial data $x \mapsto \mathbf{v}(-x)\mathbf{1}_{(-x_0,+\infty)}$ is precisely $(t, x) \mapsto \mathbf{u}(t, -x)$ ($\mathbf{u}$ being the solution with initial data $\mathbf{v}\mathbf{1}_{(-\infty,x_0)}$). This gives the expected symmetrical spreading result (the solution with initial data $x \mapsto \mathbf{v}(-x)\mathbf{1}_{(-x_0,+\infty)}$ spreads on the left at speed $-c^\star$). Moreover, since these two spreading results with front-like initial data actually cover compactly supported $\mathbf{v}$, we also get straightforwardly the spreading result for compactly supported initial data (the solution spreads on the right at speed $c^\star$ and on the left at speed $-c^\star$).

Consequently, $c^\star$ is also called the spreading speed associated with front-like or compactly supported initial data. We recall that for generic KPP problems these two spreading speeds are different as soon as the spatial domain is multidimensional. In such a case, the spreading speed associated with front-like initial data generically coincides with the minimal wave speed whereas the spreading speed associated with compactly supported initial data is smaller.

Theorem 5.7. *[Characterization and estimates for $c^\star$] Assume $\lambda_{PF}(\mathbf{L}) > 0$. We have*

$$c^\star = \min_{\mu>0} \frac{\lambda_{PF}\left(\mu^2\mathbf{D} + \mathbf{L}\right)}{\mu}$$

4. Let us emphasize once and for all that the vector field $\mathbf{c}$ is not to be confused with the real number c. The former is named after “competition” whereas the latter is traditionally named after “celerity”.

and this minimum is attained at a unique $\mu_{c^\star} > 0$.

Consequently, if we assume (without loss of generality)

$$d_1 \leq d_2 \leq \ldots \leq d_N,$$

the following estimates hold.

1. *We have*
$$2\sqrt{d_1 \lambda_{PF}(\mathbf{L})} \leq c^\star \leq 2\sqrt{d_N \lambda_{PF}(\mathbf{L})}.$$
If $d_1 < d_N$*, both inequalities are strict. If* $d_1 = d_N$*, both inequalities are equalities.*
2. *For all* $i \in [N]$ *such that* $l_{i,i} > 0$*, we have*
$$c^\star > 2\sqrt{d_i l_{i,i}}.$$
3. *Let* $\mathbf{r} \in \mathbb{R}^N$ *and* $\mathbf{M} \in \mathsf{M}$ *be given by the unique decomposition of* $\mathbf{L}$ *of the form*
$$\mathbf{L} = diag\mathbf{r} + \mathbf{M} \text{ with } \mathbf{M}^T \mathbf{1}_{N,1} = \mathbf{0}.$$
Let $(\langle d \rangle, \langle r \rangle) \in (0, +\infty) \times \mathbb{R}$ *be defined as*
$$\begin{cases} \langle d \rangle = \frac{\mathbf{d}^T \mathbf{n}_{PF}\left(\mu_{c^\star}^2 \mathbf{D} + \mathbf{L}\right)}{\mathbf{1}_{1,N} \mathbf{n}_{PF}\left(\mu_{c^\star}^2 \mathbf{D} + \mathbf{L}\right)}, \\ \langle r \rangle = \frac{\mathbf{r}^T \mathbf{n}_{PF}\left(\mu_{c^\star}^2 \mathbf{D} + \mathbf{L}\right)}{\mathbf{1}_{1,N} \mathbf{n}_{PF}\left(\mu_{c^\star}^2 \mathbf{D} + \mathbf{L}\right)}. \end{cases}$$
If $\langle r \rangle \geq 0$*, then*
$$c^\star \geq 2\sqrt{\langle d \rangle \langle r \rangle}.$$

5.1.3.2 General definition of multidimensional KPP nonlinearity

The set of assumptions (H_1)–(H_4) supplemented with $\lambda_{PF}(\mathbf{L}) > 0$ can be seen as a particular case of the following definition, which we expect to be optimal with respect to the preceding collection of theorems.

Definition 5.8. A nonlinear function $\mathbf{f} \in \mathscr{C}^1\left(\mathbb{R}^N, \mathbb{R}^N\right)$ is a *KPP nonlinearity* if:

1. $\mathbf{f}(\mathbf{0}) = \mathbf{0}$;
2. $D\mathbf{f}(\mathbf{0})$ is essentially nonnegative, irreducible and $\lambda_{PF}(D\mathbf{f}(\mathbf{0})) > 0$;
3. $D\mathbf{f}(\mathbf{0})\mathbf{v} \geq \mathbf{f}(\mathbf{v})$ for all $\mathbf{v} \in \mathsf{K}$;
4. the semiflow induced by $\partial_t \mathbf{u} = \mathbf{D}\partial_{xx}\mathbf{u} + \mathbf{f}[\mathbf{u}]$ with globally bounded, sufficiently regular initial data admits an absorbing set bounded in $\mathscr{L}^\infty(\mathbb{R})$.

Let us explain more precisely how this definition differs from (H_1)–(H_4) supplemented with $\lambda_{PF}(\mathbf{L}) > 0$. Defining

$$\mathbf{L} = D\mathbf{f}(\mathbf{0}),$$

$$\mathbf{c} : \mathbf{v} \mapsto \begin{cases} \left(\frac{1}{v_i}\left((\mathbf{L}\mathbf{v})_i - f_i(\mathbf{v})\right)\right)_{i \in [N]} & \text{if } \mathbf{v} \neq \mathbf{0} \\ \mathbf{0} & \text{if } \mathbf{v} = \mathbf{0} \end{cases},$$

we find

$$\mathbf{f}(\mathbf{v}) = \mathbf{L}\mathbf{v} - \mathbf{c}(\mathbf{v}) \circ \mathbf{v} \text{ for all } \mathbf{v} \in \mathbb{R}^N.$$

On one hand, the irreducibility and essential nonnegativity of $\mathbf{L}$ (H_1), the positivity of its Perron–Frobenius eigenvalue, as well as the nonnegativity of $\mathbf{c}$ on K (H_2) with $\mathbf{c}(\mathbf{0}) = \mathbf{0}$ (H_3) follow directly. On the other hand, the $\mathscr{C}^1$ regularity of $\mathbf{c}$ at $\mathbf{0}$ and its specific growth at infinity (H_4) are not satisfied in general.

These two properties are satisfied indeed for the applications we have in mind (which will be exposed in a moment). However it might be mathematically interesting to consider the case where at least one of them fails. For instance, let us discuss briefly (H_4).

The only forthcoming result whose proof depends directly on (H_4) is Lemma 5.11 (which is remarkably one of the main assumptions of a related paper by Barles, Evans and Souganidis [10, (F3)]). It is easily seen that if $\mathbf{c}$ grows sublinearly, we cannot hope in general to recover Lemma 5.11 (in other words, under some reasonable assumptions, Barles–Evans–Souganidis's (F3) is satisfied if and only if (H_4); of course this makes (H_4) even more interesting).

Nevertheless, this lemma is not a result in itself but a tool used for the proofs of Theorem 5.2 as well as the existence results of Theorem 5.4 and Theorem 5.5. Hence relaxing (H_4) mainly means finding new proofs of these results.

Now, without entering into too much details, we point out that if there exists $\eta > 0$ such that the following dissipative assumption:

$$(H_{diss,\eta}) \quad \begin{cases} \exists C_1 \geq 0 \quad \forall \mathbf{v} \in \mathbb{R}^N \quad (\mathbf{f}(\mathbf{v}) + \eta \mathbf{v})^T \mathbf{v} \leq C_1 \\ \exists C_2 \geq 0 \quad \forall \mathbf{v} \in \mathbb{R}^N \quad D\mathbf{f}(\mathbf{v}) + \eta \mathbf{I} \leq C_2 \mathbf{1} \\ \exists (C_3, p) \in [0, +\infty)^2 \quad \forall \mathbf{v} \in \mathbb{R}^N \quad |\mathbf{f}(\mathbf{v}) + \eta \mathbf{v}| \leq C_3 (1 + |\mathbf{v}|^p), \end{cases}$$

holds, then the semiflow induced by $\partial_t \mathbf{u} = \mathbf{D}\partial_{xx}\mathbf{u} + \mathbf{f}[\mathbf{u}]$ admits an attractor in some locally uniform topology which is bounded in $\mathscr{C}_b(\mathbb{R}, \mathbb{R}^N)$ (see Zelik [147]). If the semiflow leaves K invariant and if we only consider nonnegative initial data, then the quantifiers $\forall \mathbf{v} \in \mathbb{R}^N$ above can all be replaced by $\forall \mathbf{v} \in \mathsf{K}$.

In particular, $\mathbf{v} \mapsto \mathbf{L}\mathbf{v} - \mathbf{c}(\mathbf{v}) \circ \mathbf{v}$ supplemented with (H_1)–(H_3) and

$$(H_4') \quad \lim_{|\mathbf{v}| \to +\infty, \mathbf{v} \in \mathsf{K}} |\mathbf{c}(\mathbf{v})| = +\infty \text{ with at most algebraic growth}$$

satisfies $(H_{diss,\eta})$ for any $\eta > 0$. (Clearly, $(H_4) \cup (H_4')$ contains every choice of $\mathbf{c}$ such that $\lim_{|\mathbf{v}| \to +\infty, \mathbf{v} \in \mathsf{K}} |\mathbf{c}(\mathbf{v})| = +\infty$.)

Consequently, dissipative theory provides for some slowly decaying KPP nonlinearities a proof of Theorem 5.2. It should also provide a proof of Proposition 5.15, which is the key estimate to derive the existence of traveling waves, as well as a proof of the existence result of Theorem 5.4 . With these proofs at hand, all our results would be recovered.

5.1.4 Related results in the literature

5.1.4.1 Cooperative or almost cooperative systems

The bibliography about weakly and fully coupled elliptic and parabolic linear systems is of course extensive. It is possible, for instance, to define principal eigenvalues and eigenfunctions (Sweers *et al.* [24, 136]), to prove the weak maximum principle (the classical theorems of Protter–Weinberger [129] were refined in the more involved elliptic case by Figueiredo *et al.* [53, 54] and Sweers [136]) or Harnack inequalities (Chen–Zhao [37] or Arapostathis–Gosh–Marcus [5] for the elliptic case [5], Földes–Poláčik [73] for the parabolic case) and to use the super- and sub-solution method to deduce existence of solutions (Pao [125] among others). In some sense, weakly and fully coupled systems form the "right", or at least the most straightforward, generalization of scalar equations.

5. They both prove the same type of results but we will refer hereafter only to the latter because the former does not cover, as stated, the one-dimensional space case.

For (possibly nonlinear) cooperative systems, results analogous to Theorem 5.5 1, 3, Theorem 5.6 and Theorem 5.7 were established by Lewis, Li and Weinberger [110, 142]. Recently, Al-Kiffai and Crooks [1] introduced a convective term into a two-species cooperative system to study its influence on linear determinacy.

For non-cooperative systems that can still be controlled from above and from below by weakly and fully coupled systems whose linearizations at $\mathbf{0}$ coincide with that of the non-cooperative system, Wang [140] recovered the results of Lewis–Li–Weinberger by comparison arguments. Before going any further, let us point out that we will use extensively comparison arguments as well, nevertheless we will not need equality of the linearizations at $\mathbf{0}$. This is a crucial difference between the two sets of assumptions. To illustrate this claim, let us present an explicit example of system covered by our assumptions and not by Wang's ones: take any $N \geq 3$, $r > 0$, $\mu \in \left(0, \frac{r}{2}\right)$ and define $\mathbf{L}$ and $\mathbf{c}$ as follows:

$$\mathbf{L} = r\mathbf{I} + \mu \begin{pmatrix} -1 & 1 & 0 & \dots & 0 \\ 1 & -2 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & -2 & 1 \\ 0 & \dots & 0 & 1 & -1 \end{pmatrix},$$

$$\mathbf{c} : \mathbf{v} \mapsto \mathbf{1v}.$$

On one hand, (H_1)–(H_4) are easily verified, but on the other hand, the function $\mathbf{f} : \mathbf{v} \mapsto \mathbf{Lv} - \mathbf{c}[\mathbf{v}] \circ \mathbf{v}$ is such that, for all $i \in [N] \setminus \{1, N\}$ and all $\mathbf{v} \in \mathsf{K}^{++}$,

$$\frac{\partial \mathbf{f}_i}{\partial v_j}(\mathbf{v}) = -v_i < 0 \text{ for all } j \in [N] \setminus \{i-1, i, i+1\}.$$

Consequently, the application $v \mapsto \mathbf{f}_i(v\mathbf{e}_j)$ is decreasing in $[0, +\infty)$. This clearly violates Wang's assumptions: this instance of (E_{KPP}) cannot be controlled from below by a cooperative system whose linearization at $\mathbf{0}$ is $\partial_t \mathbf{u} - \mathbf{D}\partial_{xx}\mathbf{u} = \mathbf{Lu}$.

Even if $\mathbf{L}$ is essentially positive and the cooperative functions $\mathbf{f}^-, \mathbf{f}^+$ satisfying

$$\begin{cases} \mathbf{f}^-(\mathbf{v}) \leq \mathbf{Lv} - \mathbf{c}(\mathbf{v}) \circ \mathbf{v} \leq \mathbf{f}^+(\mathbf{v}) \\ \mathbf{f}^-(\mathbf{0}) = \mathbf{f}^+(\mathbf{0}) = \mathbf{0} \\ D\mathbf{f}^-(\mathbf{0}) = D\mathbf{f}^+(\mathbf{0}) = \mathbf{L} \end{cases}$$

are constructible, in general it is difficult to verify that $\mathbf{f}^-$ and $\mathbf{f}^+$ have each a minimal positive zero (another requirement of Wang). Our setting needs not such a verification.

Furthermore, even if these minimal zeros exist, several results presented here are still new.

1. Theorem 5.5 1 adds to [140, Theorem 2.1 iii)–v)] the existence of a critical traveling wave (Wang obtained the existence of a bounded non-constant nonnegative solution traveling at speed $c^\star$ but the limit at $+\infty$ of its profile was not addressed).
2. Theorem 5.1, Theorem 5.2, Theorem 5.3 and Theorem 5.4 as well as Theorem 5.5 2, 4 rely more deeply on the KPP structure and are completely new to the best of our knowledge.

5.1.4.2 KPP systems

Regarding weakly coupled systems equipped with KPP nonlinearities, as far as we know most related works assume the essential positivity of $\mathbf{L}$, some even requiring its positivity. Our results tend to show that this collection of results should be generalizable to the whole class of irreducible and essentially nonnegative $\mathbf{L}$ (H_1) provided $\lambda_{PF}(\mathbf{L}) > 0$.

Dockery, Hutson, Mischaikow and Pernarowski [58] studied in a celebrated paper the solutions of (S_{KPP}) in a bounded and smooth domain with Neumann boundary conditions. Their matrix $\mathbf{L}$ had the specific form $a(x)\mathbf{I} + \mu\mathbf{M}$ where a is a non-constant function of the space variable and with minimal assumptions on the constant matrix $\mathbf{M}$. They also assumed strict ordering of the components of $\mathbf{d}$, explicit and symmetric Lotka–Volterra competition, vanishingly small μ. They proved the existence of a unique positive steady state, globally attractive for the Cauchy problem with positive initial data, and which converges as $\mu \to 0$ to a steady state where only u_1 persists.

More recently, the solutions of (S_{KPP}), still in a bounded and smooth domain with Neumann boundary conditions, were studied under the assumptions of essential positivity of $\mathbf{L}$ and small Lipschitz constant of $\mathbf{v} \mapsto \mathbf{c}(\mathbf{v}) \circ \mathbf{v}$ by Hei and Wu [91]. They established by means of super- and sub-solutions the equivalence between the negativity of the principal eigenvalue of $-\mathbf{D}\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \mathbf{L}$ and the existence of a positive steady state.

Provided the positivity of $\mathbf{L}$, the vanishing viscosity limit of (E_{KPP}) is the object of a work by Barles, Evans and Souganidis [10]. Although their paper and the present one differ both in results and in techniques, they share the same ambition: describing the spreading phenomenon for KPP systems. Therefore our feeling is that together they give a more complete answer to the problem.

For two-component systems with explicit Lotka–Volterra competition, $\mathbf{D} = \mathbf{I}_2$ and symmetric and positive $\mathbf{L}$, Theorem 5.4 and Theorem 5.5 1, 3, 4 reduce to the results of Griette and Raoul [82] (see Alfaro–Griette [2] for a partial extension to space-periodic media). Their paper uses very different arguments (topological degree, explicit computations involving in particular the sum of the equations, weak mutation limit, phase plane analysis) but was our initial motivation to work on this question: our intent is really to extend their result to a larger setting by changing the underlying mathematical techniques. Let us emphasize that they obtained an algebraic formula for the minimal wave speed, $c^\star = 2\sqrt{\lambda_{PF}(\mathbf{L})}$, that we are able to generalize (Theorem 5.7). The case $\mathbf{D} \neq \mathbf{I}_2$ has been investigated heuristically and numerically by Elliott and Cornell [65], who considered the weak mutation limit as well and obtained further results.

Let us point out that the problem of the spreading speed for the Cauchy problem for the two-component system with explicit Lotka–Volterra competition was formulated but left open by Elliott and Cornell [65] as well as by Cosner [39] and not considered by Griette and Raoul [82]. This problem is completely solved here (see Theorem 5.6).

Just after the submission of this paper, a paper by Morris, Börger and Crooks [115] submitted concurrently and devoted to the analytical confirmation of Elliott and Cornell's numerical observations was brought to our attention. By applying's successfully Wang's framework, they obtained the existence of traveling waves as well as the spreading speed for the Cauchy problem. However, in order to apply Wang's framework, they had to make additional assumptions (roughly speaking, small interphenotypic competition and small mutations) and which are in fact, in view of our results, unnecessary. They also obtained very interesting results regarding the dependency on the mutation rate μ of the spreading speed

$$\lambda_{PF}\left(\mu_{c^\star}\mathbf{D} + \mu_{c^\star}^{-1}\left(\mathrm{diag}\mathbf{r} + \mu\mathbf{M}\right)\right)$$

and the associated distribution

$$\mathbf{n}_{PF}\left(\mu_{c^\star}\mathbf{D} + \mu_{c^\star}^{-1}\left(\mathrm{diag}\mathbf{r} + \mu\mathbf{M}\right)\right).$$

5.1.5 From systems to non-local equations, from mathematics to applications

It is well-known that systems can be seen as discretizations of continuous models. In this subsection, we present briefly some equations structured not only in time and space but also with a third variable and whose natural discretizations are particular instances of our system (E_{KPP}) satisfying the criterion $\lambda_{PF}(\mathbf{L}) > 0$. Our results bring therefore indirect insight into the spreading properties of these equations.

Since these examples provide also examples of biomathematical applications of our results, this subsection gives us the opportunity to present more precisely these applications, to explain how non-cooperative KPP systems arise in modeling situations and finally to comment our results from this application point of view. Several fields of biology are concerned: evolutionary invasion analysis (also known as adaptive dynamics), population dynamics, epidemiology. Applications in other sciences might also exist.

5.1.5.1 The cane toads equation with non-local competition

Recall the definition of the discrete laplacian in a finite domain of cardinal N,

$$\mathbf{M}_{Lap,N} = \begin{pmatrix} -1 & 1 & 0 & \dots & 0 \\ 1 & -2 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & -2 & 1 \\ 0 & \dots & 0 & 1 & -1 \end{pmatrix} \text{ if } N \geq 3,$$

$$\mathbf{M}_{Lap,2} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \text{ if } N = 2.$$

With this notation, the Lotka–Volterra mutation–competition–diffusion system exhibited earlier reads

$$\partial_t \mathbf{u} - \mathbf{D}\partial_{xx}\mathbf{u} = \operatorname{diag}(\mathbf{r})\,\mathbf{u} + \mu \mathbf{M}_{Lap}\mathbf{u} - (\mathbf{C}\mathbf{u}) \circ \mathbf{u}.$$

An especially interesting instance of it is the system where:

— for all $i \in [N]$, $d_{N,i} = \underline{\theta} + (i-1)\,\delta\theta$ with $\delta\theta = \frac{\overline{\theta}-\underline{\theta}}{N-1}$ and with some fixed $\overline{\theta} > \underline{\theta} > 0$;

— $\mathbf{r}_N = r\mathbf{1}_{N,1}$ with some fixed $r > 0$;

— $\mu_N = \frac{\alpha}{\delta\theta^2}$ with some fixed $\alpha > 0$;

— $\mathbf{C}_N = \delta\theta \mathbf{1}_N$.

Since $\lambda_{PF}(\mathbf{M}_{Lap,N}) = 0$ (because $\mathbf{M}_{Lap,N}\mathbf{1}_{N,1} = \mathbf{0}$), the Perron–Frobenius eigenvalue of $\mathbf{L}$ is positive indeed:

$$\lambda_{PF}\left(r\mathbf{I}_N + \frac{\alpha}{\delta\theta^2}\mathbf{M}_{Lap,N}\right) = r + \lambda_{PF}\left(\frac{\alpha}{\delta\theta^2}\mathbf{M}_{Lap,N}\right) = r > 0.$$

As $N \to +\infty$, this system converges (at least formally) to the cane toads equation with non-local competition and bounded phenotypes:

$$\begin{cases} \partial_t n - \theta\partial_{xx} n - \alpha\partial_{\theta\theta} n = n(t,x,\theta)\left(r - \int_{\underline{\theta}}^{\overline{\theta}} n(t,x,\theta')\,\mathrm{d}\theta'\right) \\ \partial_\theta n(t,x,\underline{\theta}) = \partial_\theta n\left(t,x,\overline{\theta}\right) = 0 \text{ for all } (t,x) \in \mathbb{R}^2 \end{cases}$$

where n is a function of (t, x, θ), $\theta \in [\underline{\theta}, \overline{\theta}]$ is the motility trait, α is the mutation rate and $\int_{\underline{\theta}}^{\overline{\theta}} n(t, x, \theta') \, \mathrm{d}\theta'$ is the total population present at (t, x).

This equation is named after an invasive species currently invading Australia. A startling ecologic fact is that this invasion is accelerating whereas biological invasions usually occur at a constant speed (as predicted by the KPP equation). However this issue is solved when the phenotypical structure is taken into account and the following spatial sorting phenomenon is understood: the fastest toads lead the invasion, reproduce at the edge of the front, give birth to a new generation of toads among which faster and slower toads can be found (as a result of mutations), and the new fastest toads take the lead of the invasion.

The introduction of a motility trait θ with a local mutation term $\alpha \partial_{\theta\theta} n$ into the scalar KPP equation is then a way of verifying this theory: does it lead to accelerating invasions? The answer is positive (transitory acceleration up to a constant asymptotic speed if $\overline{\theta} < +\infty$, constant acceleration if $\overline{\theta} = +\infty$) and this is why the cane toads equation achieved some fame (we refer for instance to [11, 26, 27, 28], where more detailed modeling explanations can also be found).

The overcrowding effect, which is nowadays standardly taken into account in population biology modeling, is modeled by the term $-n(t, x, \theta) \int_{\underline{\theta}}^{\overline{\theta}} n(t, x, \theta') \, \mathrm{d}\theta'$ which basically considers that one given toad competes with all other toads surrounding it, independently of their phenotype, and does not compete with distant toads. Mathematically, this term is the only responsible for the nonlinearity, non-locality and non-cooperativity of the model: it could be tempting to neglect it. However, linear growth models (which go back to Malthus) generically lead to exponential blow-up. The basic idea of the literature about the cane toads equation is then exactly the same as the one we are going to use in the forthcoming proofs: point out and use the KPP nature of the problem.

The results of the present paper are consistent with the ones for the cane toads equation with bounded phenotypes. Therefore it might be possible, in a future sequel providing new estimates uniform with respect to N, to rigorously derive the cane toads equation as the continuous limit of a family of KPP systems. Since the discrete version is easier to study, new results might be unfolded by this approach. However, let us stress that the problem of finding these new uniform estimates is not to be underestimated and is expected to be a very difficult one. At least regarding biologists, whose field measurements somehow always produce discrete classes of phenotypes instead of a continuum of phenotypes, our results bring forth an interesting new lead to address the general problem of adaptive dynamics.

Let us point out that if, instead of phenotypes of cane toads, the components of $\mathbf{u}$ model different strains of virus, then we obtain an epidemiological model representing the invasion of a population of sane individuals by a structured population of infected individuals (Griette–Raoul [82]).

Notice that this cane toads equation is only the first step of a larger research program: a more realistic model should replace clonal reproduction by sexual reproduction and should take into account the possibility of non-constant coefficients α and r as well as that of a more general competition term (logistic with a non-constant weight or even non-logistic). It is also interesting to consider non-local spatial or phenotypical dispersion.

5.1.5.2 The cane toads equation with non-local mutations and competition

Actually, historically, the cane toads equation comes from a doubly non-local model due to Prévost *et al.* [6, 128] (see also the earlier individual-based model by Champagnat and Méléard [36]). Since the non-local mutation operator is too difficult to handle mathematically, the cane toads equation with local mutations was favored as a simplified first approach. However it remains unsatisfying from the modeling point of view and non-local kernels, which could take into account large mutations, are the real aim.

Defining as above $\delta\theta = \frac{\overline{\theta}-\underline{\theta}}{N-1}$ and $(\theta_i)_{i\in[N]} = (\underline{\theta} + (i-1)\,\delta\theta)_{i\in[N]}$, the natural discretization of the doubly non-local cane toads equation,

$$\partial_t n - d(\theta)\,\partial_{xx} n = rn + \alpha\,(K \star_\theta n - n) - n\int_{\underline{\theta}}^{\overline{\theta}} n(t,x,\theta')\,\mathrm{d}\theta'$$

with $d \in \mathscr{C}\left(\left[\underline{\theta},\overline{\theta}\right],(0,+\infty)\right)$ and $K \in \mathscr{C}(\mathbb{R},[0,+\infty))$, is

$$\partial_t \mathbf{u} - \mathbf{D}_N \partial_{xx}\mathbf{u} = \mathbf{L}_N\mathbf{u} - (\delta\theta\mathbf{1}_N\mathbf{u}) \circ \mathbf{u},$$

with

$$\mathbf{d}_N = (d(\theta_i))_{i\in[N]},$$

$$\begin{aligned}\mathbf{L}_N &= r\mathbf{I}_N + \alpha\left(\delta\theta\,(K(\theta_i-\theta_j))_{(i,j)\in[N]^2} - \mathbf{I}_N\right)\\ &= (r-\alpha)\,\mathbf{I}_N + \alpha\delta\theta\,(K((i-j)\,\theta_N))_{(i,j)\in[N]^2}.\end{aligned}$$

The assumptions on $\mathbf{c}$ (H_2)–(H_4) are obviously satisfied and, as soon as, say, K is positive, the assumption on $\mathbf{L}(H_1)$ is satisfied as well. Subsequently, $\lambda_{PF}(\mathbf{L}_N) \geq r-\alpha$, whence $r>\alpha$ is a sufficient condition to ensure $\lambda_{PF}(\mathbf{L}_N) > 0$ for all $N\in\mathbb{N}$.

More generally, the system corresponding to the following equation (see Prévost *et al.* [6, 128]):

$$\begin{aligned}\partial_t n - d(\theta)\,\partial_{xx} n = r(\theta)\,n(t,x,\theta) &+ \int_{\underline{\theta}}^{\overline{\theta}} n(t,x,\theta')\,K(\theta,\theta')\,\mathrm{d}\theta'\\ &- n(t,x,\theta)\int_{\underline{\theta}}^{\overline{\theta}} n(t,x,\theta')\,C(\theta,\theta')\,\mathrm{d}\theta'\end{aligned}$$

with $d \in \mathscr{C}\left(\left[\underline{\theta},\overline{\theta}\right],(0,+\infty)\right)$, $r \in \mathscr{C}\left(\left[\underline{\theta},\overline{\theta}\right],[0,+\infty)\right)$, $K, C \in \mathscr{C}\left(\left[\underline{\theta},\overline{\theta}\right]^2,[0,+\infty)\right)$ is

$$\partial_t \mathbf{u} - \mathbf{D}_N \partial_{xx}\mathbf{u} = \mathbf{L}_N\mathbf{u} - (\mathbf{C}_N\mathbf{u}) \circ \mathbf{u},$$

with

$$\begin{aligned}\mathbf{d}_N &= (d(\theta_i))_{i\in[N]},\\ \mathbf{L}_N &= \mathrm{diag}\,(r(\theta_i))_{i\in[N]} + \delta\theta\,(K(\theta_i,\theta_j))_{(i,j)\in[N]^2},\\ \mathbf{C}_N &= \delta\theta\,(C(\theta_i,\theta_j))_{(i,j)\in[N]^2}.\end{aligned}$$

Again, (H_3) and (H_4) are clearly satisfied, (H_2) is satisfied if C is nonnegative and both (H_1) and $\lambda_{PF}(\mathbf{L}_N) > 0$ are satisfied if, say, K is positive.

In both cases, of course, the positivity of K is a far from necessary condition and might be relaxed.

To the best of our knowledge, these doubly non-local equations have been the object of no study apart from [6, 128] and are therefore still very poorly understood. In particular, the traveling wave problem as well as the spreading problem are completely open. Consequently, our results are highly valuable when applied to this system. For mathematicians, they motivate the future work on the limit $N \to +\infty$. For biologists, they provide new insight into these modeling problems and show for instance how two different mutation strategies can be compared and how the spreading speed can be evaluated.

5.1.5.3 The Gurtin–MacCamy equation with diffusion and overcrowding effect

In view of the preceding two examples, it is natural to investigate the existence of completely different applications, that is applications not concerned at all with evolutionary biology. Such applications exist indeed, as shown by this third example.

Consider the following age-structured equation with diffusion:

$$\begin{cases} \partial_t n + \partial_a n - d(a)\,\partial_{xx} n = -n(t,x,a)\left(r(a) + \int_0^A n(t,x,a')\,C(a,a')\,\mathrm{d}a'\right) \\ n(t,x,0) = \int_{a_m}^A n(t,x,a')\,K(a')\,\mathrm{d}a' \text{ for all } (t,x)\in\mathbb{R}^2 \\ n(t,x,A) = 0 \text{ for all } (t,x)\in\mathbb{R}^2 \end{cases}$$

where n is a function of (t,x,a), $a\in[0,A]$ is the age variable, $a_m\geq 0$ is the maturation age, $A > a_m$ is the maximal age, $d\in\mathscr{C}([0,A],(0,+\infty))$ is the diffusion rate, $r\in\mathscr{C}([0,A],(0,+\infty))$ is the mortality rate, $C\in\mathscr{C}\left([0,A]^2,[0,+\infty)\right)$ is the competition kernel and $K\in\mathscr{C}([0,A],[0,+\infty))$ is the birth rate. This equation is well-known, at least if $C=0$, and detailed modeling explanations can be found in the classical Gurtin–MacCamy references [87, 86].

Defining

$$\delta a = \frac{A}{N},$$

$$(a_i)_{i\in[N]} = ((i-1)\,\delta a)_{i\in[N]},$$

$$j_{m,N} = \min\{j\in[N] \mid a_j\geq a_m\},$$

$$\mathbf{u}(t,x) = (n(t,x,a_i))_{i\in[N]},$$

$$\mathbf{d}_N = (d(a_i))_{i\in[N]},$$

$$\mathbf{L}_{mortality,N} = -\mathrm{diag}\left(r(a_i)_{i\in[N]}\right),$$

$$\mathbf{L}_{birth,N} = \delta a \begin{pmatrix} 0 & \dots & 0 & K\left(a_{j_{m,N}}\right) & \dots & K(a_N) \\ 0 & & & \dots & & 0 \\ \vdots & & & & & \vdots \\ 0 & & & \dots & & 0 \end{pmatrix},$$

$$\mathbf{L}_{aging,N} = \frac{1}{\delta a}\begin{pmatrix} 0 & 0 & \dots & \dots & 0 \\ 1 & -1 & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 & -1 \end{pmatrix},$$

$$\mathbf{L}_N = \mathbf{L}_{mortality,N} + \mathbf{L}_{birth,N} + \mathbf{L}_{aging,N},$$

$$\mathbf{C}_N = \delta a\,(C(a_i,a_j))_{(i,j)\in[N]^2},$$

it follows again that

$$\partial_t\mathbf{u} - \mathbf{D}_N\partial_{xx}\mathbf{u} = \mathbf{L}_N\mathbf{u} - (\mathbf{C}_N\mathbf{u})\circ\mathbf{u}$$

is the natural discretization with (H_3) and (H_4) automatically satisfied. K nonnegative nonzero and C nonnegative are sufficient conditions to enforce (H_1) and (H_2).

Since we have

$$\lambda_{PF}(\mathbf{L}_N) \geq \lambda_{PF}(\mathbf{L}_{birth,N} + \mathbf{L}_{aging,N}) - \max_{[0,A]} r$$

and since $\lambda_{PF}(\mathbf{L}_{birth,N} + \mathbf{L}_{aging,N})$ is bounded from below by a positive constant independent of N (the proof of this claim being deliberately not detailed here for the sake of brevity), if $\max_{[0,A]} r$ is small enough, then $\lambda_{PF}(\mathbf{L}_N) > 0$ for all $N \in \mathbb{N}$.

We point out that this KPP system differs noticeably from the Lotka–Volterra mutation–competition–diffusion system presented up to now as the main instance of KPP system: here, the matrix $\mathbf{L}$ is highly non-symmetric. This should have important qualitative consequences, numerically observable. It might even be unexpected that these two systems share important properties and this makes our theorems even more interesting.

As far as we know, the traveling wave problem and the spreading problem for the continuous age-structured problem are completely open. Therefore the earlier remarks concerning the impact of our results on the doubly non-local cane toads equation apply here as well.

5.2 Strong positivity

Theorem 5.1 is mainly straightforward and follows from the following local result.

Proposition 5.9. *Let $\mathsf{Q} \subset \mathbb{R}^2$ be a bounded parabolic cylinder and $\mathbf{u}$ be a classical solution of (E_{KPP}) set in Q.*

If $\mathbf{u}$ is nonnegative on $\partial_P \mathsf{Q}$, then it is either null or positive in Q.

Proof. Let $K = \max_{\overline{\mathsf{Q}}} |\mathbf{u}|$ and observe that, for all $i \in [N]$ and all $(t,x) \in \mathsf{Q}$,

$$|l_{i,i} - c_i(\mathbf{u}(t,x))| \leq |l_{i,i}| + \max_{\mathbf{v} \in \overline{\mathsf{B}(\mathbf{0},K)}} |c_i(\mathbf{v})| .$$

Then, define

$$\mathbf{A} : (t,x) \mapsto \mathbf{L} - \operatorname{diag}(\mathbf{c}(\mathbf{u}(t,x))) .$$

By the irreducibility and the essential nonnegativity of $\mathbf{L}$ (H_1), $\mathbf{A}(t,x)$ has these two properties as well for all $(t,x) \in \overline{\mathsf{Q}}$. By the boundedness of $\mathbf{u}$ in $\overline{\mathsf{Q}}$, $\mathbf{A}$ is bounded in $\overline{\mathsf{Q}}$ as well.

Therefore $\mathbf{u}$ is a solution of the following linear weakly and fully coupled system with bounded coefficients:

$$\partial_t \mathbf{u} - \mathbf{D}\partial_{xx}\mathbf{u} - \mathbf{A}\mathbf{u} = \mathbf{0}.$$

By virtue of Protter–Weinberger's strong maximum principle [129, Chapter 3, Theorem 13], $\mathbf{u}$ is indeed either null or positive in Q. ☐

Actually, noticing that the previous proof remains true without any modification if we add to (E_{KPP}) a diagonal drift term $\mathbf{b} \circ \partial_x \mathbf{u}$ with $\mathbf{b} \in \mathbb{R}^N$, we state right now a corollary that will be quite useful later on.

Corollary 5.10. *Let $(a,b,c) \in \mathbb{R}^3$ such that $a < b$. Let $\mathbf{u}$ be a nonnegative classical solution of*

$$-\mathbf{D}\mathbf{u}'' - c\mathbf{u}' = \mathbf{L}\mathbf{u} - \mathbf{c}[\mathbf{u}] \circ \mathbf{u} \text{ in } (a,b) .$$

Then $\mathbf{u}$ is either null or positive in (a,b).

Remark. This statement does not establish the non-negativity of all solutions of $-\mathbf{D}\mathbf{u}'' - c\mathbf{u}' = \mathbf{L}\mathbf{u} - \mathbf{c}[\mathbf{u}] \circ \mathbf{u}$; it only enforces the interior positivity of the nonnegative nonzero solutions. Regarding the weak maximum principle, we refer among others to Figueiredo [53], Figueiredo–Mitidieri [54], Sweers [136]. In view of what is known in the simpler scalar case, it is to be expected that, for small $|c|$ and large enough intervals (a,b), sign-changing solutions exist.

5.3 Absorbing set and upper estimates

On the contrary, Theorem 5.2 requires some work.

5.3.1 Saturation of the reaction term

For all $i \in [N]$, let $\mathsf{H}_i \subset \mathbb{R}^N$ be the closed half-space defined as

$$\mathsf{H}_i = \left\{ \mathbf{v} \in \mathbb{R}^N \mid (\mathbf{L}\mathbf{v})_i \geq 0 \right\}.$$

Lemma 5.11. *There exists $\mathbf{k} \in \mathsf{K}^{++}$ such that, for all $i \in [N]$ and for all $\mathbf{v} \in \mathsf{K} \backslash \mathbf{e}_i^{\perp}$,*

$$\left(\mathbf{L} \left(\mathbf{v} + k_i \mathbf{e}_i \right) - \mathbf{c} \left(\mathbf{v} + k_i \mathbf{e}_i \right) \circ \left(\mathbf{v} + k_i \mathbf{e}_i \right) \right)_i < 0.$$

Proof. Let $i \in [N]$ and let

$$\mathsf{F}_i = \left(\mathsf{S}^+ \left(\mathbf{0}, 1 \right) \cap \mathsf{H}_i \right) \backslash \mathbf{e}_i^{\perp}.$$

Let

$$\begin{array}{rccc} f_i : & (0, +\infty) \times \mathsf{S}(\mathbf{0}, 1) & \to & \mathbb{R} \\ & (\alpha, \mathbf{n}) & \mapsto & \sum\limits_{j=1}^{N} l_{i,j} n_j - c_i (\alpha \mathbf{n}) n_i. \end{array}$$

Notice that for all $\mathbf{n} \in \mathsf{S}^+ (\mathbf{0}, 1) \backslash \mathsf{F}_i$, either $\sum\limits_{j=1}^{N} l_{i,j} n_j < 0$ and then $f_i (\alpha, \mathbf{n}) < 0$ for all $\alpha > 0$ or $n_i = 0$ and then $f_i (\alpha, \mathbf{n}) = \sum\limits_{j=1}^{N} l_{i,j} n_j \geq 0$ does not depend on α.

Let $\mathbf{n} \in \mathsf{F}_i$. By virtue of the behavior of $\mathbf{c}$ as $\alpha \to +\infty$ (H_4) and since $\mathbf{n} \notin \mathbf{e}_i^{\perp}$,

$$\lim_{\alpha \to +\infty} f_i (\alpha, \mathbf{n}) = -\infty.$$

Therefore the following quantity is finite and nonnegative:

$$\alpha_{i,\mathbf{n}} = \inf \left\{ \alpha \geq 0 \mid \forall \alpha' \in (\alpha, +\infty) \quad f_i (\alpha', \mathbf{n}) < 0 \right\}.$$

Now, the set

$$\{\alpha_{i,\mathbf{n}} n_i \mid \mathbf{n} \in \mathsf{F}_i\} = \{\alpha_{i,\mathbf{n}} n_i \mid \mathbf{n} \in \mathsf{F}_i,\ \alpha_{i,\mathbf{n}} > \underline{\alpha}\} \cup \{\alpha_{i,\mathbf{n}} n_i \mid \mathbf{n} \in \mathsf{F}_i,\ \alpha_{i,\mathbf{n}} \leq \underline{\alpha}\}$$

is bounded if and only if the set $\{\alpha_{i,\mathbf{n}} n_i \mid \mathbf{n} \in \mathsf{F}_i,\ \alpha_{i,\mathbf{n}} > \underline{\alpha}\}$ is bounded. Recall the definition of $\underline{\alpha} \geq 1$ and $\delta \geq 1$ (H_4). For all $\mathbf{n} \in \mathsf{F}_i$ such that $\alpha_{i,\mathbf{n}} > \underline{\alpha}$, thanks to (H_4), we have by virtue of the discrete Cauchy–Schwarz inequality

$$\begin{aligned} |\alpha_{i,\mathbf{n}} n_i| &= \alpha_{i,\mathbf{n}} n_i \\ &\leq \alpha_{i,\mathbf{n}}^{\delta} n_i \\ &\leq \frac{\sum_{j=1}^{N} l_{i,j} n_j}{\underline{c}_i} \\ &\leq \frac{\left| (l_{i,j})_{j \in [N]} \right|}{\underline{c}_i}, \end{aligned}$$

whence the finiteness of

$$k_i = \sup \{\alpha_{i,\mathbf{n}} n_i \mid \mathbf{n} \in \mathsf{F}_i\}$$

is established. Its positivity follows from the fact that $\mathbf{c}$ vanishes at $\mathbf{0}$ (H_3) which implies that for all $\mathbf{n} \in \mathrm{int}\mathsf{F}_i$, $\alpha_{i,\mathbf{n}} > 0$.

The result about $\mathbf{v} + k_i\mathbf{e}_i$ with $\mathbf{v} \in \mathsf{K}\backslash\mathbf{e}_i^\perp$ is a direct consequence. □

Assuming in addition strict monotonicity of $\alpha \mapsto c_i(\alpha\mathbf{n})$ (which is for instance satisfied if $\mathbf{c}(\mathbf{v}) = \mathbf{C}\mathbf{v}$ with $\mathbf{C} \gg \mathbf{0}$, that is in the Lotka–Volterra competition case), we can obtain the following more precise geometric description of the reaction term. The proof is quite straightforward and is not detailed here.

Lemma 5.12. *Assume in addition that $\alpha \mapsto c_i(\alpha\mathbf{n})$ is increasing for all $\mathbf{n} \in \mathsf{H}_i$.*

Then there exists a collection of connected $\mathscr{C}^1$-hypersurfaces

$$(\mathsf{Z}_i)_{i\in[N]} \subset \prod_{i=1}^{N} \left(\left(\mathsf{K}^+ \cap \mathsf{H}_i\right)\backslash\mathbf{e}_i^\perp\right)$$

such that, for any $i \in [N]$ and any $\mathbf{v} \in (\mathsf{K}^+ \cap \mathsf{H}_i)\backslash\mathbf{e}_i^\perp$,

$$(\mathbf{L}\mathbf{v} - \mathbf{c}(\mathbf{v}) \circ \mathbf{v})_i = 0 \text{ if and only if } \mathbf{v} \in \mathsf{Z}_i.$$

For all $i \in [N]$, Z_i satisfies the following properties.

1. *For all $\mathbf{n} \in (\mathsf{S}^+(\mathbf{0},1) \cap \mathsf{H}_i)\backslash\mathbf{e}_i^\perp$, $\mathsf{Z}_i \cap \mathbb{R}\mathbf{n}$ is a singleton.*
2. *The function $\mathbf{z}_i$ which associates with any $\mathbf{n} \in (\mathsf{S}^+(\mathbf{0},1) \cap \mathsf{H}_i)\backslash\mathbf{e}_i^\perp$ the unique element of $\mathsf{Z}_i \cap \mathbb{R}\mathbf{n}$ is continuous and is a $\mathscr{C}^1$-diffeomorphism of $(\mathsf{S}^{++}(\mathbf{0},1) \cap int\mathsf{H}_i)\backslash\mathbf{e}_i^\perp$ onto $int\mathsf{Z}_i$.*
3. *For any $\mathbf{v} \in \mathsf{K}^+\backslash\mathbf{e}_i^\perp$, $(\mathbf{L}\mathbf{v} - \mathbf{c}(\mathbf{v}) \circ \mathbf{v})_i > 0$ if and only if*

$$\mathbf{v} \in \mathsf{H}_i \text{ and } |\mathbf{v}| < \left|\mathbf{z}_i\left(\frac{\mathbf{v}}{|\mathbf{v}|}\right)\right|.$$

5.3.2 Absorbing set and upper estimates

Define for all $i \in [N]$

$$\begin{array}{rccc} g_i : & [0,+\infty) & \to & (0,+\infty) \\ & \mu & \mapsto & \max(\mu, k_i). \end{array}$$

The function g_i is non-decreasing and piecewise affine (whence Lipschitz-continuous).

The following local in space $\mathscr{L}^\infty$ estimate for the parabolic problem is due to Barles–Evans–Souganidis [10]. We repeat its proof for the sake of completeness.

Lemma 5.13. *Let $\mathsf{Q} \subset \mathbb{R}^2$ be a parabolic cylinder bounded in space and bounded from below in time.*

Let $\mathbf{u}$ be a nonnegative classical solution of (E_{KPP}) set in Q such that

$$\mathbf{u}_{|\partial_P\mathsf{Q}} \in \mathscr{L}^\infty\left(\partial_P\mathsf{Q}, \mathbb{R}^N\right).$$

Then we have

$$\left(\sup_{\mathsf{Q}} u_i\right)_{i\in[N]} \leq \left(g_i\left(\sup_{\partial_P\mathsf{Q}} u_i\right)\right)_{i\in[N]}.$$

Proof. Let $t_0 \in \mathbb{R}$, $T \in (0,+\infty]$ and $(a,b) \in \mathbb{R}^2$ such that $\mathsf{Q} = (t_0, t_0+T) \times (a,b)$. Let $i \in [N]$.
Define a smooth convex function $\eta : \mathbb{R} \to \mathbb{R}$ which satisfies

$$\begin{cases} \eta(u) = 0 & \text{if } u \in \left(-\infty, g_i\left(\sup_{\partial_P \mathsf{Q}} u_i\right)\right] \\ \eta(u) > 0 & \text{otherwise.} \end{cases}$$

For all $t \in (t_0, t_0+T)$, let

$$\Xi_i(t) = \left\{ x \in (a,b) \mid u_i(t,x) > g_i\left(\sup_{\partial_P \mathsf{Q}} u_i\right)\right\}.$$

This set is measurable and, by integration by parts, for all $t \in (t_0, t_0+T)$,

$$\begin{aligned}
\partial_t \left(\int_a^b \eta(u_i(t,x))\,\mathrm{d}x \right) &= \int_a^b \eta'(u_i(t,x))\,\partial_t u_i(t,x)\,\mathrm{d}x \\
&= -d_i \int_a^b \eta''(u_i(t,x))\,(\partial_x u_i(t,x))^2\,\mathrm{d}x \\
&+ \int_a^b \eta'(u_i(t,x)) \left(\sum_{j=1}^N l_{i,j} u_j(t,x) - c_i(\mathbf{u}(t,x))\,u_i(t,x) \right) \mathrm{d}x \\
&= -d_i \int_{\Xi_i(t)} \eta''(u_i(t,x))\,(\partial_x u_i(t,x))^2\,\mathrm{d}x \\
&+ \int_{\Xi_i(t)} \eta'(u_i(t,x)) \left(\sum_{j=1}^N l_{i,j} u_j(t,x) - c_i(\mathbf{u}(t,x))\,u_i(t,x) \right) \mathrm{d}x \\
&\leq 0
\end{aligned}$$

Since $\int_a^b \eta(u_i(t_0,x))\,\mathrm{d}x = 0$, we deduce

$$u_i \leq g_i\left(\sup_{\partial_P \mathsf{Q}} u_i\right) \text{ in } \mathsf{Q},$$

whence

$$\sup_{\mathsf{Q}} u_i \leq g_i\left(\sup_{\partial_P \mathsf{Q}} u_i\right).$$

□

As a corollary of this local estimate, we get Theorem 5.2.

Proposition 5.14. *Let* $\mathbf{u}_0 \in \mathscr{C}_b(\mathbb{R}, \mathsf{K})$*. Then the unique classical solution* $\mathbf{u}$ *of* (E_{KPP}) *set in* $(0,+\infty) \times \mathbb{R}$ *with initial data* $\mathbf{u}_0$ *satisfies*

$$\left(\sup_{(0,+\infty)\times\mathbb{R}} u_i \right)_{i\in[N]} \leq \left(g_i\left(\sup_{\mathbb{R}} u_{0,i}\right)\right)_{i\in[N]}$$

and furthermore

$$\left(\limsup_{t\to+\infty} \sup_{x\in\mathbb{R}} u_i(t,x) \right)_{i\in[N]} \leq \mathbf{g}(0).$$

Consequently, all bounded nonnegative classical solutions of (S_{KPP}) *are valued in*

$$\prod_{i=1}^{N}\left[0,g_i\left(0\right)\right].$$

Proof. To get the global in space $\mathscr{L}^\infty$ estimate, apply the local one to the family $(\mathbf{u}_R)_{R>0}$, where $\mathbf{u}_R$ is the solution of (E_{KPP}) set in $(0,+\infty)\times(-R,R)$ with

$$\begin{cases}\mathbf{u}_R\left(0,x\right)=\mathbf{u}_0\left(x\right) & \text{for all } x\in\left[-R,R\right],\\ \mathbf{u}_R\left(t,\pm R\right)=\mathbf{u}_0\left(\pm R\right) & \text{for all } t\geq 0,\end{cases}$$

and recall that, by classical parabolic estimates (Lieberman [111]) and a diagonal extraction process, $(\mathbf{u}_R)_{R>0}$ converges up to extraction in $\mathscr{C}^1_{loc}\left((0,+\infty),\mathscr{C}^2_{loc}\left(\mathbb{R},\mathbb{R}^N\right)\right)$ to the solution of (E_{KPP}) set in $(0,+\infty)\times\mathbb{R}$ with initial data $\mathbf{u}_0$.

Next, let us prove that the invariant set

$$\prod_{i=1}^{N}\left[0,g_i\left(0\right)\right]=\prod_{i=1}^{N}\left[0,k_i\right]$$

is in fact an absorbing set.

Assume by contradiction that there exists a bounded nonnegative classical solution $\mathbf{u}$ of (E_{KPP}) set in $(0,+\infty)\times\mathbb{R}$ such that there exists $i\in[N]$ such that

$$\limsup_{t\to+\infty}\sup_{x\in\mathbb{R}}u_i\left(t,x\right)>g_i\left(0\right).$$

Since $[0,g_i(0)]$ is invariant, it implies directly

$$\sup_{x\in\mathbb{R}}u_i\left(t,x\right)>g_i\left(0\right)\ \text{ for all } t\geq 0.$$

Using the classical second order condition at any local maximum, it is easily seen that at any local maximum in space of u_i, the time derivative is negative. At any $t>0$ such that there is no local maximum in space, by $\mathscr{C}^1$ regularity of u_i, $x\mapsto u_i(t,x)$ is either strictly monotonic or piecewise strictly monotonic with one unique local minimum and consequently it converges to some constant as $x\to\pm\infty$. At least one of these constants is $\sup_{x\in\mathbb{R}}u_i(t,x)$. For instance, assume it is the limit at $+\infty$. By classical parabolic estimates and a diagonal extraction process, there exists $(x_n)_{n\in\mathbb{N}}\in\mathbb{R}^{\mathbb{N}}$ such that $x_n\to+\infty$ and such that the following sequence converges in $\mathscr{C}^1_{loc}\left((0,+\infty),\mathscr{C}^2_{loc}\left(\mathbb{R}\right)\right)$:

$$\left(\left(t',x\right)\mapsto u_i\left(t+t',x+x_n\right)\right)_{n\in\mathbb{N}}.$$

Let v be its limit; by construction,

$$v\left(0,x\right)=\sup_{x\in\mathbb{R}}u_i\left(t,x\right)\ \text{ for all } x\in\mathbb{R},$$

so that

$$\partial_{xx}v\left(0,x\right)=0 \text{ for all } x\in\mathbb{R}.$$

Using the equation satisfied by u_i, we obtain

$$\partial_t v\left(0,x\right)<0 \text{ for all } x\in\mathbb{R}.$$

Since this argument does not depend on the choice of the sequence $(x_n)_{n\in\mathbb{N}}$, we deduce

$$\limsup_{x\to+\infty} \partial_t u_i(t,x) < 0.$$

In all cases,

$$t \mapsto \|x \mapsto u_i(t,x)\|_{\mathscr{L}^\infty(\mathbb{R})}$$

is a decreasing function, and using the global $\mathscr{L}^\infty$ estimate derived earlier, we deduce that

$$t \mapsto \|u_i\|_{\mathscr{L}^\infty((t,+\infty)\times\mathbb{R})}$$

is a decreasing function as well. Therefore

$$\limsup_{t\to+\infty} \sup_{x\in\mathbb{R}} u_i(t,x) = \liminf_{t\to+\infty} \sup_{x\in\mathbb{R}} u_i(t,x) = \lim_{t\to+\infty} \sup_{x\in\mathbb{R}} u_i(t,x) > g_i(0).$$

Now, the sequence

$$((t,x) \mapsto u_i(t+n,x))_{n\in\mathbb{N}}$$

being uniformly bounded in $\mathscr{L}^\infty((0,+\infty)\times\mathbb{R})$, by classical parabolic estimates and a diagonal extraction process, it converges up to extraction in $\mathscr{C}^1_{loc}\left((0,+\infty), \mathscr{C}^2_{loc}(\mathbb{R})\right)$ to some limit $u_{\infty,i} \in \mathscr{C}^1\left((0,+\infty), \mathscr{C}^2(\mathbb{R})\right)$.

On one hand, by construction, the function

$$t \mapsto \|x \mapsto u_{\infty,i}(t,x)\|_{\mathscr{L}^\infty(\mathbb{R})}$$

is constant and larger than $g_i(0)$. But on the other hand, passing also to the limit the other components of $(t,x) \mapsto \mathbf{u}(t+n,x)$ and then repeating the argument used earlier to prove the strict monotonicity of

$$t \mapsto \|x \mapsto u_i(t,x)\|_{\mathscr{L}^\infty(\mathbb{R})},$$

we deduce the strict monotonicity of

$$t \mapsto \|x \mapsto u_{\infty,i}(t,x)\|_{\mathscr{L}^\infty(\mathbb{R})},$$

which is an obvious contradiction. □

Quite similarly, we can establish an $\mathscr{L}^\infty$ estimate for (S_{KPP}), set in a strip, and with an additional drift.

Proposition 5.15. *Let $(a,b,c) \in \mathbb{R}^3$ such that $a < b$ and $\mathbf{u}$ be a nonnegative classical solution of*

$$-\mathbf{D}\mathbf{u}'' - c\mathbf{u}' = \mathbf{L}\mathbf{u} - \mathbf{c}[\mathbf{u}] \circ \mathbf{u} \text{ in } (a,b).$$

Then

$$\left(\max_{[a,b]} u_i\right)_{i\in[N]} \leq \left(g_i\left(\max_{\{a,b\}} u_i\right)\right)_{i\in[N]}.$$

Proof. Assume by contradiction that there exists $i \in [N]$ such that

$$\max_{[a,b]} u_i > g_i\left(\max_{\{a,b\}} u_i\right).$$

Then there exists $x_0 \in (a,b)$ such that

$$\max_{[a,b]} u_i = u_i(x_0) > k_i.$$

There exists $(x_1, x_2) \in (a,b)^2$ such that $x_1 < x_0 < x_2$ and

$$\begin{cases} u_i(x) > k_i & \text{for all } x \in (x_1, x_2) \\ u_i(x) = \frac{1}{2}(k_i + u_i(x_0)) & \text{for all } x \in \{x_1, x_2\}. \end{cases}$$

But then we find the inequality

$$-d_i u_i'' - c u_i' \ll 0 \text{ in } (x_1, x_2)$$

which contradicts the existence of an interior maximum at $x_0 \in (x_1, x_2)$. □

5.4 Extinction and persistence

This section is devoted to the proof of Theorem 5.3. The extinction case is mainly straightforward but, because of the lack of comparison principle, the persistence case is more involved.

5.4.1 Extinction

Proposition 5.16. *Assume $\lambda_{PF}(\mathbf{L}) < 0$.*

Then all bounded nonnegative classical solutions of (E_{KPP}) set in $(0,+\infty)\times\mathbb{R}$ converge asymptotically in time, exponentially fast, and uniformly in space to $\mathbf{0}$.

Proof. It suffices to notice that if $\mathbf{u}$ is a nonnegative bounded solution of (E_{KPP}), then $\mathbf{v} : (t,x) \mapsto \mathrm{e}^{\lambda_{PF}(\mathbf{L})t}\mathbf{n}_{PF}(\mathbf{L})$ satisfies by virtue of the nonnegativity of $\mathbf{c}$ on K (H_2)

$$\partial_t(\mathbf{v}-\mathbf{u}) - \mathbf{D}\partial_{xx}(\mathbf{v}-\mathbf{u}) - \mathbf{L}(\mathbf{v}-\mathbf{u}) = \mathbf{c}[\mathbf{u}] \circ \mathbf{u} \geq \mathbf{0}.$$

Hence, up to a multiplication of $\mathbf{v}$ by a large constant, the comparison principle (Protter–Weinberger [129, Chapter 3, Theorem 13]) applied to the linear weakly and fully coupled operator $\partial_t - \mathbf{D}\partial_{xx} - \mathbf{L}$ in $(0,+\infty)\times\mathbb{R}$ implies that $\mathbf{0} \leq \mathbf{u} \leq \mathbf{v}$. The limit easily follows. □

We recall that the critical case $\lambda_{PF}(\mathbf{L}) = 0$ is solved in the appendix of the current chapter.

5.4.2 Persistence

The first step toward the persistence result is giving some rigorous meaning to the statement "if $\lambda_{PF}(\mathbf{L}) > 0$, then $\mathbf{0}$ is unstable".

5.4.2.1 Slight digression: generalized principal eigenvalues and eigenfunctions for weakly and fully coupled elliptic systems

Theorem 5.17. *Let $(n,n') \in \mathbb{N}\cap[1,+\infty)\times\mathbb{N}\cap[2,+\infty)$ and $\mathscr{L} : \mathscr{C}^2\left(\mathbb{R}^n, \mathbb{R}^{n'}\right) \to \mathscr{C}\left(\mathbb{R}^n, \mathbb{R}^{n'}\right)$ be a second-order elliptic operator, weakly and fully coupled, with continuous and bounded coefficients.*

Let

$$\lambda_1(-\mathscr{L}) = \sup\left\{\lambda \in \mathbb{R} \mid \exists \mathbf{v} \in \mathscr{C}^2\left(\mathbb{R}^n, \mathsf{K}_{n'}^{++}\right) \quad -\mathscr{L}\mathbf{v} \geq \lambda\mathbf{v}\right\} \in \overline{\mathbb{R}}.$$

Then

$$\lim_{R\to+\infty} \lambda_{1,Dir}(-\mathscr{L}, \mathsf{B}_n(\mathbf{0},R)) = \lambda_1(-\mathscr{L}).$$

Furthermore, $\lambda_1(-\mathscr{L})$ is in fact a finite maximum and there exists a generalized principal eigenfunction, that is a positive solution of

$$-\mathscr{L}\mathbf{v} = \lambda_1(-\mathscr{L})\mathbf{v}.$$

Remark. The convergence of the Dirichlet principal eigenvalue to the aforementioned generalized principal eigenvalue as $R \to +\infty$ as well as the existence of a generalized principal eigenfunction are well-known for scalar elliptic equations (see Berestycki–Rossi [21]), but as far as we know these results do not explicitly appear in the literature regarding elliptic systems. Still, the proof of Berestycki–Rossi [21] uses arguments developed in the celebrated article by Berestycki–Nirenberg–Varadhan [20] and which have been generalized to weakly and fully coupled elliptic systems already in order to prove the existence of a Dirichlet principal eigenvalue in non-necessarily smooth but bounded domains by Birindelli–Mitidieri–Sweers [24]. Hence we only briefly outline here the proof so that it can be checked that the generalization to unbounded domains is straightforward.

It begins with the standard verification of the equality between the generalized principal eigenvalue as defined above and the Dirichlet principal eigenvalue for bounded smooth domains (whose existence was proved for instance by Sweers [136]). Then, since the generalized principal eigenvalue is, by definition, non-increasing with respect to the inclusion of the domains, we get that the limit of the Dirichlet principal eigenvalues as $R \to +\infty$ exists and is larger than or equal to the generalized principal eigenvalue. It remains to prove that it is also smaller than or equal to it. This is done thanks to the family of Dirichlet eigenfunctions $(\mathbf{v}_R)_{R>0}$ associated with the family of Dirichlet principal eigenvalues normalized by

$$\min_{i\in[n']} v_{i,R}(\mathbf{0}) = 1.$$

Thanks to Arapostathis–Gosh–Marcus's Harnack inequality [5] applied to the operator $\mathscr{L}$, we obtain a locally uniform $\mathscr{L}^\infty$ estimate, whence, by virtue of classical elliptic estimates (Gilbarg–Trudinger [80]) and a diagonal extraction process, the existence of a limit, up to extraction, for the family $(\mathbf{v}_R)_{R>0}$ as $R \to +\infty$. This limit $\mathbf{v}_\infty$ is nonnegative nonzero and satisfies

$$-\mathscr{L}\mathbf{v}_\infty = \left[\lim_{R\to+\infty} \lambda_{1,Dir}\left(-\mathscr{L}, \mathsf{B}_n(\mathbf{0}, R)\right)\right]\mathbf{v}_\infty.$$

Thanks again to Arapostathis–Gosh–Marcus's Harnack inequality, $\mathbf{v}_\infty$ is in fact positive in $\mathbb{R}^n$. Thus, by definition of the generalized principal eigenvalue, the limit as $R \to +\infty$ is indeed smaller than or equal to it, and in the end the equality is proved as well as the existence of a generalized principal eigenfunction $\mathbf{v}_\infty$.

5.4.2.2 Local instability and persistence

Let $\gamma \in [0,1]$. On one hand, as a direct result of Dancer [44] or Lam–Lou [105],

$$\lim_{\varepsilon\to 0} \lambda_{1,Dir}\left(-\varepsilon^2\mathbf{D}\frac{\mathrm{d}^2}{\mathrm{d}x^2} - (\mathbf{L} - \gamma\lambda_{PF}(\mathbf{L})\mathbf{I}), \mathsf{B}(\mathbf{0},1)\right) = -(1-\gamma)\lambda_{PF}(\mathbf{L}).$$

On the other hand, by a standard change of variable,

$$\frac{\lim_{\varepsilon\to 0} \lambda_{1,Dir}\left(-\varepsilon^2\mathbf{D}\frac{\mathrm{d}^2}{\mathrm{d}x^2} - (\mathbf{L} - \gamma\lambda_{PF}(\mathbf{L})\mathbf{I}), \mathsf{B}(\mathbf{0},1)\right)}{\lim_{R\to+\infty} \lambda_{1,Dir}\left(-\mathbf{D}\frac{\mathrm{d}^2}{\mathrm{d}x^2} - (\mathbf{L} - \gamma\lambda_{PF}(\mathbf{L})\mathbf{I}), \mathsf{B}(\mathbf{0},R)\right)} = 1.$$

Therefore, in view of Theorem 5.17,

$$\lambda_1\left(-\mathbf{D}\frac{\mathrm{d}^2}{\mathrm{d}x^2} - (\mathbf{L} - \gamma\lambda_{PF}(\mathbf{L})\mathbf{I})\right) = -(1-\gamma)\lambda_{PF}(\mathbf{L}).$$

This equality deserves some attention: the generalized principal eigenvalue of $\mathbf{D}\frac{\mathrm{d}^2}{\mathrm{d}x^2}+(\mathbf{L}-\gamma\lambda_{PF}(\mathbf{L})\mathbf{I})$ does not depend on $\mathbf{D}$. Of course, this is reminiscent of the scalar case, where the equality

$$\lambda_1\left(-d\frac{\mathrm{d}^2}{\mathrm{d}x^2}-r\right)=-r$$

is well-known (and follows for instance from a direct computation of $\lambda_{1,Dir}\left(-d\frac{\mathrm{d}^2}{\mathrm{d}x^2}-r,(-R,R)\right)$ or from the equality with the periodic principal eigenvalue $\lambda_{1,per}\left(-d\frac{\mathrm{d}^2}{\mathrm{d}x^2}-r\right)$).

As a corollary, we get the following lemma.

Lemma 5.18. *Assume $\lambda_{PF}(\mathbf{L})>0$. Then there exists $\left(R_0,R_{1/2}\right)\in(0,+\infty)^2$ such that*

$$\lambda_{1,Dir}\left(-\mathbf{D}\frac{d^2}{dx^2}-\mathbf{L},(-R_0,R_0)\right)<0,$$

$$\lambda_{1,Dir}\left(-\mathbf{D}\frac{d^2}{dx^2}-\left(\mathbf{L}-\frac{\lambda_{PF}(\mathbf{L})}{2}\mathbf{I}\right),\left(-R_{1/2},R_{1/2}\right)\right)<0.$$

Remark. In fact, much more precisely, it can be shown that, for all $\gamma\in[0,1]$,

$$R\mapsto\lambda_{1,Dir}\left(-\mathbf{D}\frac{\mathrm{d}^2}{\mathrm{d}x^2}-(\mathbf{L}-\gamma\lambda_{PF}(\mathbf{L})\mathbf{I}),(-R,R)\right)$$

is a decreasing homeomorphism from $(0,+\infty)$ onto $(-(1-\gamma)\lambda_{PF}(\mathbf{L}),+\infty)$.

By continuity of $\mathbf{c}$ and the fact that it vanishes at $\mathbf{0}$ (H_3), as soon as $\lambda_{PF}(\mathbf{L})>0$, the quantity

$$\alpha_{1/2}=\max\left\{\alpha>0\mid\forall\mathbf{v}\in[0,\alpha]^N\quad\mathbf{c}(\mathbf{v})\leq\frac{\lambda_{PF}(\mathbf{L})}{2}\mathbf{1}_{N,1}\right\}$$

is well-defined in $\mathbb{R}$ and is positive. The pair $\left(R_{1/2},\alpha_{1/2}\right)$ will be used repeatedly up to the end of this section.

Lemma 5.19. *Assume $\lambda_{PF}(\mathbf{L})>0$. For all $\mu\in\left(0,\alpha_{1/2}\right)$, let*

$$T_\mu=\frac{\ln\alpha_{1/2}-\ln\mu}{-\lambda_{1,Dir}\left(-\mathbf{D}\frac{d^2}{dx^2}-\left(\mathbf{L}-\frac{\lambda_{PF}(\mathbf{L})}{2}\mathbf{I}\right),\left(-R_{1/2},R_{1/2}\right)\right)}>0.$$

For all $(t_0,T,a,b)\in\mathbb{R}\times(0,+\infty)\times\mathbb{R}^2$ such that $\frac{b-a}{2}=R_{1/2}$ and for all nonnegative classical solutions $\mathbf{u}$ of (E_{KPP}) set in the bounded parabolic cylinder $(t_0,t_0+T)\times(a,b)$, if

$$\min_{i\in[N]}\min_{x\in[a,b]}u_i(t_0,x)=\mu,$$

$$\max_{i\in[N]}\max_{[t_0,t_0+T]\times[a,b]}u_i\leq\alpha_{1/2},$$

then $T<T_\mu$.

Proof. Let

$$\Lambda=\lambda_{1,Dir}\left(-\mathbf{D}\frac{\mathrm{d}^2}{\mathrm{d}x^2}-\left(\mathbf{L}-\frac{\lambda_{PF}(\mathbf{L})}{2}\mathbf{I}\right),\left(-R_{1/2},R_{1/2}\right)\right)<0.$$

Let $\mathbf{n}$ be the principal eigenfunction associated with the preceding Dirichlet principal eigenvalue normalized so that

$$\max_{i\in[N]}\max_{\left[-R_{1/2},R_{1/2}\right]} n_i = 1.$$

By definition, we have in $\left(-R_{1/2},R_{1/2}\right)$

$$-\left(-\mathbf{D}\mathbf{n}''-\left(\mathbf{L}-\frac{\lambda_{PF}\left(\mathbf{L}\right)}{2}\mathbf{I}\right)\mathbf{n}\right)=-\Lambda\mathbf{n}\gg\mathbf{0}.$$

By definition of $\alpha_{1/2}$ and by the nonnegativity of $\mathbf{c}$ on K (H_2), for all $\mathbf{v}\in\left[0,\alpha_{1/2}\right]^N$,

$$\mathbf{c}\left(\mathbf{v}\right)\circ\mathbf{v}\leq\frac{\lambda_{PF}\left(\mathbf{L}\right)}{2}\mathbf{v},$$

whence

$$-\left(\mathbf{L}\mathbf{v}-\mathbf{c}\left(\mathbf{v}\right)\circ\mathbf{v}\right)\leq-\left(\mathbf{L}-\frac{\lambda_{PF}\left(\mathbf{L}\right)}{2}\mathbf{I}\right)\mathbf{v}.$$

Now, fix $(t_0,T,a,b)\in\mathbb{R}\times(0,+\infty)\times\mathbb{R}^2$ such that $\frac{b-a}{2}=R_{1/2}$ and $T\geq T_\mu$. Assume by contradiction that there exists a nonnegative solution $\mathbf{u}:(t_0,t_0+T)\times(a,b)\to\mathsf{K}$ of (E_{KPP}) such that the following properties hold

$$\mu=\min_{i\in[N]}\min_{x\in[a,b]}u_i\left(t_0,x\right)>0,$$

$$\max_{i\in[N]}\max_{[t_0,t_0+T]\times[a,b]}u_i\leq\alpha_{1/2}.$$

In particular, since $\mu>0$, $\mathbf{u}$ is nonnegative nonzero.

To simplify the notations, hereafter we assume that $t_0=0$ and $\frac{a+b}{2}=0$. The general case is only a matter of straightforward translations.

Define the function

$$\mathbf{v}:(t,x)\mapsto\mu\mathrm{e}^{-\Lambda t}\mathbf{n}\left(x\right).$$

Clearly

$$\mathbf{v}\left(0,x\right)\leq\mathbf{u}\left(0,x\right)\ \text{ for all } x\in[a,b].$$

It is easily verified as well that $\mathbf{v}$ satisfies in $(0,T_\mu)\times\left(-R_{1/2},R_{1/2}\right)$

$$-\left(\partial_t\mathbf{v}-\mathbf{D}\partial_{xx}\mathbf{v}-\left(\mathbf{L}-\frac{\lambda_{PF}\left(\mathbf{L}\right)}{2}\mathbf{I}\right)\mathbf{v}\right)\geq\mathbf{0},$$

whence, by construction of $\alpha_{1/2}$, $\mathbf{w}=\mathbf{u}-\mathbf{v}$ satisfies

$$\partial_t\mathbf{w}-\mathbf{D}\partial_{xx}\mathbf{w}-\left(\mathbf{L}-\frac{\lambda_{PF}\left(\mathbf{L}\right)}{2}\mathbf{I}\right)\mathbf{w}\geq\partial_t\mathbf{u}-\mathbf{D}\partial_{xx}\mathbf{u}-\mathbf{L}\mathbf{u}+\mathbf{c}\left[\mathbf{u}\right]\circ\mathbf{u}=\mathbf{0}.$$

Most importantly, since by construction

$$T_\mu=\max\left\{t>0\mid\max_{i\in[N]}\max_{x\in\left[-R_{1/2},R_{1/2}\right]}v_i\left(t,x\right)\leq\alpha_{1/2}\right\},$$

there exists $t^\star\leq T_\mu\leq T$ and $x^\star\in\left(-R_{1/2},R_{1/2}\right)$ such that $\mathbf{w}\gg\mathbf{0}$ in $[0,t^\star)\times\left(-R_{1/2},R_{1/2}\right)$ and $\mathbf{w}\left(t^\star,x^\star\right)\in\partial\mathsf{K}$.

The strong maximum principle applied to the weakly and fully coupled linear operator $\partial_t-\mathbf{D}\partial_{xx}-\left(\mathbf{L}-\frac{\lambda_{PF}(\mathbf{L})}{2}\mathbf{I}\right)$ proves then that $\mathbf{w}=\mathbf{0}$ in $[0,t^\star)\times\left(-R_{1/2},R_{1/2}\right)$, which contradicts $\mathbf{w}\left(0,\pm R_{1/2}\right)\gg\mathbf{0}$. □

The persistence result follows.

Proposition 5.20. *Assume* $\lambda_{PF}(\mathbf{L}) > 0$.

There exists $\nu > 0$ *such that all bounded nonnegative nonzero classical solutions* $\mathbf{u}$ *of* (E_{KPP}) *set in* $(0,+\infty)\times\mathbb{R}$ *satisfy, for all bounded intervals* $I \subset \mathbb{R}$,

$$\left(\liminf_{t\to+\infty}\inf_{x\in I} u_i(t,x)\right)_{i\in[N]} \geq \nu\mathbf{1}_{N,1}.$$

Consequently, all bounded nonnegative classical solutions of (S_{KPP}) *are valued in*

$$\prod_{i=1}^{N}[\nu, g_i(0)].$$

Proof. Let $\mathbf{u}$ be a bounded nonnegative nonzero classical solution of (E_{KPP}) set in $(0,+\infty)\times\mathbb{R}$. In view of Proposition 5.14, for all $\varepsilon > 0$ there exists $t_\varepsilon \in (0,+\infty)$ such that

$$\mathbf{u} \leq \left(\max_{i\in[N]}(g_i(0)) + \varepsilon\right)\mathbf{1}_{N,1} \text{ in } (t_\varepsilon,+\infty)\times\mathbb{R}.$$

Let $I \subset \mathbb{R}$ be a bounded interval. Fix temporarily $\varepsilon > 0$ and $x \in I$ and define $I_x = \left(x - R_{1/2}, x + R_{1/2}\right)$.

A first application of Lemma 5.19 establishes that there exists $\hat{t}_x \in [t_\varepsilon,+\infty)$ such that

$$\max_{i\in[N]}\max_{y\in\overline{I_x}} u_i\left(\hat{t}_x, y\right) = \alpha_{1/2}$$

and that there exists $\tau > 0$ such that

$$\max_{i\in[N]}\max_{y\in\overline{I_x}} u_i(t,y) > \alpha_{1/2} \text{ for all } t \in \left(\hat{t}_x, \hat{t}_x + \tau\right).$$

Hence the following quantity is well-defined in $\left[\hat{t}_x + \tau, +\infty\right]$:

$$t_1 = \inf\left\{t \geq \hat{t}_x + \tau \mid \max_{i\in[N]}\max_{y\in\overline{I_x}} u_i(t,y) < \alpha_{1/2}\right\}.$$

Assume first $t_1 < +\infty$. Then by continuity,

$$\max_{i\in[N]}\max_{y\in\overline{I_x}} u_i(t_1,y) = \alpha_{1/2}.$$

Let

$$A_{\mathbf{L},\mathbf{c},\varepsilon} = \max_{(i,j)\in[N]^2}|l_{i,j}| + \max_{i\in[N]}\max_{\mathbf{w}\in\left[0,\max_{i\in[N]}(g_i(0))+\varepsilon\right]^N} c_i(\mathbf{w}).$$

By virtue of Földes–Poláčik's Harnack inequality [73], there exists $\overline{\kappa} > 0$, dependent only on N, $R_{1/2}$, $\min_{i\in[N]} d_i$, $\max_{i\in[N]} d_i$ and $A_{\mathbf{L},\mathbf{c},\varepsilon}$ such that, for all

$$\mathbf{w} \in \mathscr{C}_b\left((0,+\infty)\times\mathbb{R}, \left[0, \max_{i\in[N]}(g_i(0)) + \varepsilon\right]^N\right),$$

all nonnegative classical solutions $\mathbf{v}$ of the linear weakly and fully coupled system with bounded coefficients

$$\partial_t \mathbf{v} - \mathbf{D}\partial_{xx}\mathbf{v} - (\mathbf{L} - \operatorname{diag}(\mathbf{c}[\mathbf{w}]))\,\mathbf{v} = \mathbf{0} \text{ in } I_x$$

satisfy

$$\min_{i\in[N]} \min_{y\in\overline{I_x}} v_i(t_1+1,y) \geq \overline{\kappa} \max_{i\in[N]} \max_{y\in\overline{I_x}} v_i(t_1,y).$$

We stress that $\overline{\kappa}$ does not depend on $\mathbf{w}$. In particular, taking $\mathbf{w}=\mathbf{v}=\mathbf{u}$, we deduce

$$\min_{i\in[N]} \min_{y\in\overline{I_x}} u_i(t_1+1,y) \geq \overline{\kappa}\alpha_{1/2}.$$

Of course, up to a shrink of $\overline{\kappa}$, we can assume without loss of generality $\overline{\kappa}\in(0,1)$. Then let

$$T = \frac{-\ln\overline{\kappa}}{-\lambda_{1,Dir}\left(-\mathbf{D}\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \left(\mathbf{L} - \frac{\lambda_{PF}(\mathbf{L})}{2}\mathbf{I}\right), I_x\right)} > 0.$$

T does not depend on the choice of $\mathbf{u}$.

A second application of Lemma 5.19 establishes

$$\max_{i\in[N]} \max_{y\in\overline{I_x}} u_i(t_1+1+T,y) > \alpha_{1/2}.$$

Hence, defining the sequence $(t_n)_{n\in\mathbb{N}}$ by the recurrence relation

$$t_{n+1} = \inf\left\{ t \geq t_n + 1 + T \mid \max_{i\in[N]} \max_{y\in\overline{I_x}} u_i(t,y) < \alpha_{1/2} \right\}$$

and repeating by induction the process, we deduce that any connected component of

$$\left\{ t \in \left(\hat{t}_x, +\infty\right) \mid \max_{i\in[N]} \max_{y\in\overline{I_x}} u_i(t,y) < \alpha_{1/2} \right\}$$

is an interval of length smaller than $1+T$.

A second application of Földes–Poláčik's Harnack inequality shows that there exists $\overline{\sigma_\varepsilon} > 0$, dependent only on N, $R_{1/2}$, T, $\min_{i\in[N]} d_i$, $\max_{i\in[N]} d_i$ and $A_{\mathbf{L},\mathbf{c},\varepsilon}$ such that, for all $t\in\left(\hat{t}_x,+\infty\right)$,

$$\min_{i\in[N]} \min_{y\in\overline{I_x}} u_i(t+T+2,y) \geq \overline{\sigma_\varepsilon} \max_{i\in[N]} \max_{(t',y)\in[t,t+T+1]\times\overline{I_x}} u_i(t',y),$$

whence

$$\min_{i\in[N]} \min_{y\in\overline{I_x}} u_i(t,y) \geq \overline{\sigma_\varepsilon}\alpha_{1/2} \text{ for all } t\in\left(\hat{t}_x + T + 2, +\infty\right).$$

Assume next $t_1 = +\infty$. Then

$$\max_{i\in[N]} \max_{y\in\overline{I_x}} u_i(t,y) \geq \alpha_{1/2} \text{ for all } t\in\left(\hat{t}_x, +\infty\right),$$

and consequently

$$\min_{i\in[N]} \min_{y\in\overline{I_x}} u_i(t,y) \geq \overline{\sigma_\varepsilon}\alpha_{1/2} \text{ for all } t\in\left(\hat{t}_x + T + 2, +\infty\right).$$

Since I is bounded and $x \mapsto \hat{t}_x$ can be assumed continuous in $\mathbb{R}$ without loss of generality, it follows

$$\min_{i\in[N]} \inf_{y\in I} u_i(t,y) \geq \overline{\sigma_\varepsilon}\alpha_{1/2} \text{ for all } t \in \left(\max_{x\in\overline{I}}\left(\hat{t}_x\right) + T + 2, +\infty\right),$$

whence

$$\liminf_{t\to+\infty} \min_{i\in[N]} \inf_{y\in I} u_i(t,y) \geq \overline{\sigma_\varepsilon}\alpha_{1/2}$$

with $\overline{\sigma_\varepsilon}\alpha_{1/2}$ dependent only on ε. The conclusion follows of course by setting

$$\nu = \sup_{\varepsilon>0}\left(\overline{\sigma_\varepsilon}\right)\alpha_{1/2}.$$

□

Remark. We point out that $\max_{x\in\overline{I}} \hat{t}_x$ is finite because I is bounded. Of course, in $I = \mathbb{R}$, this problem becomes a spreading problem (see Proposition 5.40).

5.5 Existence of positive steady states

This section is devoted to the proof of Theorem 5.4 .

Proposition 5.21. *Assume $\lambda_{PF}(\mathbf{L}) < 0$. Then there exists no positive classical solution of (S_{KPP}).*

Proof. Recall that the Dirichlet principal eigenvalue is non-increasing with respect to the zeroth order coefficient.

On one hand, by virtue of the nonnegativity of $\mathbf{c}$ on K (H_2), we have for all $R > 0$ and all $\mathbf{v} \in \mathscr{C}_b(\mathbb{R}, \mathsf{K}^{++})$,

$$\lambda_{1,Dir}\left(-\mathbf{D}\frac{\mathrm{d}^2}{\mathrm{d}x^2} - (\mathbf{L} - \mathrm{diag}\mathbf{c}[\mathbf{v}]), (-R,R)\right) \geq \lambda_{1,Dir}\left(-\mathbf{D}\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \mathbf{L}, (-R,R)\right),$$

whence, as $R \to +\infty$,

$$\lambda_1\left(-\mathbf{D}\frac{\mathrm{d}^2}{\mathrm{d}x^2} - (\mathbf{L} - \mathrm{diag}\mathbf{c}[\mathbf{v}])\right) \geq -\lambda_{PF}(\mathbf{L}) > 0.$$

On the other hand, any positive steady state $\mathbf{v}$ is also a generalized principal eigenfunction for the generalized principal eigenvalue

$$\lambda_1\left(-\mathbf{D}\frac{\mathrm{d}^2}{\mathrm{d}x^2} - (\mathbf{L} - \mathrm{diag}\mathbf{c}[\mathbf{v}])\right) = 0.$$

□

Proposition 5.22. *Assume $\lambda_{PF}(\mathbf{L}) = 0$ and*

$$\mathit{span}(\mathbf{n}_{PF}(\mathbf{L})) \cap \mathsf{K} \cap \mathbf{c}^{-1}(\{\mathbf{0}\}) = \{\mathbf{0}\}.$$

Then there exists no bounded positive classical solution of (S_{KPP}).

Remark. The forthcoming argument is quite standard in the scalar setting. We detail it for the sake of completeness.

Proof. Assume by contradiction that there exists a bounded positive classical solution $\mathbf{v}$ of (S_{KPP}).

By boundedness of $\mathbf{v}$, there exists $\kappa \in (0,+\infty)$ such that $\kappa \mathbf{n}_{PF}(\mathbf{L}) - \mathbf{v} \geq \mathbf{0}$ in $\mathbb{R}$. Let

$$\kappa^\star = \inf \left\{ \kappa \in (0,+\infty) \mid \kappa \mathbf{n}_{PF}(\mathbf{L}) - \mathbf{v} \geq \mathbf{0} \text{ in } \mathbb{R} \right\}.$$

By positivity of $\mathbf{v}$, $\kappa^\star > 0$. Let $(\kappa_n)_{n\in\mathbb{N}} \in (0,\kappa^\star)^{\mathbb{N}}$ which converges from below to $\kappa^\star$. For all $n \in \mathbb{N}$, there exists $x_n \in \mathbb{R}$ such that

$$\kappa_n \mathbf{n}_{PF}(\mathbf{L}) - \mathbf{v}(x_n) < \mathbf{0}.$$

Let

$$\mathbf{v}_n : x \mapsto \mathbf{v}(x + x_n) \text{ for all } n \in \mathbb{N}.$$

By virtue of the global boundedness of $\mathbf{v}$, Arapostathis–Gosh–Marcus's Harnack inequality [5] applied to the linear weakly and fully coupled operator with bounded coefficients

$$\mathbf{D}\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} + c\frac{\mathrm{d}}{\mathrm{d}\xi} + \left(\mathbf{L} - \mathrm{diag}\left(\mathbf{c}\left[\mathbf{v}_n\right]\right)\right)$$

and classical elliptic estimates (Gilbarg–Trudinger [80]), $(\mathbf{v}_n)_{n\in\mathbb{N}}$ converges up to a diagonal extraction in $\mathscr{C}^2_{loc}$ as $n \to +\infty$ to a nonnegative solution $\mathbf{v}^\star$ of (S_{KPP}). Moreover, $\mathbf{v}^\star$ satisfies

$$\mathbf{v}^\star \leq \kappa^\star \mathbf{n}_{PF}(\mathbf{L}) \text{ in } \mathbb{R},$$

$$\kappa^\star \mathbf{n}_{PF}(\mathbf{L}) - \mathbf{v}^\star(0) \in \partial \mathsf{K},$$

$$-\left(\mathbf{D}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \mathbf{L}\right)\left(\kappa^\star \mathbf{n}_{PF}(\mathbf{L}) - \mathbf{v}^\star\right) = \mathbf{c}\left[\mathbf{v}^\star\right] \circ \mathbf{v}^\star \geq \mathbf{0} \text{ in } \mathbb{R}.$$

Applying Arapostathis–Gosh–Marcus's Harnack inequality [5] to $\mathbf{D}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \mathbf{L}$, we deduce

$$\kappa^\star \mathbf{n}_{PF}(\mathbf{L}) = \mathbf{v}^\star \text{ in } \mathbb{R}$$

and subsequently

$$\mathbf{c}\left(\kappa^\star \mathbf{n}_{PF}(\mathbf{L})\right) \circ \kappa^\star \mathbf{n}_{PF}(\mathbf{L}) = -\left(\mathbf{D}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \mathbf{L}\right)\mathbf{0} = \mathbf{0},$$

whence $\mathbf{c}\left(\kappa^\star \mathbf{n}_{PF}(\mathbf{L})\right) = \mathbf{0}$, which contradicts directly $\kappa^\star > 0$. □

Finally, recall that if $\lambda_{PF}(\mathbf{L}) > 0$, then the following quantity is well-defined and positive:

$$\alpha_{1/2} = \max\left\{\alpha > 0 \mid \forall \mathbf{v} \in [0,\alpha]^N \quad \mathbf{c}(\mathbf{v}) \leq \frac{\lambda_{PF}(\mathbf{L})}{2}\mathbf{1}_{N,1}\right\}.$$

Proposition 5.23. *Assume $\lambda_{PF}(\mathbf{L}) > 0$. Then there exists a solution $\mathbf{v} \in \mathsf{K}^{++}$ of*

$$\mathbf{L}\mathbf{v} = \mathbf{c}(\mathbf{v}) \circ \mathbf{v}.$$

Proof. By virtue of the Perron–Frobenius theorem, $\mathbf{n}_{PF}\left(\mathbf{L}^T\right) \in \mathsf{K}^{++}$.

There exists $\eta > 0$ such that, for all $\mathbf{v} \in \mathsf{K}$, if $\mathbf{n}_{PF}\left(\mathbf{L}^T\right)^T \mathbf{v} = \eta$, then $\mathbf{v} \in \left[0, \alpha_{1/2}\right]^N$. Defining

$$\mathsf{A} = \left\{\mathbf{v} \in \mathsf{K} \mid \mathbf{n}_{PF}\left(\mathbf{L}^T\right)^T \mathbf{v} = \eta\right\},$$

it follows that for all $\mathbf{v} \in \mathsf{A}$,

$$\mathbf{n}_{PF}\left(\mathbf{L}^T\right)^T(\mathbf{c}(\mathbf{v}) \circ \mathbf{v}) \leq \frac{\lambda_{PF}(\mathbf{L})}{2}\eta,$$

whence

$$\begin{aligned}\mathbf{n}_{PF}\left(\mathbf{L}^T\right)^T(\mathbf{Lv} - \mathbf{c}(\mathbf{v}) \circ \mathbf{v}) &= \lambda_{PF}\left(\mathbf{L}^T\right)\eta - \mathbf{n}_{PF}\left(\mathbf{L}^T\right)^T(\mathbf{c}(\mathbf{v}) \circ \mathbf{v}) \\ &\geq \frac{\lambda_{PF}(\mathbf{L})}{2}\eta,\end{aligned}$$

which is positive if $\lambda_{PF}(\mathbf{L}) > 0$ is assumed indeed.

Then, defining the convex compact set

$$\mathsf{C} = \left\{\mathbf{v} \in \mathsf{K} \mid \mathbf{n}_{PF}\left(\mathbf{L}^T\right)^T\mathbf{v} \geq \eta \text{ and } \mathbf{v} \leq \mathbf{k} + \mathbf{1}_{N,1}\right\},$$

it can easily be verified that, for all $\mathbf{v} \in \partial\mathsf{C}$,

$$\mathbf{n}_{\mathbf{v}}^T(\mathbf{Lv} - \mathbf{c}(\mathbf{v}) \circ \mathbf{v}) < 0$$

where $\mathbf{n}_{\mathbf{v}}$ is the outward pointing normal. In particular, there is no solution of $\mathbf{Lv} = \mathbf{c}(\mathbf{v}) \circ \mathbf{v}$ in $\partial\mathsf{C}$. Also, by convexity, for all $\mathbf{v} \in \partial\mathsf{C}$, there exists a unique $\delta_{\mathbf{v}} > 0$ such that

$$\mathbf{v} + \delta_{\mathbf{v}}(\mathbf{Lv} - \mathbf{c}(\mathbf{v}) \circ \mathbf{v}) \in \partial\mathsf{C}.$$

Assume by contradiction that there is no solution of $\mathbf{Lv} = \mathbf{c}(\mathbf{v}) \circ \mathbf{v}$ in $\text{int}\mathsf{C}$. Consequently and by convexity again, for all $\mathbf{v} \in \text{int}\mathsf{C}$, there exists a unique $\delta_{\mathbf{v}} > 0$ such that

$$\mathbf{v} + \delta_{\mathbf{v}}(\mathbf{Lv} - \mathbf{c}(\mathbf{v}) \circ \mathbf{v}) \in \partial\mathsf{C}.$$

The function

$$\begin{array}{ccc}\mathsf{C} & \to & (0, +\infty) \\ \mathbf{v} & \mapsto & \delta_{\mathbf{v}}\end{array}$$

is continuous and so is the function

$$\begin{array}{ccc}\mathsf{C} & \to & \partial\mathsf{C} \\ \mathbf{v} & \mapsto & \mathbf{v} + \delta_{\mathbf{v}}(\mathbf{Lv} - \mathbf{c}(\mathbf{v}) \circ \mathbf{v}).\end{array}$$

According to the Brouwer fixed point theorem, this function has a fixed point, which of course contradicts the assumption.

Hence there exists indeed a solution in $\text{int}\mathsf{C} \subset \mathsf{K}^{++}$ of

$$\mathbf{Lv} = \mathbf{c}(\mathbf{v}) \circ \mathbf{v}.$$

□

5.6 Traveling waves

In this section, we assume $\lambda_{PF}(\mathbf{L}) > 0$ and prove Theorem 5.5.

Notice as a preliminary that, for any $(\mathbf{p}, c) \in \mathscr{C}^2\left(\mathbb{R}, \mathbb{R}^N\right) \times [0, +\infty)$,

$$\mathbf{u} : (t, x) \mapsto \mathbf{p}(x - ct)$$

is a classical solution of (E_{KPP}) if and only if $\mathbf{p}$ is a classical solution of

$$-\mathbf{D}\mathbf{p}'' - c\mathbf{p}' = \mathbf{Lp} - \mathbf{c}[\mathbf{p}] \circ \mathbf{p} \text{ in } \mathbb{R}. \quad (TW[c])$$

5.6.1 The linearized equation

As usual in KPP-type problems, the linearized equation near $\mathbf{0}$:

$$-\mathbf{D}\mathbf{p}'' - c\mathbf{p}' = \mathbf{L}\mathbf{p} \text{ in } \mathbb{R} \quad (TW_0\,[c])$$

will bring forth the main informations we need in order to construct and study the traveling wave solutions. Hence we devote this first subsection to its detailed study.

Lemma 5.24. *Let* $(c,\lambda)\in\mathbb{R}^2$.
If there exists a classical positive solution $\mathbf{p}$ *of*

$$-\mathbf{D}\mathbf{p}'' - c\mathbf{p}' - (\mathbf{L}+\lambda\mathbf{I})\,\mathbf{p} = \mathbf{0} \textit{ in } \mathbb{R}, \quad (TW_0\,[c,\lambda])$$

then there exists $(\mu,\mathbf{n})\in\mathbb{R}\times\mathsf{K}^{++}$ *such that* $\mathbf{q}:\xi\mapsto e^{-\mu\xi}\mathbf{n}$ *is a classical solution of* $(TW_0\,[c,\lambda])$.

Remark. This is of course to be related with the notions of generalized principal eigenvalue and generalized principal eigenfunction (see Theorem 5.17). The mere existence of $\mathbf{p}$ enforces

$$\lambda_1\left(-\mathbf{D}\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} - c\frac{\mathrm{d}}{\mathrm{d}\xi} - (\mathbf{L}+\lambda\mathbf{I})\right)\geq 0.$$

The following proof is inspired by Berestycki–Hamel–Roques [17, Lemma 3.1].

Proof. Let $\mathbf{p}$ be a classical positive solution of $(TW_0\,[c,\lambda])$.

Let $\mathbf{v}=\left(\frac{p_i'}{p_i}\right)_{i\in[N]}$. By virtue of Arapostathis–Gosh–Marcus's Harnack inequality [5] applied to the operator $\mathbf{D}\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}+c\frac{\mathrm{d}}{\mathrm{d}\xi}+(\mathbf{L}+\lambda\mathbf{I})$, classical elliptic estimates (Gilbarg–Trudinger [80]) and invariance by translation of $(TW_0\,[c,\lambda])$, $\mathbf{v}$ is globally bounded. Let

$$\Lambda_i = \limsup_{\xi\to+\infty} v_i(\xi) \text{ for all } i\in[N],$$

$$\overline{\Lambda} = \max_{i\in[N]}\Lambda_i,$$

so that

$$\left(\limsup_{\xi\to+\infty} v_i(\xi)\right)_{i\in[N]} \leq \overline{\Lambda}\mathbf{1}_{N,1}.$$

Let $(\xi_n)_{n\in\mathbb{N}}\in\mathbb{R}^{\mathbb{N}}$ such that $\xi_n\to+\infty$ and such that there exists $\bar{i}\in[N]$ such that

$$v_{\bar{i}}(\xi_n)\to\overline{\Lambda}.$$

On one hand, let

$$\hat{\mathbf{p}}_n : \xi\mapsto \frac{\mathbf{p}(\xi+\xi_n)}{p_{\bar{i}}(\xi_n)} \text{ for all } n\in\mathbb{N}.$$

Once more by virtue of Arapostathis–Gosh–Marcus's Harnack inequality, the sequence $(\hat{\mathbf{p}}_n)_{n\in\mathbb{N}}$ is locally uniformly bounded. Since all $\hat{\mathbf{p}}_n$ solve $(TW_0\,[c,\lambda])$, by classical elliptic estimates, $(\hat{\mathbf{p}}_n)_{n\in\mathbb{N}}$ converges up to a diagonal extraction as $n\to+\infty$ in $\mathscr{C}^2_{loc}$. Let $\hat{\mathbf{p}}_\infty$ be its limit. Notice by linearity of $(TW_0\,[c,\lambda])$ that $\hat{\mathbf{p}}_\infty$ is in fact smooth and all its derivatives satisfy $(TW_0\,[c,\lambda])$ as well.

On the other hand, let

$$\mathbf{w}_n = \overline{\Lambda}\hat{\mathbf{p}}_n - \hat{\mathbf{p}}'_n \text{ for all } n \in \mathbb{N} \cup \{+\infty\}.$$

Notice the following equality:

$$\mathbf{w}_n(\xi) = \hat{\mathbf{p}}_n(\xi) \circ \left(\overline{\Lambda}\mathbf{1}_{N,1} - \mathbf{v}(\xi + \xi_n)\right) \text{ for all } n \in \mathbb{N} \text{ and } \xi \in \mathbb{R}.$$

Fix $\xi \in \mathbb{R}$. Recalling

$$\left(\limsup_{n\to+\infty} v_i(\xi + \xi_n)\right)_{i\in[N]} \leq \left(\limsup_{\zeta\to+\infty} v_i(\zeta)\right)_{i\in[N]} \leq \overline{\Lambda}\mathbf{1}_{N,1},$$

it follows that for all $\varepsilon > 0$ there exists $n_{\xi,\varepsilon} \in \mathbb{N}$ such that for all $n \geq n_{\xi,\varepsilon}$,

$$\left(\overline{\Lambda} + \varepsilon\right)\mathbf{1}_{N,1} \geq \mathbf{v}(\xi + \xi_n),$$

whence, for all $n \geq n_{\xi,\varepsilon}$,

$$\begin{aligned}\mathbf{w}_n(\xi) &\geq -\varepsilon\left(\sup_{m\geq n_{\xi,\varepsilon}} \hat{p}_{m,i}(\xi)\right)_{i\in[N]} \\ &\geq -\varepsilon\left(\sup_{m\in\mathbb{N}} \hat{p}_{m,i}(\xi)\right)_{i\in[N]},\end{aligned}$$

and consequently, passing to the limit $n \to +\infty$ and then $\varepsilon \to 0$, we obtain the non-negativity of $\mathbf{w}_\infty(\xi)$.

Hence $\mathbf{w}_\infty$ is a nonnegative solution of $(TW_0[c,\lambda])$ satisfying in addition

$$w_{\infty,\bar{i}}(0) = \hat{p}_{\infty,\bar{i}}(0)\left(\overline{\Lambda} - \lim_{n\to+\infty} v_{\bar{i}}(\xi_n)\right) = 0,$$

whence, again by Arapostathis–Gosh–Marcus's Harnack inequality, $\mathbf{w}_\infty$ is in fact the null function.

Consequently, $\overline{\Lambda}\hat{\mathbf{p}}_\infty = \hat{\mathbf{p}}'_\infty$, that is $\hat{\mathbf{p}}_\infty$ has exactly the form

$$\xi \mapsto e^{\overline{\Lambda}\xi}\mathbf{n} \text{ with } \mathbf{n} \in \mathbb{R}^N.$$

Since $\hat{\mathbf{p}}_\infty$ is nonnegative with $\hat{p}_{\infty,\bar{i}}(0) = 1$ by construction, $\mathbf{n} \in \mathsf{K}^+$, and since any nonnegative nonzero solution of $(TW_0[c,\lambda])$ is positive (Corollary 5.10), $\mathbf{n} \in \mathsf{K}^{++}$. The proof is ended with $\mu = -\overline{\Lambda}$. □

For all $\mu \in \mathbb{R}$, the matrix $\mu^2\mathbf{D} + \mathbf{L}$ is essentially nonnegative irreducible. Define $\kappa_\mu = -\lambda_{PF}\left(\mu^2\mathbf{D} + \mathbf{L}\right)$ and $\mathbf{n}_\mu = \mathbf{n}_{PF}\left(\mu^2\mathbf{D} + \mathbf{L}\right)$.

Of course, the interest of the pair $(\kappa_\mu, \mathbf{n}_\mu)$ lies in the preceding lemma: for all $(\mu, \mathbf{n}) \in \mathbb{R}\times\mathsf{K}^{++}$, $\xi \mapsto e^{-\mu\xi}\mathbf{n}$ is a solution of $(TW_0[c])$ if and only if

$$-\mu^2\mathbf{D}\mathbf{n} + \mu c\mathbf{n} - \mathbf{L}\mathbf{n} = \mathbf{0},$$

that is, thanks to the Perron–Frobenius theorem, if and only if $\mu c = -\kappa_\mu$ and $\frac{\mathbf{n}}{|\mathbf{n}|} = \mathbf{n}_\mu$. This most important observation leads naturally to the following study of the equation $c = -\frac{\kappa_\mu}{\mu}$.

Lemma 5.25. *The quantity*

$$c^\star = \min_{\mu>0}\left(-\frac{\kappa_\mu}{\mu}\right)$$

is well-defined and positive.

Let $c \in [0,+\infty)$. In $(-\infty,0)$, the equation $-\frac{\kappa_\mu}{\mu} = c$ admits no solution. In $(0,+\infty)$, it admits exactly:

1. *no solution if $c < c^\star$;*
2. *one solution $\mu_{c^\star} > 0$ if $c = c^\star$;*
3. *two solutions $(\mu_{1,c}, \mu_{2,c})$ if $c > c^\star$, which satisfy moreover*

$$0 < \mu_{1,c} < \mu_{c^\star} < \mu_{2,c}.$$

Remark. $c^\star$ does not depend on $\mathbf{c}$ and is entirely determined by $\mathbf{D}$ and $\mathbf{L}$. It will be the minimal speed of traveling waves and this kind of dependency is strongly reminiscent of the scalar Fisher–KPP case, where $c^\star = 2\sqrt{rd}$. In fact the following proof is mostly a generalization of scalar arguments.

Proof. Of course, $\mu \mapsto -\frac{\kappa_\mu}{\mu}$ is odd in $\mathbb{R}\backslash\{0\}$. It is also positive in $(0,+\infty)$:

$$-\frac{\kappa_\mu}{\mu} = \frac{1}{\mu}\lambda_{PF}\left(\mu^2\mathbf{D}+\mathbf{L}\right) > \frac{1}{\mu}\lambda_{PF}\left(\mathbf{L}\right) > 0.$$

Therefore it is negative in $(-\infty,0)$ and in particular there is no solution of $-\frac{\kappa_\mu}{\mu} = c \geq 0$ in $(-\infty,0)$.

We recall Nussbaum's theorem [122] which proves the convexity of the function $\mu \mapsto \rho\left(\mathbf{A}_\mu\right)$ provided:

— the matrix $\mathbf{A}_\mu$ is irreducible,

— its diagonal entries are convex functions of μ,

— its off-diagonal entries are nonnegative log-convex functions of μ.

These conditions are easily verified for $\mu^2\mathbf{D}+\mathbf{L}$ and $\mu\mathbf{D}+\frac{1}{\mu}\mathbf{L}$ (actually, for all $\mu^{-\gamma}\left(\mu^2\mathbf{D}+\mathbf{L}\right)$ provided $\gamma \in [0,2]$). Their spectral radii being respectively $-\kappa_\mu$ and $-\frac{\kappa_\mu}{\mu}$, these are therefore convex functions of μ. Moreover, Nussbaum's result also proves that these convexities are actually strict. Therefore $\mu \mapsto -\kappa_\mu$ and $\mu \mapsto -\frac{\kappa_\mu}{\mu}$ are strictly convex functions in $(0,+\infty)$.

Now, we investigate the behavior of $-\frac{\kappa_\mu}{\mu}$ as $\mu \to 0$ and $\mu \to +\infty$.

By continuity,

$$\kappa_\mu \to \kappa_0 \text{ as } \mu \to 0,$$

whence $-\frac{\kappa_\mu}{\mu} \to +\infty$ as $\mu \to 0$.

Since $\mu \mapsto -\frac{\kappa_\mu}{\mu}$ is convex and positive, either it is bounded in a neighborhood of $+\infty$ and then it converges to some nonnegative constant, either it is unbounded in a neighborhood of $+\infty$ and then it converges to $+\infty$. Assume that it converges to a finite constant. Notice

$$\lim_{\mu\to+\infty}\frac{1}{\mu^2}\left(\mu^2\mathbf{D}+\mathbf{L}\right) = \mathbf{D}.$$

There exists a family of Perron–Frobenius eigenvectors of $\mu\mathbf{D}+\frac{1}{\mu}\mathbf{L}$, $\left(\mathbf{m}_\mu\right)_{\mu>0}$, normalized so that $\max_{i\in[N]} m_{\mu,i} = 1$ for all $\mu > 0$. Thanks to classical compactness arguments in $\mathbb{R}$ and $\mathbb{R}^N$, we can extract a sequence $\left(\mu_n\right)_{n\in\mathbb{N}}$ such that $\mu_n \to +\infty$, $-\frac{\kappa_{\mu_n}}{\mu_n^2}$ converges to 0 and $\mathbf{m}_{\mu_n}$ converges to some $\mathbf{m} \in \mathsf{K}^+$. We point out that we do not know if $\mathbf{m} \in \mathsf{K}^{++}$, but from the normalizations, we do know that $\mathbf{m} \in \mathsf{K}^+$. Since $\mathbf{m}$ satisfies $\mathbf{Dm} = \mathbf{0}$ and since $\mathbf{D}$ is invertible, we get a contradiction. Thus

$$\lim_{\mu\to+\infty} -\frac{\kappa_\mu}{\mu} = +\infty.$$

Hence $\mu \mapsto -\frac{\kappa_\mu}{\mu}$ is a strictly convex positive function which goes to $+\infty$ as $\mu \to 0$ or $\mu \to +\infty$: it admits necessarily a unique global minimum in $(0,+\infty)$. The quantity $c^\star$ is well-defined.

Define $\mu_{c^\star} > 0$ such that

$$c^\star = -\frac{\kappa_{\mu_{c^\star}}}{\mu_{c^\star}}.$$

The quantity $\mu_{c^\star}$ is uniquely defined by strict convexity. The function $\mu \mapsto -\frac{\kappa_\mu}{\mu}$ is bijective from $(0,\mu_{c^\star})$ to $(c^\star,+\infty)$ and from $(\mu_{c^\star},+\infty)$ to $(c^\star,+\infty)$ as well. This ends the proof. □

Putting together Lemma 5.24 and Lemma 5.25, we get the following important result.

Corollary 5.26. *For all $c \in [0,+\infty)$, the set of nonnegative nonzero classical solutions of $(TW_0\,[c])$ is empty if and only if $c \in [0,c^\star)$.*

We can also get the exact values of c for which $\mathbf{0}$ is an unstable steady state of $(TW_0\,[c])$, in the sense of Lemma 5.18.

Lemma 5.27. *Let $c \in [0,+\infty)$. Then*

$$\lambda_1\left(-\mathbf{D}\frac{d^2}{dx^2} - c\frac{d}{dx} - \mathbf{L}\right) = \sup_{\mu\in\mathbb{R}}\left(\kappa_\mu + \mu c\right).$$

Furthermore:

1. $\sup_{\mu\in\mathbb{R}}\left(\kappa_\mu + \mu c\right) = \max_{\mu\geq 0}\left(\kappa_\mu + \mu c\right)$*;*
2. $\max_{\mu\geq 0}\left(\kappa_\mu + \mu c\right) < 0$ *if and only if* $c < c^\star$*.*

Remark. Just as in the case $c = 0$, it can be shown that, for all $c \in [0,+\infty)$,

$$R \mapsto \lambda_{1,Dir}\left(-\mathbf{D}\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} - c\frac{\mathrm{d}}{\mathrm{d}\xi} - \mathbf{L}, (-R,R)\right)$$

is a decreasing homeomorphism from $(0,+\infty)$ onto $\left(\lambda_1\left(-\mathbf{D}\frac{\mathrm{d}^2}{\mathrm{d}x^2} - c\frac{\mathrm{d}}{\mathrm{d}x} - \mathbf{L}\right), +\infty\right)$.

Proof. The fact that $\sup_{\mu\in\mathbb{R}}\left(\kappa_\mu + \mu c\right)$ is finite and actually a maximum attained in $[0,+\infty)$ is a direct consequence of:

— the evenness of $\mu \mapsto \kappa_\mu$ (whence, for all $\mu > 0$, $\kappa_{-\mu} + (-\mu)\,c < \kappa_\mu + \mu c$);

— $\kappa_0 < 0$;

— $\frac{\kappa_\mu}{\mu} + c \to -\infty$ as $\mu \to +\infty$ (see the proof of Lemma 5.25).

In addition, the sign of this maximum depending on the sign $c - c^\star$ is given by Lemma 5.25.

Hence it only remains to prove

$$\lambda_1\left(-\mathbf{D}\frac{\mathrm{d}^2}{\mathrm{d}x^2} - c\frac{\mathrm{d}}{\mathrm{d}x} - \mathbf{L}\right) = \max_{\mu\geq 0}\left(\kappa_\mu + \mu c\right).$$

To do so, we use and adapt a well-known strategy of proof (see for instance Nadin [116]).

We recall from Theorem 5.17 the definition of the generalized principal eigenvalue:

$$\lambda_1\left(-\mathbf{D}\frac{\mathrm{d}^2}{\mathrm{d}x^2}-c\frac{\mathrm{d}}{\mathrm{d}x}-\mathbf{L}\right)=\sup\left\{\lambda\in\mathbb{R}\mid\exists\mathbf{n}\in\mathscr{C}^2\left(\mathbb{R},\mathsf{K}^{++}\right)\quad-\mathbf{D}\mathbf{n}''-c\mathbf{n}'-\mathbf{L}\mathbf{n}\geq\lambda\mathbf{n}\right\}.$$

Also, there exists a generalized principal eigenfunction. We recall from Lemma 5.24 that if there exists a generalized principal eigenfunction, then there exists a generalized principal eigenfunction of the form $\xi\mapsto\mathrm{e}^{-\mu^\star\xi}\mathbf{m}$ with some constant $\mu^\star\geq 0$ and $\mathbf{m}\in\mathsf{K}^{++}$.

Now, $(\mu^\star,\mathbf{m})\in[0,+\infty)\times\mathsf{K}^{++}$ satisfies

$$-\left(\mu^\star\right)^2\mathbf{D}\mathbf{m}+c\mu^\star\mathbf{m}-\mathbf{L}\mathbf{m}=\lambda_1\left(-\mathbf{D}\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}-c\frac{\mathrm{d}}{\mathrm{d}\xi}-\mathbf{L}\right)\mathbf{m},$$

that is

$$-\left(\left(\mu^\star\right)^2\mathbf{D}+\mathbf{L}\right)\mathbf{m}=\left(\lambda_1\left(-\mathbf{D}\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}-c\frac{\mathrm{d}}{\mathrm{d}\xi}-\mathbf{L}\right)-c\mu^\star\right)\mathbf{m},$$

or in other words

$$\lambda_1\left(-\mathbf{D}\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}-c\frac{\mathrm{d}}{\mathrm{d}\xi}-\mathbf{L}\right)=\kappa_{\mu^\star}+c\mu^\star\text{ and }\frac{\mathbf{m}}{|\mathbf{m}|}=\mathbf{n}_{\mu^\star}.$$

Finally, the suitable test function to verify

$$\lambda_1\left(-\mathbf{D}\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}-c\frac{\mathrm{d}}{\mathrm{d}\xi}-\mathbf{L}\right)\geq\kappa_\mu+\mu c\text{ for all }\mu\geq 0$$

is of course $\mathbf{v}_\mu:\xi\mapsto\mathrm{e}^{-\mu\xi}\mathbf{n}_\mu$ itself, which satisfies precisely

$$-\mathbf{D}\mathbf{v}_\mu''-c\mathbf{v}_\mu'-\mathbf{L}\mathbf{v}_\mu=\left(\kappa_\mu+\mu c\right)\mathbf{v}_\mu.$$

□

Corollary 5.28. *The quantity $c^\star$ is characterized by*

$$\begin{aligned}c^\star&=\sup\left\{c\geq 0\mid\lambda_1\left(-\mathbf{D}\frac{d^2}{d\xi^2}-c\frac{d}{d\xi}-\mathbf{L}\right)<0\right\}\\&=\inf\left\{c\geq 0\mid\lambda_1\left(-\mathbf{D}\frac{d^2}{d\xi^2}-c\frac{d}{d\xi}-\mathbf{L}\right)>0\right\}.\end{aligned}$$

5.6.2 Qualitative properties of the traveling solutions

Thanks to Lemma 5.24 and Corollary 5.26, we are now in position to establish a few interesting properties that have direct consequences but will also be used at the end of the construction of the traveling waves.

Lemma 5.29. *Let $c\in[0,+\infty)$ and $\mathbf{p}$ be a bounded nonnegative nonzero classical solution of $(TW[c])$.*

If $\left(\liminf_{\xi\to+\infty}p_i(\xi)\right)_{i\in[N]}\in\partial\mathsf{K}$, then $c\geq c^\star$.

Remark. The following proof is analogous to that of Berestycki–Nadin–Perthame–Ryzhik [19, Lemma 3.8] for the non-local KPP equation.

Proof. Let $(\zeta_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ such that, as $n \to +\infty$, $\zeta_n \to +\infty$ and at least one component of $(\mathbf{p}(\zeta_n))_{n\in\mathbb{N}}$ converges to 0. Define

$$\mathbf{p}_n : \xi \mapsto \mathbf{p}(\xi + \zeta_n)$$

and observe that $\mathbf{p}_n$ satisfies $(TW[c])$ as well. By virtue of Arapostathis–Gosh–Marcus's Harnack inequality [5] applied to the linear operator

$$\mathbf{D}\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} + c\frac{\mathrm{d}}{\mathrm{d}\xi} + (\mathbf{L} - \mathrm{diag}(\mathbf{c}[\mathbf{p}_n])),$$

classical elliptic estimates (Gilbarg–Trudinger [80]), $(\mathbf{p}_n)_{n\in\mathbb{N}}$ converges up to a diagonal extraction in $\mathscr{C}^2_{loc}$ to $\mathbf{0}$. This proves that there is no limit point of $\mathbf{p}$ at $+\infty$ in $\partial\mathsf{K}\setminus\{\mathbf{0}\}$.

Next, define

$$\tilde{\mathbf{p}}_n : \xi \mapsto \frac{\mathbf{p}(\xi + \zeta_n)}{|\mathbf{p}(\zeta_n)|}$$

and notice, again by Arapostathis–Gosh–Marcus's Harnack inequality, that $(\tilde{\mathbf{p}}_n)_{n\in\mathbb{N}}$ is locally uniformly bounded. Since, for all $n \in \mathbb{N}$, $\tilde{\mathbf{p}}_n$ solves

$$-\mathbf{D}\tilde{\mathbf{p}}_n'' - c\tilde{\mathbf{p}}_n' = \mathbf{L}\tilde{\mathbf{p}}_n - \mathbf{c}[\mathbf{p}_n] \circ \tilde{\mathbf{p}}_n,$$

with, thanks to the fact that $\mathbf{c}$ vanishes at $\mathbf{0}$ (H_3), $\mathbf{c}[\mathbf{p}_n] \to \mathbf{0}$ locally uniformly, up to extraction $(\tilde{\mathbf{p}}_n)_{n\in\mathbb{N}}$ converges in $\mathscr{C}^2_{loc}$ to a nonnegative solution $\tilde{\mathbf{p}}$ of $(TW_0[c])$. Since $\tilde{\mathbf{p}}_n(0) \in \mathsf{S}^{++}(\mathbf{0}, 1)$ for all $n \in \mathbb{N}$, $\tilde{\mathbf{p}}$ is nonnegative nonzero, whence positive (Corollary 5.10).

Now, from Corollary 5.26, we deduce indeed that $c \geq c^\star$. □

This result implies the nonexistence half of Theorem 5.5 1.

Corollary 5.30. *For all $c \in [0, c^\star)$, there is no traveling wave solution of (E_{KPP}) with speed c.*

Now, with Proposition 5.14, $c \geq c^\star > 0$ and the fact that $(t, x) \mapsto \mathbf{p}(x - ct)$ solves (E_{KPP}), we can straightforwardly derive the uniform upper bound Theorem 5.5 2, which is interestingly independent of c.

Corollary 5.31. *All profiles $\mathbf{p}$ satisfy*

$$\mathbf{p} \leq \mathbf{g}(0) \quad \text{in } \mathbb{R}.$$

Subsequently, using Proposition 5.20 and again $c \geq c^\star > 0$ and the fact that $(t, x) \mapsto \mathbf{p}(x - ct)$ solves (E_{KPP}), we get Theorem 5.5 3, independent of c as well.

Corollary 5.32. *All profiles $\mathbf{p}$ satisfy*

$$\left(\liminf_{\xi\to-\infty} p_i(\xi)\right)_{i\in[N]} \geq \nu \mathbf{1}_{N,1}.$$

Now, we establish Theorem 5.5 4. Its proof is actually mostly a repetition of that of Lemma 5.24.

Proposition 5.33. *Let $(\mathbf{p}, c)$ be a traveling wave solution of (E_{KPP}).*

Then there exists $\overline{\xi} \in \mathbb{R}$ such that $\mathbf{p}$ is component-wise decreasing in $[\overline{\xi}, +\infty)$.

Proof. Let $\mathbf{v} = \left(\frac{p_i'}{p_i}\right)_{i\in[N]}$. By virtue of Arapostathis–Gosh–Marcus's Harnack inequality [5], classical elliptic estimates (Gilbarg–Trudinger [80]) and invariance by translation of $(TW\,[c])$, $\mathbf{v}$ is globally bounded. Define for all $i \in [N]$

$$\Lambda_i = \limsup_{\xi\to+\infty} v_i\left(\xi\right).$$

Let $\overline{\Lambda} = \max_{i\in[N]} \Lambda_i$, so that

$$\left(\limsup_{\xi\to+\infty} v_i\left(\xi\right)\right)_{i\in[N]} \leq \overline{\Lambda}\mathbf{1}_{N,1}.$$

Let $\left(\xi_n\right)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ such that $\xi_n \to +\infty$ and such that there exists $\overline{i} \in [N]$ such that

$$v_{\overline{i}}\left(\xi_n\right) \to \overline{\Lambda} \text{ as } n \to +\infty.$$

Let

$$\hat{\mathbf{p}}_n : \xi \mapsto \frac{\mathbf{p}\left(\xi + \xi_n\right)}{p_{\overline{i}}\left(\xi_n\right)} \text{ for all } n \in \mathbb{N}.$$

and notice, again by Arapostathis–Gosh–Marcus's Harnack inequality, that $\left(\hat{\mathbf{p}}_n\right)_{n\in\mathbb{N}}$ is locally uniformly bounded. Since, for all $n \in \mathbb{N}$, $\hat{\mathbf{p}}_n$ solves

$$-\mathbf{D}\hat{\mathbf{p}}_n'' - c\hat{\mathbf{p}}_n' = \mathbf{L}\hat{\mathbf{p}}_n - \mathbf{c}\left[p_{\overline{i}}\left(\xi_n\right)\hat{\mathbf{p}}_n\right] \circ \hat{\mathbf{p}}_n,$$

and, thanks to the fact that $\mathbf{c}$ vanishes at $\mathbf{0}$ (H_3) and the asymptotic behavior of $\mathbf{p}$ at $+\infty$, $\mathbf{c}\left[p_{\overline{i}}\left(\xi_n\right)\hat{\mathbf{p}}_n\right]$ converges locally uniformly to $\mathbf{0}$ as $n \to +\infty$, up to a diagonal extraction process, $\left(\hat{\mathbf{p}}_n\right)_{n\in\mathbb{N}}$ converges in $\mathscr{C}^2_{loc}$ to a nonnegative solution $\hat{\mathbf{p}}_\infty$ of $(TW_0\,[c])$.

Now we repeat the second part of the proof of Lemma 5.24 and we deduce in the end from Lemma 5.25 that $\hat{\mathbf{p}}_\infty$ has exactly the form

$$\xi \mapsto A\mathrm{e}^{-\mu_c\xi}\mathbf{n}_{\mu_c},$$

with $\mu_c \in \{\mu_{1,c}, \mu_{2,c}\}$ if $c > c^\star$, $\mu_c = \mu_{c^\star}$ if $c = c^\star$, $A > 0$ and, most importantly, with $\mu_c = -\overline{\Lambda}$.

Thus $\overline{\Lambda} < 0$. This implies that there exists $\overline{\xi} \in \mathbb{R}$ such that, for all $\xi \geq \overline{\xi}$,

$$\mathbf{v}\left(\xi\right) \leq -\frac{\left|\overline{\Lambda}\right|}{2}\mathbf{1}_{N,1},$$

whence, by positivity of $\mathbf{p}$,

$$\mathbf{p}'\left(\xi\right) \leq -\frac{\left|\overline{\Lambda}\right|}{2}\mathbf{p}\left(\xi\right).$$

The right-hand side being negative, $\mathbf{p}$ is component-wise decreasing indeed. □

Lemma 5.34. *Let $c \in [0, +\infty)$ and $\mathbf{p}$ be a bounded nonnegative nonzero classical solution of $(TW\,[c])$.*

If $\left(\liminf_{\xi\to+\infty} p_i\left(\xi\right)\right)_{i\in[N]} \in \partial\mathsf{K}$, then $\lim_{\xi\to+\infty}\mathbf{p}\left(\xi\right) = \mathbf{0}$.

Proof. Let $(\zeta_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ such that, as $n \to +\infty$, $\zeta_n \to +\infty$ and at least one component of $(\mathbf{p}(\zeta_n))_{n\in\mathbb{N}}$ converges to 0. The proof of Lemma 5.29 shows that $(\mathbf{p}_n)_{n\in\mathbb{N}}$, defined by $\mathbf{p}_n : \xi \mapsto \mathbf{p}(\xi + \zeta_n)$, converges up to extraction in $\mathscr{C}^2_{loc}$ to $\mathbf{0}$.

Now, defining

$$\mathbf{v}_n : \xi \mapsto \left(\frac{p'_{n,i}(\xi)}{p_{n,i}(\xi)}\right)_{i\in[N]},$$

$$\Lambda_i = \limsup_{n\to+\infty} \max_{[-1,1]} v_{n,i},$$

$$\overline{\Lambda} = \max_{i\in[N]} \Lambda_i,$$

$$\overline{i} \in [N] \text{ such that } \Lambda_{\overline{i}} = \overline{\Lambda},$$

and $(n_m)_{m\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ an increasing sequence such that $v_{n_m,\overline{i}}(0) \to \overline{\Lambda}$ as $m \to +\infty$, we can repeat once more the argument of the proof of Lemma 5.24 and obtain

$$\overline{\Lambda}\hat{\mathbf{p}}_\infty = \hat{\mathbf{p}}'_\infty \text{ in } (-1,1)$$

(notice that, contrarily to the proof of Lemma 5.24 where this equality was proved in $\mathbb{R}$, here it only holds locally). This brings forth $\overline{\Lambda} = -\mu_c < 0$, as in the proof of Proposition 5.33, whence $\mathbf{p}_n$ is component-wise decreasing in $[-1,1]$ provided n is large enough.

Now, assuming by contradiction

$$\left(\limsup_{\xi\to+\infty} p_i(\xi)\right)_{i\in[N]} \in \mathsf{K}^+,$$

that is

$$\left(\limsup_{\xi\to+\infty} p_i(\xi)\right)_{i\in[N]} \in \mathsf{K}^{++},$$

we deduce from the $\mathscr{C}^1$ regularity of $\mathbf{p}$ that, for any $i \in [N]$, there exists a sequence $(\zeta'_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ such that:

— $\zeta'_n \to +\infty$ as $n \to +\infty$,

— $p_i(\zeta'_n)$ is a local minimum of p_i,

— $p_i(\zeta'_n) \to 0$ as $n \to +\infty$.

Since this directly contradicts the preceding argument, we get indeed

$$\left(\limsup_{\xi\to+\infty} p_i(\xi)\right)_{i\in[N]} = \mathbf{0} = \left(\liminf_{\xi\to+\infty} p_i(\xi)\right)_{i\in[N]}.$$

□

Lemma 5.35. *Let $c \in [0,+\infty)$. There exists $\eta_c > 0$ such that, for all bounded nonnegative classical solutions $\mathbf{p}$ of $(TW[c])$, exactly one of the following properties holds:*

1. $\lim_{\xi\to+\infty} \mathbf{p}(\xi) = \mathbf{0}$;
2. $\left(\inf_{(0,+\infty)} p_i\right)_{i\in[N]} \geq \eta_c \mathbf{1}_{N,1}$.

Remark. The following proof is again analogous to that of Berestycki–Nadin–Perthame–Ryzhik [19, Lemma 3.4] for the non-local KPP equation.

Proof. Recall from Corollary 5.10 and Lemma 5.34 that $\left(\inf_{(0,+\infty)} p_i\right)_{i\in[N]} \in \partial\mathsf{K}$ if and only if $\lim_{\xi\to+\infty} \mathbf{p}(\xi) = \mathbf{0}$. Hence, defining Σ as the set of all bounded nonnegative classical solutions $\mathbf{p}$ of $(TW[c])$ such that

$$\min_{i\in[N]} \inf_{(0,+\infty)} p_i > 0,$$

this set containing at least one positive constant vector by virtue of Theorem 5.4, it only remains to show the positivity of

$$\eta_c = \inf\left\{\min_{i\in[N]} \inf_{(0,+\infty)} p_i \mid \mathbf{p}\in\Sigma\right\}.$$

We assume by contradiction the existence of a sequence $(\mathbf{p}_n)_{n\in\mathbb{N}} \in \Sigma^{\mathbb{N}}$ such that

$$\lim_{n\to+\infty} \min_{i\in[N]} \inf_{(0,+\infty)} p_{n,i} = 0.$$

For all $n\in\mathbb{N}$, define

$$\beta_n = \min_{i\in[N]} \inf_{(0,+\infty)} p_{n,i} > 0,$$

fix $\xi_n \in (0,+\infty)$ such that

$$\min_{i\in[N]} p_{n,i}(\xi_n) \in \left[\beta_n, \beta_n + \frac{1}{n}\right],$$

and define finally

$$\mathbf{v}_n : \xi \mapsto \frac{1}{\beta_n}\mathbf{p}_n(\xi+\xi_n).$$

By virtue of Arapostathis–Gosh–Marcus's Harnack inequality [5], classical elliptic estimates (Gilbarg–Trudinger [80]) and invariance by translation of $(TW[c])$, $(\mathbf{v}_n)_{n\in\mathbb{N}}$ is locally uniformly bounded and, up to a diagonal extraction process, converges in $\mathscr{C}^2_{loc}$ to some bounded limit $\mathbf{v}_\infty$. As in the proof of Lemma 5.24, it is easily verified that $\mathbf{v}_\infty$ is a bounded positive classical solution of $(TW_0[c])$. Furthermore, by definition of $(\mathbf{v}_n)_{n\in\mathbb{N}}$,

$$\mathbf{v}_\infty \geq \mathbf{1}_{N,1} \text{ in } (0,+\infty).$$

Repeating once more the argument of the proof of Lemma 5.24, we deduce that $\mathbf{v}_\infty$ is component-wise decreasing in a neighborhood of $+\infty$. Thus its limit at $+\infty$, say $\mathbf{m}\geq\mathbf{1}_{N,1}$, is well-defined. By classical elliptic estimates, $\mathbf{m}$ satisfies $\mathbf{Lm}=\mathbf{0}$, which obviously contradicts $\lambda_{PF}(\mathbf{L})>0$. □

5.6.3 Existence of traveling waves

This whole subsection is devoted to the adaptation of a proof of existence due to Berestycki, Nadin, Perthame and Ryzhik [19] and originally applied to the non-local KPP equation.

Remark. There is a couple of slight mistakes in the aforementioned proof.

1. Using the notations of [19], the sub-solution is defined as $\overline{r}_c = \max(0, r_c)$, with r_c chosen so that
$$-cr_c' \leq r_c'' + \mu r_c - \mu \overline{q}_c (\phi \star \overline{q}_c)$$
and it is claimed that $\overline{r}_c$ satisfies as well this inequality, in the distributional sense. This is false: in an interval where $\overline{r}_c = 0$, we have
$$-c\overline{r}_c' - \overline{r}_c'' - \mu \overline{r}_c = 0 > -\mu \overline{q}_c (\phi \star \overline{q}_c).$$
As we will show, the correct sub-solution is $\overline{r}_c = \max(0, r_c)$ with r_c chosen so that
$$-cr_c' \leq r_c'' + \mu r_c - \mu r_c (\phi \star \overline{q}_c).$$
Fortunately, the function r_c constructed by the authors satisfies this inequality as well.
2. Later on, Φ_a is defined as the mapping which maps u_0 to the solution of
$$-cu' = u'' + \mu u_0 (1 - \phi \star u_0).$$
This mapping does not leave invariant the set of functions R_a defined with the correct sub-solution. It is necessary to change Φ_a and to define it as the mapping which maps u_0 to the solution of
$$-cu' = u'' + \mu u (1 - \phi \star u_0).$$
Consequently, in order to establish that the set of functions R_a is invariant by Φ_a, the elliptic maximum principle is applied not to $u \mapsto -cu' - u''$ but to
$$u \mapsto -u'' - cu' - \mu u$$
on one hand and to
$$u \mapsto -u'' - cu' - \mu (1 - \phi \star \overline{q}_c) u$$
on the other hand.

During the first three subsubsections, we fix $c > c^\star$.

5.6.3.1 Super-solution

We will use $\overline{\mathbf{p}} : \xi \mapsto \mathrm{e}^{-\mu_{1,c}\xi} \mathbf{n}_{\mu_{1,c}}$ as a super-solution (recall from Lemma 5.25 that it is a solution of $(TW_0 [c])$).

5.6.3.2 Sub-solution

Proposition 5.36. *There exist $\overline{\varepsilon} > 0$ such that, for any $\varepsilon \in (0, \overline{\varepsilon})$, there exists $A_\varepsilon \in (0, +\infty)$ such that the function*
$$\underline{\mathbf{p}} : \xi \mapsto \left(\max\left(e^{-\mu_{1,c}\xi} n_{\mu_{1,c},i} - A_\varepsilon e^{-(\mu_{1,c}+\varepsilon)\xi} n_{\mu_{1,c}+\varepsilon,i}, 0\right)\right)_{i\in[N]},$$
satisfies
$$-\mathbf{D}\underline{\mathbf{p}}'' - c\underline{\mathbf{p}}' - \mathbf{L}\underline{\mathbf{p}} \leq -\mathbf{c}[\overline{\mathbf{p}}] \circ \underline{\mathbf{p}} \text{ in } \mathscr{H}^{-1}\left(\mathbb{R}, \mathbb{R}^N\right).$$

Remark. Notice that, in the right-hand side of the inequality above, we find $\mathbf{c}[\overline{\mathbf{p}}]$ and not $\mathbf{c}[\underline{\mathbf{p}}]$. This is of course related to the lack of comparison principle for (E_{KPP}).

During the forthcoming quite technical proof, in order to ease the reading, we denote $\langle \bullet, \bullet \rangle_1$ and $\langle \bullet, \bullet \rangle_N$ the duality pairings of $\mathscr{H}^1(\mathbb{R},\mathbb{R})$ and $\mathscr{H}^1(\mathbb{R},\mathbb{R}^N)$ respectively, the latter being of course defined by:

$$\langle \mathbf{f}, \mathbf{g} \rangle_{\mathscr{H}^{-1}(\mathbb{R},\mathbb{R}^N)\times\mathscr{H}^1(\mathbb{R},\mathbb{R}^N)} = \sum_{i=1}^{N} \langle f_i, g_i \rangle_{\mathscr{H}^{-1}(\mathbb{R})\times\mathscr{H}^1(\mathbb{R})}.$$

The speed c being fixed, we also omit the subscript c in the notations $\mu_{1,c}$ and $\mu_{2,c}$.

Proof. For the moment, let $A, \varepsilon > 0$ (they will be made precise during the course of the proof) and define

$$\begin{aligned}
\mathbf{v} &: \xi \mapsto \mathrm{e}^{-\mu_1 \xi}\mathbf{n}_{\mu_1} - A\mathrm{e}^{-(\mu_1+\varepsilon)\xi}\mathbf{n}_{\mu_1+\varepsilon},\\
\underline{\mathbf{p}} &: \xi \mapsto \left(\max\left(\mathrm{e}^{-\mu_1\xi} n_{\mu_1,i} - A\mathrm{e}^{-(\mu_1+\varepsilon)\xi} n_{\mu_1+\varepsilon,i}, 0\right)\right)_{i\in[N]},\\
\Xi_+ &= \underline{\mathbf{p}}^{-1}\left(\mathsf{K}^{++}\right),\\
\Xi_0 &= \underline{\mathbf{p}}^{-1}(\mathbf{0}),\\
\Xi_\# &= \mathbb{R}\setminus(\Xi_+ \cup \Xi_0).
\end{aligned}$$

Notice that $\Xi_\#$ is a connected compact set.

Fix a positive test function $\varphi \in \mathscr{H}^1(\mathbb{R}, \mathsf{K}^{++})$. We have to verify that

$$\left\langle -\mathbf{D}\underline{\mathbf{p}}'' - c\underline{\mathbf{p}}' - \mathbf{L}\underline{\mathbf{p}}, \varphi \right\rangle_N \leq \left\langle -\mathbf{c}[\overline{\mathbf{p}}] \circ \underline{\mathbf{p}}, \varphi \right\rangle_N.$$

To this end, we distinguish three cases: $\mathrm{supp}\varphi \subset \Xi_+$, $\mathrm{supp}\varphi \subset \Xi_0$ and $\mathrm{supp}\varphi \cap \Xi_\# \neq \emptyset$. The case $\mathrm{supp}\varphi \subset \Xi_0$ is trivial, with the inequality above satisfied in the classical sense.

Regarding the case $\mathrm{supp}\varphi \subset \Xi_+$, we only have to verify the inequality in the classical sense in Ξ_+ for the regular function $\mathbf{v}$.

Fix temporarily $\xi \in \Xi_+$. We have

$$-\mathbf{D}\mathbf{v}''(\xi) - c\mathbf{v}'(\xi) - \mathbf{L}\mathbf{v}(\xi) = A\mathrm{e}^{-(\mu_1+\varepsilon)\xi}\left((\mu_1+\varepsilon)^2\mathbf{D} - c(\mu_1+\varepsilon)\mathbf{I} + \mathbf{L}\right)\mathbf{n}_{\mu_1+\varepsilon},$$

$$(-\mathbf{c}[\overline{\mathbf{p}}]\circ\mathbf{v})(\xi) = -\mathrm{e}^{-\mu_1\xi}\mathbf{c}\left(\mathrm{e}^{-\mu_1\xi}\mathbf{n}_{\mu_1}\right)\circ\left(\mathbf{n}_{\mu_1} - A\mathrm{e}^{-\varepsilon\xi}\mathbf{n}_{\mu_1+\varepsilon}\right).$$

From

$$\left((\mu_1+\varepsilon)^2\mathbf{D} + \mathbf{L}\right)\mathbf{n}_{\mu_1+\varepsilon} = -\kappa_{\mu_1+\varepsilon}\mathbf{n}_{\mu_1+\varepsilon},$$

$$-c(\mu_1+\varepsilon)\mathbf{n}_{\mu_1+\varepsilon} = \frac{\kappa_{\mu_1}}{\mu_1}(\mu_1+\varepsilon)\mathbf{n}_{\mu_1+\varepsilon},$$

and the following direct consequence of the nonnegativity of $\mathbf{c}$ on K (H_2),

$$-\mathbf{c}\left(\mathrm{e}^{-\mu_1\xi}\mathbf{n}_{\mu_1}\right)\circ\left(\mathbf{n}_{\mu_1} - A\mathrm{e}^{-\varepsilon\xi}\mathbf{n}_{\mu_1+\varepsilon}\right) \geq -\mathbf{c}\left(\mathrm{e}^{-\mu_1\xi}\mathbf{n}_{\mu_1}\right)\circ\mathbf{n}_{\mu_1},$$

it follows that it suffices to find A and ε such that

$$A\mathrm{e}^{-\varepsilon\xi}(\mu_1+\varepsilon)\left(-\frac{\kappa_{\mu_1+\varepsilon}}{\mu_1+\varepsilon} + \frac{\kappa_{\mu_1}}{\mu_1}\right)\mathbf{n}_{\mu_1+\varepsilon} \leq -\mathbf{c}\left(\mathrm{e}^{-\mu_1\xi}\mathbf{n}_{\mu_1}\right)\circ\mathbf{n}_{\mu_1}.$$

The right-hand side above being nonnegative ($\mu \mapsto \frac{\kappa_\mu}{\mu}$ is positive and convex in $(0,+\infty)$, as detailed in the proof of Lemma 5.25), it follows clearly that such an inequality is never satisfied if $\mu_1+\varepsilon>\mu_2$, whence a first necessary condition on ε is $\varepsilon\leq\mu_2-\mu_1$ (notice that if $\varepsilon=\mu_2-\mu_1$, then the inequality above holds if and only if $\mathbf{c}\left(\mathrm{e}^{-\mu_1\xi}\mathbf{n}_{\mu_1}\right)=\mathbf{0}$, which is in general not true). Thus from now on we assume $\varepsilon<\mu_2-\mu_1$. This ensures that $\frac{\kappa_{\mu_1+\varepsilon}}{\mu_1+\varepsilon}-\frac{\kappa_{\mu_1}}{\mu_1}>0$, whence we now search for A and ε such that

$$A\mathbf{n}_{\mu_1+\varepsilon}>\frac{\mathrm{e}^{\varepsilon\xi}}{(\mu_1+\varepsilon)\left(\frac{\kappa_{\mu_1+\varepsilon}}{\mu_1+\varepsilon}-\frac{\kappa_{\mu_1}}{\mu_1}\right)}\mathbf{c}\left(\mathrm{e}^{-\mu_1\xi}\mathbf{n}_{\mu_1}\right)\circ\mathbf{n}_{\mu_1}.$$

Define $\overline{\xi}=\min\Xi_{\#}$, so that any $\xi\in\Xi_+$ satisfies necessarily $\xi>\overline{\xi}$. Remark that there exists $\overline{i}\in[N]$ such that

$$\overline{\xi}=\frac{1}{\varepsilon}\left(\ln A+\ln\left(\frac{n_{\mu_1+\varepsilon,\overline{i}}}{n_{\mu_1,\overline{i}}}\right)\right).$$

Now, defining $\alpha:\xi\mapsto\mathrm{e}^{-\mu_1\xi}$, if

$$A\geq\max_{i\in[N]}\left(\frac{n_{\mu_1+\varepsilon,i}}{n_{\mu_1,i}}\right),$$

then $\overline{\xi}\geq 0$ and $\alpha(\xi)\leq 1$ in $\left(\overline{\xi},+\infty\right)$. Moreover, we have

$$\mathrm{e}^{\varepsilon\xi}=(\alpha(\xi))^{-\frac{\varepsilon}{\mu_1}},$$

whence, for all $i\in[N]$,

$$\mathrm{e}^{\varepsilon\xi}c_i\left(\mathrm{e}^{-\mu_1\xi}\mathbf{n}_{\mu_1}\right)=\frac{c_i\left(\alpha(\xi)\mathbf{n}_{\mu_1}\right)}{(\alpha(\xi))^{\frac{\varepsilon}{\mu_1}}},$$

and from the $\mathscr{C}^1$ regularity of $\mathbf{c}$ as well as the fact that it vanishes at $\mathbf{0}$ (H_3), the above function of ξ is globally bounded in $\left(\overline{\xi},+\infty\right)$, provided $\frac{\varepsilon}{\mu_1}\leq 1$, by the positive constant

$$\begin{aligned}M_i&=\sup_{\xi\in\left(\overline{\xi},+\infty\right)}\frac{c_i\left(\alpha(\xi)\mathbf{n}_{\mu_1}\right)}{\alpha(\xi)}\\&=\sup_{\alpha\in(0,1)}\frac{c_i\left(\alpha\mathbf{n}_{\mu_1}\right)}{\alpha}.\end{aligned}$$

Subsequently, if A and ε satisfy also

$$\varepsilon\leq\mu_1,$$

$$A\geq\max_{i\in[N]}\left(\frac{M_i n_{\mu_1,i}}{(\mu_1+\varepsilon)\left(\frac{\kappa_{\mu_1+\varepsilon}}{\mu_1+\varepsilon}-\frac{\kappa_{\mu_1}}{\mu_1}\right)n_{\mu_1+\varepsilon,i}}\right),$$

then the inequality is established indeed in Ξ_+. Hence we define

$$\overline{\varepsilon}=\min\left(\mu_2-\mu_1,\mu_1\right)$$

and, for any $\varepsilon\in(0,\overline{\varepsilon})$,

$$A_\varepsilon=\max_{i\in[N]}\max\left(\frac{n_{\mu_1+\varepsilon,i}}{n_{\mu_1,i}},\frac{M_i n_{\mu_1,i}}{(\mu_1+\varepsilon)\left(\frac{\kappa_{\mu_1+\varepsilon}}{\mu_1+\varepsilon}-\frac{\kappa_{\mu_1}}{\mu_1}\right)n_{\mu_1+\varepsilon,i}}\right)$$

and we assume from now on $\varepsilon \in (0, \overline{\varepsilon})$ and $A = A_\varepsilon$.

Let us point out here a fact which is crucial for the next step: choosing $\overline{\xi} = \min \Xi_\#$ instead of $\overline{\xi} = \max \Xi_\#$ (which might seem more natural at first view) implies that the differential inequality

$$-\mathbf{D}\mathbf{v}'' - c\mathbf{v}' - \mathbf{L}\mathbf{v} \leq -\mathbf{c}\left[\overline{\mathbf{p}}\right] \circ \mathbf{v}$$

holds classically in $\Xi_\# \cup \Xi_+$.

To conclude, let us verify the case $\operatorname{supp}\varphi \cap \Xi_\# \neq \emptyset$. In order to ease the following computations, we actually assume $\varphi \in \mathscr{D}\left(\mathbb{R}, \mathbb{R}^N\right)$ (the result with $\varphi \in \mathscr{H}^1\left(\mathbb{R}, \mathbb{R}^N\right)$ can be recovered as usual by density). By definition,

$$\left\langle -\mathbf{D}\underline{\mathbf{p}}'' - c\underline{\mathbf{p}}' - \mathbf{L}\underline{\mathbf{p}} + \mathbf{c}\left[\overline{\mathbf{p}}\right] \circ \underline{\mathbf{p}}, \varphi \right\rangle_N = \sum_{i=1}^N \left\langle -d_i \underline{p}_i'' - c\underline{p}_i' - \sum_{j=1}^N l_{i,j}\underline{p}_j + c_i\left[\overline{\mathbf{p}}\right]\underline{p}_i, \varphi_i \right\rangle_1.$$

Fix $i \in [N]$ and define $\xi_{0,i}$ as the unique element of $v_i^{-1}(\{0\})$ and

$$\Psi_i = \left\langle -d_i \underline{p}_i'' - c\underline{p}_i' - \sum_{j=1}^N l_{i,j}\underline{p}_j + c_i\left[\overline{\mathbf{p}}\right]\underline{p}_i, \varphi_i \right\rangle_1.$$

Classical integrations by parts yield

$$\int_{\mathbb{R}} \underline{p}_i''\varphi_i = \int_{\xi_{0,i}}^{+\infty} v_i''\varphi_i + v_i'(\xi_{0,i})\varphi_i(\xi_{0,i}) \geq \int_{\xi_{0,i}}^{+\infty} v_i''\varphi_i,$$

$$\int_{\mathbb{R}} \underline{p}_i'\varphi_i = \int_{\xi_{0,i}}^{+\infty} v_i'\varphi_i,$$

whence

$$\Psi_i \leq \int_{\xi_{0,i}}^{+\infty} \left(-d_i v_i'' - cv_i' + c_i\left[\overline{\mathbf{p}}\right] v_i\right)\varphi_i - \sum_{j=1}^N l_{i,j}\int_{\xi_{0,j}}^{+\infty} v_j\varphi_i.$$

As was pointed out previously, from the construction of ε and A, we know that

$$-\mathbf{D}\mathbf{v}'' - c\mathbf{v}' + \mathbf{c}\left[\overline{\mathbf{p}}\right] \circ \mathbf{v} \leq \mathbf{L}\mathbf{v} \text{ in } \Xi_\#,$$

whence, with $J_i = \{j \in [N] \mid \xi_{0,j} < \xi_{0,i}\}$,

$$\Psi_i \leq -\sum_{j \in J_i}\int_{\xi_{0,j}}^{\xi_{0,i}} l_{i,j}v_j\varphi_i + \sum_{j \in [N]\setminus J_i}\int_{\xi_{0,i}}^{\xi_{0,j}} l_{i,j}v_j\varphi_i.$$

Finally, recalling that $v_j(\xi) > 0$ if $\xi > \xi_{0,j}$ and $v_j(\xi) < 0$ if $\xi < \xi_{0,j}$, the inequality above yields $\Psi_i \leq 0$, which ends the proof. □

5.6.3.3 The finite domain problem

Let $R > 0$ and define the following truncated problem:

$$\begin{cases} -\mathbf{D}\mathbf{p}'' - c\mathbf{p}' = \mathbf{L}\mathbf{p} - \mathbf{c}\left[\mathbf{p}\right] \circ \mathbf{p} & \text{in } (-R, R), \\ \mathbf{p}(\pm R) = \underline{\mathbf{p}}(\pm R). \end{cases} \qquad (TW[R,c])$$

Lemma 5.37. *Assume*

$$D\mathbf{c}(\mathbf{v}) \geq \mathbf{0} \text{ for all } \mathbf{v} \in \mathsf{K}.$$

Then there exists a nonnegative nonzero classical solution $\mathbf{p}_R$ *of* $(TW[R,c])$.

Remark. The new assumption made here ensures that the vector field $\mathbf{c}$ is non-decreasing in K, in the following natural sense: if $\mathbf{0} \leq \mathbf{v} \leq \mathbf{w}$, then $\mathbf{0} \leq \mathbf{c}(\mathbf{v}) \leq \mathbf{c}(\mathbf{w})$.

Proof. Fix arbitrarily $\varepsilon \in (0, \overline{\varepsilon})$, define consequently $\underline{\mathbf{p}}$ and then define the following convex set of functions:

$$\mathscr{F} = \left\{ \mathbf{v} \in \mathscr{C}\left([-R,R], \mathbb{R}^N\right) \mid \underline{\mathbf{p}} \leq \mathbf{v} \leq \overline{\mathbf{p}} \right\}.$$

Recall that Figueiredo–Mitidieri [54] establishes that the elliptic weak maximum principle holds for a weakly and fully coupled elliptic operator with null Dirichlet boundary conditions if this operator admits a positive strict super-solution. Since, for all $\mathbf{v} \in \mathscr{C}\left([-R,R], \mathbb{R}^N\right)$ such that $\mathbf{0} \leq \mathbf{v} \leq \overline{\mathbf{p}}$, we have by the nonnegativity of $\mathbf{c}$ on K (H_2)

$$-\mathbf{D}\overline{\mathbf{p}}'' - c\overline{\mathbf{p}}' - \mathbf{L}\overline{\mathbf{p}} + \mathbf{c}[\mathbf{v}] \circ \overline{\mathbf{p}} \geq -\mathbf{D}\overline{\mathbf{p}}'' - c\overline{\mathbf{p}}' - \mathbf{L}\overline{\mathbf{p}} \geq \mathbf{0},$$

$$\overline{\mathbf{p}}(\pm R) \gg \mathbf{0},$$

it follows that every operator of the family

$$\left(\mathbf{D}\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} + c\frac{\mathrm{d}}{\mathrm{d}\xi} + (\mathbf{L} - \operatorname{diag}\mathbf{c}[\mathbf{v}])\right)_{\mathbf{0} \leq \mathbf{v} \leq \overline{\mathbf{p}}}$$

supplemented with null Dirichlet boundary conditions at $\pm R$ satisfies the weak maximum principle in $(-R,R)$.

Define the map $\mathbf{f}$ which associates with some $\mathbf{v} \in \mathscr{F}$ the unique classical solution $\mathbf{f}[\mathbf{v}]$ of:

$$\begin{cases} -\mathbf{D}\mathbf{p}'' - c\mathbf{p}' = \mathbf{L}\mathbf{p} - \mathbf{c}[\mathbf{v}] \circ \mathbf{p} & \text{in } (-R,R) \\ \quad \mathbf{p}(\pm R) = \underline{\mathbf{p}}(\pm R). \end{cases}$$

The map $\mathbf{f}$ is compact by classical elliptic estimates (Gilbarg–Trudinger [80]).

Let $\mathbf{v} \in \mathscr{F}$. By monotonicity of $\mathbf{c}$, the function $\mathbf{w} = \mathbf{f}[\mathbf{v}] - \underline{\mathbf{p}}$ satisfies

$$\begin{aligned} -\mathbf{D}\mathbf{w}'' - c\mathbf{w}' - \mathbf{L}\mathbf{w} &\geq -\mathbf{c}[\mathbf{v}] \circ \mathbf{f}[\mathbf{v}] + \mathbf{c}[\overline{\mathbf{p}}] \circ \underline{\mathbf{p}} \\ &\geq -\mathbf{c}[\mathbf{v}] \circ \mathbf{f}[\mathbf{v}] + \mathbf{c}[\mathbf{v}] \circ \underline{\mathbf{p}} \\ &\geq -\mathbf{c}[\mathbf{v}] \circ \mathbf{w} \end{aligned}$$

with null Dirichlet boundary conditions at $\pm R$. Therefore, by virtue of the weak maximum principle applied to $\mathbf{D}\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} + c\frac{\mathrm{d}}{\mathrm{d}\xi} + (\mathbf{L} - \operatorname{diag}\mathbf{c}[\mathbf{v}])$, $\mathbf{f}[\mathbf{v}] \geq \underline{\mathbf{p}}$ in $(-R,R)$. Next, since it is now established that $\mathbf{f}[\mathbf{v}] \geq \mathbf{0}$, we also have by (H_2)

$$\begin{aligned} -\mathbf{D}\overline{\mathbf{p}}'' - c\overline{\mathbf{p}}' - \mathbf{L}\overline{\mathbf{p}} &= \mathbf{0} \\ &\geq -\mathbf{c}[\mathbf{v}] \circ \mathbf{f}[\mathbf{v}] \\ &= -\mathbf{D}\mathbf{f}[\mathbf{v}]'' - c\mathbf{f}[\mathbf{v}]' - \mathbf{L}\mathbf{f}[\mathbf{v}], \end{aligned}$$

$$\overline{\mathbf{p}}(\pm R) \geq \underline{\mathbf{p}}(\pm R) = \mathbf{f}[\mathbf{v}](\pm R),$$

whence $\overline{\mathbf{p}} \geq \mathbf{f}[\mathbf{v}]$ follows from the weak maximum principle applied this time to $\mathbf{D}\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} + c\frac{\mathrm{d}}{\mathrm{d}\xi} + \mathbf{L}$.

Thus $\underline{\mathbf{p}} \leq \mathbf{f}[\mathbf{v}] \leq \overline{\mathbf{p}}$ and consequently $\mathbf{f}(\mathscr{F}) \subset \mathscr{F}$.

Finally, by virtue of the Schauder fixed point theorem, $\mathbf{f}$ admits a fixed point $\mathbf{p}_R \in \mathscr{F}$, which is indeed a classical solution of $(TW[R,c])$ by elliptic regularity. □

5.6.3.4 The infinite domain limit and the minimal wave speed

The speed c is not fixed anymore.

The following uniform upper estimate is a direct consequence of Proposition 5.15.

Corollary 5.38. *There exists $R^\star > 0$ such that, for any $c > c^\star$, any $R \geq R^\star$ and any nonnegative classical solution $\mathbf{p}$ of $(TW[R,c])$,*

$$\left(\max_{[-R,R]} p_i\right)_{i\in[N]} \leq \mathbf{g}(0).$$

We are now in position to prove the second half of Theorem 5.5 1.

Proposition 5.39. *Assume*

$$D\mathbf{c}(\mathbf{v}) \geq \mathbf{0} \text{ for all } \mathbf{v} \in \mathsf{K}.$$

Then for all $c \geq c^\star$, there exists a traveling wave solution of (E_{KPP}) with speed c.

Remark. Of course, it would be interesting to exhibit other additional assumptions on $\mathbf{c}$ sufficient to ensure existence of traveling waves for all $c \geq c^\star$. In view of known results about scalar multistable reaction–diffusion equations (we refer for instance to Fife–McLeod [71]), some additional assumption should in any case be necessary.

Proof. Hereafter, for all $c > c^\star$ and all $R > 0$, the triplet $\left(\overline{\mathbf{p}}, \underline{\mathbf{p}}, \mathbf{p}_R\right)$ constructed in the preceding subsections is denoted $\left(\overline{\mathbf{p}}_c, \underline{\mathbf{p}}_c, \mathbf{p}_{R,c}\right)$.

For all $c > c^\star$, thanks to Corollary 5.38, the family $\left(\mathbf{p}_{R,c}\right)_{R>0}$ is uniformly globally bounded. By classical elliptic estimates (Gilbarg–Trudinger [80]) and a diagonal extraction process, we can extract a sequence $\left(R_n, \mathbf{p}_{R_n,c}\right)_{n\in\mathbb{N}}$ such that, as $n \to +\infty$, $R_n \to +\infty$ and $\mathbf{p}_{R_n,c}$ converges to some limit $\mathbf{p}_c$ in $\mathscr{C}^2_{loc}$. As expected, $\mathbf{p}_c$ is a bounded nonnegative classical solution of $(TW[c])$. The fact that its limit as $\xi \to +\infty$ is $\mathbf{0}$, as well as the fact that $\mathbf{p}_c$ is nonzero whence positive (Corollary 5.10), are obvious thanks to the inequality $\underline{\mathbf{p}}_c \leq \mathbf{p}_c \leq \overline{\mathbf{p}}_c$. At the other end of the real line, Corollary 5.30 clearly enforces

$$\left(\liminf_{\xi\to-\infty} p_{c,i}(\xi)\right)_{i\in[N]} \in \mathsf{K}^{++} \subset \mathsf{K}^{+}.$$

Thus $(\mathbf{p}_c, c)$ is a traveling wave solution.

In order to construct a critical traveling wave $(\mathbf{p}_{c^\star}, c^\star)$, we consider a decreasing sequence $(c_n)_{n\in\mathbb{N}} \in (c^\star, +\infty)^{\mathbb{N}}$ such that $c_n \to c^\star$ as $n \to +\infty$ and intend to apply a compactness argument to a normalized version of the sequence $(\mathbf{p}_{c_n})_{n\in\mathbb{N}}$.

By Corollary 5.32,

$$\liminf_{\xi\to-\infty} \min_{i\in[N]} p_{c_n,i}(\xi) \geq \nu \text{ for all } n \in \mathbb{N}.$$

Recall from Lemma 5.35 the definition of $\eta_c > 0$. For all $n \in \mathbb{N}$ the following quantity is well-defined and finite:

$$\xi_n = \inf\left\{\xi \in \mathbb{R} \mid \min_{i\in[N]} p_{c_n,i}(\xi) < \min\left(\frac{\nu}{2}, \frac{\eta_{c^\star}}{2}\right)\right\}.$$

We define then the sequence of normalized profiles

$$\tilde{\mathbf{p}}_{c_n} : \xi \mapsto \mathbf{p}_{c_n}(\xi + \xi_n) \text{ for all } n \in \mathbb{N}.$$

A translation of a profile of traveling wave being again a profile of traveling wave, $(\tilde{\mathbf{p}}_{c_n}, c_n)_{n\in\mathbb{N}}$ is again a sequence of traveling wave solutions. Notice the following two immediate consequences of the normalization:

$$\min_{i\in[N]} \tilde{p}_{c_n,i}(0) = \min\left(\frac{\nu}{2}, \frac{\eta_{c^\star}}{2}\right) \text{ for all } n \in \mathbb{N},$$

$$\inf_{\xi\in(-\infty,0)} \min_{i\in[N]} \tilde{p}_{c_n,i}(\xi) \geq \min\left(\frac{\nu}{2}, \frac{\eta_{c^\star}}{2}\right) \text{ for all } n \in \mathbb{N}.$$

We are now in position to pass to the limit $n \to +\infty$. The sequence $(\tilde{\mathbf{p}}_{c_n})_{n\in\mathbb{N}}$ being globally uniformly bounded, it admits, up to a diagonal extraction process, a bounded nonnegative limit $\mathbf{p}_{c^\star}$ in $\mathscr{C}^2_{loc}$. Since $c_n \to c^\star$, $\mathbf{p}_{c^\star}$ satisfies $(TW\,[c^\star])$. The normalization yields

$$\min_{i\in[N]} p_{c^\star,i}(0) = \min\left(\frac{\nu}{2}, \frac{\eta_{c^\star}}{2}\right),$$

$$\inf_{\xi\in(-\infty,0)} \min_{i\in[N]} p_{c^\star,i}(\xi) \geq \min\left(\frac{\nu}{2}, \frac{\eta_{c^\star}}{2}\right).$$

Consequently,

$$\left(\liminf_{\xi\to-\infty} p_{c^\star,i}(\xi)\right)_{i\in[N]} \in \mathsf{K}^{++}$$

and, according to Lemma 5.35,

$$\lim_{\xi\to+\infty} \mathbf{p}_{c^\star}(\xi) = \mathbf{0}.$$

The pair $(\mathbf{p}_{c^\star}, c^\star)$ is a traveling wave solution indeed and this ends the proof. $\square$

5.7 Spreading speed

In this section, we assume $\lambda_{PF}(\mathbf{L}) > 0$ and prove Theorem 5.6. In order to do so, we fix $\mathbf{u}_0 \in \mathscr{C}_b(\mathbb{R}, \mathbb{R}^N)$ of the form $\mathbf{u}_0 = \mathbf{v}\mathbf{1}_{(-\infty,x_0)}$ with $x_0 \in \mathbb{R}$ and $\mathbf{v}$ nonnegative nonzero and we define $\mathbf{u}$ as the unique classical solution of (E_{KPP}) set in $(0,+\infty)\times\mathbb{R}$ with initial data $\mathbf{u}_0$.

Remark. This type of spreading result, as well as its proof by means of super- and sub-solutions, is quite classical (we refer to Aronson–Weinberger [8] and Berestycki–Hamel–Nadin [15] among others). Still, we provide it to make clear that the lack of comparison principle for (E_{KPP}) is not really an issue.

Of course, for the scalar KPP equation, much more precise spreading results exist (for instance the celebrated articles by Bramson [30, 29] using probabilistic methods). Here, our aim is not to give a complete description of the spreading properties of (E_{KPP}) but rather to illustrate that it is, once more, very similar to the scalar situation and that further generalizations should be possible.

5.7.1 Upper estimate

Proposition 5.40. *Let $c > c^\star$ and $y \in \mathbb{R}$. We have*

$$\left(\lim_{t\to+\infty} \sup_{x\in(y,+\infty)} u_i(t, x+ct)\right)_{i\in[N]} = \mathbf{0}.$$

Proof. By definition of $\mathbf{u}_0$, there exists $\xi_1 \in \mathbb{R}$ such that

$$\overline{\mathbf{p}} : \xi \mapsto \mathrm{e}^{-\mu_{c^\star}(\xi-\xi_1)}\mathbf{n}_{\mu_{c^\star}}$$

(which is a positive solution of $(TW_0[c^\star])$ by Lemma 5.25) satisfies $\overline{\mathbf{p}} \geq \mathbf{u}_0$. Then, defining $\overline{\mathbf{u}} : (t,x) \mapsto \overline{\mathbf{p}}(x - c^\star t)$, we obtain by the nonnegativity of $\mathbf{c}$ on K (H_2)

$$\begin{aligned}\partial_t \overline{\mathbf{u}} - \mathbf{D}\partial_{xx}\overline{\mathbf{u}} - \mathbf{L}\overline{\mathbf{u}} &= \mathbf{0} \\ &\geq -\mathbf{c}[\mathbf{u}] \circ \mathbf{u} \\ &= \partial_t \mathbf{u} - \mathbf{D}\partial_{xx}\mathbf{u} - \mathbf{L}\mathbf{u}\end{aligned}$$

and then, applying the parabolic strong maximum principle to the operator $\partial_t - \mathbf{D}\partial_{xx} - \mathbf{L}$, we deduce that $\overline{\mathbf{u}} - \mathbf{u}$ is nonnegative in $[0,+\infty) \times \mathbb{R}$. Consequently, for all $x \in \mathbb{R}$, $t > 0$ and $c > c^\star$,

$$\mathbf{0} \leq \mathbf{u}(t, x + ct) \leq \overline{\mathbf{p}}(x + (c - c^\star)t),$$

and by component-wise monotonicity of $\overline{\mathbf{p}}$, for all $y \in \mathbb{R}$ and all $x \geq y$,

$$\mathbf{0} \leq \mathbf{u}(t, x + ct) \leq \overline{\mathbf{p}}(y + (c - c^\star)t),$$

which gives the result. □

5.7.2 Lower estimate

Proposition 5.41. *Let $c \in [0, c^\star)$ and $I \subset \mathbb{R}$ be a bounded interval. We have*

$$\left(\liminf_{t\to+\infty} \inf_{x\in I} u_i(t, x + ct)\right)_{i\in[N]} \in \mathsf{K}^{++}.$$

Proof. Recall Lemma 5.27 and define

$$\lambda_c = -\max_{\mu\geq 0}(\kappa_\mu + \mu c) > 0$$

($-\lambda_c$ being the generalized principal eigenvalue of $-\mathbf{D}\frac{\mathrm{d}^2}{\mathrm{d}x^2} - c\frac{\mathrm{d}}{\mathrm{d}x} - \mathbf{L}$) and, using the fact that $\mathbf{c}$ vanishes at $\mathbf{0}$ (H_3),

$$\alpha_c = \max\left\{\alpha > 0 \mid \forall \mathbf{v} \in [0,\alpha]^N \quad \mathbf{c}(\mathbf{v}) \leq \frac{\lambda_c}{2}\mathbf{1}_{N,1}\right\}.$$

Let R_c be a sufficiently large radius satisfying

$$\lambda_{1,Dir}\left(-\mathbf{D}\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} - c\frac{\mathrm{d}}{\mathrm{d}\xi} - \left(\mathbf{L} - \frac{\lambda_c}{2}\mathbf{I}\right), (-R_c, R_c)\right) < 0.$$

Let $\mathbf{u}_c : (t,y) \mapsto \mathbf{u}(t, y + ct)$. It is a solution of

$$\partial_t \mathbf{u}_c - \mathbf{D}\partial_{yy}\mathbf{u}_c - c\partial_y \mathbf{u}_c = \mathbf{L}\mathbf{u}_c - \mathbf{c}[\mathbf{u}_c] \circ \mathbf{u}_c \text{ in } (0,+\infty) \times \mathbb{R}$$

with initial data $\mathbf{u}_0$. Just as in the proof of Proposition 5.20, we can use R_c, α_c and Földes–Poláčik's Harnack inequality [73] to deduce the existence of $\nu_c > 0$ such that

$$\left(\liminf_{t\to+\infty} \inf_{x\in I} u_i(t, x + ct)\right)_{i\in[N]} \geq \nu_c \mathbf{1}_{N,1}.$$

This ends the proof. □

Remark. We point out that $R_c \to +\infty$ as $c \to c^\star$. Hence the proof above cannot be used directly to obtain a lower bound uniform with respect to c. Although we expect indeed the existence of such a bound, we do not know how to obtain it.

5.8 Estimates for the minimal wave speed

In this section, we assume $\lambda_{PF}(\mathbf{L}) > 0$,

$$d_1 \leq d_2 \leq \ldots \leq d_N,$$

and prove the estimates provided by Theorem 5.7. Recall the equality

$$c^\star = \min_{\mu>0}\left(-\frac{\kappa_\mu}{\mu}\right).$$

Recall as a preliminary that for all $r > 0$ and $d > 0$, the following equality holds:

$$2\sqrt{rd} = \min_{\mu>0}\left(\mu d + \frac{r}{\mu}\right).$$

Proposition 5.42. *We have*

$$2\sqrt{d_1\lambda_{PF}(\mathbf{L})} \leq c^\star \leq 2\sqrt{d_N\lambda_{PF}(\mathbf{L})}.$$

If $d_1 < d_N$, both inequalities are strict. If $d_1 = d_N$, both inequalities are equalities.

Proof. Since $d_1\mathbf{1}_{N,1} \leq \mathbf{d} \leq d_N\mathbf{1}_{N,1}$, we have, for all $\mu > 0$,

$$\mu d_1 + \frac{1}{\mu}\lambda_{PF}(\mathbf{L}) \leq \lambda_{PF}\left(\mu\mathbf{D} + \frac{1}{\mu}\mathbf{L}\right) \leq \mu d_N + \frac{1}{\mu}\lambda_{PF}(\mathbf{L}),$$

whence we deduce

$$2\sqrt{d_1\lambda_{PF}(\mathbf{L})} \leq c^\star \leq 2\sqrt{d_N\lambda_{PF}(\mathbf{L})}.$$

On one hand, it is well-known that if $d_1 < d_N$, then the above inequalities are strict. On the other hand, if $d_1 = d_N$, we have

$$\lambda_{PF}\left(\mu\mathbf{D} + \frac{1}{\mu}\mathbf{L}\right) = \mu d_1 + \frac{1}{\mu}\lambda_{PF}(\mathbf{L}),$$

whence the equality. □

Recall from Lemma 5.25 that $\mathbf{n}_{\mu_{c^\star}} = \mathbf{n}_{PF}\left(\mu_{c^\star}^2\mathbf{D} + \mathbf{L}\right)$.

Proposition 5.43. *For all $i \in [N]$ such that $l_{i,i} > 0$, we have*

$$c^\star > 2\sqrt{d_i l_{i,i}}.$$

Proof. Let $i \in [N]$. The characterization of $c^\star$ (see Lemma 5.25) yields

$$\mu_{c^\star} d_i + \frac{l_{i,i}}{\mu_{c^\star}} = c^\star - \frac{1}{\mu_{c^\star}}\sum_{j\in[N]\setminus\{i\}} l_{i,j}\frac{n_{\mu_{c^\star},j}}{n_{\mu_{c^\star},i}},$$

whence, if $l_{i,i} > 0$,

$$c^\star \geq 2\sqrt{d_i l_{i,i}} + \frac{1}{\mu_{c^\star}}\sum_{j\in[N]\setminus\{i\}} l_{i,j}\frac{n_{\mu_{c^\star},j}}{n_{\mu_{c^\star},i}}.$$

From the irreducibility and essential nonnegativity of $\mathbf{L}$ (H_1), there exists $j \in [N]\setminus\{i\}$ such that $l_{i,j} > 0$, whence $c^\star > 2\sqrt{d_i l_{i,i}}$. □

Recall the existence of a unique decomposition of $\mathbf{L}$ of the form

$$\mathbf{L} = \operatorname{diag}\mathbf{r} + \mathbf{M} \text{ with } \mathbf{r} \in \mathbb{R}^N \text{ and } \mathbf{M}^T \mathbf{1}_{N,1} = \mathbf{0}.$$

Remark. Regarding the Lotka–Volterra mutation–competition–diffusion ecological model, the decomposition $\mathbf{L} = \operatorname{diag}\mathbf{r} + \mathbf{M}$ is ecological meaningful: $\mathbf{r}$ is the vector of the growth rates of the phenotypes whereas $\mathbf{M}$ describes the mutations between the phenotypes.

Proposition 5.44. *Let* $(\langle d \rangle, \langle r \rangle) \in (0, +\infty) \times \mathbb{R}$ *be defined as*

$$\begin{cases} \langle d \rangle = \frac{\mathbf{d}^T \mathbf{n}_{PF}\left(\mu_{c^\star}^2 \mathbf{D} + \mathbf{L}\right)}{\mathbf{1}_{1,N} \mathbf{n}_{PF}\left(\mu_{c^\star}^2 \mathbf{D} + \mathbf{L}\right)}, \\ \langle r \rangle = \frac{\mathbf{r}^T \mathbf{n}_{PF}\left(\mu_{c^\star}^2 \mathbf{D} + \mathbf{L}\right)}{\mathbf{1}_{1,N} \mathbf{n}_{PF}\left(\mu_{c^\star}^2 \mathbf{D} + \mathbf{L}\right)}. \end{cases}$$

If $\langle r \rangle \geq 0$, *then*

$$c^\star \geq 2\sqrt{\langle d \rangle \langle r \rangle}.$$

Proof. Using $(\mathbf{r}, \mathbf{M})$, the characterization of $c^\star$ (see Lemma 5.25) is rewritten as

$$\left(\mu_{c^\star}^2 \mathbf{D} + \operatorname{diag}\mathbf{r}\right) \mathbf{n}_{\mu_{c^\star}} + \mathbf{M}\mathbf{n}_{\mu_{c^\star}} = \mu_{c^\star} c^\star \mathbf{n}_{\mu_{c^\star}}.$$

Summing the lines of this system, dividing by $\sum_{i=1}^{N} n_{\mu_{c^\star}, i}$ and defining $\langle d \rangle$ and $\langle r \rangle$ as in the statement, we find

$$\mu_{c^\star}^2 \langle d \rangle + \langle r \rangle = \mu_{c^\star} c^\star.$$

The equation $\langle d \rangle \mu^2 - c^\star \mu + \langle r \rangle = 0$ admits a real positive solution μ if and only if $(c^\star)^2 - 4 \langle d \rangle \langle r \rangle \geq 0$. □

Acknowledgments

The author thanks Grégoire Nadin for the attention he paid to this work, Vincent Calvez for fruitful discussions on the cane toads equation, Cécile Taing for pointing out the related work by Wang and an anonymous reviewer for a detailed report thanks to which the original manuscript was largely improved.

5.A Extinction in the critical case

Adrian Lam pointed out after the publication of the preceding article that the argument used to establish the upper estimates of Theorem 5.2 can actually be used again to solve the extinction case. The proof is included here for the sake of completeness.

Theorem 5.45. *[Extinction, critical case] Assume* $\lambda_{PF}(\mathbf{L}) = 0$ *and*

$$\mathit{span}\left(\mathbf{n}_{PF}(\mathbf{L})\right) \cap \mathsf{K} \cap \mathbf{c}^{-1}(\{\mathbf{0}\}) = \{\mathbf{0}\}.$$

Then all bounded nonnegative classical solutions of (E_{KPP}) *set in* $(0, +\infty) \times \mathbb{R}$ *converge asymptotically in time and uniformly in space to* $\mathbf{0}$.

Proof. Let $\mathbf{u}$ be a bounded nonnegative classical solution. Using its boundedness, we can define for all $T > 0$

$$C_T = \inf\left\{C > 0 \mid C\mathbf{n}_{PF}\left(\mathbf{L}\right) \geq \mathbf{u}\left(T, x\right) \text{ for all } x \in \mathbb{R}\right\}.$$

Subsequently, fix temporarily $T > 0$ and let $\mathbf{v} = C_T\mathbf{n}_{PF}\left(\mathbf{L}\right) - \mathbf{u}$. Since $\lambda_{PF}\left(\mathbf{L}\right) = 0$, $\mathbf{v}$ satisfies

$$\mathbf{v}\left(T, x\right) \geq \mathbf{0} \text{ for all } x \in \mathbb{R},$$

$$\partial_t\mathbf{v} - \mathbf{D}\partial_{xx}\mathbf{v} - \mathbf{L}\mathbf{v} = \mathbf{c}\left[\mathbf{u}\right] \circ \mathbf{u} \text{ in } (0, +\infty) \times \mathbb{R}.$$

Since the nonnegativity of $\mathbf{c}$ (H_2) implies $\mathbf{c}\left[\mathbf{u}\right] \circ \mathbf{u} \geq \mathbf{0}$, it follows from the maximum principle that

$$\mathbf{v} \geq \mathbf{0} \text{ in } [T, +\infty) \times \mathbb{R},$$

that is

$$C_T\mathbf{n}_{PF}\left(\mathbf{L}\right) \geq \mathbf{u} \text{ in } [T, +\infty) \times \mathbb{R},$$

whence $C_{T'} \leq C_T$ for all $T' \geq T$. In other words, $\left(C_T\right)_{T>0}$ is a nonincreasing family.

Next, let us verify that it is in fact a decreasing family if and only if $\mathbf{u}$ is nonzero.

Of course, if $\mathbf{u} = \mathbf{0}$, then $C_T = 0$ for all $T > 0$.

Now, assume that there exist $T > 0$ and $T' > T$ such that, for all $t \in [T, T']$, $C_t = C_T$. Let $\mathbf{v} = C_T\mathbf{n}_{PF}\left(\mathbf{L}\right) - \mathbf{u}$. By optimality of $C_{T'}$,

$$\min_{i\in[N]} \inf_{x\in\mathbb{R}} v_i\left(T', x\right) = 0.$$

If there exists $x \in \mathbb{R}$ such that

$$\mathbf{v}\left(T', x\right) = C_{T'}\mathbf{n}_{PF}\left(\mathbf{L}\right) - \mathbf{u}\left(T', x\right) \in \partial\mathsf{K},$$

then by the strong maximum principle

$$\mathbf{v} = \mathbf{0} \text{ in } [T, T'] \times \mathbb{R}.$$

This leads to $\mathbf{c}\left[\mathbf{u}\right] \circ \mathbf{u} = \mathbf{0}$ on one hand and to $\mathbf{u} = C_T\mathbf{n}_{PF}\left(\mathbf{L}\right)$ on the other hand, whence by assumption on $\mathbf{c}$ we deduce $C_T = 0$. Therefore $\mathbf{u} = \mathbf{0}$ in $[T, T'] \times \mathbb{R}$ and then in $(0, +\infty) \times \mathbb{R}$.

On the contrary, if such an $x \in \mathbb{R}$ does not exist, then by optimality of $C_{T'}$, there exists $\mathbf{w} \in \partial\mathsf{K}$ and $\left(x_n\right)_{n\in\mathbb{N}}$ such that, as $n \to +\infty$,

$$x_n \to \pm\infty \text{ and } \mathbf{v}\left(T', x_n\right) \to \mathbf{w}.$$

Defining the sequence

$$\mathbf{u}_n : (t, x) \mapsto \mathbf{u}\left(t, x + x_n\right)$$

and using classical parabolic estimates to extract a locally uniform limit $\mathbf{u}_\infty$, we find that $\mathbf{v}_\infty = C_T\mathbf{n}_{PF}\left(\mathbf{L}\right) - \mathbf{u}_\infty$ satisfies

$$\partial_t\mathbf{v}_\infty - \mathbf{D}\partial_{xx}\mathbf{v}_\infty - \mathbf{L}\mathbf{v}_\infty \geq \mathbf{0} \text{ in } [T, T'] \times \mathbb{R},$$

$$\mathbf{v}_\infty \geq \mathbf{0} \text{ in } [T, T'] \times \mathbb{R},$$

$$\mathbf{v}_\infty\left(T', 0\right) = \mathbf{w} \in \partial\mathsf{K},$$

and then again by the strong maximum principle we find $\mathbf{v}_\infty = \mathbf{0}$ and subsequently $\mathbf{u} = \mathbf{0}$.

Hence either $\mathbf{u} = \mathbf{0}$ or the family $\left(C_T\right)_{T>0}$ is decreasing. Let

$$C_\infty = \lim_{T\to+\infty} C_T \geq 0.$$

Assuming by contradiction that $C_\infty > 0$, defining the sequence

$$\mathbf{u}_n : (t,x) \mapsto \mathbf{u}(t+n,x)$$

and its locally uniform limit $\mathbf{u}_\infty$, we can repeat the argument and obtain that the family $(D_T)_{T>0}$, where

$$D_T = \inf\left\{D > 0 \mid D\mathbf{n}_{PF}(\mathbf{L}) \geq \mathbf{u}_\infty(T,x) \text{ for all } x \in \mathbb{R}\right\},$$

is decreasing, which directly contradicts the fact that $D_T = C_\infty$ for all $T > 0$. This ends the proof. □

Chapitre 6

Systèmes de Fisher – KPP non-monotones : comportement asymptotique des ondes progressives

Résumé

Ce chapitre est la suite directe du précédent, dans lequel l'existence d'ondes progressives connectant l'état nul à un compact de l'intérieur du cône positif pour les systèmes KPP non-coopératifs persistants a été prouvée. L'objet de ce chapitre est la recherche d'une description plus précise des profils de ces ondes.

Ce chapitre a fait l'objet d'une publication sous le titre *Non-cooperative Fisher–KPP systems : asymptotic behavior of traveling waves* dans *Mathematical Models and Methods in Applied Sciences* [Gir18a].

6.1 Introduction

This paper is a sequel to a previous paper by the same author [Gir18b] where the so-called *KPP systems* were investigated. The prototypical and, arguably, most famous KPP system is the *Lotka–Volterra mutation–competition–diffusion system*:

$$\frac{\partial \mathbf{u}}{\partial t} - \operatorname{diag}(\mathbf{d})\,\Delta_x \mathbf{u} = \operatorname{diag}(\mathbf{r})\,\mathbf{u} + \mathbf{M}\mathbf{u} - \operatorname{diag}(\mathbf{u})\,\mathbf{C}\mathbf{u},$$

where $\mathbf{u}$ is a nonnegative vector containing phenotypical densities, $\mathbf{d}$ and $\mathbf{r}$ are positive vectors containing respectively diffusion rates and growth rates, $\mathbf{M}$ is an essentially nonnegative irreducible matrix with null Perron–Frobenius eigenvalue containing mutation rates (typically a discrete Neumann Laplacian) and $\mathbf{C}$ is a positive matrix containing competition rates. Although the Lotka–Volterra competition–diffusion system (without mutations) is a very classical research subject, mutations can dramatically influence some of its properties and their overall effect is still poorly understood.

More generally, KPP systems as defined in [Gir18b] are non-cooperative (or non-monotone, i.e. they do not satisfy a comparison principle; see Protter–Weinberger [129, Chapter 3, Section 8]) and have started to attract attention relatively recently. Their study requires innovative ideas and the literature is limited; a detailed bibliography can be found in [Gir18b].

By adapting proofs and methods well-known in the context of the scalar KPP equation,

$$\frac{\partial u}{\partial t} - d\Delta_x u = ru - cu^2,$$

first studied by Fisher [72] and Kolmogorov, Petrovsky and Piskunov [104], various properties of these systems were established in [Gir18b]. In particular, a KPP system equipped with a reaction term sufficiently analogous to $u - u^2$ admits traveling wave solutions with a half-line of possible speeds and a positive minimal speed $c^\star$. These traveling waves are defined in a very general way: it is merely required that they describe the invasion of $\mathbf{0}$ by a positive population density. A very natural subsequent question is that of the evolution of the distribution $\mathbf{u}$ during the invasion. Which components lead the invasion? Which components settle once the invasion is over?

Having in mind that the waves traveling at speed $c^\star$ should attract front-like and compactly supported initial data (although this statement has yet to be proven, since [Gir18b] only established the equality between $c^\star$ and the spreading speed associated with such initial data, and it is expected to be a very difficult problem), a more general question is then: given a class of initial data, what is the long-time distribution of the solution?

In the rest of the introduction, we present more precisely the problem and state our main results. Sections 2, 3 and 4 are dedicated to the proofs of these results. Finally, open questions, interesting remarks and numerical experiments are discussed in Section 5.

6.1.1 The non-cooperative KPP system

From now on, an integer $N \geq 2$ is fixed.

A positive vector $\mathbf{d} \in \mathsf{K}^{++}$, a square matrix $\mathbf{L} \in \mathsf{M}$ and a vector field $\mathbf{c} \in \mathscr{C}^1\left(\mathbb{R}^N, \mathbb{R}^N\right)$ are fixed. We denote for the sake of brevity $\mathbf{D} = \operatorname{diag}(\mathbf{d})$.

We consider the following semilinear parabolic system:

$$\partial_t \mathbf{u} - \mathbf{D}\partial_{xx}\mathbf{u} = \mathbf{L}\mathbf{u} - \mathbf{c}[\mathbf{u}] \circ \mathbf{u}, \quad (E_{KPP})$$

with $\mathbf{u} : (t,x) \in \mathbb{R}^2 \mapsto \mathbf{u}(t,x) \in \mathbb{R}^N$ as unknown. In order to ease the notations, we only consider one-dimensional spaces, however all forthcoming results could be applied directly to traveling plane waves in multidimensional spaces (these solutions being in fact one-dimensional).

When restricted to solutions $\mathbf{u} : \mathbb{R} \to \mathbb{R}^N$ which are constant in space, (E_{KPP}) reduces to

$$\mathbf{u}' = \mathbf{L}\mathbf{u} - \mathbf{c}[\mathbf{u}] \circ \mathbf{u}. \quad \left(E^0_{KPP}\right)$$

When restricted to solutions $\mathbf{u} : \mathbb{R} \to \mathbb{R}^N$ which are constant in time, (E_{KPP}) reduces to

$$-\mathbf{D}\mathbf{u}'' = \mathbf{L}\mathbf{u} - \mathbf{c}[\mathbf{u}] \circ \mathbf{u}. \quad (S_{KPP})$$

When restricted to traveling solutions of the form $\mathbf{u} : (t,x) \mapsto \mathbf{p}(x-ct)$ with $c \in \mathbb{R}$, (E_{KPP}) reduces to

$$-\mathbf{D}\mathbf{p}'' - c\mathbf{p}' = \mathbf{L}\mathbf{p} - \mathbf{c}[\mathbf{p}] \circ \mathbf{p}. \quad (TW[c])$$

6.1.1.1 Basic KPP assumptions

The basic assumptions introduced in [Gir18b] are the following ones.

(H_1) $\mathbf{L}$ is essentially nonnegative and irreducible.

(H_2) $\mathbf{c}(\mathsf{K}) \subset \mathsf{K}$.

(H_3) $\mathbf{c}(\mathbf{0}) = \mathbf{0}$.

(H_4) There exists

$$(\underline{\alpha}, \delta, \underline{\mathbf{c}}) \in [1,+\infty)^2 \times \mathsf{K}^{++}$$

such that

$$\sum_{j=1}^{N} l_{i,j} n_j \geq 0 \implies \alpha^{\delta} \underline{c}_i \leq c_i(\alpha \mathbf{n})$$

for all

$$(\mathbf{n}, \alpha, i) \in \mathsf{S}^+(\mathbf{0},1) \times [\underline{\alpha}, +\infty) \times [N].$$

The assumption (H_4) loosely means that $\mathbf{c}$ grows at least linearly at infinity. The precise condition means, however, that in the set $\{\mathbf{v} \in \mathsf{K} \mid (\mathbf{L}\mathbf{v})_i < 0\}$ (which is nonempty if and only if $l_{i,i} < 0$ and contains in such a case the open half-line $\operatorname{span}(\mathbf{e}_i) \cap \mathsf{K}^+$), the growth of c_i is not important. Anyway, (H_4) includes the Lotka–Volterra form of competition (linear and positive $\mathbf{c}$) as well as more general forms (see for instance Gilpin–Ayala [81]).

Recall from the Perron–Frobenius theorem that if $\mathbf{L}$ is nonnegative and irreducible, its spectral radius $\rho(\mathbf{L})$ is also its dominant eigenvalue, called the *Perron–Frobenius eigenvalue* $\lambda_{PF}(\mathbf{L})$, and is the unique eigenvalue associated with a positive eigenvector. Recall also that if $\mathbf{L}$ is essentially nonnegative and irreducible, the Perron–Frobenius theorem can still be applied. In such a case, the unique eigenvalue of $\mathbf{L}$ associated with a positive eigenvector is $\lambda_{PF}(\mathbf{L}) = \rho\left(\mathbf{L} - \min_{i\in[N]}(l_{i,i})\mathbf{I}_N\right) + \min_{i\in[N]}(l_{i,i})$. Any eigenvector associated with $\lambda_{PF}(\mathbf{L})$ is referred to as a *Perron–Frobenius eigenvector* and the unit one is denoted $\mathbf{n}_{PF}(\mathbf{L})$.

In view of [Gir18b, Theorems 1.3, 1.4, 1.5], in order to study traveling waves and non-trivial long-time behavior, the following assumption is also necessary.

(H_5) $\lambda_{PF}(\mathbf{L}) > 0$.

The collection (H_1)–(H_5) is always assumed from now on. Notice that, although this does not bring any new result, the scalar KPP equation could be seen as a particular KPP system (understanding the pair (H_1) and (H_5) as $r > 0$). Biological interpretations of these assumptions can be found in [Gir18b, Section 1.5].

6.1.1.2 Traveling waves

Traveling waves are defined in [Gir18b] as follows.

Definition. A *traveling wave solution* of (E_{KPP}) is a *profile–speed* pair

$$(\mathbf{p}, c) \in \mathscr{C}^2\left(\mathbb{R}, \mathbb{R}^N\right) \times [0, +\infty)$$

which satisfies:

— $\mathbf{u} : (t, x) \mapsto \mathbf{p}(x - ct)$ is a bounded positive classical solution of (E_{KPP});

— $\left(\liminf_{\xi \to -\infty} p_i(\xi)\right)_{i \in [N]} > \mathbf{0}$;

— $\lim_{\xi \to +\infty} \mathbf{p}(\xi) = \mathbf{0}$.

By construction, a traveling wave solution $(\mathbf{p}, c)$ solves $(TW[c])$.

The set of all profiles associated with some speed c is denoted $\mathscr{P}_c$. By [Gir18b, Theorems 1.5, 1.7], $\mathscr{P}_c$ is empty if

$$c < c^\star = \min_{\mu > 0} \frac{\lambda_{PF}\left(\mu^2 \mathbf{D} + \mathbf{L}\right)}{\mu}.$$

The converse statement (existence of a profile if $c \geq c^\star$) is likely false in general but is true provided $\mathbf{c}$ is monotonic in the following sense:

$$D\mathbf{c}(\mathbf{v}) \geq \mathbf{0} \text{ for all } \mathbf{v} \in \mathsf{K}.$$

6.1.2 Results: at the edge of the fronts

The distribution of the profiles near $+\infty$ follows the "rule of thumb" unfolded in [Gir18b]: for several standard problems, KPP systems can be addressed exactly as KPP equations and the results are analogous.

Recall from [Gir18b, Lemma 6.2] the notation $\mathbf{n}_\mu = \mathbf{n}_{PF}\left(\mu^2 \mathbf{D} + \mathbf{L}\right)$ for all $\mu \in \mathbb{R}$. Recall also that the equation

$$\frac{\lambda_{PF}\left(\mu^2 \mathbf{D} + \mathbf{L}\right)}{\mu} = c$$

admits no real solution if $c < c^\star$, exactly one real solution $\mu_{c^\star} > 0$ if $c = c^\star$ and exactly two real solutions $\mu_{2,c} > \mu_{1,c} > 0$ if $c > c^\star$. Define subsequently for all $c \geq c^\star$ the quantity

$$\mu_c = \min\left\{\mu > 0 \mid \frac{\lambda_{PF}\left(\mu^2 \mathbf{D} + \mathbf{L}\right)}{\mu} = c\right\} = \begin{cases} \mu_{c^\star} & \text{if } c = c^\star, \\ \mu_{1,c} & \text{if } c > c^\star. \end{cases}$$

Theorem 6.1. *Let*

$$k_c = \begin{cases} 0 & \text{if } c > c^\star, \\ 1 & \text{if } c = c^\star. \end{cases}$$

For all traveling wave solutions $(\mathbf{p}, c)$, there exists $A > 0$ such that, as $\xi \to +\infty$,

$$\begin{cases} \mathbf{p}(\xi) \sim A\xi^{k_c} e^{-\mu_c \xi} \mathbf{n}_{\mu_c}, \\ \mathbf{p}'(\xi) \sim -\mu_c \mathbf{p}(\xi), \\ \mathbf{p}''(\xi) \sim \mu_c^2 \mathbf{p}(\xi). \end{cases}$$

In particular, if $\mathbf{d} = \mathbf{1}_{N,1}$,

$$\mathbf{p}(\xi) \sim A\xi^{k_c} e^{-\frac{1}{2}\left(c - \sqrt{c^2 - 4\lambda_{PF}(\mathbf{L})}\right)\xi} \mathbf{n}_{PF}(\mathbf{L}).$$

This result is proved in Section 2.

Recall that up to a well-known change of variable x, we can always assume without loss of generality $\max_{i\in[N]} d_i = 1$.

If we have in mind the mutation–competition–diffusion system, then the ecological interpretation of this result is the following: at the leading edge of the invasion, the normalized distribution in phenotypes is $\mathbf{n}_{\mu_c}$ and the total population is proportional to $(x-ct)^{k_c}\,\mathrm{e}^{-\mu_c(x-ct)}$.

In the special case $c = c^\star$, this theorem answers positively a conjecture of Morris, Börger and Crooks [115, Section 4].

Recall that, for the scalar KPP equation, the analogous result on exponential decays has two common proofs, one using ODE arguments and especially phase-plane analysis and the other one using elliptic arguments and especially the comparison principle. Although we could prove the above result by phase-plane analysis indeed, the proof we will provide uses a third technique relying upon the monotonicity of the profiles near $+\infty$, bilateral Laplace transforms and a Ikehara theorem. In our opinion, this technique of proof has independent interest: on one hand, it does not require the comparison principle and, on the other hand, it might be generalizable to non-ODE settings (space-periodic media and pulsating fronts, for instance).

6.1.3 Results: at the back of the fronts

On the contrary, the distribution of the profiles near $-\infty$ is a much more intricate question, where the multidimensional and non-cooperative structure of the KPP system become preponderant.

Given a positive classical solution $\mathbf{u}$ of (S_{KPP}), a *traveling wave connecting* $\mathbf{0}$ *to* $\mathbf{u}$ is a traveling wave whose profile $\mathbf{p}$ converges to $\mathbf{u}$ as $\xi \to -\infty$. The general aim is to prove that all traveling waves connect $\mathbf{0}$ to some positive classical solution of (S_{KPP}) and, when several solutions can be connected to $\mathbf{0}$, to determine somehow which connection prevails. However, as will be explained in Subsection 6.5.1 (and was first pointed out in Barles–Evans–Souganidis [10]), a general and precise treatment of this problem is likely impossible. It is necessary to focus on special cases. Looking at the literature, we find two frameworks commonly assumed to be mathematically tractable:

— competition terms $c_i(\mathbf{v})$ with separated dependencies on i and on $\mathbf{v}$ (Coville–Fabre [40], Dockery–Hutson–Mischaikow–Pernarowski [58], Griette–Raoul [82], Leman–Méléard–Mirrahimi [106]),

— two-component systems with linear competition and vanishingly small mutations (Dockery–Hutson–Mischaikow–Pernarowski [58], Griette–Raoul [82], Morris–Börger–Crooks [115]).

6.1.3.1 Separated competition

(H_6) There exist $\mathbf{a} \in \mathsf{K}^{++}$ and $b : \mathbb{R}^N \to \mathbb{R}$ such that:

— $\mathbf{c}(\mathbf{v}) = b(\mathbf{v})\,\mathbf{a}$ for all $\mathbf{v} \in \mathsf{K}$;

— the function $w \mapsto b(w\mathbf{e}_i + \mathbf{v})$ is increasing in $(0,+\infty)$ for all $\mathbf{v} \in \mathsf{K}$ and all $i \in [N]$.

By monotonicity of $\mathbf{c}$, supplementing (H_1)–(H_5) with (H_6) implies the existence of a profile $\mathbf{p} \in \mathscr{P}_c$ for all $c \geq c^\star$. The decomposition $\mathbf{c} = b\mathbf{a}$ is unique up to a multiplicative normalization and we will assume for instance $\max_{i\in[N]} a_i = 1$. We denote $\mathbf{A} = \mathrm{diag}(\mathbf{a})$ (so that $\mathbf{c}(\mathbf{v})\circ\mathbf{v} = b(\mathbf{v})\,\mathbf{A}\mathbf{v}$).

An especially interesting subcase is the intersection between (H_6) and the Lotka–Volterra competition form, where b is a linear functional, that is where there exists $\mathbf{b} \in \mathsf{K}^{++}$ such that

$$b(\mathbf{v}) = \mathbf{b}^T\mathbf{v} \text{ for all } \mathbf{v} \in \mathsf{K}.$$

The system (E_{KPP}) then reads

$$\partial_t \mathbf{u} - \mathbf{D}\partial_{xx}\mathbf{u} = \mathbf{L}\mathbf{u} - \left(\mathbf{b}^T\mathbf{u}\right)\mathbf{A}\mathbf{u}.$$

The systems studied in Dockery–Hutson–Mischaikow–Pernarowski [58] and in Griette–Raoul [82] correspond respectively to

$$\mathbf{a} = \mathbf{b} = \mathbf{1}_{N,1}$$

and to

$$(\mathbf{a}, \mathbf{b}) = \left(\left(\frac{K}{r}, 1\right)^T, \frac{r}{K}\mathbf{1}_{2,1}\right).$$

The matrix $\mathbf{A}^{-1}\mathbf{L}$ being essentially nonnegative and irreducible, the following eigenpair is well-defined:

$$(\lambda_{\mathbf{a}}, \mathbf{n}_{\mathbf{a}}) = \left(\lambda_{PF}\left(\mathbf{A}^{-1}\mathbf{L}\right), \mathbf{n}_{PF}\left(\mathbf{A}^{-1}\mathbf{L}\right)\right).$$

Applying [Gir18b, Theorem 1.4] to the following two pairs of parameters $(\mathbf{L}, \mathbf{c})$:

$$\left(\mathbf{L}, \mathbf{v} \mapsto \left(\mathbf{1}_{1,N}\mathbf{v}\right)\mathbf{a}\right),$$

$$\left(\mathbf{A}^{-1}\mathbf{L}, \mathbf{v} \mapsto \left(\mathbf{1}_{1,N}\mathbf{v}\right)\mathbf{1}_{N,1}\right),$$

it is easily deduced that $\lambda_{PF}(\mathbf{L}) > 0$ if and only if $\lambda_{\mathbf{a}} > 0$. By strict monotonicity of $\alpha \mapsto b(\alpha\mathbf{n}_{\mathbf{a}})$, we can define $\alpha^\star > 0$ as the unique solution of $b(\alpha\mathbf{n}_{\mathbf{a}}) = \lambda_{\mathbf{a}}$. It follows easily that $\mathbf{v}^\star = \alpha^\star\mathbf{n}_{\mathbf{a}}$ is the unique positive constant solution of (S_{KPP}). In particular, if b is a linear functional, then

$$\mathbf{v}^\star = \frac{\lambda_{\mathbf{a}}}{\mathbf{b}^T\mathbf{n}_{\mathbf{a}}}\mathbf{n}_{\mathbf{a}}.$$

Theorem 6.2. *Assume (H_6), $\mathbf{d} = \mathbf{1}_{N,1}$ and $\mathbf{a} = \mathbf{1}_{N,1}$.*

For all $c \in [c^\star, +\infty)$, let $p_c \in \mathscr{C}^2(\mathbb{R})$ such that (p_c, c) is the unique traveling wave solution of the scalar equation

$$\partial_t u - \partial_{xx}u = \lambda_{PF}(\mathbf{L})\,u - b\left(u\mathbf{n}_{PF}(\mathbf{L})\right)u$$

connecting 0 to $\alpha^\star$ and satisfying $p_c(0) = \frac{\alpha^\star}{2}$.

Then all $\mathbf{p} \in \mathscr{P}_c$ have the form

$$\mathbf{p} : \xi \mapsto p_c(\xi - \xi_0)\,\mathbf{n}_{PF}(\mathbf{L}) \ \text{ with } \xi_0 \in \mathbb{R}.$$

Consequently, $\mathbf{p} \in \mathscr{P}_c$ is unique up to translation and connects $\mathbf{0}$ to $\mathbf{v}^\star$.

This result is proved in Section 3.2.

This theorem establishes that the set of assumptions (H_6), $\mathbf{d} = \mathbf{1}_{N,1}$, $\mathbf{a} = \mathbf{1}_{N,1}$ is so restrictive that the multidimensional problem can in fact be reduced to the scalar one. This is really the strongest result we could hope for.

Notice that it shows that the following two mutation–competition–diffusion systems:

$$\partial_t \mathbf{u} - \partial_{xx}\mathbf{u} = r\mathbf{u} + \mathbf{M}_1\mathbf{u} - \left(\mathbf{b}^T\mathbf{u}\right)\mathbf{u},$$

$$\partial_t \mathbf{u} - \partial_{xx}\mathbf{u} = r\mathbf{u} + \mathbf{M}_2\mathbf{u} - \left(\mathbf{b}^T\mathbf{u}\right)\mathbf{u},$$

with $r > 0$ and $\mathbf{M}_1$ and $\mathbf{M}_2$ essentially nonnegative irreducible with null Perron–Frobenius eigenvalues and equal Perron–Frobenius eigenvectors, have the exact same traveling wave solutions. In other words, all else being equal (*neutral* internal structure), the mutation strategy does not matter. In the absence of mutations, *neutral genetic diversity* has been studied recently in a collection of papers by Garnier, Hamel, Roques and others (for instance, we refer to [25, 78]). In view of their results on pulled fronts, the preceding theorem indicates that the presence of mutations is a necessary and sufficient condition to ensure the preservation of the genetic diversity during the invasion.

As a side note (slightly off topic), we can use the reduction to the scalar problem to prove the following generalization of a result due to Coville and Fabre [40, Theorem 1.1].

Theorem 6.3. *Assume* (H_6) *and* $\mathbf{a} = \mathbf{1}_{N,1}$.

All positive classical solutions of $\left(E^0_{KPP}\right)$ *set in* $(0, +\infty)$ *converge as* $t \to +\infty$ *to* $\mathbf{v}^\star$.

Furthermore, if $\mathbf{d} = \mathbf{1}_{N,1}$*, then, for all bounded intervals* $I \subset \mathbb{R}$*, all bounded positive classical solutions* $\mathbf{u}$ *of* (E_{KPP}) *set in* $(0, +\infty) \times \mathbb{R}$ *satisfy*

$$\lim_{t \to +\infty} \sup_{x \in I} |\mathbf{u}(t, x) - \mathbf{v}^\star| = 0.$$

Consequently, if $\mathbf{d} = \mathbf{1}_{N,1}$*, the set of bounded nonnegative classical solutions of* (S_{KPP}) *is exactly* $\{\mathbf{0}, \mathbf{v}^\star\}$.

This result is proved in Section 3.3.

We believe that the preceding two theorems are robust, in that they should remain true in a neighborhood of $(\mathbf{d}, \mathbf{a}) = (\mathbf{1}_{N,1}, \mathbf{1}_{N,1})$. In particular, Theorem 6.2 could be extended by showing with the implicit function theorem that no solution of $(TW[c])$ bifurcates from $\mathbf{v}^\star$ at $(\mathbf{d}, \mathbf{a}) = (\mathbf{1}_{N,1}, \mathbf{1}_{N,1})$. Theorem 6.3 could be extended thanks to Conley index theory and a Morse decomposition, exactly as in Dockery–Hutson–Mischaikow–Pernarowski [58, Section 4]. For the sake of brevity, we do not address these questions.

6.1.3.2 Two-component systems with linear competition and small mutations

(H_7) $N = 2$, there exists $\mathbf{C} \gg \mathbf{0}$ such that

$$\mathbf{c}(\mathbf{v}) = \mathbf{C}\mathbf{v} \text{ for all } \mathbf{v} \in \mathsf{K},$$

and the vector $\mathbf{r} \in \mathbb{R}^N$ given by the unique decomposition of $\mathbf{L}$ of the form

$$\mathbf{L} = \operatorname{diag}(\mathbf{r}) + \mathbf{M} \text{ with } \mathbf{1}_{1,N}\mathbf{M} = \mathbf{0}$$

is positive.

By monotonicity of $\mathbf{c}$, supplementing (H_1)–(H_5) with (H_7) implies the existence of a profile $\mathbf{p} \in \mathscr{P}_c$ for all $c \geq c^\star$.

When (H_7) is satisfied, we denote $\mathbf{R} = \operatorname{diag}(\mathbf{r})$ and define $(\eta, \mathbf{m}) \in (0, +\infty) \times \mathsf{S}^{++}(\mathbf{0}, 1)$ such that

$$\mathbf{M} = \eta \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \operatorname{diag}(\mathbf{m}).$$

The quantity η is unique and commonly referred to as the *mutation rate*.

In other words, we are considering the following system:

$$\begin{cases} \partial_t u_1 - d_1 \partial_{xx} u_1 = r_1 u_1 - (c_{1,1} u_1 + c_{1,2} u_2) u_1 + \eta m_1 (u_2 - u_1) \\ \partial_t u_2 - d_2 \partial_{xx} u_2 = r_2 u_2 - (c_{2,1} u_1 + c_{2,2} u_2) u_2 + \eta m_2 (u_1 - u_2) \end{cases}.$$

The idea is to assume that η is small compared to $\mathbf{r}$ so that the mutation–competition–diffusion system is close to the pure competition–diffusion system

$$\begin{cases} \partial_t u_1 - d_1 \partial_{xx} u_1 = r_1 u_1 - (c_{1,1} u_1 + c_{1,2} u_2) u_1 \\ \partial_t u_2 - d_2 \partial_{xx} u_2 = r_2 u_2 - (c_{2,1} u_1 + c_{2,2} u_2) u_2 \end{cases}. \qquad (E_{KPP})_0$$

Indeed, two-component competition–diffusion systems being cooperative up to the change of unknowns $v = \frac{r_2}{c_{2,2}} - u_2$, the maximum principle then simplifies noticeably the characterization of the asymptotic behaviors. In particular, defining $\alpha_i = \frac{r_i}{c_{i,i}}$ for all $i \in \{1,2\}$ and, if $\det \mathbf{C} \neq 0$,

$$\mathbf{v}_m = \frac{1}{\det \mathbf{C}} \begin{pmatrix} r_1 c_{2,2} - r_2 c_{1,2} \\ r_2 c_{1,1} - r_1 c_{2,1} \end{pmatrix},$$

the asymptotic behavior of the solutions of the spatially homogeneous competitive system

$$\mathbf{u}' = \mathbf{R}\mathbf{u} - (\mathbf{C}\mathbf{u}) \circ \mathbf{u}$$

is well-known.

1. [Extinction of u_2] If $\frac{r_1}{r_2} \geq \max\left(\frac{c_{1,1}}{c_{2,1}}, \frac{c_{1,2}}{c_{2,2}}\right)$ and $\frac{r_1}{r_2} > \min\left(\frac{c_{1,1}}{c_{2,1}}, \frac{c_{1,2}}{c_{2,2}}\right)$, then $\alpha_1 \mathbf{e}_1$ is globally asymptotically stable in $\mathsf{K}^{++} \cup (\mathrm{span}(\mathbf{e}_1) \cap \mathsf{K}^{+})$ and $\alpha_2 \mathbf{e}_2$ is globally asymptotically stable in $\mathrm{span}(\mathbf{e}_2) \cap \mathsf{K}^{+}$.
2. [Coexistence] If $\frac{c_{1,2}}{c_{2,2}} < \frac{r_1}{r_2} < \frac{c_{1,1}}{c_{2,1}}$, then $\mathbf{v}_m \in \mathsf{K}^{++}$, $\mathbf{v}_m$ is globally asymptotically stable in K^{++} and, for all $i \in \{1,2\}$, $\alpha_i \mathbf{e}_i$ is globally asymptotically stable in $\mathrm{span}(\mathbf{e}_i) \cap \mathsf{K}^{+}$.
3. [Competitive exclusion] If $\frac{c_{1,2}}{c_{2,2}} > \frac{r_1}{r_2} > \frac{c_{1,1}}{c_{2,1}}$, then $\mathbf{v}_m \in \mathsf{K}^{++}$ and a one-dimensional curve S, referred to as the separatrix, induces a partition $\left(\mathsf{K}_1^{+}, \mathsf{S}, \mathsf{K}_2^{+}\right)$ of K^{+} such that $\alpha_i \mathbf{e}_i$ is globally asymptotically stable in K_i^{+} for all $i \in \{1,2\}$ and $\mathbf{v}_m$ is globally asymptotically stable in S.
4. [Extinction of u_1] If $\frac{r_1}{r_2} \leq \min\left(\frac{c_{1,1}}{c_{2,1}}, \frac{c_{1,2}}{c_{2,2}}\right)$ and $\frac{r_1}{r_2} < \max\left(\frac{c_{1,1}}{c_{2,1}}, \frac{c_{1,2}}{c_{2,2}}\right)$, then $\alpha_2 \mathbf{e}_2$ is globally asymptotically stable in $\mathsf{K}^{++} \cup (\mathrm{span}(\mathbf{e}_2) \cap \mathsf{K}^{+})$ and $\alpha_1 \mathbf{e}_1$ is globally asymptotically stable in $\mathrm{span}(\mathbf{e}_1) \cap \mathsf{K}^{+}$.

The cases 1, 2 and 4 are *monostable* whereas the case 3 is *bistable*. The case $\frac{r_1}{r_2} = \frac{c_{1,1}}{c_{2,1}} = \frac{c_{1,2}}{c_{2,2}}$ is degenerate and is usually discarded.

In the forthcoming statements, η is understood as a positive parameter which can be passed to the limit $\eta \to 0$ (notice that for all $\eta > 0$, (H_1)–(H_5) is satisfied indeed). The system (E_{KPP}) and the objects $\mathscr{P}_c$ and $c^\star$ depend on η and might be denoted respectively $(E_{KPP})_\eta$, $\mathscr{P}_{c,\eta}$ and $c^\star_\eta$. We define subsequently $\mathscr{E}$ as the set of all $(\eta, \mathbf{p}, c) \in (0,+\infty) \times \mathscr{C}^2\left(\mathbb{R}, \mathbb{R}^2\right) \times (0,+\infty)$ such that $(\mathbf{p}, c)$ is a traveling wave solution of $(E_{KPP})_\eta$. Contrarily to the case $\eta > 0$, a traveling wave solution of the limiting system $(E_{KPP})_0$ has no prescribed asymptotic behaviors.

We point out that Morris–Börger–Crooks [115] showed that the limit $c_0^\star$ of $\left(c_\eta^\star\right)_{\eta>0}$ as $\eta \to 0$ is well-defined and satisfies as expected

$$c_0^\star \geq 2\sqrt{\max_{i\in\{1,2\}} (d_i r_i)},$$

with, quite interestingly, strict inequality if

$$1 + \sqrt{1 + \frac{\alpha_i}{\alpha_{3-i}}} < \frac{2c_{3-i,3-i}}{c_{i,3-i}} \text{ and } \frac{d_i}{d_{3-i}} + \frac{r_i}{r_{3-i}} > 2 \text{ for all } i \in \{1,2\}.$$

However, they did not characterize the limiting profiles. This is what we intend to do here (but will only partially achieve).

In the following conjecture, stability is to be understood as local asymptotic stability with respect to $\left(E^0_{KPP}\right)$.

Conjecture 6.4. *Assume* (H_7)*. Let* $(\mathbf{p}_\eta)_{\eta>0}$ *and* $(c_\eta)_{\eta\geq 0}$ *such that*

$$\begin{cases} (\eta, \mathbf{p}_\eta, c_\eta) \in \mathscr{E} & \text{for all } \eta > 0, \\ c_0 = \lim_{\eta\to 0} c_\eta. \end{cases}$$

1. *Assume that both* $\alpha_1\mathbf{e}_1$ *and* $\alpha_2\mathbf{e}_2$ *are stable and that* $c_{\alpha_1\mathbf{e}_1\to\alpha_2\mathbf{e}_2} \neq 0$*. Then there exists* $(\xi_\eta)_{\eta>0}$ *such that* $(\xi \mapsto \mathbf{p}_\eta(\xi+\xi_\eta), c_\eta)_{\eta>0}$ *converges in*

$$\left(\mathscr{C}^2_{loc}\left(\mathbb{R},\mathbb{R}^2\right) \cap \mathscr{L}^\infty\left(\mathbb{R},\mathbb{R}^2\right)\right) \times \mathbb{R}$$

as $\eta \to 0$ *to a semi-extinct traveling wave solution* $(p_0\mathbf{e}_i, c_0)$ *of* $(E_{KPP})_0$ *connecting* $\mathbf{0}$ *to* $\alpha_i\mathbf{e}_i$ *with*

$$i = \begin{cases} 1 & \text{if } c_{\alpha_1\mathbf{e}_1\to\alpha_2\mathbf{e}_2} > 0, \\ 2 & \text{if } c_{\alpha_1\mathbf{e}_1\to\alpha_2\mathbf{e}_2} < 0. \end{cases}$$

2. *Assume that there is a unique stable state* $\mathbf{v}_s \in \{\alpha_1\mathbf{e}_1, \alpha_2\mathbf{e}_2, \mathbf{v}_m\}$*. Then one and only one of the following two properties holds true.*
   a) *There exists* $(\xi_\eta)_{\eta>0}$ *such that* $(\xi \mapsto \mathbf{p}_\eta(\xi+\xi_\eta), c_\eta)_{\eta>0}$ *converges in*

   $$\left(\mathscr{C}^2_{loc}\left(\mathbb{R},\mathbb{R}^2\right) \cap \mathscr{L}^\infty\left(\mathbb{R},\mathbb{R}^2\right)\right) \times \mathbb{R}$$

   as $\eta \to 0$ *to a component-wise monotonic traveling wave solution* $(\mathbf{p}_0, c_0)$ *of* $(E_{KPP})_0$ *connecting* $\mathbf{0}$ *to* $\mathbf{v}_s$*.*
   b) *There exist* $(\xi^1_\eta)_{\eta>0}$*,* $(\xi^2_\eta)_{\eta>0}$ *and a unique* $i \in \mathsf{I}_u$ *such that, as* $\eta \to 0$*:*
      — $\xi^2_\eta - \xi^1_\eta \to +\infty$*;*
      — $\left(\xi \mapsto \mathbf{p}_\eta\left(\xi+\xi^2_\eta\right), c_\eta\right)_{\eta>0}$ *converges in* $\mathscr{C}^2_{loc}\left(\mathbb{R},\mathbb{R}^2\right)\times\mathbb{R}$ *to a semi-extinct traveling wave solution* $(p_{front}\mathbf{e}_i, c_0)$ *of* $(E_{KPP})_0$ *connecting* $\mathbf{0}$ *to* $\alpha_i\mathbf{e}_i$*;*
      — $\left(\xi \mapsto \mathbf{p}_\eta\left(\xi+\xi^1_\eta\right), c_\eta\right)_{\eta>0}$ *converges in* $\mathscr{C}^2_{loc}\left(\mathbb{R},\mathbb{R}^2\right)\times\mathbb{R}$ *to a component-wise monotonic traveling wave solution* $(\mathbf{p}_{back}, c_0)$ *of* $(E_{KPP})_0$ *connecting* $\alpha_i\mathbf{e}_i$ *to* $\mathbf{v}_s$*.*

We emphasize once more that traveling waves with minimal speed $c^\star_\eta$ do not, in general, converge to a traveling wave with minimal speed. In particular, Figure 6.5.1 illustrates an interesting case of invasion driven by the fast phenotype u_2 but where the only settler is the slow phenotype u_1. This is reminiscent of Griette–Raoul [82], where an analogous result was established analytically under a stronger scaling.

Conjecture 6.4, 1 is expected to be a very difficult problem and seems to be beyond our reach. We leave it as an open problem.

On the contrary, regarding Conjecture 6.4, 2, a partial confirmation is within reach. On one hand, we point out that the special case

$$\frac{c_{1,1}}{c_{2,1}} = \frac{c_{1,2}}{c_{2,2}} = 1 \text{ and } \mathbf{d} = \mathbf{1}_{2,1}$$

is somehow solved by Theorem 6.2 without any assumption on $\mathbf{r}$. On the other hand, we also have the following general theorem which concerns all monostable cases apart from

$$\frac{c_{1,1}}{c_{2,1}} < \frac{c_{1,2}}{c_{2,2}} = \frac{r_1}{r_2},$$

$$\frac{r_1}{r_2} = \frac{c_{1,1}}{c_{2,1}} < \frac{c_{1,2}}{c_{2,2}}.$$

Theorem 6.5. *Assume* (H_7) *and the existence of* $i \in \{1,2\}$ *such that*

$$\frac{r_i}{r_{3-i}} > \frac{c_{i,3-i}}{c_{3-i,3-i}}.$$

Let

$$\mathbf{v}_s = \begin{cases} \alpha_i \mathbf{e}_i & \text{if } \frac{r_i}{r_{3-i}} \geq \frac{c_{i,i}}{c_{3-i,i}}, \\ \mathbf{v}_m & \text{if } \frac{r_i}{r_{3-i}} < \frac{c_{i,i}}{c_{3-i,i}}. \end{cases}$$

For all $(\mathbf{p}_\eta)_{\eta>0}$ *and* $(c_\eta)_{\eta\geq 0}$ *such that*

$$\begin{cases} (\eta, \mathbf{p}_\eta, c_\eta) \in \mathscr{E} & \text{for all } \eta > 0, \\ c_0 = \lim_{\eta \to 0} c_\eta, \end{cases}$$

there exists $(\zeta_\eta)_{\eta>0}$ *such that, as* $\eta \to 0$, $(\xi \mapsto \mathbf{p}_\eta(\xi + \zeta_\eta), c_\eta)_{\eta>0}$ *converges up to extraction in* $\mathscr{C}^2_{loc}(\mathbb{R}, \mathbb{R}^2) \times \mathbb{R}$ *to a traveling wave solution* $(\mathbf{p}, c_0)$ *of* $(E_{KPP})_0$ *achieving one of the following connections:*

1. $\mathbf{0}$ *to* $\mathbf{v}_s$*,*
2. $\alpha_{3-i}\mathbf{e}_{3-i}$ *to* $\mathbf{v}_s$*,*
3. $\mathbf{0}$ *to* $\alpha_i \mathbf{e}_i$ *with* $\mathbf{p}$ *semi-extinct.*

This result is proved in Section 4.

Let us clarify how this result confirms partially Conjecture 6.4, 2 and what are the remaining open questions.

— Assume $\mathbf{v}_s = \mathbf{v}_m$. Up to the component-wise monotonicity of the profile in the first and second cases, the three connections above correspond exactly to the three possible limiting profiles of Conjecture 6.4, 2. Moreover we can apply the theorem with $i = 1$ and $i = 2$ and obtain two limiting profiles. However, at this point, the normalizations $(\zeta^1_\eta)_{\eta>0}$ and $(\zeta^2_\eta)_{\eta>0}$ are unrelated and nine possible pairs of profiles seem to exist. We do not know how to prove that only the three following situations actually occur: $\mathbf{0}$ to $\mathbf{v}_m$ and $\mathbf{0}$ to $\mathbf{v}_m$ with $(\zeta^2_\eta - \zeta^1_\eta)_{\eta>0}$ bounded, semi-extinct $\mathbf{0}$ to $\alpha_1\mathbf{e}_1$ and $\alpha_1\mathbf{e}_1$ to $\mathbf{v}_m$ with $\zeta^2_\eta - \zeta^1_\eta \to -\infty$, semi-extinct $\mathbf{0}$ to $\alpha_2\mathbf{e}_2$ and $\alpha_2\mathbf{e}_2$ to $\mathbf{v}_m$ with $\zeta^2_\eta - \zeta^1_\eta \to +\infty$.

— Assume $\mathbf{v}_s = \alpha_i\mathbf{e}_i$. The third connection above is actually a subcase of the first one and the normalization $(\zeta_\eta)_{\eta>0}$ is unable to track the semi-extinct limiting profile connecting $\mathbf{0}$ to $\alpha_{3-i}\mathbf{e}_{3-i}$. This is not a question of optimality of the proof: the normalization $(\zeta_\eta)_{\eta>0}$ is precisely chosen so that p_i is always non-zero. Hence $(\zeta_\eta)_{\eta>0}$ corresponds either to $(\xi_\eta)_{\eta>0}$ or to $(\xi^1_\eta)_{\eta>0}$. The construction of the normalization $(\xi^2_\eta)_{\eta>0}$ of Conjecture 6.4, 2 is a completely open problem. Of course, once this problem is solved, it remains to relate the limiting profiles and the normalizations, as in the case $\mathbf{v}_s = \mathbf{v}_m$.

6.2 The edge of the fronts

In this section, we fix a traveling wave $(\mathbf{p}, c)$ and we prove Theorem 6.1.

6.2.1 Preparatory lemmas and the Ikehara theorem

Lemma 6.6. *For all $i \in [N]$,*

$$\left\{ \liminf_{+\infty} \frac{-p_i'}{p_i}, \limsup_{+\infty} \frac{-p_i'}{p_i} \right\} \subset \left\{ \mu \in (0,+\infty) \mid \frac{\lambda_{PF}\left(\mu^2 \mathbf{D} + \mathbf{L}\right)}{\mu} = c \right\},$$

$$\left\{ \liminf_{+\infty} \frac{p_i''}{p_i}, \limsup_{+\infty} \frac{p_i''}{p_i} \right\} \subset \left\{ \mu^2 \in (0,+\infty) \mid \frac{\lambda_{PF}\left(\mu^2 \mathbf{D} + \mathbf{L}\right)}{\mu} = c \right\}.$$

Consequently, there exists $\tilde{\xi} \in \mathbb{R}$ such that $\mathbf{p}$ is component-wise strictly convex in $[\tilde{\xi}, +\infty)$.

Proof. The proof of

$$\min_{i\in[N]} \liminf_{+\infty} \frac{-p_i'}{p_i} \in \left\{ \mu > 0 \mid \frac{\lambda_{PF}\left(\mu^2 \mathbf{D} + \mathbf{L}\right)}{\mu} = c \right\}$$

can be found in [Gir18b, Proposition 6.10]. The proof also directly yields that for any sequence $(\xi_n)_{n\in\mathbb{N}}$ such that $\xi_n \to +\infty$ and such that there exists $j \in [N]$ satisfying

$$\lim_{n\to+\infty} \frac{-p_j'(\xi_n)}{p_j(\xi_n)} = \min_{i\in[N]} \liminf_{+\infty} \frac{-p_i'}{p_i},$$

convergence occurs in the following sense:

$$\lim_{n\to+\infty} \left(\frac{-p_i'(\xi_n)}{p_i(\xi_n)} \right)_{i\in[N]} = \left(\min_{i\in[N]} \liminf_{+\infty} \frac{-p_i'}{p_i} \right) \mathbf{1}_{N,1}.$$

The proof of

$$\max_{i\in[N]} \limsup_{+\infty} \frac{-p_i'}{p_i} \in \left\{ \mu > 0 \mid \frac{\lambda_{PF}\left(\mu^2 \mathbf{D} + \mathbf{L}\right)}{\mu} = c \right\}$$

is a slight modification of the preceding proof, where the quantity

$$\overline{\Lambda} = \max_{i\in[N]} \limsup_{\xi\to+\infty} \frac{p_i'(\xi)}{p_i(\xi)}$$

is replaced by

$$\underline{\Lambda} = \min_{i\in[N]} \liminf_{\xi\to+\infty} \frac{p_i'(\xi)}{p_i(\xi)}.$$

Similarly, we also obtain directly that for any sequence $(\xi_n)_{n\in\mathbb{N}}$ such that $\xi_n \to +\infty$ and such that there exists $j \in [N]$ satisfying

$$\lim_{n\to+\infty} \frac{-p_j'(\xi_n)}{p_j(\xi_n)} = \max_{i\in[N]} \limsup_{+\infty} \frac{-p_i'}{p_i},$$

convergence occurs in the following sense:

$$\lim_{n\to+\infty} \left(\frac{-p_i'(\xi_n)}{p_i(\xi_n)} \right)_{i\in[N]} = \left(\max_{i\in[N]} \limsup_{+\infty} \frac{-p_i'}{p_i} \right) \mathbf{1}_{N,1}.$$

The statements regarding $\left(\frac{p_i''}{p_i}\right)_{i\in[N]}$ are again established very similarly. The quantity

$$\overline{\Lambda} = \max_{i\in[N]} \limsup_{\xi\to+\infty} \frac{p_i'(\xi)}{p_i(\xi)}$$

is replaced by

$$\underline{\Theta} = \min_{i\in[N]} \liminf_{\xi\to+\infty} \frac{p_i''(\xi)}{p_i(\xi)}$$

and

$$\overline{\Theta} = \max_{i\in[N]} \liminf_{\xi\to+\infty} \frac{p_i''(\xi)}{p_i(\xi)}$$

respectively, and the function

$$\mathbf{w}_n = \overline{\Lambda}\hat{\mathbf{p}}_n - \hat{\mathbf{p}}_n'$$

is replaced by

$$\mathbf{w}_n = \underline{\Theta}\hat{\mathbf{p}}_n - \hat{\mathbf{p}}_n''$$

and

$$\mathbf{w}_n = \overline{\Theta}\hat{\mathbf{p}}_n - \hat{\mathbf{p}}_n''$$

respectively. Since $\hat{\mathbf{p}}_\infty$ is nonnegative nonzero and $\mathbf{w}_\infty = \mathbf{0}$, necessarily $\underline{\Theta} > 0$ and $\overline{\Theta} > 0$ and then, as in [Gir18b, Proposition 6.10], both quantities have the form μ^2 with μ solution of $\frac{\lambda_{PF}(\mu^2\mathbf{D}+\mathbf{L})}{\mu} = c$.

Finally, the strict convexity in a neighborhood of $+\infty$ is deduced exactly as the monotonicity in the proof of [Gir18b, Proposition 6.10]. □

We will also need the Ikehara theorem [34, Proposition 2.3], commonly used in such problems (see for instance Guo–Wu [85]), as well as a lemma due to Volpert, Volpert and Volpert [139, Chapter 5, Lemma 4.1].

Theorem 6.7. *[Ikehara] Let $f : (0,+\infty) \to (0,+\infty)$ be a decreasing function. Assume that there exist $\overline{\lambda} \in (0,+\infty)$, $k \in (-1,+\infty)$ and an analytic function*

$$h : \left(0,\overline{\lambda}\right] + i\mathbb{R} \to (0,+\infty)$$

such that

$$\int_0^{+\infty} e^{\lambda x} f(x)\, dx = \frac{h(\lambda)}{\left(\overline{\lambda}-\lambda\right)^{k+1}} \text{ for all } \lambda \in \left(0,\overline{\lambda}\right).$$

Then

$$\lim_{x\to+\infty} f(x) \frac{e^{\overline{\lambda}x}}{x^k} = \frac{h\left(\overline{\lambda}\right)}{\Gamma\left(\overline{\lambda}+1\right)}.$$

Lemma 6.8. *[Volpert–Volpert–Volpert] Let $\mathbf{A}$ be an essentially nonnegative matrix and let $\mathbf{z} \in \mathbb{C}^N$.*

If

$$\begin{cases} sp\mathbf{A} \subset (-\infty,0) + i\mathbb{R}, \\ (Re(z_k))_{k\in[N]} \leq \mathbf{0}, \end{cases}$$

then

$$sp\left(\mathbf{A} + diag(\mathbf{z})\right) \subset (-\infty,0) + i\mathbb{R}.$$

6.2.2 Convergence at the edge

Let
$$k_c = \begin{cases} 0 & \text{if } c > c^\star, \\ 1 & \text{if } c = c^\star. \end{cases}$$

Proposition 6.9. *There exists $A > 0$ such that, as $\xi \to +\infty$,*
$$\begin{cases} \mathbf{p}(\xi) \sim A\xi^{k_c} e^{-\mu_c \xi} \mathbf{n}_{\mu_c}, \\ \mathbf{p}'(\xi) \sim -\mu_c \mathbf{p}(\xi), \\ \mathbf{p}''(\xi) \sim \mu_c^2 \mathbf{p}(\xi). \end{cases}$$

Proof. Fix temporarily $\mu \in (0, \mu_c) + i\mathbb{R}$. In view of Lemma 6.6 and of the Gronwall lemma,
$$\xi \mapsto \mathrm{e}^{\mu\xi} \mathbf{p}(\xi) \in \mathscr{L}^1\left(\mathbb{R}, \mathbb{C}^N\right),$$
$$\xi \mapsto \mathrm{e}^{\mu\xi} \mathbf{c}(\mathbf{p}(\xi)) \circ \mathbf{p}(\xi) \in \mathscr{L}^1\left(\mathbb{R}, \mathbb{C}^N\right).$$
Multiplying $(TW[c])$ by $\mathrm{e}^{\mu\xi}$, integrating by parts over $\mathbb{R}$ and defining
$$\mathbf{f}_+(\mu) = \int_0^{+\infty} \mathrm{e}^{\mu\xi} \mathbf{p}(\xi) \, \mathrm{d}\xi,$$
$$\mathbf{f}_-(\mu) = \int_{-\infty}^{0} \mathrm{e}^{\mu\xi} \mathbf{p}(\xi) \, \mathrm{d}\xi,$$
$$\mathbf{f}_\mathbf{c}(\mu) = \int_\mathbb{R} \mathrm{e}^{\mu\xi} \mathbf{c}(\mathbf{p}(\xi)) \circ \mathbf{p}(\xi) \, \mathrm{d}\xi,$$
we get easily
$$\left(\mu^2 \mathbf{D} - c\mu \mathbf{I} + \mathbf{L}\right)\left(\mathbf{f}_+(\mu) + \mathbf{f}_-(\mu)\right) = \mathbf{f}_\mathbf{c}(\mu),$$
whence, denoting $\mathrm{adj}\left(\mu^2 \mathbf{D} - c\mu \mathbf{I} + \mathbf{L}\right)$ the adjugate matrix of $\mu^2 \mathbf{D} - c\mu \mathbf{I} + \mathbf{L}$, we find
$$\det\left(\mu^2 \mathbf{D} - c\mu \mathbf{I} + \mathbf{L}\right) \mathbf{f}_+(\mu) = \mathrm{adj}\left(\mu^2 \mathbf{D} - c\mu \mathbf{I} + \mathbf{L}\right) \mathbf{f}_\mathbf{c}(\mu) - \det\left(\mu^2 \mathbf{D} - c\mu \mathbf{I} + \mathbf{L}\right) \mathbf{f}_-(\mu).$$

The functions $\mathbf{f}_+$, $\mathbf{f}_-$ and $\mathbf{f}_\mathbf{c}$ defined above are respectively analytic in $(0, \mu_c) + i\mathbb{R}$, $(0, +\infty) + i\mathbb{R}$ and $(0, 2\mu_c) + i\mathbb{R}$ (by local Lipschitz-continuity of $\mathbf{c}$, (H_2) and global boundedness of $\mathbf{p}$).

The function
$$\begin{array}{ccc} \mathbb{C} & \to & \mathbb{C} \\ \mu & \mapsto & \det\left(\mu^2 \mathbf{D} - c\mu \mathbf{I} + \mathbf{L}\right) \end{array}$$
is polynomial (whence analytic). Let $\mathsf{Z} \subset \mathbb{C}$ be the finite set of its roots, counted with algebraic multiplicity. In particular, $\mu_c \in \mathsf{Z}$ with multiplicity $k_c + 1$.

For all $\mu \in ((0, \mu_c) + i\mathbb{R}) \backslash \mathsf{Z}$,
$$\mathbf{f}_+(\mu) = \left(\mu^2 \mathbf{D} - c\mu \mathbf{I} + \mathbf{L}\right)^{-1} \mathbf{f}_\mathbf{c}(\mu) - \mathbf{f}_-(\mu).$$
The function
$$\mu \mapsto \left(\mu^2 \mathbf{D} - c\mu \mathbf{I} + \mathbf{L}\right)^{-1} \mathbf{f}_\mathbf{c}(\mu)$$
is well-defined and analytic in $((0, \mu_c) + i\mathbb{R}) \backslash \mathsf{Z}$, where it coincides with $\mathbf{f}_+ + \mathbf{f}_-$ which is analytic in $(0, \mu_c) + i\mathbb{R}$.

Define the analytic function
$$\begin{array}{cccc} \mathbf{h}: & (0, \mu_c) + i\mathbb{R} & \to & \mathbb{R}^N \\ & \mu & \mapsto & (\mu_c - \mu)^{k_c+1} \mathbf{f}_+(\mu) \end{array}$$

so that

$$\mathbf{f}_+(\mu) = \frac{\mathbf{h}(\mu)}{(\mu_c-\mu)^{k_c+1}} \text{ for all } \mu \in (0,\mu_c) + i\mathbb{R}.$$

Since, for all $\mu \in ((0,\mu_c)+i\mathbb{R})\setminus \mathsf{Z}$,

$$\mathbf{h}(\mu) = \frac{(\mu_c-\mu)^{k_c+1}}{\det(\mu^2\mathbf{D}-c\mu\mathbf{I}+\mathbf{L})}\operatorname{adj}\left(\mu^2\mathbf{D}-c\mu\mathbf{I}+\mathbf{L}\right)\mathbf{f_c}(\mu) - (\mu_c-\mu)^{k_c+1}\mathbf{f}_-(\mu)$$

the function $\mathbf{h}$ can be analytically extended on $(0,\mu_c]+i\mathbb{R}$ if and only if

$$\mu \mapsto \frac{(\mu_c-\mu)^{k_c+1}}{\det(\mu^2\mathbf{D}-c\mu\mathbf{I}+\mathbf{L})}$$

has no pole in $\{\mu_c\}+i\mathbb{R}$.

Let $\theta \in \mathbb{R}\setminus\{0\}$. In view of

$$(\mu_c+i\theta)^2\mathbf{D} - c(\mu_c+i\theta)\mathbf{I}+\mathbf{L} = \mu_c^2\mathbf{D}-c\mu_c\mathbf{I}+\mathbf{L}-\theta^2\mathbf{D}+i\theta(2\mu_c\mathbf{D}-c\mathbf{I})$$

and

$$\begin{aligned}\lambda_{PF}\left(\mu_c^2\mathbf{D}-c\mu_c\mathbf{I}+\mathbf{L}-\theta^2\mathbf{D}\right) &\leq \lambda_{PF}\left(\mu_c^2\mathbf{D}-c\mu_c\mathbf{I}+\mathbf{L}-\theta^2\min_{k\in[N]}d_k\right)\\ &= \lambda_{PF}\left(\mu_c^2\mathbf{D}-c\mu_c\mathbf{I}+\mathbf{L}\right)-\theta^2\min_{k\in[N]}d_k\\ &= -\theta^2\min_{k\in[N]}d_k\end{aligned}$$

Lemma 6.8 yields that

$$\operatorname{sp}\left(\left(\mu_c^2\mathbf{D}-c\mu_c\mathbf{I}+\mathbf{L}-\theta^2\mathbf{D}\right)+\operatorname{diag}\left(i\theta(2\mu_c d_k-c)\right)_{k\in[N]}\right)\subset(-\infty,0)+i\mathbb{R}.$$

Hence $\mu\mapsto\frac{(\mu_c-\mu)^{k_c+1}}{\det(\mu^2\mathbf{D}-c\mu\mathbf{I}+\mathbf{L})}$ has no pole in $\{\mu_c\}+i(\mathbb{R}\setminus\{0\})$ and then it has no pole in $\{\mu_c\}+i\mathbb{R}$ indeed.

We are now in position to apply the Ikehara theorem component-wise and to deduce from it the existence of $\mathbf{n}\in\mathsf{S}^+(\mathbf{0},1)$ and $A\geq 0$ such that

$$\lim_{\xi\to+\infty}\mathbf{p}(\xi)\frac{e^{\mu_c\xi}}{\xi^{k_c}} = A\mathbf{n}.$$

In particular, for all $k\in[N]$ such that $n_k>0$,

$$\lim_{\zeta\to+\infty}\frac{\mathbf{p}(\xi+\zeta)}{p_k(\zeta)}e^{\mu_c\xi} = \frac{1}{n_k}\mathbf{n}.$$

However, back to the proof of Lemma 6.6, there exists $k\in[N]$ and a sequence $(\xi_n)_{n\in\mathbb{N}}$ such that $\xi_n\to+\infty$, $\left(\frac{-p_k'(\xi_n)}{p_k(\xi_n)}\right)_{n\in\mathbb{N}}$ converges to

$$\mu = \max_{k\in[N]}\limsup_{+\infty}\frac{-p_k'}{p_k},$$

and

$$\left(\xi\mapsto\frac{\mathbf{p}(\xi+\zeta_n)}{p_k(\zeta_n)}\right)_{n\in\mathbb{N}}$$

converges in $\mathscr{C}^2_{loc}$ to

$$\xi \mapsto \frac{1}{n_{\mu,k}} \mathrm{e}^{-\mu\xi} \mathbf{n}_{\mu}.$$

This clearly implies $\mu = \mu_c$ and $\mathbf{n} = \mathbf{n}_{\mu_c}$.

Consequently, $A > 0$,

$$\lim_{\xi \to +\infty} \mathbf{p}(\xi) \frac{\mathrm{e}^{\mu_c \xi}}{\xi^{k_c}} = A\mathbf{n}_{\mu_c},$$

and, by Lemma 6.6,

$$\mu_c \leq \min_{k \in [N]} \liminf_{+\infty} \frac{-p_k'}{p_k} \leq \max_{k \in [N]} \limsup_{+\infty} \frac{-p_k'}{p_k} = \mu_c,$$

that is

$$\lim_{+\infty} \left(\frac{-p_k'}{p_k} \right)_{k \in [N]} = \mu_c.$$

Quite similarly, we also obtain

$$\lim_{+\infty} \left(\frac{p_k''}{p_k} \right)_{k \in [N]} = \mu_c^2.$$

□

If $\mathbf{d} = \mathbf{1}_{N,1}$, the quantities at hand are:

$$\begin{aligned}(\mu_c, \mathbf{n}_{\mu_c}) &= \left(\min \left\{ \mu > 0 \mid \lambda_{PF}\left(\mu^2 \mathbf{I} + \mathbf{L}\right) = c\mu \right\}, \mathbf{n}_{PF}\left(\mu_c^2 \mathbf{I} + \mathbf{L}\right) \right) \\ &= \left(\frac{1}{2} \left(c - \sqrt{c^2 - 4\lambda_{PF}(\mathbf{L})} \right), \mathbf{n}_{PF}(\mathbf{L}) \right)\end{aligned}$$

and an obvious corollary follows.

6.3 The back of the fronts: separated competition

In this section, we assume (H_6) and $\mathbf{a} = \mathbf{1}_{N,1}$ and prove Theorem 6.2 and Theorem 6.3.

6.3.1 Main tools: Jordan normal form and Perron–Frobenius projection

Let $m \in [N]$ be the number of pairwise distinct eigenvalues of $\mathbf{L}$ ($\lambda_{PF}(\mathbf{L})$ being simple, $m \geq 2$) and let $(\lambda_k)_{k \in [m]} \in \mathbb{C}^m$ be the pairwise distinct complex eigenvalues of $\mathbf{L}$ ordered so that $(\mathrm{Re}(\lambda_k))_{k \in [m]}$ is a nondecreasing family (in particular, $\lambda_m = \lambda_{PF}(\mathbf{L})$ and $\mathrm{Re}(\lambda_{m-1}) < \lambda_{PF}(\mathbf{L})$).

Let $\mathbf{P} \in \mathsf{GL}(\mathbb{C})$ be such that $\mathbf{J} = \mathbf{PLP}^{-1}$ is the Jordan normal form of $\mathbf{L}$:

$$\mathbf{J} = \begin{pmatrix} \lambda_{PF}(\mathbf{L}) & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_{m-1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{J}_1 \end{pmatrix},$$

where, for all $k \in [m-1]$, $\mathbf{J}_k$ is the (upper triangular) Jordan block associated with the eigenvalue λ_k.

Noticing that

$$\mathbf{LP}^{-1}\mathbf{e}_1 = \mathbf{P}^{-1}\mathbf{J}\mathbf{e}_1 = \lambda_{PF}\left(\mathbf{L}\right)\mathbf{P}^{-1}\mathbf{e}_1,$$
$$\mathbf{e}_1^T\mathbf{PL} = \mathbf{e}_1^T\mathbf{JP} = \lambda_{PF}\left(\mathbf{L}\right)\mathbf{e}_1^T\mathbf{P},$$

it follows that $\mathbf{P}^{-1}\mathbf{e}_1 \in \mathrm{span}\mathbf{n}_{PF}\left(\mathbf{L}\right)$ and $\mathbf{e}_1^T\mathbf{P} \in \mathrm{span}\mathbf{n}_{PF}\left(\mathbf{L}^T\right)^T$. In particular, we can normalize without loss of generality $\mathbf{P}$ so that $\mathbf{P}^{-1}\mathbf{e}_1 = \mathbf{n}_{PF}\left(\mathbf{L}\right)$ and then deduce from $\mathbf{e}_1^T\mathbf{P}\mathbf{n}_{PF}\left(\mathbf{L}\right) = 1$ that

$$\mathbf{e}_1^T\mathbf{P} = \frac{1}{\mathbf{n}_{PF}\left(\mathbf{L}^T\right)^T\mathbf{n}_{PF}\left(\mathbf{L}\right)}\mathbf{n}_{PF}\left(\mathbf{L}^T\right)^T.$$

From the preceding equality, it follows directly that the Perron–Frobenius projection, defined as

$$\mathbf{\Pi}_{PF}\left(\mathbf{L}\right) = \frac{\mathbf{n}_{PF}\left(\mathbf{L}\right)\mathbf{n}_{PF}\left(\mathbf{L}^T\right)^T}{\mathbf{n}_{PF}\left(\mathbf{L}^T\right)^T\mathbf{n}_{PF}\left(\mathbf{L}\right)},$$

satisfies

$$\mathbf{P}\mathbf{\Pi}_{PF}\left(\mathbf{L}\right)\mathbf{P}^{-1} = \mathrm{diag}\left(\mathbf{e}_1\right).$$

6.3.2 Uniqueness up to translation of the profile

In this subsection, we assume $\mathbf{d} = \mathbf{1}_{N,1}$, we fix $c \geq c^\star$ and we prove Theorem 6.2. The scalar front p_c is defined as in the statement of the theorem.

Proposition 6.10. *All $\mathbf{p} \in \mathscr{P}_c$ have the form*

$$\mathbf{p} : \xi \mapsto p_c\left(\xi - \xi_0\right)\mathbf{n}_{PF}\left(\mathbf{L}\right) \text{ with } \xi_0 \in \mathbb{R}.$$

Proof. Let $\mathbf{p} \in \mathscr{P}_c$ and

$$\mathbf{q} = \mathbf{Pp} \in \mathscr{C}^2\left(\mathbb{R}, \mathbb{C}^N\right) \cap \mathscr{L}^\infty\left(\mathbb{R}, \mathbb{C}^N\right).$$

Multiplying $(TW\left[c\right])$ on the left by $\mathbf{P}$, we get

$$-\mathbf{q}'' - c\mathbf{q}' = \mathbf{Jq} - b\left[\mathbf{P}^{-1}\mathbf{q}\right]\mathbf{q} \text{ in } \mathbb{R},$$

and in particular

$$-q_1'' - cq_1' = \left(\lambda_{PF}\left(\mathbf{L}\right) - b\left[\mathbf{P}^{-1}\mathbf{q}\right]\right)q_1 \text{ in } \mathbb{R}.$$

Since

$$\begin{aligned}\left(\mathbf{\Pi}_{PF}\left(\mathbf{L}\right)\mathbf{p}\right)^T\mathbf{n}_{PF}\left(\mathbf{L}\right) &= \left(\mathbf{P}^{-1}\mathrm{diag}\left(\mathbf{e}_1\right)\mathbf{q}\right)^T\mathbf{n}_{PF}\left(\mathbf{L}\right)\\ &= q_1\left(\mathbf{P}^{-1}\mathbf{e}_1\right)^T\mathbf{n}_{PF}\left(\mathbf{L}\right)\\ &= q_1,\end{aligned}$$

q_1 is real-valued and in fact positive in $\mathbb{R}$.

First, let us verify that $\frac{q_k}{q_1}$ is globally bounded in $\mathbb{R}$ for all $k \in [N]\setminus\{1\}$. It is bounded in $(-\infty, 0]$ since $\inf\limits_{(-\infty,0]} q_1 > 0$ by [Gir18b, Theorem 1.5, iii)]. It is bounded in $[0, +\infty)$ since a left-multiplication of the first equivalent of Theorem 6.1 by $\mathbf{P}$ yields

$$\mathbf{q}\left(\xi\right) \sim A\xi^k \mathrm{e}^{-\frac{1}{2}\left(c-\sqrt{c^2-4\lambda_{PF}(\mathbf{L})}\right)\xi}\mathbf{e}_1$$

whence

$$\limsup_{+\infty}\left|\frac{q_k}{q_1}\right| = 0.$$

Next, let us show by induction that $q_{N+1-k} = 0$ in $\mathbb{R}$ for all $k \in [N-1]$.

— Basis: $k=1$. Due to the special form of $\mathbf{J}$, the equation satisfied by q_N is

$$-q_N''-cq_N'=\left(\lambda_1-b\left[\mathbf{P}^{-1}\mathbf{q}\right]\right)q_N \text{ in } \mathbb{R}.$$

Define $z=\frac{q_N}{q_1}$ and $w=|z|^2$. The function w is nonnegative and globally bounded. From

$$z'=\frac{q_N'}{q_1}-\frac{q_1'}{q_1}z,$$
$$z''=\frac{q_N''}{q_1}-\frac{q_1''}{q_1}z-\frac{2q_1'}{q_1}z',$$

it follows

$$-z''-\frac{q_1c+2q_1'}{q_1}z'-\frac{q_1''+cq_1'}{q_1}z=\left(\lambda_1-b\left[\mathbf{P}^{-1}\mathbf{q}\right]\right)z \text{ in } \mathbb{R}.$$

Using the equality satisfied by q_1, this equation reads:

$$-z''-\frac{q_1c+2q_1'}{q_1}z'+\left(\lambda_{PF}\left(\mathbf{L}\right)-\lambda_1\right)z=0 \text{ in } \mathbb{R}.$$

Now, multiplying by $\overline{z}$, taking the real part, defining

$$\gamma=2\left(\lambda_{PF}\left(\mathbf{L}\right)-\operatorname{Re}\left(\lambda_1\right)\right)>0$$

and using the obvious equality

$$\begin{aligned}\operatorname{Re}\left(z''\overline{z}\right)&=\operatorname{Re}\left(z\right)''\operatorname{Re}\left(z\right)+\operatorname{Im}\left(z\right)''\operatorname{Im}\left(z\right)\\&=\frac{1}{2}w''-\left(\operatorname{Re}\left(z\right)'\right)^2-\left(\operatorname{Im}\left(z\right)'\right)^2,\end{aligned}$$

it follows

$$-w''-\frac{q_1c+2q_1'}{q_1}w'+\gamma w\leq 0 \text{ in } \mathbb{R}.$$

This inequality implies the nonexistence of local maxima of w. Since $w\in\mathscr{C}^1\left(\mathbb{R}\right)$, there exists consequently $\xi_0\in\overline{\mathbb{R}}$ such that w is decreasing on $(-\infty,\xi_0)$ and increasing on $(\xi_0,+\infty)$. Therefore w has well-defined limits at $\pm\infty$ and since $w\in\mathscr{L}^\infty\left(\mathbb{R}\right)$, these limits are finite. By classical elliptic regularity and the Harnack inequality (see Gilbarg–Trudinger [80]) applied to the equation satisfied by q_1, $\frac{q_1'}{q_1}$ is bounded in $\mathbb{R}$. By elliptic regularity again, applied this time to the equation

$$-w''-\frac{q_1c+2q_1'}{q_1}w'+\gamma w=-2\left(\operatorname{Re}\left(z\right)'\right)^2-2\left(\operatorname{Im}\left(z\right)'\right)^2,$$

the limits of w have to be null, whence w itself is null, and then q_N is null.

— Inductive step: let $k\in[N-1]\setminus\{1\}$ and assume $q_{N+1-k}=0$. Defining

$$\lambda=j_{N-k,N-k}\in\operatorname{sp}\mathbf{L}\setminus\left\{\lambda_{PF}\left(\mathbf{L}\right)\right\},$$

the equation satisfied by $q_{N+1-(k+1)}=q_{N-k}$ is

$$-q_{N-k}''-cq_{N-k}'=\left(\lambda-b\left[\mathbf{P}^{-1}\mathbf{q}\right]\right)q_{N-k} \text{ in } \mathbb{R}.$$

Repeating the argument detailed in the previous step shows similarly that q_{N-k} is null.

Hence the proof by induction is ended and yields indeed $\mathbf{q}=q_1\mathbf{e}_1$ in $\mathbb{R}$. Now, back to the equation satisfied by q_1, we find

$$-q_1''-cq_1'=\left(\lambda_{PF}\left(\mathbf{L}\right)-b\left[q_1\mathbf{n}_{PF}\left(\mathbf{L}\right)\right]\right)q_1 \text{ in } \mathbb{R},$$

which implies in view of well-known results on the traveling wave equation for the scalar KPP equation the existence of $\xi_0\in\mathbb{R}$ such that q_1 coincides with $\xi\mapsto p_c\left(\xi-\xi_0\right)$. □

6.3.3 Global asymptotic stability

The auxiliary functions used in the proof of Proposition 6.10 can be used again to prove the global asymptotic stability of $\mathbf{v}^\star$ as stated in Theorem 6.3. In particular, the following lemma will be used repeatedly.

Lemma 6.11. *There exists $\gamma > 0$ such that all bounded positive classical solutions $\mathbf{u}$ of (E_{KPP}) set in $(0,+\infty)\times\mathbb{R}$ satisfying*

$$\inf_{(t,x)\in(0,+\infty)\times\mathbb{R}} \mathbf{n}_{PF}(\mathbf{L})^T \mathbf{\Pi}_{PF}(\mathbf{L})\,\mathbf{u}(t,x) > 0$$

satisfy also

$$\lim_{t\to+\infty}\left(e^{\gamma t}\sup_{x\in\mathbb{R}}\left|\left(\mathbf{I}-\mathbf{\Pi}_{PF}(\mathbf{L})\right)\mathbf{u}(t,x)\right|\right) = 0.$$

Proof. The proof is very similar to the first part of that of Proposition 6.10. Defining $\mathbf{v} = \mathbf{P}\mathbf{u}$, the equation satisfied by v_1 is

$$\partial_t v_1 - \partial_{xx} v_1 = \left(\lambda_{PF}(\mathbf{L}) - b\left[\mathbf{P}^{-1}\mathbf{v}\right]\right) v_1 \text{ in } (0,+\infty)\times\mathbb{R}.$$

For all $k \in [N]\setminus\{1\}$, there exists $\gamma_k > 0$ such that v_k satisfies

$$\begin{cases} \partial_t\left(\left|\frac{v_k}{v_1}\right|^2\right) - \partial_{xx}\left(\left|\frac{v_k}{v_1}\right|^2\right) - \frac{2\partial_x v_1}{v_1}\partial_x\left(\left|\frac{v_k}{v_1}\right|^2\right) + \gamma_k\left|\frac{v_k}{v_1}\right|^2 \leq 0 & \text{in } (0,+\infty)\times\mathbb{R} \\ \left(\left|\frac{v_k}{v_1}\right|^2\right)_{|\{0\}\times\mathbb{R}} \in \mathscr{L}^\infty\left(\mathbb{R},[0,+\infty)\right), & \end{cases}$$

that is such that $z_k : (t,x) \mapsto \mathrm{e}^{\frac{\gamma_k}{2}t}\left|\frac{v_k}{v_1}\right|^2$ satisfies

$$\begin{cases} \partial_t z_k - \partial_{xx} z_k - \frac{2\partial_x v_1}{v_1}\partial_x z_k + \frac{\gamma_k}{2} z_k \leq 0 & \text{in } (0,+\infty)\times\mathbb{R} \\ (z_k)_{|\{0\}\times\mathbb{R}} \in \mathscr{L}^\infty\left(\mathbb{R},[0,+\infty)\right). & \end{cases}$$

Since z_k stays bounded locally in time, by a classical argument (detailed for instance in [Gir18b, Proposition 3.4]), z_k vanishes uniformly in space as $t\to+\infty$. Consequently,

$$\mathrm{e}^{\frac{\gamma_k}{4}t}\sup_{x\in\mathbb{R}}|v_k| \to 0 \text{ as } t\to+\infty.$$

The conclusion follows from $\gamma = \min_{k\in[N]}\frac{\gamma_k}{4}$ and the following obvious algebraic equality:

$$\left(\mathbf{I}-\mathbf{\Pi}_{PF}(\mathbf{L})\right)\mathbf{u} = \mathbf{P}^{-1}\left(\sum_{k=2}^N v_k\mathbf{e}_k\right).$$

□

We begin with the case of homogeneous initial data, which does not require $\mathbf{d} = \mathbf{1}_{N,1}$ since (E_{KPP}) reduces to $\left(E_{KPP}^0\right)$ in this context.

Proposition 6.12. *All positive classical solutions of $\left(E_{KPP}^0\right)$ set in $(0,+\infty)$ converge as $t\to+\infty$ to $\mathbf{v}^\star$.*

Proof. Once again, the proof is very similar to that of Proposition 6.10.

Fix a positive classical solution $\mathbf{v}$ of $\left(E^0_{KPP}\right)$. By [Gir18b, Theorem 1.1], $\mathbf{v}(1) \gg \mathbf{0}$. Hence the function $\mathbf{u} : t \mapsto \mathbf{v}(t+1)$ is a classical solution of $\left(E^0_{KPP}\right)$ set in $(0,+\infty)$ which is positive in $[0,+\infty)$ (whereas $\mathbf{v}(0)$ might have null components) and which converges to $\mathbf{v}^\star$ if and only if $\mathbf{v}$ converges to $\mathbf{v}^\star$.

The function $u = \mathbf{n}_{PF}(\mathbf{L})^T \mathbf{\Pi}_{PF}(\mathbf{L})\mathbf{u}$ satisfies

$$u' = \lambda_{PF}(\mathbf{L}) u - b[\mathbf{u}] u.$$

In order to apply Lemma 6.11, it suffices to verify

$$\inf_{t\in(0,+\infty)} u(t) > 0.$$

On one hand, since $\mathbf{u}$ is positive in $[0,+\infty)$, u is positive in $[0,+\infty)$ as well. Hence any $t > 0$ such that $u'(t) = 0$ is such that $b(\mathbf{u}(t)) = \lambda_{PF}(\mathbf{L})$ and consequently any local minimum is larger than some positive constant. On the other hand, $\liminf_{t\to+\infty} u > 0$ is a direct consequence of the persistence result [Gir18b, Theorem 1.3].

Since b is Lipschitz-continuous on the compact set $\{\mathbf{v} \in \mathsf{K} \mid \mathbf{v} \leq \mathbf{k}\}$, there exists $C_1 > 0$ such that

$$|b[u\mathbf{n}_{PF}(\mathbf{L})] - b[\mathbf{u}]| \leq C_1 |(\mathbf{I} - \mathbf{\Pi}_{PF}(\mathbf{L}))\mathbf{u}| \text{ in } [0,+\infty),$$

Now u satisfies

$$u' = \lambda_{PF}(\mathbf{L}) u - b[u\mathbf{n}_{PF}(\mathbf{L})] u + (b[u\mathbf{n}_{PF}(\mathbf{L})] - b[\mathbf{u}]) u,$$

with, by Lemma 6.11,

$$(b[u\mathbf{n}_{PF}(\mathbf{L})] - b[\mathbf{u}]) u = o(u) \text{ as } t \to +\infty.$$

It follows easily (see for instance [106]) that u converges to the unique constant $\alpha^\star > 0$ such that $\lambda_{PF}(\mathbf{L}) = b[\alpha^\star \mathbf{n}_{PF}(\mathbf{L})]$, which precisely means

$$\lim_{t\to+\infty} \mathbf{u}(t) = \mathbf{v}^\star.$$

□

Finally, at the expense of assuming $\mathbf{d} = \mathbf{1}_{N,1}$, we extend the previous result to non-homogeneous initial data.

Proposition 6.13. *Assume $\mathbf{d} = \mathbf{1}_{N,1}$. Then, for all bounded intervals $I \subset \mathbb{R}$, all bounded positive classical solutions $\mathbf{u}$ of (E_{KPP}) set in $(0,+\infty)\times\mathbb{R}$ satisfy*

$$\lim_{t\to+\infty} \sup_{x\in I} |\mathbf{u}(t,x) - \mathbf{v}^\star| = 0.$$

Consequently, if $\mathbf{d} = \mathbf{1}_{N,1}$, the set of bounded nonnegative classical solutions of (S_{KPP}) is exactly $\{\mathbf{0}, \mathbf{v}^\star\}$.

Proof. Let $(t_n)_{n\in\mathbb{N}} \in (0,+\infty)^{\mathbb{N}}$ such that $\lim_{n\to+\infty} t_n = +\infty$. Then, by classical parabolic estimates (Lieberman [111]) and a diagonal extraction process, the sequence

$$(\mathbf{u}_n)_{n\in\mathbb{N}} = ((t,x) \mapsto \mathbf{u}(t+t_n, x))_{n\in\mathbb{N}}$$

converges up to extraction to an entire classical solution of (E_{KPP}) valued in $\prod_{i=1}^{N}[\nu, g_i(0)]$ (see [Gir18b, Theorems 1.2 and 1.3]).

Now let us prove that $\mathbf{v}^\star$ is the unique bounded entire classical solution $\tilde{\mathbf{u}}$ of (E_{KPP}) satisfying

$$\left(\inf_{\mathbb{R}^2}\tilde{u}_i\right)_{i\in[N]} \gg \mathbf{0}.$$

Let $\tilde{\mathbf{u}}$ be such a solution. The function $\tilde{u} = \mathbf{n}_{PF}(\mathbf{L})^T\mathbf{\Pi}_{PF}(\mathbf{L})\tilde{\mathbf{u}}$ satisfies

$$\partial_t\tilde{u} - \partial_{xx}\tilde{u} = \lambda_{PF}(\mathbf{L})\tilde{u} - b[\tilde{u}\mathbf{n}_{PF}(\mathbf{L})]\tilde{u} + (b[\tilde{u}\mathbf{n}_{PF}(\mathbf{L})] - b[\tilde{\mathbf{u}}])\tilde{u}.$$

For all $\tau\in\mathbb{R}$,

$$\inf_{(t,x)\in(0,+\infty)\times\mathbb{R}}\tilde{u}(t+\tau,x) > 0.$$

By Lemma 6.11, there exists $C>0$ such that, for all $t>0$ and all $\tau\in\mathbb{R}$,

$$\sup_{x\in\mathbb{R}}|\tilde{u}(t+\tau,x)\mathbf{n}_{PF}(\mathbf{L}) - \tilde{\mathbf{u}}(t+\tau,x)| \leq C\mathrm{e}^{-\gamma t}.$$

It follows that for all $t>0$,

$$\sup_{(t',x)\in\mathbb{R}^2}|\tilde{u}(t',x)\mathbf{n}_{PF}(\mathbf{L}) - \tilde{\mathbf{u}}(t',x)| \leq C\mathrm{e}^{-\gamma t}$$

and then passing the right-hand side to the limit $t\to+\infty$, we find

$$\tilde{u}(t',x)\mathbf{n}_{PF}(\mathbf{L}) = \tilde{\mathbf{u}}(t',x) \text{ for all } (t',x)\in\mathbb{R}^2.$$

Consequently, $\tilde{u}$ satisfies

$$\partial_t\tilde{u} - \partial_{xx}\tilde{u} = \lambda_{PF}(\mathbf{L})\tilde{u} - b[\tilde{u}\mathbf{n}_{PF}(\mathbf{L})]\tilde{u}.$$

By standard results on the scalar KPP equation, $\tilde{u} = \alpha^\star$ in $\mathbb{R}^2$, that is $\tilde{\mathbf{u}} = \mathbf{v}^\star$.

A standard compactness argument ends the proof. □

6.4 The back of the fronts: vanishingly small mutations in monostable two-component systems

In this section, we assume (H_7) and recall the existence and uniqueness of $(\mathbf{r},\eta,\mathbf{m})\in\mathsf{K}^{++}\times(0,+\infty)\times\mathsf{S}^{++}(\mathbf{0},1)$ such that

$$\mathbf{L} = \mathbf{R} + \eta\begin{pmatrix}-1 & 1\\ 1 & -1\end{pmatrix}\mathbf{M} \text{ with } (\mathbf{R},\mathbf{M}) = (\operatorname{diag}(\mathbf{r}),\operatorname{diag}(\mathbf{m})).$$

The various objects and notations of the problem now depend *a priori* on η and a subscript $_\eta$ might be added accordingly. The following definitions are recalled:

$$\alpha_i = \frac{r_i}{c_{i,i}} \text{ for all } i\in\{1,2\},$$

$$\mathbf{v}_m = \frac{1}{\det\mathbf{C}}\begin{pmatrix}r_1c_{2,2} - r_2c_{1,2}\\ r_2c_{1,1} - r_1c_{2,1}\end{pmatrix} \text{ if } \det\mathbf{C}\neq 0,$$

$$\mathscr{E} = \left\{(\eta,\mathbf{p},c)\in(0,+\infty)^2\times\mathscr{C}^2(\mathbb{R},\mathbb{R}^2) \mid \mathbf{p}\in\mathscr{P}_{c,\eta},\ c\geq c^\star_\eta\right\},$$

$$\partial_t\mathbf{u} - \mathbf{D}\partial_{xx}\mathbf{u} = \mathbf{R}\mathbf{u} - (\mathbf{C}\mathbf{u})\circ\mathbf{u}. \quad (E_{KPP})_0$$

6.4.1 Preparatory lemmas

The proof of Theorem 6.5 will use the following lemmas which are of independent interest.

Lemma 6.14. *Let $i \in \{1,2\}$, $j = 3-i$ and*

$$\eta \in \left(0, \frac{r_i c_{i,j}}{m_j c_{i,i}}\right].$$

Then for all traveling wave solutions $(\mathbf{p}, c)$ of $(E_{KPP})_\eta$,

$$p_i \leq \alpha_i \text{ in } \mathbb{R}.$$

Remark. This lemma is straightforwardly generalizable to the case $N > 2$.

Proof. Having in mind the proof of [Gir18b, Theorem 1.5, ii)], it suffices to investigate the sign of

$$r_i p_i - \eta m_i p_i + \eta m_j p_j - (c_{i,i} p_i + c_{i,j} p_j) p_i = p_i (r_i - \eta m_i - c_{i,i} p_i) + p_j (\eta m_j - c_{i,j} p_i).$$

This quantity is nonpositive provided

$$p_i \geq \max\left(\frac{r_i - \eta m_i}{c_{i,i}}, \frac{\eta m_j}{c_{i,j}}\right).$$

Since

$$\frac{r_i}{c_{i,i}} \geq \frac{r_i - \eta m_i}{c_{i,i}} \text{ for all } \eta \geq 0,$$
$$\frac{r_i}{c_{i,i}} \geq \frac{\eta m_j}{c_{i,j}} \text{ for all } \eta \leq \frac{r_i c_{i,j}}{m_j c_{i,i}},$$

we deduce indeed $p_i \leq \frac{r_i}{c_{i,i}}$. □

Lemma 6.15. *Let $i \in \{1,2\}$, $j = 3-i$ and assume*

$$\frac{r_i}{r_j} > \frac{c_{i,j}}{c_{j,j}}.$$

Let

$$\overline{\eta}_i = \frac{1}{2} \min\left(\frac{r_j c_{j,i}}{m_i c_{j,j}}, \frac{r_j}{m_i}\left(\frac{r_i}{r_j} - \frac{c_{i,j}}{c_{j,j}}\right)\right),$$

$$\rho_i = \frac{1}{2} \frac{r_j}{c_{i,i}} \left(\frac{r_i}{r_j} - \frac{c_{i,j}}{c_{j,j}}\right).$$

Then for all $\rho \in (0, \rho_i]$, all $\eta \in (0, \overline{\eta}_i)$ and all traveling wave solutions $(\mathbf{p}, c)$ of $(E_{KPP})_\eta$, there exists a unique

$$\xi_\rho \in p_i^{-1}(\{\rho\}).$$

Furthermore p_i is decreasing in $(\xi_\rho, +\infty)$ and $p_i - \rho$ is positive in $(-\infty, \xi_\rho)$.

Remark. The following proof is mostly due to Griette–Raoul [82, Proposition 5.1].

Proof. Let $\zeta \in \mathbb{R}$ such that $p_i(\zeta)$ is a local minimum of p_i. Then

$$r_i p_i(\zeta) - \eta m_i p_i(\zeta) + \eta m_j p_j(\zeta) - (c_{i,i} p_i(\zeta) + c_{i,j} p_j(\zeta)) p_i(\zeta) \leq 0.$$

This implies

$$r_i p_i(\zeta) - \eta m_i p_i(\zeta) - (c_{i,i} p_i(\zeta) + c_{i,j} p_j(\zeta)) p_i(\zeta) < 0,$$

whence

$$r_i - \eta m_i < c_{i,i} p_i(\zeta) + c_{i,j} p_j(\zeta),$$

whence by Lemma 6.14

$$r_i - \eta m_i < c_{i,i} p_i(\zeta) + c_{i,j} \frac{r_j}{c_{j,j}},$$

and then

$$\begin{aligned} p_i(\zeta) &> \frac{1}{c_{i,i}} \left(r_i - \frac{r_j c_{i,j}}{c_{j,j}} \right) - \frac{\eta m_i}{c_{i,i}} \\ &> \frac{r_j}{c_{i,i}} \left(\frac{r_i}{r_j} - \frac{c_{i,j}}{c_{j,j}} \right) - \frac{\overline{\eta}_i m_i}{c_{i,i}} \\ &\geq \frac{1}{2} \frac{r_j}{c_{i,i}} \left(\frac{r_i}{r_j} - \frac{c_{i,j}}{c_{j,j}} \right) \\ &= \rho_i. \end{aligned}$$

Now let $\rho \in (0, \rho_i]$ and $\xi_\rho \in p_i^{-1}(\{\rho\})$.

Since $p_i(\xi_\rho)$ cannot be a local minimum, there exists a neighborhood of ξ_ρ in which p_i is strictly monotonic. Assume it is increasing. Then by continuity of p_i' and the previous estimate on local minima, p_i is increasing in $(-\infty, \xi_\rho)$. By classical elliptic regularity, $\mathbf{p}$ converges as $\xi \to -\infty$ to a solution of $\mathbf{Lv} = \mathbf{Cv} \circ \mathbf{v}$, and by [Gir18b, Theorem 1.5, iii)], this solution is positive. But in view of the preceding estimates, necessarily

$$\lim_{\xi \to -\infty} p_i(\xi) > \rho_i \geq p_i(\xi_\rho),$$

which contradicts the monotonicity of p_i in $(-\infty, \xi_\rho)$. Hence p_i is decreasing in a neighborhood of ξ_ρ and then in $(\xi_\rho, +\infty)$. Consequently,

$$p_i^{-1}(\{\rho\}) = \{\xi_\rho\}.$$

This holds for all $\rho \in (0, \rho_i]$ and therefore ends the proof. □

6.4.2 Convergence at the back

Let $i \in \{1, 2\}$, $j = 3 - i$, $(c_\eta)_{\eta \geq 0}$ and $(\mathbf{p}_\eta)_{\eta > 0}$ such that

$$\begin{cases} (\eta, \mathbf{p}_\eta, c_\eta) \in \mathscr{E} & \text{for all } \eta > 0, \\ c_0 = \lim_{\eta \to 0} c_\eta, \end{cases}$$

and assume from now on that

$$\frac{r_i}{r_j} > \frac{c_{i,j}}{c_{j,j}}$$

so that the assumptions of Theorem 6.5 are satisfied. Define subsequently

$$\mathbf{v}_s = \begin{cases} \alpha_i \mathbf{e}_i & \text{if } \frac{r_i}{r_j} \geq \frac{c_{i,i}}{c_{j,i}}, \\ \mathbf{v}_m & \text{if } \frac{r_i}{r_j} < \frac{c_{i,i}}{c_{j,i}}. \end{cases}$$

Proposition 6.16. *There exists $(\zeta_\eta)_{\eta>0}$ such that, as $\eta \to 0$, $(\xi \mapsto \mathbf{p}_\eta(\xi+\zeta_\eta), c_\eta)_{\eta>0}$ converges up to extraction in $\mathscr{C}^2_{loc}(\mathbb{R},\mathbb{R}^2)\times\mathbb{R}$ to a traveling wave solution $(\mathbf{p}_{back}, c_0)$ of $(E_{KPP})_0$ achieving one of the following connections:*

1. $\mathbf{0}$ *to* $\mathbf{v}_s$,
2. $\alpha_j\mathbf{e}_j$ *to* $\mathbf{v}_s$,
3. $\mathbf{0}$ *to* $\alpha_i\mathbf{e}_i$ *with* $\mathbf{p}$ *semi-extinct.*

Proof. Let $\rho = \min(\rho_i, v_{s,i})$. By virtue of Lemma 6.15, for all $\eta > 0$, there exists a unique ζ_η such that:

— $p_{\eta,i}$ is decreasing in $(\zeta_\eta, +\infty)$,

— $p_{\eta,i}(\zeta_\eta) = \rho$,

— $p_{\eta,i} - \rho$ is positive in $(-\infty, \zeta_\eta)$.

By Lemma 6.14, classical elliptic estimates (Gilbarg–Trudinger [80]) and a diagonal extraction process, $(\xi \mapsto \mathbf{p}_\eta(\xi+\xi_\eta))_{\eta>0}$ converges in $\mathscr{C}^2_{loc}$ up to extraction. Let $\mathbf{p}$ be its limit. We have directly $\mathbf{0} \leq \mathbf{p} \leq \boldsymbol{\alpha}$ in $\mathbb{R}$. In view of the normalization, we also have:

— p_i is nonincreasing in $(0, +\infty)$,

— $p_i(0) = \rho$,

— $p_i - \rho$ is nonnegative in $(-\infty, 0)$.

Let $(\xi_n)_{n\in\mathbb{N}}$ such that $\xi_n \to -\infty$ as $n \to +\infty$. Defining

$$\hat{\mathbf{p}}_n : \xi \mapsto \mathbf{p}(\xi+\xi_n) \text{ for all } n \in \mathbb{N},$$

by classical elliptic estimates and a diagonal extraction process again, $(\hat{\mathbf{p}}_n)_{n\in\mathbb{N}}$ converges up to extraction in $\mathscr{C}^2_{loc}$ to a function $\hat{\mathbf{p}}$ satisfying

$$-\mathbf{D}\hat{\mathbf{p}}'' - c\hat{\mathbf{p}}' = \mathbf{R}\hat{\mathbf{p}} - (\mathbf{C}\hat{\mathbf{p}})\circ\hat{\mathbf{p}}$$

and such that

$$(\rho, 0) \leq (\hat{p}_i, \hat{p}_j) \leq (\alpha_i, \alpha_j).$$

In particular, $\hat{\mathbf{p}}$ is a stationary solution of

$$\begin{cases} \partial_t\mathbf{u} - \partial_{xx}\mathbf{u} - c_0\partial_x\mathbf{u} = \mathbf{R}\mathbf{u} - (\mathbf{C}\mathbf{u})\circ\mathbf{u} & \text{in } (0,+\infty)\times\mathbb{R} \\ \mathbf{u}(0,x) = \hat{\mathbf{p}}(x) & \text{for all } x \in \mathbb{R}. \end{cases}$$

Applying the comparison principle for two-components competitive parabolic systems to $\hat{\mathbf{p}}$ and to the solution of

$$\begin{cases} \partial_t\mathbf{u} - \partial_{xx}\mathbf{u} - c_0\partial_x\mathbf{u} = \mathbf{R}\mathbf{u} - (\mathbf{C}\mathbf{u})\circ\mathbf{u} & \text{in } (0,+\infty)\times\mathbb{R} \\ (u_i, u_j)(0,x) = (\rho, \sup\hat{p}_j) & \text{for all } x \in \mathbb{R}, \end{cases}$$

which is homogeneous in space and is therefore the solution of

$$\begin{cases} \partial_t\mathbf{u} = \mathbf{R}\mathbf{u} - (\mathbf{C}\mathbf{u})\circ\mathbf{u} & \text{in } (0,+\infty)\times\mathbb{R} \\ (u_i, u_j)(0,x) = (\rho, \sup\hat{p}_j) & \text{for all } x \in \mathbb{R}, \end{cases}$$

we directly obtain $\hat{\mathbf{p}} = \mathbf{v}_s$ if $\sup\hat{p}_j > 0$ and $\hat{\mathbf{p}} = \alpha_i\mathbf{e}_i$ if $\sup\hat{p}_j = 0$. In other words, if $\mathbf{v}_s = \alpha_i\mathbf{e}_i$, $\hat{\mathbf{p}} = \alpha_i\mathbf{e}_i$, and if $\mathbf{v}_s = \mathbf{v}_m$, $\hat{\mathbf{p}} \in \{\mathbf{v}_s, \alpha_i\mathbf{e}_i\}$. Since $\mathbf{v}_s$ and $\alpha_i\mathbf{e}_i$ are isolated steady states and $\mathbf{p}$ is continuous, the last diagonal extraction was not necessary and $(\hat{\mathbf{p}}_n)_{n\in\mathbb{N}}$ converges indeed to $\hat{\mathbf{p}}$, that is

$$\lim_{-\infty} \mathbf{p} \in \{\mathbf{v}_s, \alpha_i \mathbf{e}_i\}.$$

Since p_i is nonincreasing in $(0,+\infty)$, it converges as $\xi \to +\infty$. By classical elliptic regularity,

$$\lim_{+\infty} (-d_i p_i'' - c_0 p_i') = 0,$$

whence either

$$\lim_{+\infty} p_i = 0$$

or p_j converges as well, its limit being

$$\lim_{+\infty} p_j = \frac{1}{c_{i,j}} \left(r_i - c_{i,i} \lim_{+\infty} p_i \right).$$

In the second case, using $-d_j p_j'' - c_0 p_j' \to 0$, $p_i(0) = \rho$ and the monotonicity of p_i in $(0,+\infty)$, we find $\lim_{+\infty} \mathbf{p} \in \{\alpha_j \mathbf{e}_j, \mathbf{0}\}$, which contradicts directly $\lim_{+\infty} p_i > 0$. Hence p_i converges to 0.

Subsequently, since p_j is positive, every local minimum of p_j satisfies

$$r_j \leq c_{j,j} p_j(\xi) + c_{j,i} p_i(\xi),$$

which proves that for all sequences $(\xi_n)_{n\in\mathbb{N}}$ such that $\xi_n \to +\infty$ and $p_j(\xi_n)$ is a local minimum of p_j, $p_j(\xi_n)$ converges to α_j. But then, by $\mathscr{C}^1$ regularity, either p_j is monotonic in a neighborhood of $+\infty$ or there exists a sequence $(\xi_n)_{n\in\mathbb{N}}$ such that $\xi_n \to +\infty$, $p_j(\xi_n)$ is a local minimum of p_j and $(p_j(\xi_n))_{n\in\mathbb{N}}$ converges to $\liminf_{+\infty} p_j$. It turns out that in both cases p_j converges, the possible limits being 0 and α_j.

Therefore $\mathbf{p}$ is a traveling wave achieving exactly one of the following connections:

1. $\mathbf{0}$ to $\mathbf{v}_s$,
2. $\alpha_j \mathbf{e}_j$ to $\mathbf{v}_s$,
3. $\mathbf{0}$ to $\alpha_i \mathbf{e}_i$ with $\alpha_i \mathbf{e}_i \neq \mathbf{v}_s$,
4. $\alpha_j \mathbf{e}_j$ to $\alpha_i \mathbf{e}_i$ with $\alpha_i \mathbf{e}_i \neq \mathbf{v}_s$.

It remains to show that the third case is semi-extinct and the fourth case is impossible. We will actually prove both statements simultaneously by proving that $\lim_{-\infty} \mathbf{p} = \alpha_i \mathbf{e}_i \neq \mathbf{v}_s$ implies $p_j = 0$ in $\mathbb{R}$.

Assume $\lim_{-\infty} \mathbf{p} = \alpha_i \mathbf{e}_i$ and $\mathbf{v}_s = \mathbf{v}_m$. Assume also by contradiction that p_j is positive in $\mathbb{R}$.

Multiplying the equation

$$-d_j p_j'' - c_0 p_j' = (r_j - c_{j,j} p_j - c_{j,i} p_i) p_j,$$

by the function

$$\varphi : \xi \mapsto \mathrm{e}^{\frac{c_0}{d_j}\xi},$$

we find

$$-d_j \left(\varphi p_j'\right)' = (r_j - c_{j,j} p_j - c_{j,i} p_i) \varphi p_j.$$

Recall that $\mathbf{v}_s = \mathbf{v}_m$ implies $\frac{r_i}{r_j} < \frac{c_{i,i}}{c_{j,i}}$, that is $r_j - c_{j,i}\alpha_i > 0$. Therefore the quantity

$$\overline{\xi} = \sup \{\xi \in \mathbb{R} \mid \forall \zeta \in (-\infty, \xi) \quad r_j - c_{j,j} p_j(\zeta) - c_{j,i} p_i(\zeta) > 0\}$$

is well-defined in $\mathbb{R}\cup\{+\infty\}$. In $\left(-\infty,\overline{\xi}\right)$, $\varphi p_j'$ is decreasing. Since on one hand $\lim\limits_{-\infty}\varphi=0$ and on the other hand $\lim\limits_{-\infty}p_j'=0$ by classical elliptic regularity, the limit of $\varphi p_j'$ itself is 0. Consequently, $\varphi p_j'$ is negative in $\left(-\infty,\overline{\xi}\right)$. It follows that p_j itself is decreasing in $\left(-\infty,\overline{\xi}\right)$. But then $\lim\limits_{-\infty}p_j=0$ implies that p_j is negative in $\left(-\infty,\overline{\xi}\right)$, which obviously contradicts the positivity of p_j. This ends the proof. □

6.5 Discussion

6.5.1 Why is it likely hopeless to search for a general result on the behavior at the back of the front?

First of all, the linearization of (S_{KPP}) at $\mathbf{0}$ being cooperative, it is natural to wonder whether the dynamics of (E_{KPP}) near some constant positive solution $\mathbf{u}$ of (S_{KPP}) might be purely competitive or cooperative. In general, neither is the case. The linearized reaction term at any constant solution $\mathbf{u}$ of (S_{KPP}) is

$$\mathbf{L_u}=\mathbf{L}-\operatorname{diag}\left(\mathbf{c}\left(\mathbf{u}\right)\right)-\left(\mathbf{u}\mathbf{1}_{1,N}\right)\circ D\mathbf{c}\left(\mathbf{u}\right).$$

In the Lotka–Volterra case where there exists $\mathbf{C}\gg\mathbf{0}$ such that $\mathbf{c}\left(\mathbf{v}\right)=\mathbf{C}\mathbf{v}$, it reads

$$\mathbf{L_u}=\mathbf{L}-\operatorname{diag}\left(\mathbf{C}\mathbf{u}\right)-\left(\mathbf{u}\mathbf{1}_{1,N}\right)\circ\mathbf{C}.$$

On one hand, it is clear that if there exists $(i,j)\in[N]^2$ such that $l_{i,j}=0$, then $l_{\mathbf{u},i,j}<0$. On the other hand, assuming that there exists $i\in[N]$ such that $l_{i,i}\leq 0$, we find

$$-l_{i,i}u_i+u_ic_{i,i}u_i>0.$$

Since $\mathbf{L}\mathbf{u}-\left(\mathbf{C}\mathbf{u}\right)\circ\mathbf{u}=\mathbf{0}$, it follows

$$\sum_{j\in[N]\setminus\{i\}}\left(l_{i,j}u_j-u_ic_{i,j}u_j\right)>0,$$

whence there exists $j\in[N]\setminus\{i\}$ such that $l_{i,j}u_j-u_ic_{i,j}u_j>0$, that is such that

$$l_{\mathbf{u},i,j}=l_{i,j}-u_ic_{i,j}>0.$$

Hence the competitive dynamics and the cooperative dynamics are indeed intertwined near $\mathbf{u}$.

Next, in view of the literature on non-cooperative KPP systems, it could be tempting to conjecture the uniqueness and the local stability of the constant positive solution of (S_{KPP}) (see for instance Dockery–Hutson–Mischaikow–Pernarowski [58] or Morris–Börger–Crooks [115]). However, if $\mathbf{c}$ is linear as before and if

$$(N,\mathbf{L},\mathbf{C})=\left(2,\mathbf{I}_2+\frac{1}{5}\begin{pmatrix}-1 & 1\\ 1 & -1\end{pmatrix},\frac{1}{10}\begin{pmatrix}1 & 9\\ 9 & 1\end{pmatrix}\right),$$

then this property fails. Indeed, straightforward computations show that the set of constant positive solutions of (S_{KPP}) is

$$\left\{\begin{pmatrix}3-\sqrt{\frac{15}{2}}\\ 3+\sqrt{\frac{15}{2}}\end{pmatrix},\mathbf{1}_{2,1},\begin{pmatrix}3+\sqrt{\frac{15}{2}}\\ 3-\sqrt{\frac{15}{2}}\end{pmatrix}\right\}.$$

From the associated linearizations, it is easily found that, with respect to $\left(E^0_{KPP}\right)$, the symmetric solution $\mathbf{1}_{2,1}$ is a saddle point whereas the other two solutions are stable nodes.

Last, we also point out that if $\mathbf{d} = \mathbf{1}_{2,1}$ then the preceding counter-example admits a family of traveling waves connecting $\mathbf{0}$ to the saddle point $\mathbf{1}_{2,1}$. Indeed, looking for profiles $\mathbf{p}$ of the form $\xi \mapsto p(\xi)\,\mathbf{1}_{2,1}$, $(TW[c])$ reduces to

$$-p'' - cp' = p - p^2,$$

which, by virtue of well-known results on the scalar KPP equation, admits solutions connecting 0 to 1 if and only if $c \geq 2$. Hence we cannot hope to prove that all traveling waves connect $\mathbf{0}$ to a stable steady state.

6.5.2 What about the general separated competition case, with $\mathbf{d}$ and $\mathbf{a}$ possibly different from $\mathbf{1}_{N,1}$?

The general case might be more subtle than expected, even regarding the ODE system $\left(E^0_{KPP}\right)$: although the linearization at $\mathbf{v}^\star$,

$$\mathbf{L}_{\mathbf{v}^\star} = \mathbf{L} - \lambda_{\mathbf{a}}\mathbf{A} - \mathbf{A}\mathbf{v}^\star\left(\nabla b\left(\mathbf{v}^\star\right)^T\right),$$

seems to be adequately described as a matrix of the form $-\mathbf{P} - \mathbf{Q}$ with $\mathbf{P} = \lambda_{\mathbf{a}}\mathbf{A} - \mathbf{L}$ a singular M-matrix and $\mathbf{Q} = \mathbf{A}\mathbf{v}^\star\left(\nabla b\left(\mathbf{v}^\star\right)^T\right)$ a positive rank-one matrix, a recent paper by Bierkens and Ran [23] highlights thanks to a counter-example that such matrices can have eigenvalues with positive real part (and there is in addition a counter-example with irreducible $-\mathbf{P}$, so that irreducibility is not a sufficient condition to ensure all eigenvalues are negative). Therefore it is unclear whether $\mathbf{v}^\star$ is always locally asymptotically stable with respect to $\left(E^0_{KPP}\right)$. Actually, the main purpose of the study of Bierkens and Ran is to establish several conditions sufficient to guarantee that all eigenvalues have a negative real part (conditions among which we find $N = 2$ and, of course, $\mathbf{a} = \mathbf{1}_{N,1}$).

In the case $N = 2$, classical calculations show that the system (E_{KPP}) is not subjected to Turing instabilities with respect to periodic perturbations. Therefore it might be fruitful to investigate more thoroughly the two-component system. Nevertheless, to this day we do not have any further result.

6.5.3 Where does Conjecture 6.4 come from?

Let us bring forth some insight into the limiting problem. What are the spreading properties of $(E_{KPP})_0$ with respect to front-like initial data? What are the propagating solutions of $(E_{KPP})_0$ invading the null state?

Concerning the bistable case, we have at our disposal a recent result by Carrère [35] which can be summed up as follows. Consider the Cauchy problem where $(-\infty, 0)$ is initially inhabited mostly but not only (in a sense made rigorous by Carrère) by u_1 and $(0, +\infty)$ is completely uninhabited. Let $c_{\alpha_1\mathbf{e}_1\to\alpha_2\mathbf{e}_2}$ be the speed of the bistable front equal to $\alpha_1\mathbf{e}_1$ at $-\infty$ and to $\alpha_2\mathbf{e}_2$ at $+\infty$, as given by Kan-On [100] and Gardner [77]. Recall that the following bounds hold true:

$$-2\sqrt{d_2 r_2} < c_{\alpha_1\mathbf{e}_1\to\alpha_2\mathbf{e}_2} < 2\sqrt{d_1 r_1}.$$

Carrère's theorem is then:

1. if $2\sqrt{d_1 r_1} > 2\sqrt{d_2 r_2}$ and $c_{\alpha_1\mathbf{e}_1\to\alpha_2\mathbf{e}_2} > 0$, then asymptotically in time, u_2 is extinct and u_1 spreads at speed $2\sqrt{d_1 r_1}$;

2. if $2\sqrt{d_1r_1} < 2\sqrt{d_2r_2}$ and $c_{\alpha_1\mathbf{e}_1\to\alpha_2\mathbf{e}_2} > 0$, then asymptotically in time, u_2 spreads on the right at speed $2\sqrt{d_2r_2}$ but is then replaced by u_1 at speed $c_{\alpha_1\mathbf{e}_1\to\alpha_2\mathbf{e}_2}$;
3. if $2\sqrt{d_1r_1} < 2\sqrt{d_2r_2}$ and $c_{\alpha_1\mathbf{e}_1\to\alpha_2\mathbf{e}_2} < 0$, then asymptotically in time, u_2 chases u_1 on the left at speed $c_{\alpha_1\mathbf{e}_1\to\alpha_2\mathbf{e}_2}$ and spreads on the right at speed $2\sqrt{d_2r_2}$.

This result was long-awaited but, as far as we know, Carrère's proof is the first one.

Up to the sign of $c_{\alpha_1\mathbf{e}_1\to\alpha_2\mathbf{e}_2}$, the second and the third cases above are identical. Recall that the sign of $c_{\alpha_1\mathbf{e}_1\to\alpha_2\mathbf{e}_2}$ is in general a tough problem, although recently some particular cases have been successfully solved (strong competition in Girardin–Nadin [GN15], special choices of parameter values in Guo–Lin [83], perturbation of the standing wave in Risler [130]).

A natural conjecture in view of Carrère's result is the long-time convergence, in the first case, to a traveling wave connecting $\mathbf{0}$ to $\alpha_1\mathbf{e}_1$ at speed $2\sqrt{d_1r_1}$ and with a semi-extinct profile $\mathbf{p} = p\mathbf{e}_1$. However, in the second and third cases, a more complex limit seems to arise.

The entire solutions connecting three or more stationary states with decreasingly ordered speeds were first described in the scalar setting by Fife and McLeod [71] and are referred to as *propagating terraces*, or simply *terraces*, since the work of Ducrot, Giletti and Matano [62]. A terrace with $n-1$ intermediate states is defined as a finite family of traveling waves $((\mathbf{p}_i, c_i))_{i\in[n]}$ such that $\mathbf{p}_i(-\infty) = \mathbf{p}_{i+1}(+\infty)$ for all $i \in [n-1]$ and such that $(c_i)_{i\in[n]}$ is decreasing. Provided the uniqueness (up to translation of the profile) of the traveling wave connecting $\mathbf{v}_i = \mathbf{p}_i(+\infty)$ to $\mathbf{v}_{i+1} = \mathbf{p}_i(-\infty)$ at speed c_i, the terrace is equivalently defined as the family $\left((\mathbf{v}_i, c_i)_{i\in[n]}, \mathbf{v}_{n+1}\right)$. However, in general, this family only defines a family of terraces that will be denoted hereafter $\mathscr{T}\left((\mathbf{v}_i, c_i)_{i\in[n]}, \mathbf{v}_{n+1}\right)$.

In terms of this definition, the expected limits in the second and third cases studied by Carrère are terraces belonging to

$$\mathscr{T}\left(\mathbf{0}, 2\sqrt{d_2r_2}, \alpha_2\mathbf{e}_2, c_{\alpha_1\mathbf{e}_1\to\alpha_2\mathbf{e}_2}, \alpha_1\mathbf{e}_1\right)$$

with a semi-extinct first profile.

The obvious conjecture is then that all propagating solutions invading $\mathbf{0}$ apart from semi-extinct monostable traveling waves belong to

$$\bigcup_{i\in\{1,2\}} \bigcup_{c\geq 2\sqrt{d_ir_i}} \mathscr{T}\left(\mathbf{0}, c, \alpha_i\mathbf{e}_i, c_{\alpha_{3-i}\mathbf{e}_{3-i}\to\alpha_i\mathbf{e}_i}, \alpha_{3-i}\mathbf{e}_{3-i}\right)$$

and have a semi-extinct first profile.

The bistable case being more or less understood, we now turn our attention to the monostable case. Let $\mathbf{v}_s \in \{\alpha_1\mathbf{e}_1, \alpha_2\mathbf{e}_2, \mathbf{v}_m\}$ be the unique stable state, $\mathbf{v}_u \in \{\mathbf{0}, \alpha_1\mathbf{e}_1, \alpha_2\mathbf{e}_2\}$ be an unstable state and consider the Cauchy problem with compactly supported perturbations of $\mathbf{v}_u$ as initial data. Although the case $\mathbf{v}_u = \alpha_i\mathbf{e}_i$ with

$$i \in \mathsf{I}_u = \{j \in \{1,2\} \mid \alpha_j\mathbf{e}_j \neq \mathbf{v}_s\}.$$

is well understood (Lewis, Li and Weinberger proved the uniqueness of the spreading speed $c^\star_{\mathbf{v}_s\to\alpha_i\mathbf{e}_i}$ [108, 142]), the case $\mathbf{v}_u = \mathbf{0}$ is much more intricate: in particular, for $\mathbf{v}_s = \mathbf{v}_m$, a recent theorem analogous to that of Carrère and due to Lin and Li [112] shows that if $d_2r_2 > d_1r_1$, then u_2 will invade first at speed $2\sqrt{d_2r_2}$ and then be chased by u. Although straightforward comparisons show that the replacement occurs somewhere in $\left[c^\star_{\mathbf{v}_m\to\alpha_2\mathbf{e}_2}t, 2\sqrt{d_1r_1}t\right]$, the exact speed of u is a delicate question, unsettled in the paper of Lin and Li.

Tang and Fife [137] established by phase-plane analysis that traveling waves connecting $\mathbf{0}$ to $\mathbf{v}_s$ exist if and only if the speed c satisfies $c \geq c^{TW}_{\mathbf{v}_s \to \mathbf{0}}$, where

$$c^{TW}_{\mathbf{v}_s \to \mathbf{0}} = 2\sqrt{\max_{i \in \{1,2\}} d_i r_i}$$

is linearly determinate.

Terraces connecting $\mathbf{0}$ to $\mathbf{v}_s$ through an intermediate unstable state $\alpha_i \mathbf{e}_i$ with $i \in \mathsf{I}_u$ should involve semi-extinct monostable traveling waves connecting $\mathbf{0}$ to $\alpha_i \mathbf{e}_i$ and monostable traveling waves connecting $\alpha_i \mathbf{e}_i$ to $\mathbf{v}_s$. Again, there exists a minimal wave speed $c^{TW}_{\mathbf{v}_s \to \alpha_i \mathbf{e}_i}$, as proved for instance by Kan–On [101] or Lewis–Li–Weinberger [110]. Recall that $c^{TW}_{\mathbf{v}_s \to \alpha_i \mathbf{e}_i}$ is not linearly determinate in general, however it is bounded from below by the linear speed:

$$c^{TW}_{\mathbf{v}_s \to \alpha_i \mathbf{e}_i} \geq 2\sqrt{d_{3-i} r_{3-i} \left(1 - \frac{c_{3-i,i} r_i}{c_{i,i} r_{3-i}}\right)}.$$

In any case, it is natural to expect that for all $i \in \mathsf{I}_u$, terraces belonging to $\mathscr{T}\left(\mathbf{0}, c, \alpha_i \mathbf{e}_i, c', \mathbf{v}_s\right)$ with a semi-extinct first profile exist if and only if

$$\begin{cases} c_{\mathbf{v}_s \to \alpha_i \mathbf{e}_i} \leq c' \\ 2\sqrt{d_i r_i} \leq c \\ c' < c. \end{cases}$$

Consequently, the conjecture is that all propagating solutions invading $\mathbf{0}$ apart from (possibly semi-extinct) monostable traveling waves belong to

$$\bigcup_{i \in \mathsf{I}_u} \bigcup_{c \geq 2\sqrt{d_i r_i}} \bigcup_{c' \geq c_{\mathbf{v}_s \to \alpha_i \mathbf{e}_i}} \mathscr{T}\left(\mathbf{0}, c, \alpha_i \mathbf{e}_i, c', \mathbf{v}_s\right)$$

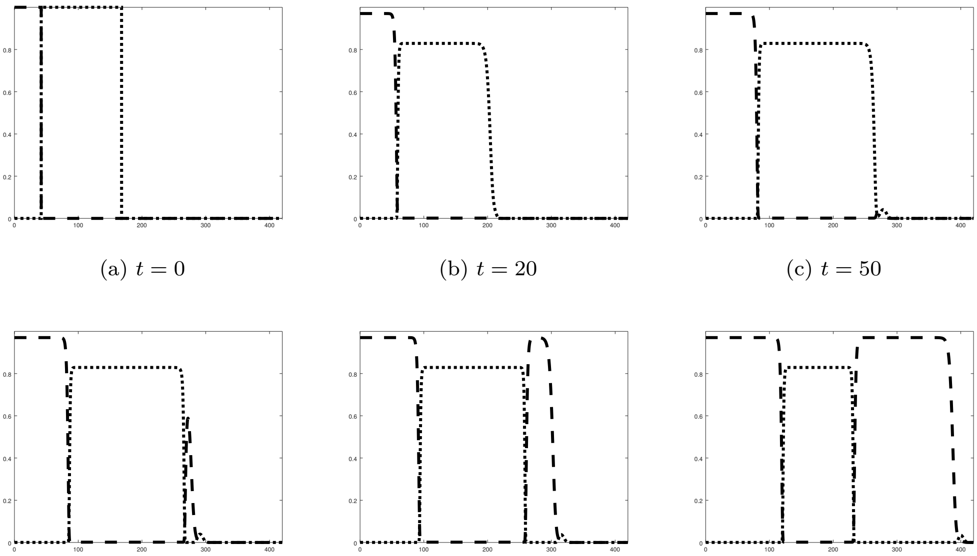
and have a semi-extinct first profile.

Having these conjectures in mind, we introduce small mutations and wonder how they affect the outcome. An heuristic answer due to Elliott and Cornell [65] suggests that “the only role of mutations is to ensure that both morphs travel at the same speed”. Therefore, there might exist functions $\mathbf{u}^0 : \mathbb{R} \to \mathsf{K}$ such that the solutions $\left(\mathbf{u}_\eta\right)_{\eta \geq 0}$ of the Cauchy problem associated with $\left(E_{KPP}\right)_\eta$ with initial data $\mathbf{u}^0$ admit as long-time asymptotic a traveling wave if $\eta > 0$ and a terrace of $\mathscr{T}\left(\mathbf{0}, c, \alpha_i \mathbf{e}_i, c', \mathbf{v}\right)$ if $\eta = 0$. We refer hereafter to such traveling waves as quasi-$\mathscr{T}\left(\mathbf{0}, c, \alpha_i \mathbf{e}_i, c', \mathbf{v}\right)$ traveling waves.

In order to study these special traveling waves, we resort to numerical simulations. We find two completely different behaviors.

- In the bistable case (Figure 6.5.1), quasi-$\mathscr{T}\left(\mathbf{0}, 2\sqrt{d_i r_i}, \alpha_i \mathbf{e}_i, c_{\alpha_j \mathbf{e}_j \to \alpha_i \mathbf{e}_i}, \alpha_j \mathbf{e}_j\right)$ traveling waves (with $i \in \{1,2\}$ and $j = 3 - i$) converge as $\eta \to 0$ to a semi-extinct traveling wave connecting $\mathbf{0}$ to $\alpha_j \mathbf{e}_j$ if $c_{\alpha_j \mathbf{e}_j \to \alpha_i \mathbf{e}_i} > 0$ and to $\alpha_i \mathbf{e}_i$ if $c_{\alpha_j \mathbf{e}_j \to \alpha_i \mathbf{e}_i} < 0$.

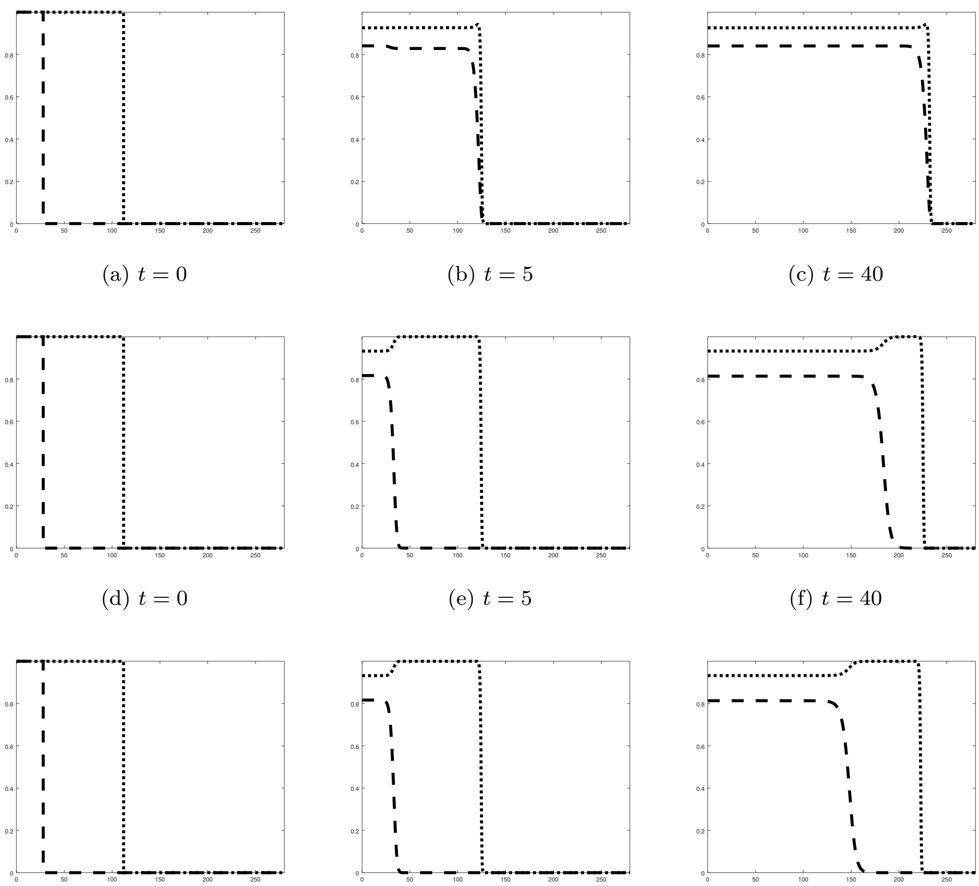
- In the monostable case (Figure 6.5.2), for all $i \in \mathsf{I}_u$, quasi-$\mathscr{T}\left(\mathbf{0}, 2\sqrt{d_i r_i}, \alpha_i \mathbf{e}_i, c', \mathbf{v}_s\right)$ traveling waves connect $\mathbf{0}$ to $\mathbf{v}_s$ through an intermediate bump of u_i. As $\eta \to 0$, the amplitude of this bump tends to α_i while its length tends slowly to $+\infty$ (seemingly like $\ln \eta$). Therefore, depending on the normalization, the limit of the profiles as $\eta \to 0$ is either a semi-extinct connection between $\mathbf{0}$ and $\alpha_i \mathbf{e}_i$ or a monostable connection between $\alpha_i \mathbf{e}_i$ and $\mathbf{v}_s$.



(a) $t = 0$ (b) $t = 20$ (c) $t = 50$

(d) $t = 55$ (e) $t = 65$ (f) $t = 100$

Figure 6.5.1 – Numerical simulation of the bistable case with initial data corresponding to a competition–diffusion terrace (u_1 dashed line, u_2 dotted line, x as horizontal axis). Parameter values: $\mathbf{d} = (1, 1.5125)^T$, $\mathbf{r} = \mathbf{1}_{2,1}$, $\mathbf{m} = \mathbf{1}_{2,1}$, $\eta = 0.025$, $c_{1,1} = c_{2,2} = 1$, $c_{1,2} = 20$, $c_{2,1} = 110$, so that [GN15] $c_{\alpha_1 \mathbf{e}_1 \to \alpha_2 \mathbf{e}_2} > 0$.
The traveling wave which is on the right at $t = 100$, driven by a very small bump of u_2 but dominated at the back by u_1, is the long-time asymptotic. Indeed the u_2-dominated area in the middle shrinks from both sides at a speed close to $|c_{\alpha_1 \mathbf{e}_1 \to \alpha_2 \mathbf{e}_2}|$ and will ultimately disappear.



(a) $t = 0$ (b) $t = 5$ (c) $t = 40$

(d) $t = 0$ (e) $t = 5$ (f) $t = 40$

(g) $t = 0$ (h) $t = 5$ (i) $t = 40$

Figure 6.5.2 – Numerical simulations of the monostable case with initial data corresponding to a competition–diffusion terrace (u_1 dashed line, u_2 dotted line, x as horizontal axis).

Parameter values: $\mathbf{d} = \left(1, \frac{1}{3}\right)^T$, $\mathbf{r} = (1,6)^T$, $\mathbf{m} = \mathbf{1}_{2,1}$, $c_{1,1} = 1$, $c_{2,2} = 6$, $c_{1,2} = 0.2$, $c_{2,1} = 0.5$, $\eta = 2.5 \times 10^{-1}$ on the first line, $\eta = 2.5 \times 10^{-6}$ on the second line, $\eta = 2.5 \times 10^{-11}$ on the third line.

6.5.4 Why is Conjecture 6.4 silent about the bistable case with $c_{\alpha_1\mathbf{e}_1\to\alpha_2\mathbf{e}_2} = 0$?

In this very special case, additional asymmetry assumptions on the coefficients are necessary in order to exclude connections between $\mathbf{0}$ and the saddle-point $\mathbf{v}_m$, as indicated by the following immediate proposition, built on a counter-example given in Subsection 6.5.1.

Proposition 6.17. *Assume* (H_7), $\mathbf{d} = \mathbf{1}_{2,1}$, $\mathbf{r} = \mathbf{1}_{2,1}$, $\mathbf{m} = \frac{1}{\sqrt{2}}\mathbf{1}_{2,1}$ *and the existence of* $a \in (1,+\infty)$ *such that*

$$\mathbf{C} = \begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix}.$$

Then $\mathbf{v}_m = \frac{1}{\lambda_{PF}(\mathbf{C})}\mathbf{1}_{2,1} \in \mathsf{K}^{++}$ *is a saddle-point and, for all* $\eta \geq 0$ *and all* $c \geq 2$, *there exists a unique* $p_{c,\eta} \in \mathscr{C}^2(\mathbb{R})$ *such that*

$$\begin{cases} p_{c,\eta}\mathbf{1}_{2,1} \in \mathscr{P}_{c,\eta} \\ p_{c,\eta}(0) = \frac{1}{2\lambda_{PF}(\mathbf{C})} \\ \lim\limits_{\xi\to-\infty} p_{c,\eta}(\xi) = \frac{1}{\lambda_{PF}(\mathbf{C})}. \end{cases}$$

In particular, $(p_{c,\eta}\mathbf{1}_{2,1}, c)$ *connects* $\mathbf{0}$ *to* $\mathbf{v}_m$.

Furthermore,

$$(c,\eta) \mapsto p_{c,\eta} \in \mathscr{C}\left([2,+\infty) \times [0,+\infty), \mathscr{W}^{2,\infty}(\mathbb{R},\mathbb{R})\right).$$

Acknowledgments

The author thanks Grégoire Nadin for the attention he paid to this work and Elaine Crooks for fruitful discussions. He also thanks anonymous referees and the associate editor for very valuable comments which led to a much clearer manuscript.

Notation	Definition
$[n]$	$[1,n]\cap\mathbb{N}$
$(\mathbf{e}_{n,i})_{i\in[n]}$	canonical basis of $\mathbb{R}^n$
$\lvert\bullet\rvert_n$	Euclidean norm of $\mathbb{R}^n$
$\mathsf{B}_n(\mathbf{v},r)$, $\mathsf{S}_n(\mathbf{v},r)$	open ball and sphere of center $\mathbf{v}\in\mathbb{R}^n$ and radius $r>0$
$\geq_n$, $>_n$, $\gg_n$	$v_i\geq\hat{v}_i$ for all $i\in[n]$, $\mathbf{v}\geq_n\hat{\mathbf{v}}$ and $\mathbf{v}\neq\hat{\mathbf{v}}$, $v_i>\hat{v}_i$ for all $i\in[n]$
nonnegative, nonneg. nonzero, positive $\mathbf{v}\in\mathbb{R}^n$	$\mathbf{v}\geq_n\mathbf{0}$, $\mathbf{v}>_n\mathbf{0}$, $\mathbf{v}\gg_n\mathbf{0}$
K_n, K_n^+, K_n^{++}	sets of all nonnegative, nonneg. nonzero, positive vectors
$\mathsf{S}_n^+(\mathbf{0},1)$, $\mathsf{S}_n^{++}(\mathbf{0},1)$	$\mathsf{K}_n^+\cap\mathsf{S}_n(\mathbf{0},1)$, $\mathsf{K}_n^{++}\cap\mathsf{S}_n(\mathbf{0},1)$
$\mathsf{M}_{n,n'}$, M_n	sets of all real matrices of dimension $n\times n'$, $n\times n$
$\mathbf{I}_n$, $\mathbf{1}_{n,n'}$	identity matrix, matrix whose every entry is equal to 1
$\operatorname{diag}(\mathbf{v})$	diagonal matrix whose i-th diagonal entry is v_i
essentially nonnegative matrix	matrix $\mathbf{A}$ such that $\mathbf{A}-\min\limits_{i\in[n]}(a_{i,i})\,\mathbf{I}_n$ is nonnegative
$\mathbf{A}\circ\mathbf{B}$	Hadamard (entry-by-entry) product $(a_{i,j}b_{i,j})_{(i,j)\in[n]\times[n']}$
$\mathbf{f}\left[\hat{\mathbf{f}}\right]$	composition of the functions $\mathbf{f}$ and $\hat{\mathbf{f}}$

Table 6.1 – General notations (the subscripts depending only on 1 or N are omitted when the context is unambiguous)

Systèmes de compétition – diffusion monostables à deux espèces

« Tel est pris qui croyait prendre. »

(Dicton populaire français)

Chapitre 7

Invasion d'un territoire inoccupé par deux compétiteurs : propriétés de propagation de systèmes de compétition – diffusion monostables à deux espèces

Résumé

Dans ce chapitre, on se tourne vers des propriétés de propagation de systèmes de compétition – diffusion de Lotka – Volterra à deux espèces et monostables dont les conditions initiales sont nulles ou exponentiellement décroissantes dans une demi-droite tournée vers la droite. Grâce à une construction délicate de sur-solutions et de sous-solutions, on améliore des résultats précédemment établis et on résout des questions ouvertes. En particulier, on montre que si le compétiteur le plus faible est aussi le plus mobile, il est alors susceptible d'échapper au compétiteur fort et moins mobile en envahissant en premier un territoire inoccupé. La paire de vitesse dépend des conditions initiales. Si celles-ci sont nulles dans une demi-droite tournée vers la droite, alors la première vitesse est la vitesse KPP du compétiteur le plus mobile et la seconde vitesse est donnée par une formule exacte dépendent de la première vitesse et de la vitesse minimale des ondes progressives connectant les deux équilibres semi-triviaux. De plus, l'ensemble non-borné de paires de vitesses atteignables avec des conditions initiales exponentiellement décroissantes est caractérisé, à un ensemble négligeable près.

Ce chapitre, co-écrit avec Adrian Lam, a fait l'objet d'une soumission sous le titre *Invasion of an empty habitat by two competitors : spreading properties of monostable two-species competition–diffusion systems* dans *Proceedings of the London Mathematical Society* [GL18].

7.1 Introduction

In this paper, we are interested in some spreading properties of the classical monostable Lotka–Volterra two-species competition–diffusion system

$$\begin{cases} \partial_t u - \partial_{xx} u = u\left(1-u-av\right) & \text{in } (0,+\infty)\times\mathbb{R} \\ \partial_t v - d\partial_{xx} v = rv\left(1-v-bu\right) & \text{in } (0,+\infty)\times\mathbb{R} \\ u\left(0,x\right) = u_0\left(x\right) & \text{for all } x\in\mathbb{R} \\ v\left(0,x\right) = v_0\left(x\right) & \text{for all } x\in\mathbb{R} \end{cases} \tag{7.1.1}$$

with $d>0$, $a\in(0,1)$, $b>1$, $r>0$ and $u_0, v_0 \in \mathscr{C}\left(\mathbb{R},[0,1]\right)\setminus\{0\}$. The assumptions on a and b mean that u and v are respectively the stronger and the weaker competitor.

Recall from the classical literature [8, 72, 104] that the scalar Fisher-KPP equation

$$\begin{cases} \partial_t w - \delta\partial_{xx} w = \rho w\left(1-w\right) & \text{in } (0,+\infty)\times\mathbb{R} \\ w\left(0,x\right) = w_0\left(x\right) & \text{for all } x\in\mathbb{R} \end{cases}$$

with $\delta,\rho>0$ and $w_0\in\mathscr{C}_b\left(\mathbb{R}\right)$ with nonempty support included in $(-\infty,0]$ has the following spreading property: there exists a unique $c_{KPP}>0$ satisfying

$$\begin{cases} \lim\limits_{t\to+\infty} \sup\limits_{|x|<ct} \left|w\left(t,x\right)-1\right| = 0 \text{ for each } c<c_{KPP} \\ \lim\limits_{t\to+\infty} \sup\limits_{ct<x} \left|w\left(t,x\right)\right| = 0 \text{ for each } c>c_{KPP} \end{cases}.$$

These asymptotics describe the invasion of the unstable state 0 by the stable state 1 and c_{KPP} is consequently referred to as the spreading speed of this invasion. Furthermore, c_{KPP} coincides with the minimal speed of the traveling wave solutions, which are particular entire solutions of the form $w:(t,x)\mapsto\varphi\left(x-ct\right)$ with $\varphi\geq0$, $\varphi\left(-\infty\right)=1$ and $\varphi\left(+\infty\right)=0$. A striking result is the so-called *linear determinacy* property: there exists such a pair (φ,c) if and only if the linear equation

$$-\delta\varphi''-c\varphi'=\rho\varphi,$$

namely, the linearization at $\varphi=0$ of the semilinear equation satisfied by φ, admits a positive solution in $\mathbb{R}$. Consequently, $c_{KPP}=2\sqrt{\rho\delta}$. As far as the system (7.1.1) is concerned, this result shows that in the absence of the competitor, u and v respectively spread at speed 2 and $2\sqrt{rd}$.

Recall also from the collection of works due to Lewis, Li and Weinberger [108, 110] that the competition–diffusion system

$$\begin{cases} \partial_t u - \partial_{xx} u = u\left(1-u-av\right) & \text{in } (0,+\infty)\times\mathbb{R} \\ \partial_t v - d\partial_{xx} v = rv\left(1-v-bu\right) & \text{in } (0,+\infty)\times\mathbb{R} \\ u\left(0,x\right) = \tilde{u}_0\left(x\right) & \text{for all } x\in\mathbb{R} \\ v\left(0,x\right) = 1-\tilde{v}_0\left(x\right) & \text{for all } x\in\mathbb{R} \end{cases} \tag{7.1.2}$$

with $\tilde{u}_0$ and $\tilde{v}_0$ compactly supported and $\tilde{u}_0$ nonnegative nonzero, has an analogous spreading property: there exists a unique $c_{LLW}>0$ satisfying

$$\begin{cases} \lim\limits_{t\to+\infty} \sup\limits_{|x|<ct} \left(\left|u\left(t,x\right)-1\right|+\left|v\left(t,x\right)\right|\right) = 0 \text{ for each } c<c_{LLW} \\ \lim\limits_{t\to+\infty} \sup\limits_{ct<|x|} \left(\left|u\left(t,x\right)\right|+\left|v\left(t,x\right)-1\right|\right) = 0 \text{ for each } c>c_{LLW} \end{cases}$$

and describing the invasion of the unstable state $(0,1)$ by the stable state $(1,0)$. As in the KPP case, the spreading speed c_{LLW} is the minimal speed of the monotonic traveling wave solutions; linearizing at $(0,1)$, it is easily deduced that the linear speed is $2\sqrt{1-a}$ and that $c_{LLW} \geq 2\sqrt{1-a}$. However, contrarily to the KPP case, the converse inequality $c_{LLW} \leq 2\sqrt{1-a}$ is only sometimes true. More precisely,

— on one hand, according to Lewis–Li–Weinberger [108], linear determinacy holds if

$$d \leq 2 \text{ and } \frac{ab-1}{1-a} \leq \frac{1}{r}(2-d),$$

a result which was later on improved by Huang [93] who established that the weaker condition

$$\frac{(2-d)(1-a)+r}{rb} \geq \max\left(a, \frac{d-2}{2|d-1|}\right)$$

is sufficient;

— but on the other hand, Huang–Han [94] constructed explicit counter-examples where $c_{LLW} > 2\sqrt{1-a}$.

Anyways, a rough upper estimate of c_{LLW} can be obtained by comparison with the KPP equation satisfied by u in the absence of v: $c_{LLW} \leq 2$ (the competition always slows down the invasion of u). The strict inequality $c_{LLW} < 2$ is expected but, as far as we know, cannot be established easily. Of course, it automatically holds in case of linear determinacy.

We focus now on the system (7.1.1) and observe that, when u_0 and v_0 are both null or exponentially decaying in $[0,+\infty)$, the long-time behavior in $(0,+\infty)$ is unclear. It is the purpose of this paper to address this question.

If $rd > 1$ and u_0 and v_0 are compactly supported, then for all small $\epsilon > 0$, one expects the following statements to hold:

$$\lim_{t\to+\infty} \sup_{|x|<(c_{LLW}-\epsilon)t} \left(|u(t,x)-1| + |v(t,x)|\right) = 0,$$

$$\lim_{t\to+\infty} \sup_{(2+\epsilon)t<|x|<(2\sqrt{rd}-\epsilon)t} \left(|u(t,x)| + |v(t,x)-1|\right) = 0,$$

$$\lim_{t\to+\infty} \sup_{(2\sqrt{rd}+\epsilon)t<|x|} \left(|u(t,x)| + |v(t,x)|\right) = 0.$$

This fact, which we are going to prove in the forthcoming pages (see Proposition 7.13) by adapting very slightly arguments from the related literature and is therefore not really new, basically means the following: the empty space is first invaded by the faster competitor v at speed $2\sqrt{rd}$ and then the replacement of v by the stronger competitor u occurs somewhere in the area $c_{LLW} \leq \frac{x}{t} \leq 2$. In particular, as far as spreading speeds are concerned, the first invasion ($(0,0)$ by $(0,1)$) is not influenced by the second invasion ($(0,1)$ by $(1,0)$): the competition exerted by the exponential tail of u in the area $2 < \frac{x}{t}$ is negligible.

It is then natural to investigate whether the converse statement is true: is the second invasion influenced by the first one? Is it possible to show that the speed c_2 of the second invasion is exactly c_{LLW}, or is there on the contrary a possibility of acceleration, namely $c_2 > c_{LLW}$?

Previous works on the spreading properties of the system (7.1.1) with u_0 and v_0 supported in $(-\infty,0]$ are due to Carrère [35] and Lin and Li [112]. Carrère studied the bistable case ($a > 1$, $b > 1$). She proved that the second invasion admits a single spreading speed which is the speed of the unique bistable traveling wave connecting $(0,1)$ to $(1,0)$: the two invasions are indeed independent. We point out that the bistable case is easier to handle (based on the uniqueness of traveling wave speed and profile in the bistable case, the arguments used on the left of the second transition can be used again on its right). As a matter of fact, Lin and Li investigated the monostable case with stable coexistence ($a < 1$, $b < 1$) and compactly supported initial data but were unable to determine the second speed. All three monostable cases (stable coexistence, stable $(1,0)$, stable $(0,1)$) being handled quite similarly (see Lewis–Li–Weinberger [108] for instance), the technical obstacles they encountered should not depend on the sign of $b-1$.

In the present paper, we adopt a different point of view: we aim directly for the construction of (almost) optimal pairs of super-solutions and sub-solutions. This point of view is highly fruitful. On one hand, it brings forth a complete spreading result when the support of u_0 is included in a left half-line (namely, u_0 is Heavyside-like or compactly supported) and v_0 is compactly supported. This result shows that, surprisingly, acceleration of the second front does indeed occur in some cases.

Heuristically, there are three spreading mechanisms involved:

1. u invading a hostile environment where $v = 1$ at speed $2\sqrt{1-a}$;
2. u chasing the competitor v at speed c_{LLW};
3. u invading an environment where $v = \mathbf{1}_{\{x \leq 2\sqrt{rd}t\}}$ at some speed c.

While both c_{LLW} and c are greater than or equal to $2\sqrt{1-a}$ (by the comparison principle), the sign of $c - c_{LLW}$ can vary. One of our main results (Theorem 7.1) states that the actual invasion speed of u is the maximum of c_{LLW} and c.

The problem we consider in this paper was in fact already considered by Shigesada and Kawasaki in 1997 [133], where they illustrated numerically the hair-trigger effect (the fact that a small number of the weaker competitor eventually reaches the range front of the other species and establishes a KPP-type wave into the open space). They also gave practical estimates of the respective spreading speeds of the species, based on linearizations. Our work takes into account the possibility of failure of linear determinacy, and discovers additionally the possibility of a further accelerated invasion. Therefore, we have completely settled the mathematical questions raised by their study. Let us also point out here that, inspired by the study of Shigesada and Kawasaki, Li [109] very recently addressed similar questions in the framework of integro-difference systems.

Our approach also delivers a general existence and nonexistence result related to *propagating terraces* (succession of compatible traveling waves with decreasingly ordered speeds, first described by Fife and McLeod [71]) having the unstable steady state $(0,1)$ as intermediate steady state and corresponding to exponentially decaying initial data. As far as scalar terraces for reaction–diffusion equations are concerned, Ducrot, Giletti and Matano [62] showed quite generically that all intermediate states are stable from below (see also Poláčik [126] for a complete account in the general setting). In more sophisticated contexts (reaction–diffusion systems, nonlocal equations, etc.), propagating terraces with unstable intermediate states are observed numerically (see Nadin–Perthame–Tang [119] for the nonlocal KPP equation, Faye–Holzer [70] for a different two-component reaction–diffusion system). Rigorous analytical studies are however very difficult and have only been carried out in simple cases. For instance, a closely related paper due to Iida, Lui and Ninomiya [97] studied a monostable system of cooperatively coupled KPP equations. The comparison with that paper shows well the value of the present paper: Iida–Lui–Ninomiya only considered Heavyside-like or compactly supported initial data and obtained that all spreading speeds are in fact scalar KPP speeds (independent invasions). In this regard, the present paper is, to the best of our knowledge, unprecedented.

Our approach relies heavily upon the comparison principle. Therefore it might be appropriate for cooperative systems of arbitrary size with couplings more sophisticated than the Lotka–Volterra one considered in Iida–Lui–Ninomiya [97]. Let us point out right now that a fully coupled cooperative system, namely a cooperative system where the positivity of any one component implies the positivity of all the others, cannot admit propagating terrace solutions. Unfortunately, our approach cannot be adapted to settings devoid of comparison principle.

Finally, let us point out that our forthcoming results would still hold true if $u(1-u)$ and $rv(1-v)$ were replaced by more general KPP reaction terms. In order to ease the reading, however, we focus on the traditional logistic form.

7.1.1 Main results

Define the auxiliary function

$$f: \begin{array}{ccc} \left[2\sqrt{1-a},+\infty\right) & \to & \left(2\sqrt{a},2\left(\sqrt{1-a}+\sqrt{a}\right)\right] \\ c & \mapsto & c-\sqrt{c^2-4(1-a)}+2\sqrt{a} \end{array}. \tag{7.1.3}$$

This function is decreasing and bijective and satisfies in particular

$$f(2)=2,$$

$$f^{-1}:\tilde{c}\mapsto\frac{\tilde{c}}{2}-\sqrt{a}+\frac{2(1-a)}{\tilde{c}-2\sqrt{a}}.$$

7.1.1.1 Spreading properties of initially localized solutions

Theorem 7.1. *Let $u_0\in\mathscr{C}(\mathbb{R},[0,1])\setminus\{0\}$ with support included in a left half-line and $v_0\in\mathscr{C}(\mathbb{R},[0,1])\setminus\{0\}$ with compact support. Let (u,v) be the solution of (7.1.1).*

1. *Assume $2\sqrt{rd}<2$. Then*
$$\lim_{t\to+\infty}\sup_{x\geq 0}|v(t,x)|=0,$$
$$\lim_{t\to+\infty}\sup_{0\leq x<(2-\varepsilon)t}|u(t,x)-1|=0 \text{ for each } \varepsilon\in(0,2),$$
$$\lim_{t\to+\infty}\sup_{(2+\varepsilon)t<x}|u(t,x)|=0 \text{ for each } \varepsilon>0.$$

2. *Assume $2\sqrt{rd}\in(2,f(c_{LLW}))$ and define*
$$c_{acc}=f^{-1}\left(2\sqrt{rd}\right)=\sqrt{rd}-\sqrt{a}+\frac{1-a}{\sqrt{rd}-\sqrt{a}}\in(c_{LLW},2).$$
Then
$$\lim_{t\to+\infty}\sup_{0\leq x<(c_{acc}-\varepsilon)t}(|u(t,x)-1|+|v(t,x)|)=0 \text{ for each } \varepsilon\in(0,c_{acc}),$$
$$\lim_{t\to+\infty}\sup_{(c_{acc}+\varepsilon)t<x<\left(2\sqrt{rd}-\varepsilon\right)t}(|u(t,x)|+|v(t,x)-1|)=0 \text{ for each } \varepsilon\in\left(0,\frac{2\sqrt{rd}-c_{acc}}{2}\right),$$
$$\lim_{t\to+\infty}\sup_{\left(2\sqrt{rd}+\varepsilon\right)t<x}(|u(t,x)|+|v(t,x)|)=0 \text{ for each } \varepsilon>0.$$

3. *Assume $2\sqrt{rd}\geq f(c_{LLW})$. Then*
$$\lim_{t\to+\infty}\sup_{0\leq x<(c_{LLW}-\varepsilon)t}(|u(t,x)-1|+|v(t,x)|)=0 \text{ for each } \varepsilon\in(0,c_{LLW}),$$
$$\lim_{t\to+\infty}\sup_{(c_{LLW}+\varepsilon)t<x<\left(2\sqrt{rd}-\varepsilon\right)t}(|u(t,x)|+|v(t,x)-1|)=0 \text{ for each } \varepsilon\in\left(0,\frac{2\sqrt{rd}-c_{LLW}}{2}\right),$$
$$\lim_{t\to+\infty}\sup_{\left(2\sqrt{rd}+\varepsilon\right)t<x}(|u(t,x)|+|v(t,x)|)=0 \text{ for each } \varepsilon>0.$$

In the first case, v goes extinct. In the second case, v invades first at speed $2\sqrt{rd}$ and is then replaced by u at speed $c_{acc} > c_{LLW}$. In the third case, v invades first at speed $2\sqrt{rd}$ and is then replaced by u at speed c_{LLW}.

Notice that the limits above are chiefly concerned with $x \geq 0$. This is intentional, for the sake of brevity and clarity. In $(-\infty, 0)$, two behaviors are possible, depending on whether u_0 is compactly supported or Heavyside-like. In the former case, all inequalities above hold with x replaced by $|x|$ (and this claim is proved simply by symmetry). In the latter case, (u, v) converges uniformly to $(1, 0)$ in $(-\infty, 0)$ (and this claim can be proved by a standard comparison argument).

7.1.1.2 The set of admissible pairs of speeds for more general initial data

Define the auxiliary function

$$\begin{array}{rccc} \lambda_v : & \left[2\sqrt{rd}, +\infty\right) & \to & \left(0, \sqrt{\frac{r}{d}}\right] \\ & c & \mapsto & \frac{1}{2d}\left(c - \sqrt{c^2 - 4rd}\right) \end{array}.$$

Theorem 7.2. *Let $c_1 \in \left[2\sqrt{rd}, +\infty\right)$ and $c_2 \in [c_{LLW}, c_1]$. Let (u, v) be a solution of (7.1.1) such that*

$$c_2 = \sup\left\{c > 0 \mid \lim_{t\to+\infty} \sup_{0\leq x\leq ct} \left(|u(t,x) - 1| + |v(t,x)|\right) = 0\right\}$$

and such that at least one of the following two properties holds true:

1. *$x \mapsto v(0,x)\, e^{\lambda_v(c_1)x}$ is bounded in $\mathbb{R}$; or*
2. *c_1 satisfies*

$$c_1 \geq \inf\left\{c > 0 \mid \lim_{t\to+\infty} \sup_{x\geq ct} |v(t,x)| = 0\right\}.$$

Then $c_2 \geq f^{-1}(c_1)$.

The assumption on c_2 basically means that u spreads at speed c_2. However, in general, the spreading speed is ill-defined: the *minimal spreading speed* of u,

$$\sup\left\{c > 0 \mid \lim_{t\to+\infty} \sup_{0\leq x\leq ct} |u(t,x) - 1| = 0\right\},$$

might very well be smaller than its *maximal spreading speed*,

$$\inf\left\{c > 0 \mid \lim_{t\to+\infty} \sup_{0\leq x\leq ct} |u(t,x)| = 0\right\}.$$

On this problem, we refer to Hamel –Nadin [89].

The properties (1) and (2) above are more or less equivalent. Indeed, on one hand, (1) directly implies (2) by standard comparison; on the other hand, if (2) holds, then for all $\lambda \in (0, \lambda_v(c_1))$, there exists T_λ such that $x \mapsto v(T_\lambda, x)\,\mathrm{e}^{\lambda x}$ is bounded in $\mathbb{R}$. However the proof of the latter implication is difficult. In fact, instead of establishing it, we will directly prove the result in each case. We emphasize that although (2) might be easier to understand in that it directly relates c_1 to the spreading of v, (1) has the advantage of being easier to apply since it only requires knowledge of the initial condition.

In short, this theorem means that if v spreads no faster than c_1 and if u spreads at speed c_2, then $c_2 \geq f^{-1}(c_1)$. The next theorem shows the sharpness of this threshold: any $c_2 > f^{-1}(c_1)$ can actually be achieved.

Theorem 7.3. *Let* $c_1 \in \left(2\sqrt{rd}, +\infty\right)$ *and* $c_2 \in (c_{LLW}, c_1)$*.Assume* $c_1 > f(c_2)$*. Then there exists* $(u_{c_1,c_2}, v_{c_1,c_2}) \in \mathscr{C}\left(\mathbb{R}, [0,1]^2\right)$ *such that the solution* (u,v) *of (7.1.1) with initial value* $(u_0, v_0) = (u_{c_1,c_2}, v_{c_1,c_2})$ *satisfies*

$$\lim_{t\to+\infty} \sup_{x<(c_2-\varepsilon)t} \left(|u(t,x) - 1| + |v(t,x)|\right) = 0 \text{ for each } \varepsilon \in (0, c_2),$$

$$\lim_{t\to+\infty} \sup_{(c_2+\varepsilon)t<x<(c_1-\varepsilon)t} \left(|u(t,x)| + |v(t,x) - 1|\right) = 0 \text{ for each } \varepsilon \in \left(0, \frac{c_1 - c_2}{2}\right),$$

$$\lim_{t\to+\infty} \sup_{(c_1+\varepsilon)t<x} \left(|u(t,x)| + |v(t,x)|\right) = 0 \text{ for each } \varepsilon > 0.$$

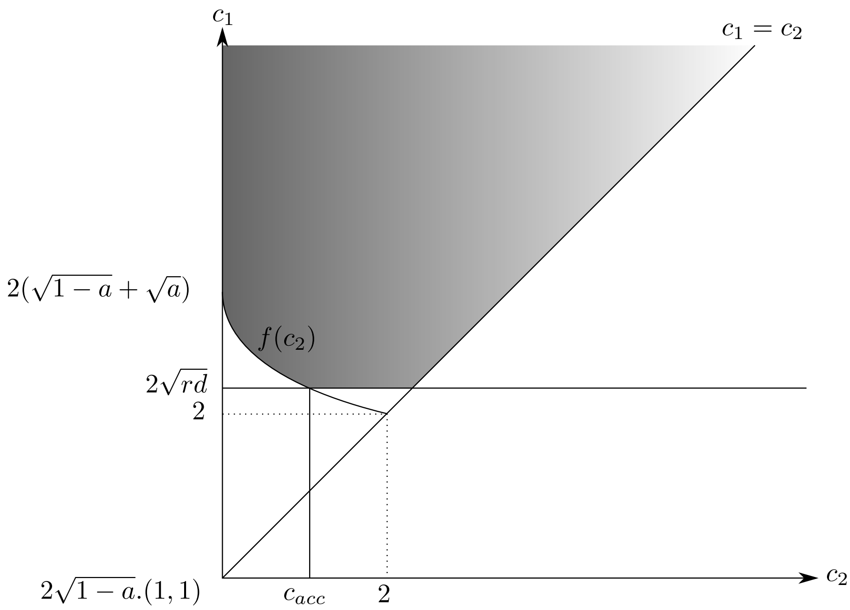


Figure 7.1.1 – Example of set of admissible pairs of speeds (c_1, c_2): $c_{LLW} = 2\sqrt{1-a}$, $2\sqrt{rd} < f(c_{LLW})$.
In this case Theorem 7.1(2) applies with $(c_1, c_2) = (2\sqrt{rd}, c_{acc})$, where $c_{acc} > c_{LLW}$.

Let us point out that this solution (u,v) is not a proper propagating terrace in the sense of Ducrot–Giletti–Matano [62]: the locally uniform convergence of the profiles is missing (as in Carrère [35]).

The fact that the set of admissible speeds is not always the maximal set

$$\left\{(c_1, c_2) \in \left[2\sqrt{rd}, +\infty\right) \times [c_{LLW}, +\infty) \mid c_1 > c_2\right\}.$$

settles completely a question raised by the first author [Gir18a].

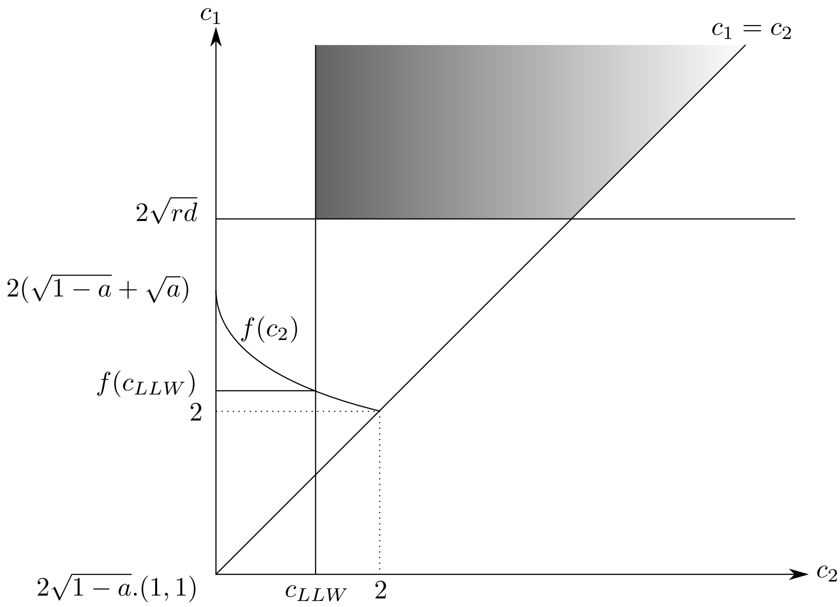


Figure 7.1.2 – Example of set of admissible pairs of speeds (c_1, c_2): $c_{LLW} > 2\sqrt{1-a}$, $2\sqrt{rd} > f(c_{LLW})$.
In this case Theorem 7.1(3) applies with $(c_1, c_2) = (2\sqrt{rd}, c_{LLW})$.

7.1.1.3 The super-solutions and sub-solutions

The preceding theorems will be proved thanks to the following three propositions, which are of independent interest and concern existence results for super-solutions and sub-solutions (the precise definition of these will be recalled in the next section).

Let

$$\lambda : \begin{array}{ccc} \left[2\sqrt{1-a}, +\infty\right) & \to & \left(0, \sqrt{1-a}\right] \\ c & \mapsto & \frac{1}{2}\left(c - \sqrt{c^2 - 4(1-a)}\right) \end{array},$$

$$\Lambda : (c, \tilde{c}) \mapsto \frac{1}{2}\left(\tilde{c} - \sqrt{\tilde{c}^2 - 4(\lambda(c)(\tilde{c} - c) + 1)}\right).$$

The domain of Λ is the set of all $(c, \tilde{c})$ such that $c \geq 2\sqrt{1-a}$ and $\tilde{c} \geq \max(c, f(c))$.

For all $c \geq c_{LLW}$ and $\tilde{c} \geq \max(c, f(c))$, $\overline{w_{c,\tilde{c}}}$ denotes the function

$$\overline{w_{c,\tilde{c}}} : (t, x) \mapsto \mathrm{e}^{-\lambda(c)(\tilde{c}-c)t} \mathrm{e}^{-\Lambda(c,\tilde{c})(x-\tilde{c}t)}.$$

Proposition 7.4. *Let $c_1 \geq 2\sqrt{rd}$, $c_2 \geq c_{LLW}$ and assume $c_2 < c_1 < f(c_2)$.*

There exist $c > c_2$, $\tilde{c} \in (c_1, f(c))$, $L > 0$ and $\delta^\star > 0$ such that, for all $\delta \in (0, \delta^\star)$, all $\kappa \in \left(0, \min\left(\frac{1-a}{2}, \frac{\delta}{2}\right)\right)$ and all $\zeta > L$, there exists $R_\delta > 0$ and a sub-solution $\left(\underline{u_{\delta,\zeta,\kappa}}, \overline{v_{\delta,\zeta}}\right)$ of (7.1.1) satisfying the following properties:

1. *$\underline{u_{\delta,\zeta,\kappa}}(0, x) \leq 1 - a$ for all $x \in \mathbb{R}$;*
2. *the support of $x \mapsto \underline{u_{\delta,\zeta,\kappa}}(0, x)$ is included in $[0, L + \zeta + 2R_\delta]$;*
3. *$\underline{u_{\delta,\zeta,\kappa}}(0, x) \leq \kappa$ for all $x \in [L, L + \zeta + 2R_\delta]$;*

4. *there exists* $X>0$ *such that* $\underline{u_{\delta,\zeta,\kappa}}$ *satisfies*

$$\partial_t \underline{u_{\delta,\zeta,\kappa}} - \partial_{xx}\underline{u_{\delta,\zeta,\kappa}} \leq (1-\delta)\,\underline{u_{\delta,\zeta,\kappa}} \text{ in } \{(t,x)\in[0,+\infty)\times\mathbb{R} \mid x > X+\tilde{c}t\}.$$

5. *there exists* $C_\delta>0$ *depending only on* δ *such that*

$$\overline{v_{\delta,\zeta}}(0,x) \geq \min\left(1, C_\delta e^{-\lambda_v(\tilde{c})(x-\zeta)}\right) \text{ for all } x\in\mathbb{R};$$

6. *the following spreading property holds true:*

$$\lim_{t\to+\infty} \sup_{L\leq x<(c-\varepsilon)t} \left| \underline{u_{\delta,\zeta,\kappa}}(t,x) - \frac{1-a}{2}\right| = 0 \text{ for all } \varepsilon\in(0,c).$$

Proposition 7.5. *Let* $c_2 \in \left(\max\left(c_{LLW}, f^{-1}\left(2\sqrt{rd}\right)\right), 2\right)$.
There exists $\delta^\star>0$ *and* $\left(c_1^\delta, c_2^\delta\right)_{\delta\in(0,\delta^\star)}$ *such that*

$$c_2 < c_2^\delta < c_1^\delta < 2\sqrt{rd} \text{ for all } \delta\in(0,\delta^\star),$$

$$\lim_{\delta\to0}\left(c_1^\delta, c_2^\delta\right) = \left(2\sqrt{rd}, c_2\right),$$

and, for all $\delta\in(0,\delta^\star)$*, there exists a super-solution* $\left(\overline{u_\delta}, \underline{v_\delta}\right)$ *of (7.1.1) satisfying the following properties:*

1. *there exists* $y_0\in\mathbb{R}$ *such that, for all* $y\geq y_0$ *and* $t\geq0$,

$$\overline{u_\delta}\left(0, x-y-\frac{\left(\Lambda\left(c_2, 2\sqrt{rd}\right)\right)^2+1}{\Lambda\left(c_2,2\sqrt{rd}\right)}t\right) \geq \min\left(1, \overline{w_{c_2,2\sqrt{rd}}}(t,x)\right) \text{ for all } x\in\mathbb{R};$$

2. $x\mapsto \underline{v_\delta}(0,x)$ *is compactly supported;*
3. $\underline{v_\delta(0,x)} \leq 1-\delta$ *for all* $x\in\mathbb{R}$*;*
4. *the following spreading property holds true:*

$$\lim_{t\to+\infty} \sup_{\left(c_2^\delta+\varepsilon\right)t<x<\left(c_1^\delta-\varepsilon\right)t} \left(\left|\overline{u_\delta}(t,x)\right| + \left|\underline{v_\delta}(t,x)-(1-2\delta)\right|\right) = 0 \text{ for all } \varepsilon\in\left(0, \frac{c_1^\delta-c_2^\delta}{2}\right).$$

Proposition 7.6. *Let* $c_1>2\sqrt{rd}$*,* $c_2>c_{LLW}$ *and assume* $c_1>\max(c_2, f(c_2))$.
There exists $\delta^\star>0$ *and* $\left(c_2^\delta\right)_{\delta\in(0,\delta^\star)}$ *such that*

$$c_2^\delta > c_2 \text{ for all } \delta\in(0,\delta^\star),$$

$$\lim_{\delta\to0} c_2^\delta = c_2,$$

and, for all $\delta\in(0,\delta^\star)$*, there exists a super-solution of (7.1.1)* $\left(\overline{u_\delta}, \underline{v_\delta}\right)$ *and a sub-solution of (7.1.1)* $\left(\underline{u_\delta}, \overline{v_\delta}\right)$ *satisfying the following properties:*

1. *there exists* $y_0\in\mathbb{R}$ *such that, for all* $y\geq y_0$ *and* $t\geq0$,

$$\overline{u_\delta}\left(0, x-y-\frac{\left(\Lambda(c_2,c_1)\right)^2+1}{\Lambda(c_2,c_1)}t\right) \geq \min\left(1, \overline{w_{c_2,c_1}}(t,x)\right) \text{ for all } x\in\mathbb{R};$$

2. *the support of* $\underline{v_\delta}$ *is a right half-line and there exists* $(y,z)\in\mathbb{R}^2$ *such that*

$$\frac{1}{2}\leq e^{\lambda_v(c_1)(x-c_1t)}\underline{v_\delta}(0,x-y)\leq 1 \text{ for all } t\geq 0 \text{ and } x\geq z;$$

3. $\underline{u_\delta}(0,x)\leq\overline{u_\delta}(0,x)$ *and* $\underline{v_\delta}(0,x)\leq\overline{v_\delta}(0,x)$ *for all* $x\in\mathbb{R}$*;*
4. *the following spreading properties hold true:*

$$\lim_{t\to+\infty}\sup_{x<(c_2-\varepsilon)t}\left|\underline{u_\delta}(t,x)-(1-a)\right|=0 \text{ for all } \varepsilon\in(0,c_2),$$

$$\lim_{t\to+\infty}\sup_{\left(c_2^\delta+\varepsilon\right)t<x<(c_1-\varepsilon)t}\left(\left|\overline{u_\delta}(t,x)\right|+\left|\underline{v_\delta}(t,x)-(1-2\delta)\right|\right)=0 \text{ for all } \varepsilon\in\left(0,\frac{c_1-c_2^\delta}{2}\right).$$

$$\lim_{t\to+\infty}\sup_{(c_1+\varepsilon)t<x}\left(\left|\overline{u_\delta}(t,x)\right|+\left|\overline{v_\delta}(t,x)\right|\right)=0 \text{ for all } \varepsilon>0,$$

The forms of the super- and sub-solutions of Proposition 7.4 and Proposition 7.5 are illustrated in Figure 7.4.4 and Figure 7.4.3 respectively. Those of Proposition 7.6 are illustrated in Figure 7.4.1 and Figure 7.4.2.

7.1.2 The quantities $f(c_2)$, $\lambda(c_2)$, $\Lambda(c_2,c_1)$

Let us explain by a heuristic argument how these three quantities come out naturally in the problem and what is their ecological meaning.

Assume that v invades the uninhabited territory at some speed $c_1\geq 2\sqrt{rd}$ and that u chases v at some speed $c_2\in[c_{LLW},c_1)$. In the area where $v\simeq 1$, u looks like the exponential tail of the monostable traveling wave connecting $(0,1)$ to $(1,0)$ at speed c_2, that is

$$u(t,x)\simeq \mathrm{e}^{-\lambda(c_2)(x-c_2t)}.$$

Accordingly, in a neighborhood of $x=\tilde{c}t$ with $\tilde{c}\in(c_2,c_1)$, we can observe non-negligible quantities only if we consider the rescaled function

$$w:(t,x)\mapsto u(t,x)\,\mathrm{e}^{\lambda(c_2)(x-c_2t)}$$

instead of u itself.

Yet, in a neighborhood of $x=\tilde{c}t$ with $\tilde{c}>c_1$, where $(u,v)\simeq(0,0)$, w satisfies at the first order

$$\partial_t w-\partial_{xx}w=(1+\lambda(c_2)(\tilde{c}-c_2))\,w$$

whence the exponential ansatz $w(t,x)=\mathrm{e}^{-\Lambda(x-\tilde{c}t)}$ leads to the equation

$$\Lambda^2-\tilde{c}\Lambda+(1+\lambda(c_2)(\tilde{c}-c_2))=0.$$

The minimal zero of this equation being precisely

$$\Lambda(c_2,\tilde{c})=\frac{1}{2}\left(\tilde{c}-\sqrt{\tilde{c}^2-4(\lambda(c_2)(\tilde{c}-c_2)+1)}\right),$$

we deduce then that $\tilde{c}$ has to satisfy

$$\tilde{c}^2-4(\lambda(c_2)(\tilde{c}-c_2)+1)\geq 0$$

that is $\tilde{c}\geq f(c_2)$. Passing to the limit $\tilde{c}\to c_1$, we find indeed $c_1\geq f(c_2)$.

7.1.3 Organization of the paper

In Section 2, we recall the comparison principle for (7.1.1) and define super-solutions and sub-solutions.

In Section 3, we prove Theorem 7.1, Theorem 7.2 and Theorem 7.3 assuming Proposition 7.4, Proposition 7.5 and Proposition 7.6 are true.

In Section 4, we prove Proposition 7.4, Proposition 7.5 and Proposition 7.6. These constructions are rather delicate and require several objects and preliminary lemmas, which we summarize in a table at the beginning of Subsection 7.4.1.

In Section 5, we comment on the results and provide some future perspectives.

7.2 Competitive comparison principle

7.2.1 Competitive comparison principle

In what follows, vectors in $\mathbb{R}^2$ are always understood as column vectors.

We define the competitive ordering $\preceq$ in $\mathbb{R}^2$ as follows: for all $(u_1, v_1) \in \mathbb{R}^2$, $(u_2, v_2) \in \mathbb{R}^2$,

$$(u_1, v_1) \preceq (u_2, v_2) \text{ if } u_1 \leq u_2 \text{ and } v_1 \geq v_2.$$

The strict competitive ordering $\prec$ is defined by

$$(u_1, v_1) \prec (u_2, v_2) \text{ if } u_1 < u_2 \text{ and } v_1 > v_2.$$

We define also the operators

$$P : (u, v) \mapsto \partial_t (u, v) - \operatorname{diag}(1, d)\, \partial_{xx} (u, v),$$

$$F : (u, v) \mapsto \begin{pmatrix} u(1-u-av) \\ rv(1-v-bu) \end{pmatrix}.$$

With these notations, (7.1.1) can be written as

$$\begin{cases} P(u,v) = F(u,v) & \text{in } (0,+\infty) \times \mathbb{R} \\ (u,v)(0,x) = (u_0, v_0)(x) & \text{for all } x \in \mathbb{R} \end{cases}.$$

Definition 7.7. A classical super-solution of (7.1.1) is a pair

$$(\overline{u}, \underline{v}) \in \mathscr{C}^1\left((0,+\infty), \mathscr{C}^2\left(\mathbb{R}, [0,1]^2\right)\right) \cap \mathscr{C}\left([0,+\infty) \times \mathbb{R}, [0,1]^2\right)$$

satisfying

$$P(\overline{u}, \underline{v}) \succeq F(\overline{u}, \underline{v}) \text{ in } (0,+\infty) \times \mathbb{R}.$$

A classical sub-solution of (7.1.1) is a pair

$$(\underline{u}, \overline{v}) \in \mathscr{C}^1\left((0,+\infty), \mathscr{C}^2\left(\mathbb{R}, [0,1]^2\right)\right) \cap \mathscr{C}\left([0,+\infty) \times \mathbb{R}, [0,1]^2\right)$$

satisfying

$$P(\underline{u}, \overline{v}) \preceq F(\underline{u}, \overline{v}) \text{ in } (0,+\infty) \times \mathbb{R}.$$

The unbounded domain $(0,+\infty) \times \mathbb{R}$ can be replaced in the above definition by a bounded parabolic cylinder $(0,T) \times (-R,R)$. In such a case, the required regularity is $\mathscr{C}^1\left((0,T), \mathscr{C}^2\left((-R,R), [0,1]^2\right)\right) \cap \mathscr{C}\left([0,T] \times [-R,R], [0,1]^2\right)$.

We also recall that it is possible to extend the theory of super- and sub-solutions to Sobolev spaces. The full extension is outside of the scope of this reminder, however a very partial extension will be necessary later on. More precisely, we will use the following definition and theorem. In what follows, $\circ$ denotes the Hadamard product $(u_1, v_1) \circ (u_2, v_2) = (u_1 u_2, v_1 v_2)$.

Definition 7.8. A generalized super-solution of (7.1.1) is a pair

$$(\overline{u}, \underline{v}) \in \mathscr{C}^{0,1}\left((0,+\infty)\times\mathbb{R}, [0,1]^2\right)$$

satisfying, for all $(U,V) \in \mathscr{D}\left((0,+\infty)\times\mathbb{R}, [0,1]^2\right)$,

$$\int \partial_t (\overline{u}, \underline{v}) \circ (U,V) + \operatorname{diag}(1,d)\, \partial_x (\overline{u}, \underline{v}) \circ \partial_x (U,V) \succeq \int F(\overline{u}, \underline{v}) \circ (U,V).$$

A generalized sub-solution of (7.1.1) is a pair

$$(\underline{u}, \overline{v}) \in \mathscr{C}^{0,1}\left((0,+\infty)\times\mathbb{R}, [0,1]^2\right)$$

satisfying, for all $(U,V) \in \mathscr{D}\left((0,+\infty)\times\mathbb{R}, [0,1]^2\right)$,

$$\int \partial_t (\underline{u}, \overline{v}) \circ (U,V) + \operatorname{diag}(1,d)\, \partial_x (\underline{u}, \overline{v}) \circ \partial_x (U,V) \preceq \int F(\underline{u}, \overline{v}) \circ (U,V).$$

Again, the unbounded domain $(0,+\infty)\times\mathbb{R}$ can be replaced by a bounded parabolic cylinder $(0,T)\times(-R,R)$. The following important theorem, that will be used many times thereafter, actually uses the local definition.

Theorem 7.9. *Let $R>0$, $T>0$, $Q=(0,T)\times(-R,R)$ and*

$$(\overline{u}_1, \overline{u}_2, \underline{v}_1, \underline{v}_2) \in \mathscr{C}^1\left([0,T], \mathscr{C}^2\left([-R,R], [0,1]^4\right)\right) \cap \mathscr{C}\left([0,T]\times[-R,R], [0,1]^4\right).$$

1. *Assume that $(\overline{u}_1, \underline{v}_1)$ and $(\overline{u}_1, \underline{v}_2)$ are both classical super-solutions in Q. Then $(\overline{u}_1, \max(\underline{v}_1, \underline{v}_2))$ is a generalized super-solution in Q.*
2. *Assume that $(\overline{u}_1, \underline{v}_1)$ and $(\overline{u}_2, \underline{v}_1)$ are both classical super-solutions in Q. Then $(\min(\overline{u}_1, \overline{u}_2), \underline{v}_1)$ is a generalized super-solution in Q.*

Remark. We state this theorem in a bounded parabolic cylinder in order to be able to construct later on more complex super- and sub-solutions, for instance super-solutions $(\overline{u}, \underline{v})$ with $\overline{u}$ of the form

$$\overline{u}(t,x) = \begin{cases} \overline{u}_1(t,x) & \text{if } x < x(t) \\ \overline{u}_2(t,x) & \text{if } x \in [x(t), y(t)] \\ \overline{u}_3(t,x) & \text{if } x > y(t) \end{cases},$$

where $x(t) < y(t)$ and $\overline{u}_1$, $\overline{u}_2$ and $\overline{u}_3$ are such that $\overline{u}_1(t,x) \leq \overline{u}_2(t,x)$ if $x < x(t)$, $\overline{u}_2(t,x) \leq \overline{u}_1(t,x)$ in a right-sided neighborhood of $x(t)$, $\overline{u}_2(t,x) \leq \overline{u}_3(t,x)$ in a left-sided neighborhood of $y(t)$ and $\overline{u}_3(t,x) \leq \overline{u}_2(t,x)$ if $x > y(t)$. Although we do not have any global information on $\overline{u}_1 - \overline{u}_2$, $\overline{u}_1 - \overline{u}_3$ and $\overline{u}_2 - \overline{u}_3$, the local theorem shows that the construction is still valid.

Proof. Since the second statement is proved similarly, we only prove the first one.

For simplicity, we only consider the special case where $\Gamma = (\underline{v}_1 - \underline{v}_2)^{-1}(\{0\})$ is a smooth hypersurface, which is always satisfied for our purposes. A proof that does not require such a regularity assumption can be found for instance in [132].

Define $\underline{v} = \max\left(\underline{v}_1, \underline{v}_2\right)$ and let $(U,V) \in \mathscr{D}\left(\overline{Q}, [0,1]^2\right)$. On one hand,

$$\partial_t \overline{u}_1 - \partial_{xx} \overline{u}_1 \geq F_1\left(\overline{u}_1, \underline{v}\right)$$

is satisfied in the classical sense (using for instance $-a\overline{u}_1\underline{v}_1 \geq -a\overline{u}_1\underline{v}$). On the other hand, we have assumed that $\Gamma = \left(\underline{v}_1 - \underline{v}_2\right)^{-1}(\{0\})$ is a smooth hypersurface, so that we may integrate by parts. Denoting $Q_1 = \left(\underline{v}_1 - \underline{v}_2\right)^{-1}([0,1])$, $Q_2 = \left(\underline{v}_2 - \underline{v}_1\right)^{-1}([0,1])$, ν the outward unit normal to Q_1, we find $\Gamma = \partial Q_1 \backslash \partial Q = \partial Q_2 \backslash \partial Q$ and $\left(\partial_x \underline{v}_1 - \partial_x \underline{v}_2\right)\nu \leq 0$ on Γ, whence

$$\begin{aligned}
&\int_Q \partial_t \underline{v} V + d\partial_x \underline{v} \partial_x V \\
&= \int_{Q_1} \partial_t \underline{v}_1 V + d\partial_x \underline{v}_1 \partial_x V + \int_{Q_2} \partial_t \underline{v}_2 V + d\partial_x \underline{v}_2 \partial_x V \\
&= \int_{Q_1} \left(\partial_t \underline{v}_1 - d\partial_{xx}\underline{v}_1\right) V + \int_{\partial Q_1} \partial_x \underline{v}_1 V \nu + \int_{Q_2} \left(\partial_t \underline{v}_2 - d\partial_{xx}\underline{v}_2\right) V + \int_{\partial Q_2} \partial_x \underline{v}_1 V(-\nu) \\
&\leq \int_{Q_1} F_2\left(\overline{u}_1, \underline{v}_1\right) V + \int_{Q_2} F_2\left(\overline{u}_1, \underline{v}_2\right) V + \int_\Gamma \left(\partial_x \underline{v}_1 - \partial_x \underline{v}_2\right) V\nu \\
&\leq \int_Q F_2\left(\overline{u}_1, \underline{v}\right) V.
\end{aligned}$$

This completes the proof. □

An inversion of the roles yields a similar statement on sub-solutions.

Theorem 7.10. *Let $R > 0$, $T > 0$, $Q = (0,T) \times (-R,R)$ and*

$$\left(\underline{u}_1, \underline{u}_2, \overline{v}_1, \overline{v}_2\right) \in \mathscr{C}^1\left([0,T], \mathscr{C}^2\left([-R,R], [0,1]^4\right)\right) \cap \mathscr{C}\left([0,T] \times [-R,R], [0,1]^4\right).$$

1. *Assume that $\left(\underline{u}_1, \overline{v}_1\right)$ and $\left(\underline{u}_1, \overline{v}_2\right)$ are both classical sub-solutions in Q. Then $\left(\underline{u}_1, \min\left(\overline{v}_1, \overline{v}_2\right)\right)$ is a generalized sub-solution in Q.*
2. *Assume that $\left(\underline{u}_1, \overline{v}_1\right)$ and $\left(\underline{u}_2, \overline{v}_1\right)$ are both classical sub-solutions in Q. Then $\left(\max\left(\underline{u}_1, \underline{u}_2\right), \overline{v}_1\right)$ is a generalized sub-solution in Q.*

Since a classical super- or sub-solution is *a fortiori* a generalized super- or sub-solution respectively, from now on, we omit the adjectives classical and generalized and always have in mind the generalized notion.

The comparison principle for (7.1.1), directly derived from the comparison principle for cooperative systems (see Protter–Weinberger [129]) via the transformation $w = 1 - v$, reads as follows.

Theorem 7.11. *Let $(\overline{u}, \underline{v})$ and $(\underline{v}, \overline{u})$ be respectively a super-solution and a sub-solution of (7.1.1). Assume that*

$$(\overline{u}, \underline{v})(0,x) \succeq (\underline{u}, \overline{v})(0,x) \text{ for all } x \in \mathbb{R}.$$

Then

$$(\overline{u}, \underline{v}) \succeq (\underline{u}, \overline{v}) \text{ in } [0,+\infty) \times \mathbb{R}.$$

Furthermore, if there exists $(T,x) \in (0,+\infty) \times \mathbb{R}$ such that $\overline{u}(T,x) = \underline{u}(T,x)$ or $\underline{v}(T,x) = \overline{v}(T,x)$, then

$$(\overline{u}, \underline{v}) = (\underline{u}, \overline{v}) \text{ in } [0,T] \times \mathbb{R}.$$

In other words, $(\overline{u}, \underline{v}) \succ (\underline{u}, \overline{v})$ holds at $t = 0$ if and only if it holds at all $t \geq 0$.

Finally, we recall an important existence result that will be used later on.

Theorem 7.12. *Let $(\overline{u}, \underline{v})$ and $(\underline{v}, \overline{u})$ be respectively a super-solution and a sub-solution of (7.1.1). Assume that for some $(u_0, v_0) \in C(\mathbb{R}, [0,1]^2)$ we have*

$$(\overline{u}, \underline{v})(0, x) \succeq (u_0, v_0)(x) \succeq (\underline{u}, \overline{v})(0, x) \quad \text{for all } x \in \mathbb{R},$$

then the solution (u, v) of (7.1.1) with initial data (u_0, v_0) satisfies

$$(\overline{u}, \underline{v}) \succeq (u, v) \succeq (\underline{u}, \overline{v}) \ \text{ in } \ [0, +\infty) \times \mathbb{R}.$$

7.3 Proofs of Theorem 7.1, Theorem 7.2 and Theorem 7.3

In this section, we assume Proposition 7.4, Proposition 7.5 and Proposition 7.6 are true.

7.3.1 Proof of Theorem 7.2

Proof. Let $c_1 \geq 2\sqrt{rd}$ and $c_2 \geq c_{LLW}$ such that $c_1 \geq c_2$ and $c_1 < f(c_2)$.

First, we consider the case where $x \mapsto v_0(x) \mathrm{e}^{\lambda_v(c_1)x}$ is globally bounded. By contradiction, assume the existence of a solution (u, v) such that both the boundedness of $x \mapsto v_0(x) \mathrm{e}^{\lambda_v(c_1)x}$ and the equality

$$c_2 = \sup \left\{ c > 0 \mid \lim_{t \to +\infty} \sup_{0 \leq x \leq ct} \left(|u(t,x) - 1| + |v(t,x)| \right) = 0 \right\}$$

are true.

Define c, $\tilde{c}$, $\delta^\star$, $\delta = \frac{\delta^\star}{2}$, R_δ, L, $\underline{u_{\delta,\zeta,\kappa}}$, $\overline{v_{\delta,\eta}}$ as in Proposition 7.4. Note that $c > c_2$, $\tilde{c} > c_1$.

In view of the equality satisfied by c_2, there exists $T \geq \frac{2L}{c_2}$ such that, for all $x \in \left[0, \frac{c_2}{2}T\right]$, $u(T, x) \geq 1 - \frac{a}{2}$.

We claim that $(t, x) \mapsto v(t, x) \mathrm{e}^{\lambda_v(c_1)(x - c_1 t)}$ is globally bounded in $[0, +\infty) \times \mathbb{R}$. To see this, it suffices to observe that, by definition of $\lambda_v(c_1)$, $Ce^{-\lambda_v(c_1)(x - c_1 t)}$ is a supersolution of the equation of v for any constant $C > 0$. Hence standard comparison implies that

$$v(t,x) \leq \left(\sup_{x \in \mathbb{R}} v(0,x) e^{\lambda_v(c_1)x} \right) e^{-\lambda_v(c_1)(x - c_1 t)}.$$

Since $c_1 < \tilde{c}$ and λ_v is decreasing, we have $\lambda_v(c_1) > \lambda_v(\tilde{c})$. Hence, there exists $\zeta > L$ such that

$$v(T, x) \leq \overline{v_{\delta,\zeta}}(0, x) \ \text{ for all } x \in \mathbb{R}.$$

Now, we fix

$$\kappa = \frac{1}{2} \min \left(\min \left(\frac{1-a}{2}, \frac{\delta}{2} \right), \min_{x \in [L, L+\zeta+2R_\delta]} u(T, x) \right).$$

It follows that

$$u(T,x) \geq \begin{cases} 1 - a & \text{for } x \in [0, L], \\ \kappa & \text{for } x \in (L, L + \zeta + 2R_\delta], \\ 0 & \text{for } x \in \mathbb{R} \setminus [0, L + \zeta + 2R_\delta], \end{cases}$$

whence $u(T, x) \geq \underline{u_{\delta,\zeta,\kappa}}(0, x)$ for $x \in \mathbb{R}$. Then

$$(\underline{u}, \overline{v}) : (t, x) \mapsto \left(\underline{u_{\delta,\zeta,\kappa}}(t - T, x), \overline{v}(t - T, x) \right)$$

is a sub-solution of (7.1.1) which satisfies $(\underline{u},\overline{v}) \preceq (u,v)$ at $t=T$, whence by the comparison principle of Theorem 7.11 it satisfies the same inequality at any time $t \geq T$.

Now, due to the spreading property satisfied by $\underline{u}$, for all $\varepsilon \in (0,c)$, there exists $T_\varepsilon \geq T$ such that, for all $t \geq T_\varepsilon$,

$$\inf_{L \leq x < (c-\varepsilon)t} u(t,x) \geq \frac{1-a}{4}.$$

Thus, by comparison of (u,v) with the spatially homogeneous sub-solution $\left(\underline{U},\overline{V}\right)$ satisfying the system

$$\begin{cases} \underline{U}'(t) = \underline{U}(t)\left(1-\underline{U}(t)-a\overline{V}(t)\right) & \text{for all } t \in (T_\varepsilon,+\infty) \\ \overline{V}'(t) = r\overline{V}(t)\left(1-\overline{V}(t)-b\underline{U}(t)\right) & \text{for all } t \in (T_\varepsilon,+\infty) \\ \underline{U}(T_\varepsilon) = \frac{1-a}{4} \\ \overline{V}(T_\varepsilon) = 1 \end{cases},$$

whose convergence to $(1,0)$ is well-known, we find

$$\lim_{t\to+\infty} \sup_{0\leq x<(c-\varepsilon)t} \left(|u(t,x)-1| + |v(t,x)|\right) = 0 \text{ for all } \varepsilon \in (0,c).$$

This means $c_2 \geq c$, and directly contradicts the chose of $c > c_2$ made at the beginning of the proof.

Next, we consider the case where

$$c_1 \geq \inf\left\{c > 0 \mid \lim_{t\to+\infty} \sup_{x\geq ct} |v(t,x)| = 0\right\}.$$

Since the proof is mostly the same, we only sketch it. Again, we argue by contradiction and use Proposition 7.4. Using the assumption on the spreading of v, we can establish the following estimate:

$$v(t,x-\hat{c}t) \leq \mathbf{1}_{y\leq y_0}(x-\hat{c}t) + \frac{\delta}{2a}\mathbf{1}_{y\geq y_0}(x-\hat{c}t),$$

for some $y_0 \in \mathbb{R}$ and with $\hat{c} = \frac{\tilde{c}+c_1}{2}$. Thanks to this, we can directly use $\eta \underline{u_{\delta,\zeta,\kappa}}$, for some small $\eta > 0$, as sub-solution for u and deduce a contradiction. We point out that in this case, we do not use the competitive comparison principle but instead use the scalar one. □

7.3.2 Proof of Theorem 7.1

7.3.2.1 Hair-trigger effect and extinction

Proposition 7.13. *Let $u_0 \in \mathscr{C}(\mathbb{R},[0,1]) \setminus \{0\}$ with support included in a left half-line and $v_0 \in \mathscr{C}(\mathbb{R},[0,1]) \setminus \{0\}$ with compact support. Let (u,v) be the solution of 7.1.1.*

1. *If $2\sqrt{rd} > 2$, then*

$$\lim_{t\to+\infty} \sup_{0\leq x<(c_{LLW}-\varepsilon)t} \left(|u(t,x)-1| + |v(t,x)|\right) = 0 \textit{ for all } \varepsilon \in (0,c_{LLW}),$$

$$\lim_{t\to+\infty} \sup_{\left(2\sqrt{rd}+\varepsilon\right)t<x} \left(|u(t,x)| + |v(t,x)|\right) = 0 \textit{ for all } \varepsilon > 0,$$

$$\lim_{t\to+\infty} \sup_{(2+\varepsilon)t<x<\left(2\sqrt{rd}-\varepsilon\right)t} \left(|u(t,x)| + |v(t,x)-1|\right) = 0 \textit{ for all } \varepsilon \in \left(0, \frac{2\sqrt{rd}-c_{LLW}}{2}\right).$$

2. *If* $2\sqrt{rd} < 2$*, then*

$$\lim_{t\to+\infty} \sup_{x\in\mathbb{R}} |v(t,x)| = 0,$$

$$\lim_{t\to+\infty} \sup_{0\leq x<(2-\varepsilon)t} |u(t,x)-1| = 0 \textit{ for all } \varepsilon \in (0,2),$$

$$\lim_{t\to+\infty} \sup_{(2+\varepsilon)t<x} |u(t,x)| = 0 \textit{ for all } \varepsilon > 0.$$

Remark. The inequality regarding $(2+\varepsilon)t < x < \left(2\sqrt{rd} - \varepsilon t\right)$ is by far the more interesting and the less trivial. It basically means that u does not exert any competition far ahead of its own territory. It was first proved by Ducrot, Giletti and Matano [61] in the case of predator–prey interactions (the conclusion being then that no predation occurs far ahead of the territory of the predator), and by Lin and Li [112] in case of two-species competition (the conclusion being that the region of coexistence falls behind the territory where the faster diffuser dominates). The proof of Ducrot *et al.* was sufficiently robust and generic to be reused by Carrère [35] in the bistable competitive case and to be reused again here, in the monostable case. Although it would certainly be interesting to write the result of Ducrot *et al.* in the most general form possible (with more than two species and minimal assumptions on the interactions), this is far beyond the scope of this paper. Therefore we simply adapt the main idea of their proof.

Proof. First, applying the comparison principle with the solution of

$$\begin{cases} \partial_t u_{KPP} - \partial_{xx} u_{KPP} = u_{KPP}(1-u_{KPP}) & \text{in } (0,+\infty)\times\mathbb{R} \\ u_{KPP}(0,x) = u_0(x) & \text{for all } x\in\mathbb{R} \end{cases},$$

we find directly $u \leq u_{KPP}$, whence

$$\lim_{t\to+\infty} \sup_{x>(2+\varepsilon)t} u(t,x) = 0 \text{ for all } \varepsilon > 0.$$

Similarly,

$$\lim_{t\to+\infty} \sup_{x>\left(2\sqrt{rd}+\varepsilon\right)t} v(t,x) = 0 \text{ for all } \varepsilon > 0.$$

Furthermore, (u,v) satisfies also $(u,v) \succeq (u_{LLW}, v_{LLW})$, where (u_{LLW}, v_{LLW}) is the solution of (7.1.1) with initial data $(u_0, 1)$, and by Lewis–Li–Weinberger [108], this yields

$$\lim_{t\to+\infty} \sup_{0\leq x<(c_{LLW}-\varepsilon)t} |u(t,x)-1| + |v(t,x)| = 0 \text{ for all } \varepsilon \in (0, c_{LLW}).$$

Next, let us prove that if $2\sqrt{rd} < 2$ and provided

$$\lim_{t\to+\infty} \sup_{\left(2\sqrt{rd}+\varepsilon\right)t<x<(2-\varepsilon)t} (|u(t,x)-1| + |v(t,x)|) = 0 \text{ for all } \varepsilon \in \left(0, \frac{2-2\sqrt{rd}}{2}\right),$$

then in fact the above limit can be reinforced as

$$\lim_{t\to+\infty} \sup_{0\leq x<(2-\varepsilon)t} (|u(t,x)-1| + |v(t,x)|) = 0 \text{ for all } \varepsilon \in (0,2).$$

Let $\varepsilon \in \left(0, \frac{2-2\sqrt{rd}}{3}\right)$. It is well-known (see Du–Lin [59, 60]) that there exists a unique solution of

$$\begin{cases} -\underline{\varphi}'' = \underline{\varphi}\left(1-a-\underline{\varphi}\right) & \text{in } (0,+\infty) \\ \underline{\varphi}(0) = 0 & \\ \underline{\varphi}(x) > 0 & \text{for all } x > 0 \end{cases}.$$

Furthermore, this solution is increasing in $(0,+\infty)$ and converges to $1-a$ at $+\infty$. In view of the assumption on the limit of (u,v) in $\left(2\sqrt{rd}+\varepsilon\right)t<x<(2-\varepsilon)\,t$, there exists $T\geq 0$ and $x_0>0$ such that

$$u\,(t,y+(2-2\varepsilon)\,t)>1-a>\underline{\varphi}(y+(2-2\varepsilon)t-x_0)\text{ for }(y,t)\in\{0\}\times[T,+\infty),$$
$$u(t,y+(2-2\varepsilon)t)\geq\underline{\varphi}(y+(2-2\varepsilon)t-x_0)\text{ for }(y,t)\in(-\infty,0]\times\{T\},$$
$$\underline{\varphi}(y+(2-2\varepsilon)t-x_0)>0\text{ when }(y,t)=(0,T).$$

Let $\tilde{u}(t,y)=u(t,y+(2-2\varepsilon)t)$ and $\underline{\tilde{u}}(t,y)=\underline{\varphi}(y+(2-2\varepsilon)t-x_0)$. Then they satisfy for all $(t,y)\in(0,+\infty)\times\mathbb{R}$,

$$\partial_t\underline{\tilde{u}}-\partial_{xx}\underline{\tilde{u}}-(2-2\varepsilon)\partial_x\underline{\tilde{u}}-\underline{\tilde{u}}(1-\underline{\tilde{u}}-av)\leq 0=\partial_{xx}\tilde{u}-(2-2\varepsilon)\partial_x\tilde{u}-\tilde{u}(1-\tilde{u}-av),$$

it follows by virtue of the scalar comparison principle and of a change of variable $y=x-(2-2\varepsilon)\,t$ that

$$\underline{u}\,(t,x)\leq u\,(t,x)\ \text{ for all }t\geq T\text{ and }x\leq(2-2\varepsilon)\,t.$$

Consequently,

$$\liminf_{t\to+\infty}\inf_{\varepsilon t<x<(2-2\varepsilon)t}u\,(t,x)\geq 1-a,$$

whence there exists $T'\geq 0$ such that

$$\inf_{\varepsilon t<x<(2-2\varepsilon)t}u\,(t,x)\geq\frac{1-a}{2}>0\text{ for all }t\geq T'.$$

By virtue of Theorem 7.11, the solution (U,V) of (7.1.1) with constant initial values $\left(\frac{1-a}{2},1\right)$ satisfies

$$(U,V)\,(t-T',x)\preceq(u,v)\,(t,x)\ \text{ for all }t\geq T'\text{ and }\varepsilon t<x<(2-2\varepsilon)\,t.$$

Since (U,V) coincides with the solution of the ODE system

$$\begin{cases}U'=U\,(1-U-aV)\\ V'=rV\,(1-V-bU)\\ (U,V)\,(0)=\left(\frac{1-a}{2},1\right)\end{cases},$$

standard theory on such systems shows that (U,V) converges to $(1,0)$, whence (u,v) itself converges to $(1,0)$ uniformly in $\varepsilon t<x<(2-2\varepsilon)\,t$. Recalling that we also have the estimate

$$\lim_{t\to+\infty}\sup_{0\leq x<(c_{LLW}-\varepsilon)t}|u\,(t,x)-1|+|v\,(t,x)|=0\text{ for all }\varepsilon\in(0,c_{LLW})\,,$$

the claim is proved.

It now remains to prove the most difficult part, namely

$$\lim_{t\to+\infty}\sup_{(2+\varepsilon)t<x<\left(2\sqrt{rd}-\varepsilon\right)t}\left(|u\,(t,x)|+|v\,(t,x)-1|\right)=0,\qquad\text{if }2\sqrt{rd}>2,$$

and

$$\lim_{t\to+\infty}\sup_{\left(2\sqrt{rd}+\varepsilon\right)t<x<(2-\varepsilon)t}\left(|u\,(t,x)-1|+|v\,(t,x)|\right)=0,\qquad\text{if }2\sqrt{rd}<2.$$

Since this is a symmetric statement and since the forthcoming proof does not rely upon the assumptions $a<1$ and $b>1$, we only do the case $2\sqrt{rd}>2$ (when v spreads faster than u) and the proof will be valid for the other case (when u spreads faster than v) as well.

Step 1: Let $\bar{u} : (t,x) \mapsto \min\left(1, e^{-(x-2t-x_1)}\right)$, where x_1 is chosen such that $\bar{u}(0,x) \geq u(0,x)$ for all $x \in \mathbb{R}$. Then, by standard scalar comparison,

$$u(t,x) \leq \bar{u}(t,x) \quad \text{for all } t \geq 0 \text{ and } x \in \mathbb{R}.$$

Step 2: We show that for each $c \in (2, 2\sqrt{rd})$, there exist positive constants δ, x_2, η_1, R such that

$$v(t', x + x_2 + ct) \geq \eta_1 \quad \text{for all } t \geq 1, x \in (-2R, 2R) \text{ and } t' \in [(1-\delta)t, (1+\delta)t]. \tag{7.3.1}$$

To show (7.3.1), fix $c \in \left(2, 2\sqrt{dr}\right)$ and fix δ so small that

$$2 < \frac{c}{1+\delta} < \frac{c}{1-\delta} < 2\sqrt{dr}.$$

Let $\eta > 0$, $R > 0$, $x_2 \in \mathbb{R}$, $\tilde{c} \in \left[\frac{c}{1+\delta}, \frac{c}{1-\delta}\right]$ and define

$$\underline{v}^{\tilde{c}} : (t,x) \mapsto \eta e^{-\frac{\tilde{c}}{2d}(x-\tilde{c}t)} \psi_{4R}(x - \tilde{c}t - x_2),$$

where $(\lambda_{4R}, \psi_{4R})$ is the Dirichlet principal eigenpair defined by

$$\begin{cases} -d\psi_{4R}'' = \lambda_{4R}\psi_{4R} & \text{in } (-4R, 4R) \\ \psi_{4R}(\pm 4R) = 0 & \\ \psi_{4R}(x) > 0 & \text{for all } x \in (-4R, 4R) \\ \max \psi_{4R} = 1 & \end{cases}.$$

The principal eigenvalue λ_{4R} is positive, vanishes as $R \to +\infty$ and ψ_{4R} is extended into $\mathbb{R}$ by setting $\psi_{4R}(x) = 0$ if $|x| > 4R$.

Obviously,

$$\partial_t \underline{v}^{\tilde{c}} - d\partial_{xx}\underline{v}^{\tilde{c}} - r\underline{v}^{\tilde{c}}\left(1 - \underline{v}^{\tilde{c}} - bu\right) \leq \partial_t \underline{v}^{\tilde{c}} - d\partial_{xx}\underline{v}^{\tilde{c}} - r\underline{v}^{\tilde{c}}\left(1 - \underline{v}^{\tilde{c}} - b\overline{u}\right),$$

whence the right-hand side above divided by $\eta e^{-\frac{\tilde{c}}{2d}(x-\tilde{c}t)}$ is *a fortiori* smaller than or equal to

$$\begin{aligned} &\frac{\tilde{c}^2}{2d}\psi_{4R} - \tilde{c}\psi_{4R}' - d\left(\psi_{4R}'' + \frac{\tilde{c}^2}{4d^2}\psi_{4R} - \frac{\tilde{c}}{d}\psi_{4R}'\right) - r\psi_{4R}\left(1 - \underline{v}^{\tilde{c}} - b\overline{u}\right) \\ &\leq \left(\frac{\tilde{c}^2}{4d} + \lambda_{4R} - r + r\left(\underline{v}^{\tilde{c}} + b\overline{u}\right)\right)\psi_{4R} \\ &\leq \left(\lambda_{4R} + r\left(\underline{v}^{\tilde{c}} + b\overline{u} - \gamma\right)\right)\psi_{4R}, \end{aligned}$$

where the last inequality holds provided we choose the constant $\gamma > 0$ so small that

$$2\sqrt{r(1-\gamma)d} > \frac{c}{1-\delta} \geq \tilde{c}.$$

Therefore, by choosing R so large that $\lambda_{4R} < r\frac{\gamma}{4}$, x_2 so large that

$$\bar{u}(t,x) \leq \frac{\gamma}{4b} \text{ for all } t \geq 0 \text{ and } x \geq 2t + x_2 - 4R$$

(which is possible by Step 1), and η so small that

$$\eta \sup_{\hat{c} \in \left[\frac{c}{1+\delta}, \frac{c}{1-\delta}\right]} \sup_{\xi \in (-4R+x_2, 4R+x_2)} \left(e^{-\frac{\tilde{c}}{2d}\xi}\psi_{4R}(\xi - x_2)\right) \leq \frac{\gamma}{4},$$

$$\eta \sup_{\hat{c}\in\left[\frac{c}{1+\delta},\frac{c}{1-\delta}\right]} \left(\mathrm{e}^{-\frac{\hat{c}}{2d}(x-\hat{c})}\psi_{4R}\left(x-\hat{c}-x_2\right)\right) \leq v\left(1,x\right) \text{ for all } x\in\mathbb{R}$$

we deduce that $\underline{v}^{\tilde{c}}$ is a sub-solution for the single parabolic equation satisfied by v.

By scalar comparison, $v(t,x)\geq \underline{v}^{\tilde{c}}(t,x)$ for all $t\geq 1$ and $x\in\mathbb{R}$. It follows then that

$$\begin{aligned} v\left(\frac{c}{\tilde{c}}t, x+x_2+ct\right) &\geq \underline{v}^{\tilde{c}}\left(\frac{c}{\tilde{c}}t, x+x_2+ct\right) \\ &= \eta e^{-\frac{\tilde{c}}{2d}(x+x_2)}\psi_{4R}(x) \\ &\geq \eta e^{-\frac{\tilde{c}}{2d}x_2}e^{-\frac{\tilde{c}}{d}R}\min_{[-2R,2R]}\psi_{4R} \end{aligned}$$

for all $t\geq 1$ and $x\in[-2R,2R]$. Noticing that the last expression on the right-hand side above is constant and denoting

$$\eta_1 = \eta \min_{[-2R,2R]}\psi_{4R} \inf_{\hat{c}\in\left[\frac{c}{1+\delta},\frac{c}{1-\delta}\right]}\left(e^{-\frac{\hat{c}}{2d}x_2}e^{-\frac{\hat{c}}{d}R}\right),$$

we may take the infimum over all $\tilde{c}\in\left[\frac{c}{1+\delta},\frac{c}{1-\delta}\right]$ and obtain indeed (7.3.1).

Step 3: We are now in position to show that, for any small $\varepsilon>0$,

$$\lim_{t\to+\infty}\sup_{(2+\varepsilon)t<x<(2\sqrt{rd}-\varepsilon)t}\left|v\left(t,x\right)-1\right| = 0.$$

Assume by contradiction the existence of sequences $(t_n)_{n\in\mathbb{N}}$, $(x_n)_{n\in\mathbb{N}}$ and of $c\in(2,2\sqrt{rd})$ such that, as $n\to+\infty$, $t_n\to+\infty$, $\frac{x_n}{t_n}\to c$ and $\limsup v\left(t_n,x_n\right)<1$.

Denote $(c_n)_{n\in\mathbb{N}} = \left(\frac{x_n}{t_n}\right)_{n\in\mathbb{N}}$, assume without loss of generality that $\left|\frac{c}{c_n}-1\right|<\delta/2$, where $\delta=\delta(c)$ is specified in Step 2.

For all $n\in\mathbb{N}$, define $\tau_n=\frac{c_n}{c}t_n=\frac{x_n}{c}$ and

$$v_n : (t,x)\mapsto v\left(t+\frac{c}{c_n}\tau_n, x+x_2+c\tau_n\right).$$

By Step 2 (with $t'=t+\frac{c}{c_n}\tau_n$ and $t=\tau_n$), we deduce that

$$v_n(t,x)\geq\eta_1 \text{ if } |x|\leq 2R \text{ and } \left|t+\left(\frac{c}{c_n}-1\right)\tau_n\right|<\delta\tau_n,$$

and hence (using $\left|\frac{c}{c_n}-1\right|<\frac{\delta}{2}$) if $|x-x_2|\leq 2R$ and $|t|<\frac{\delta}{2}\tau_n$.

By classical parabolic estimates (see Lieberman [111]), $(v_n)_{n\in\mathbb{N}}$ converges up to a diagonal extraction in $\mathscr{C}_{loc}\left(\mathbb{R}^2,[0,1]\right)$ to a limit v_∞ which satisfies (using the fact that $u(t+\frac{c}{c_n}\tau_n,x+c\tau_n)\to 0$ in $\mathscr{C}_{loc}\left(\mathbb{R}^2,[0,1]\right)$ by Step 1, since $(c\tau_n)/(\frac{c}{c_n}\tau_n)=c_n\geq\frac{c}{1+\delta}>2$ for all n)

$$\partial_t v_\infty - d\partial_{xx}v_\infty - rv_\infty\left(1-v_\infty\right) = 0 \text{ in } \mathbb{R}^2$$

and, in view of the above estimates,

$$v_\infty\left(t,x+x_2\right)\geq\eta_1 \text{ for all } t\in\mathbb{R} \text{ and } x\in\left[-2R,2R\right].$$

By standard classification of the entire solutions of the KPP equation, this implies $v_\infty\equiv 1$. In particular,

$$v(t_n,x_n) = v\left(\frac{c}{c_n}\tau_n, c\tau_n\right) = v_n(0,-x_2)\to 1.$$

This directly contradicts $\limsup v(t_n,x_n)<1$. □

In view of Proposition 7.13, in order to prove Theorem 7.1, we only have to prove that for each sufficiently small $\varepsilon > 0$,

$$\lim_{t\to+\infty} \sup_{x<(c^\star-\varepsilon)t} \left(|u(t,x)-1| + |v(t,x)|\right) = 0,$$

$$\lim_{t\to+\infty} \sup_{(c^\star+\varepsilon)t<x<\left(2\sqrt{rd}-\varepsilon\right)t} \left(|u(t,x)| + |v(t,x)-1|\right) = 0,$$

where

$$c^\star = \max\left(c_{LLW}, f^{-1}\left(2\sqrt{rd}\right)\right).$$

7.3.2.2 Proof of Theorem 7.1

We begin with an algebraic lemma.

Lemma 7.14. *Let $c_2 \geq 2\sqrt{1-a}$ and $c_1 > c_2$ such that $c_1 \geq f(c_2)$. Then*

$$\frac{\left(\Lambda(c_2,c_1)\right)^2+1}{\Lambda(c_2,c_1)} < c_1.$$

Proof. First, $\Lambda(c_2,c_1)$ is well-defined as $c_1 \geq \max\{c_2, f(c_2)\}$. Noticing that

$$\left(\Lambda(c_2,c_1)\right)^2 - c_1\Lambda(c_2,c_1) + \lambda(c_2)(c_1-c_2) + 1 = 0,$$

we find that the claimed inequality is equivalent to $-\lambda(c_2)(c_1-c_2) < 0$. □

Now we prove the remaining part of Theorem 7.1. Assume $2\sqrt{rd} > 2$, define $c^\star$ as above, fix $u_0 \in \mathscr{C}(\mathbb{R},[0,1])\setminus\{0\}$ with support included in a left half-line and $v_0 \in \mathscr{C}(\mathbb{R},[0,1])\setminus\{0\}$ with compact support and let (u,v) be the solution of equation (7.1.1).

Proof. By virtue of Theorem 7.2 and Proposition 7.13,

$$\sup\left\{c>0 \mid \lim_{t\to+\infty}\sup_{x\leq ct}\left(|u(t,x)-1|+|v(t,x)|\right)=0\right\} \geq c^\star.$$

It remains to verify that the quantity

$$\overline{c} = \inf\left\{0<c<2\sqrt{rd} \mid \lim_{t\to+\infty}\sup_{ct\leq x\leq\frac{2\sqrt{rd}+c}{2}t}\left(|u(t,x)|+|v(t,x)-1|\right)=0\right\}$$

satisfies $\overline{c} \leq c^\star$. Notice that by Proposition 7.13, $\overline{c} \leq 2$.

Assume by contradiction $\overline{c} \in (c^\star, 2]$ and let $c_2 \in (c^\star, \overline{c})$. Define $\delta^\star$ as in Proposition 7.5, let $\delta \in (0,\delta^\star)$ so small that $c_2^\delta < \overline{c}$ and define subsequently $(\overline{u_\delta}, \underline{v_\delta})$.

By standard comparison,

$$u(t,x) \leq \min\left(1, \overline{w_{c_2,2\sqrt{rd}}}(t,x)\right) \text{ for all } (t,x)\in(0,+\infty)\times\mathbb{R}.$$

By virtue of Proposition 7.5, there exists $y_0 \in \mathbb{R}$ such that, for all $y \geq y_0$ and $t \geq 0$,

$$\overline{u_\delta}\left(0, x-y-\frac{\left(\Lambda\left(c_2,2\sqrt{rd}\right)\right)^2+1}{\Lambda\left(c_2,2\sqrt{rd}\right)}t\right) \geq \min\left(1,\overline{w_{c_2,2\sqrt{rd}}}(t,x)\right) \text{ for all } x\in\mathbb{R},$$

Since

$$c_2 < \overline{c} \leq 2 < 2\sqrt{rd},$$

$$c_2 > c^\star = \max\left(c_{LLW}, f^{-1}\left(2\sqrt{rd}\right)\right) \geq \max\left(2\sqrt{1-a}, f^{-1}\left(2\sqrt{rd}\right)\right),$$

Lemma 7.14 yields

$$\frac{\left(\Lambda\left(c_2, 2\sqrt{rd}\right)\right)^2 + 1}{\Lambda\left(c_2, 2\sqrt{rd}\right)} < 2\sqrt{rd}.$$

Choose $c > 0$ such that

$$\max\left(\frac{1}{2}\left(2\sqrt{rd} + \frac{\left(\Lambda\left(c_2, 2\sqrt{rd}\right)\right)^2 + 1}{\Lambda\left(c_2, 2\sqrt{rd}\right)}\right), 2\right) < c < 2\sqrt{rd}.$$

By virtue of Proposition 7.5, $x \mapsto \underline{v_\delta}\left(0, x - y - \frac{\left(\Lambda\left(c_2, 2\sqrt{rd}\right)\right)^2+1}{\Lambda\left(c_2, 2\sqrt{rd}\right)} t\right)$ is compactly supported for all $y \geq y_0$ and $t \geq 0$. Since also $2 < c < 2\sqrt{rd}$, by virtue of Proposition 7.13, there exists $T_0 \geq 0$ such that, for all $T \geq T_0$,

$$\underline{v_\delta}\left(0, x - cT\right) \leq v\left(T, x\right) \text{ for all } x \in \mathbb{R}.$$

Now, relating the parameters y and T as follows,

$$cT = y + \frac{\left(\Lambda\left(c_2, 2\sqrt{rd}\right)\right)^2 + 1}{\Lambda\left(c_2, 2\sqrt{rd}\right)} T, \quad \text{where we have } c > \frac{\left(\Lambda\left(c_2, 2\sqrt{rd}\right)\right)^2 + 1}{\Lambda\left(c_2, 2\sqrt{rd}\right)},$$

we find the existence of $y \geq y_0$ and $T \geq T_0$ such that

$$\underline{v_\delta}\left(0, x - y - \frac{\left(\Lambda\left(c_2, 2\sqrt{rd}\right)\right)^2 + 1}{\Lambda\left(c_2, 2\sqrt{rd}\right)} T\right) \leq v\left(T, x\right) \text{ for all } x \in \mathbb{R}.$$

Then

$$\left(\overline{u}, \underline{v}\right) : (t, x) \mapsto \left(\overline{u_\delta}, \underline{v_\delta}\right)\left(t - T, x - y - \frac{\left(\Lambda\left(c_2, 2\sqrt{rd}\right)\right)^2 + 1}{\Lambda\left(c_2, 2\sqrt{rd}\right)} T\right)$$

is a super-solution of (7.1.1) which satisfies $(u, v) \preceq \left(\overline{u}, \underline{v}\right)$ at $t = T$, whence by the comparison principle of Theorem 7.11 it satisfies the same inequality at any time $t \geq T$.

A contradiction follows from Proposition 7.5 and $c_2^\delta < \overline{c}$, as in the proof of Theorem 7.2. □

7.3.3 Proof of Theorem 7.3

Let $c_1 > 2\sqrt{rd}$, $c_2 > c_{LLW}$ and assume $c_1 > \max\left(c_2, f\left(c_2\right)\right)$.

Proof. Fix $\delta^\star$, $\delta = \frac{\delta^\star}{2}$ and c_2^δ, and define the super- and sub-solutions $(\overline{u_\delta}, \underline{v_\delta})$ and $(\underline{u_\delta}, \overline{v_\delta})$ as in Proposition 7.6.

First, let $(u_0, v_0) \in \mathscr{C}\left(\mathbb{R}, [0,1]^2\right)$ be a pair such that

$$\left(\underline{u_\delta}, \overline{v_\delta}\right)(0,x) \preceq (u_0, v_0)(x) \preceq \left(\overline{u_\delta}, \underline{v_\delta}\right)(0,x) \text{ for all } x \in \mathbb{R}$$

and satisfying also

$$u_0(x) \leq \overline{w_{c_2,c_1}}(0,x) \text{ for all } x \in \mathbb{R}.$$

By virtue of Theorem 7.12, there exists a (unique) solution (u,v) of (7.1.1) such that

$$(u,v)(0,x) = (u_0,v_0)(x) \text{ for all } x \in \mathbb{R},$$

$$\left(\underline{u_\delta}, \overline{v_\delta}\right)(t,x) \preceq (u,v)(t,x) \preceq \left(\overline{u_\delta}, \underline{v_\delta}\right)(t,x) \text{ for all } t \in (0,+\infty) \text{ and } x \in \mathbb{R},$$

$$u(t,x) \leq \min\left(1, \overline{w_{c_2,c_1}}(t,x)\right) \text{ for all } t \in (0,+\infty) \text{ and } x \in \mathbb{R}.$$

Next, in view of the spreading properties of the super-solution and the sub-solution and thanks to the comparison argument with the ODE system detailed in the proof of Theorem 7.2, it only remains to show that the quantity

$$\bar{c} = \inf\left\{c > 0 \mid \lim_{t\to+\infty} \sup_{ct \leq x \leq \frac{c_1+c}{2}t} \left(|u(t,x)| + |v(t,x) - 1|\right) = 0\right\}$$

satisfies $\bar{c} \leq c_2$.

Now, the choice of super- and sub-solutions above proves that $\bar{c} \in [c_2, c_2^\delta]$. Suppose to the contrary that $\bar{c} > c_2$. Then we can fix a sufficiently small $\delta' \in (0,\delta)$ such that $c_2^{\delta'} \in (c_2, \bar{c})$. (This is possible since $c_2^{\delta'} \searrow c_2$ as $\delta' \searrow 0$, by Proposition 7.6.) Then, thanks to □

— the estimate $u \leq \min\left(1, \overline{w_{c_2,c_1}}\right)$,

— Lemma 7.14 which controls from above the speed of $\overline{w_{c_2,c_1}}$,

— the control from below of the exponential tail of v,

Proof. we can use the super-solution $\left(\overline{u_{\delta'}}, \underline{v_{\delta'}}\right)$ associated with a sufficiently small $\delta' \in (0,\delta)$ as barrier after some large time $T_{\delta'}$ to slow down the invasion of u in an impossible way. More precisely, just as in the proof of Theorem 7.1, there exist large T' and y_0 such that, for all $x \in \mathbb{R}$,

$$u(T',x) \leq \min\left(1, \overline{w_{c_2,c_1}}(T',x)\right) \leq \overline{u_{\delta'}}(0, x - y_0)$$

and

$$v(T',x) \geq \underline{v_{\delta'}}(0, x - y_0).$$

This implies that

$$\left(u(t,x), v(t,x)\right) \preceq \left(\overline{u_{\delta'}}(t - T', x - y_0), \underline{v_{\delta'}}(t - T', x - y_0)\right) \text{ for all } t \geq T' \text{ and } x \in \mathbb{R}$$

and $\bar{c} \leq c_2^{\delta'}$, which is a contradiction. This ends the proof. □

7.4 Proofs of Proposition 7.4, Proposition 7.5 and Proposition 7.6

7.4.1 Several useful objects

In this subjection, we will define components which will be used for our later constructions. For ease of reading, we suggest the readers to skip Subsection 7.4.1 and only refer to it when a specific object is being used.

List of Objects

Object(s)	*Defined in*	*Used in*	*Property*
$f(c)$	Sect. 7.1.1, (7.1.3)		$f(c) = c + 2\sqrt{a} - \sqrt{c^2 - 4(1-a)}$
c_{acc}	Theorem 7.1(2)		$c_{acc} > c_{LLW}$
$\lambda(c)$	Sect. 7.1.1.3		$\lambda(c) = \lambda_\delta(c)\big\vert_{\delta=0}$
$\Lambda(c, \tilde{c})$	Sect. 7.1.1.3; Lemma 7.14		$\Lambda(c, \tilde{c}) = \Lambda_\delta(c, \tilde{c})\big\vert_{\delta=0}$
$\overline{w_{c,\tilde{c}}}$	Sect. 7.1.1.3		
$\underline{u_{\delta,\zeta,\kappa}}, \overline{v_{\delta,\zeta}}$	Proposition 7.4	Sect. 7.3.1	Fig. 7.4.4
$\overline{u_\delta}, \underline{v_\delta}$	Proposition 7.5	Sect. 7.3.2.2	Fig. 7.4.3
$(\overline{u_\delta}, \underline{v_\delta}), (\underline{u_\delta}, \overline{v_\delta})$	Proposition 7.6	Sect. 7.3.3	Fig. 7.4.1 and Fig. 7.4.2
$\lambda_v(c)$	Sect. 7.4.1.1	Sect. 7.4.3.2 and 7.4.5.1	$d\lambda^2 - c\lambda + r = 0$
a_δ	Sect. 7.4.1.2		$a_\delta \to a$ as $\delta \to 0$
$\lambda_\delta(c)$	Sect. 7.4.1.3		$\lambda^2 - c\lambda + (1 - a_\delta) = 0$
c^δ_{LLW}	Sect. 7.4.1.4; Lemma 7.16		Minimal speed for $P(u,v) = F_\delta(u,v)$
F_δ	Sect. 7.4.1.4		
$(\overline{\varphi_{\delta,c}}, \underline{\psi_{\delta,c}})$	Sect. 7.4.1.5	Sect. 7.4.3.1 and 7.4.4.1	Monotone profile for $P(u,v) = F_\delta(u,v)$
$\underline{\theta_{\delta,c,A}}$	Sect. 7.4.1.6; Lemma 7.24	Sect. 7.4.3.1 and 7.4.4.1	(7.4.9)
$\underline{\omega_{\delta,R}}$, R^ω_δ, $x_{\delta,R}$	Sect. 7.4.1.7; Lemma 7.17	Sect. 7.4.4.1	(7.4.1)
$\underline{\pi_{\delta,c,h}}$, $h^\star$	Sect. 7.4.1.8; Lemma 7.23	Sect. 7.4.3.1	(7.4.2)
$\underline{\beta_{c,B,\eta}}, \xi_\beta, K_\beta$	Sect. 7.4.1.9; Lemma 7.22	Sect. 7.4.3.1	Exp. decay of $\underline{v}$ at $+\infty$
$\underline{\alpha_l}, L_\alpha, x_l$	Sect. 7.4.1.10; Lemma 7.18	Sect. 7.4.5.1	(7.4.3)
$\underline{\chi_c}$	Sect. 7.4.1.11	Sect. 7.4.3.2 and 7.4.5.1	(7.4.4)
$f_\delta(c)$, $\Lambda_\delta(c, \tilde{c})$	Sect. 7.4.1.12	Beginning, Sect. 7.4.4	(7.4.5)
$\overline{w_{\delta,c,\tilde{c}}}$	Sect. 7.4.1.13; Lemma 7.19	Sect. 7.4.3.1 and 7.4.4.1	(7.4.6)
$\underline{w_{c,\tilde{c},A,\eta}}, X_w$	Sect. 7.4.1.14; Lemma 7.21	Sect. 7.4.3.2	(7.4.8)
$\underline{z_{\delta,c,\tilde{c}}}, R_z$	Sect. 7.4.1.15; Lemma 7.20	Sect. 7.4.5.1	(7.4.7)
$\lambda^{-\infty}(c)$	Lemma 7.15		

List of Intersection Points

Symbol	*Defined in*	*Used in*	*Relation*
$x_0(t)$, ζ_0	Lemma 7.25	Sect. 7.4.3.2	$\underline{\chi_c}(x_0(t) - ct + \zeta_0) = \underline{w_{c,\tilde{c},A,\eta}}(t, x_0(t))$
$\xi_{1,\kappa}, \zeta_{1,\kappa}$, A_κ	Lemma 7.26	Sect. 7.4.3.1 and 7.4.4.1	$\underline{\theta_{\delta,c,A_\kappa}}(\xi_{1,\kappa}) = \underline{\psi_{\delta,c}}(\xi_{1,\kappa} - \zeta_{1,\kappa})$
$x_2(t), \zeta_2$	Lemma 7.27	Sect. 7.4.3.1 and 7.4.4.1	$\overline{\varphi_{\delta,c}}(x_2(t) - ct) = \overline{w_{\delta,c,\tilde{c}}}(t, x_2(t) - \zeta_2)$
$x_3(t), \zeta_3$	Lemma 7.28	Sect. 7.4.4.1	$\underline{\psi_{\delta,c}}(\hat{x}_3(t) - ct) = \underline{\omega_{\delta,R_\delta}}(\hat{x}_3(t) - (2\sqrt{r(1-2\delta)d} - \delta)t - \hat{\zeta}_3)$
	Lemma 7.29	Sect. 7.4.3.1	$\underline{\psi_{\delta,c}}(x_3(t) - ct) = \underline{\pi_{\delta,\tilde{c},h}}(x_3(t) - \tilde{c}t - \zeta_3)$
ξ_4, ζ_4	Lemma 7.30	Sect. 7.4.3.1	$\underline{\pi_{\delta,c,h}}(\xi_4) = \underline{\beta_{c,B,\eta}}(\xi_4 - \zeta_4)$
$x_{0,\kappa}(t), \zeta_0$	Lemma 7.31	Sect. 7.4.5.1	$\underline{\alpha_L}(x_{0,\kappa}(t)) = \underline{\chi_c}(x_{0,\kappa}(t) - ct - \zeta_0)$
$x_1(t)$	Lemma 7.32	Sect. 7.4.5.1	$\underline{\chi_c}(x_1(t) - ct) = \frac{\underline{\chi_c}(\zeta)}{\underline{z_{c,\tilde{c},\delta}}(0, X_z)} \underline{z_{c,\tilde{c},\delta}}(t, x_1(t) - \zeta)$

7.4.1.1 The function λ_v

The function λ_v is defined as

$$\lambda_v : \begin{array}{ccc} \left[2\sqrt{rd}, +\infty\right) & \to & \left(0, \sqrt{\frac{r}{d}}\right] \\ c & \mapsto & \frac{1}{2d}\left(c - \sqrt{c^2 - 4rd}\right) \end{array}.$$

7.4.1.2 The real a_δ

For all $\delta \in \left[0, \frac{1}{2}\right)$, we denote

$$a_\delta = \frac{(1-2\delta)\,a}{1+\delta}.$$

Notice that $a_0 = a$ and that $\delta \mapsto a_\delta$ is decreasing.

7.4.1.3 The function λ_δ

For all $\delta \in \left[0, \frac{1}{2}\right]$, the function λ_δ is defined as

$$\lambda_\delta : \begin{array}{ccc} \left[2\sqrt{1-a_\delta}, +\infty\right) & \to & \left(0, \sqrt{1-a_\delta}\right] \\ c & \mapsto & \frac{1}{2}\left(c - \sqrt{c^2 - 4\left(1-a_\delta\right)}\right) \end{array}.$$

The family $\left(\lambda_\delta\right)_{\delta\in\left[0,\frac{1}{2}\right]}$ is continuous and increasing in δ. Note that $\lambda_0 = \lambda$, the latter being introduced in Subsection 7.1.1.3.

7.4.1.4 The real c^δ_{LLW}

For all $\delta \in \left[0, \frac{1}{2}\right)$, c^δ_{LLW} denotes the minimal wave speed of the problem $P(u,v) = F_\delta(u,v)$, where

$$F_\delta : (u,v) \mapsto \begin{pmatrix} u\left(1+\delta-u-av\right) \\ rv\left(1-2\delta-v-bu\right) \end{pmatrix}.$$

Notice that (u,v) is a solution of $P(u,v) = F_\delta(u,v)$ if and only if

$$(U,V) : (t,x) \mapsto \left(\frac{u}{1+\delta}, \frac{v}{1-2\delta}\right)\left(\frac{t}{1+\delta}, \frac{x}{\sqrt{1+\delta}}\right)$$

is a solution of

$$P(U,V) = \begin{pmatrix} U\left(1-U-\frac{(1-2\delta)a}{1+\delta}V\right) \\ \frac{(1-2\delta)r}{1+\delta}V\left(1-V-\frac{(1+\delta)b}{1-2\delta}U\right) \end{pmatrix}.$$

Therefore

$$c^\delta_{LLW} = \sqrt{1+\delta}\,\hat{c}^\delta_{LLW},$$

where $\hat{c}^\delta_{LLW}$ is the minimal wave speed of equation (7.1.2) where (r,a,b) is replaced by $\left(\frac{(1-2\delta)r}{1+\delta}, a_\delta, \frac{(1+\delta)b}{1-2\delta}\right)$. As such, c^δ_{LLW} satisfies

$$2\sqrt{\left(1-a_\delta\right)\left(1+\delta\right)} \leq c^\delta_{LLW} \leq 2\sqrt{1+\delta}.$$

7.4.1.5 The functions $\overline{\varphi_{\delta,c}}$ and $\underline{\psi_{\delta,c}}$

For all $\delta \in \left[0, \frac{1}{2}\right)$ and $c \geq c_{LLW}^{\delta}$, $\left(\overline{\varphi_{\delta,c}}, \underline{\psi_{\delta,c}}\right)$ denotes a component-wise monotonic profile of traveling wave with speed c for the problem $P(u,v) = F_\delta(u,v)$, connecting $(1+\delta, 0)$ to $(0, 1-2\delta)$ and satisfying the normalization $\underline{\psi_{\delta,c}}(0) = \frac{1-2\delta}{2}$.

The existence of such a profile is well-known (and proved for instance in [110]). In fact, in the appendix, we will prove that any profile of traveling wave is component-wise monotonic and show that the condition

$$\frac{c + \sqrt{c^2 + 4rd}}{2d} \geq \frac{c - \sqrt{c^2 - 4(1-a)}}{2}$$

implies the uniqueness, up to translation, of the profile associated with a particular speed $c \geq c_{LLW}$. However these results are not actually required here. What is required indeed is the forthcoming Lemma 7.15.

7.4.1.6 The function $\underline{\theta_{\delta,c,A}}$

For all $\delta \in \left[0, \frac{1}{2}\right)$, $c \geq c_{LLW}^{\delta}$ and $A > 0$, define the function

$$\underline{\theta_{\delta,c,A}} : \xi \mapsto A\mathrm{e}^{\frac{1}{2d}\left(\sqrt{c^2+4rd(b-1+\delta)}-c\right)(\xi-\xi_\theta)} - \mathrm{e}^{\frac{1}{2d}\left(-\sqrt{c^2+4rd(b-1+\delta)}-c\right)(\xi-\xi_\theta)}$$

where the constant

$$\xi_\theta = \frac{d \ln A}{\sqrt{c^2 + 4rd(b-1+\delta)}}$$

is fixed so that $\underline{\theta_{\delta,c,A}}(0) = 0$. This function is increasing in $\mathbb{R}$.

7.4.1.7 The function $\underline{\omega_{\delta,R}}$ and the real R_δ^ω

For all $\delta \in [0,1)$ and all $R > 0$ large enough, $\underline{\omega_{\delta,R}} : [-R, R] \to [0, +\infty)$ denotes the unique nonnegative nonzero solution of

$$\begin{cases} -d\omega'' - \left(2\sqrt{r(1-\delta)d} - \delta\right)\omega' = r\omega(1-\delta-\omega) & \text{in } (-R,R) \\ \omega(\pm R) = 0 \end{cases}. \tag{7.4.1}$$

It is well-known that this problem admits a solution if and only if R is larger than or equal to some $R_\delta^\omega > 0$.

We extend the definition of $\underline{\omega_{\delta,R}}$ into the whole real line by setting $\underline{\omega_{\delta,R}}(\xi) = 0$ if $|\xi| > R$.

7.4.1.8 The function $\underline{\pi_{\delta,c,h}}$

For all $\delta \in [0,1)$, $c \geq 2\sqrt{rd}$ and all $h \in \mathbb{R}$, $\underline{\pi_{\delta,c,h}}$ denotes

$$\underline{\pi_{\delta,c,h}} : (\xi) \mapsto \underline{\pi_{\delta,c}}(\xi) + h\xi, \tag{7.4.2}$$

where $\underline{\pi_{\delta,c}}$ denotes the unique (decreasing) profile of traveling wave solution of

$$\partial_t v - d\partial_{xx} v = rv(1-\delta-v)$$

connecting 0 to $1-\delta$ at speed c and satisfying $\underline{\pi_{\delta,c}}(0) = \frac{1-\delta}{2}$.

7.4.1.9 The function $\underline{\beta_{c,B,\eta}}$

For all $c > 2\sqrt{rd}$, $\eta \in \left(0, \frac{1}{d}\sqrt{c^2 - 4rd}\right)$ and $B > 0$, $\underline{\beta_{c,B,\eta}}$ denotes

$$\underline{\beta_{c,B,\eta}} : \xi \mapsto \max\left(0, \mathrm{e}^{-\lambda_v(c)(\xi+\xi_\beta)} - K_\beta \mathrm{e}^{-(\lambda_v(c)+\eta)(\xi+\xi_\beta)}\right),$$

where the constants

$$K_\beta = \frac{r\,(1+bB)}{\eta\left(\sqrt{c^2-4rd} - d\eta\right)}$$

and $\xi_\beta = \frac{\ln K_\beta}{\eta}$ are fixed so that $\underline{\beta_{c,B,\eta}}$ is positive in $(0,+\infty)$ and null elsewhere.

7.4.1.10 The function $\underline{\alpha_l}$ and the real L_α

Similarly to the construction of $\underline{\omega_{\delta,R}}$ and R_δ^ω, we define $\underline{\alpha_l} : \mathbb{R} \to [0,+\infty)$ and $L_\alpha > 0$ such that, for all $l \geq L_\alpha$,

$$\begin{cases} -\underline{\alpha_l}'' = \underline{\alpha_l}\left(1 - a - \underline{\alpha_l}\right) & \text{in } (0,l) \\ \underline{\alpha_l}(0) = \underline{\alpha_l}(l) = 0 & \\ \underline{\alpha_l}(x) = 0 & \text{if } \left|x - \frac{l}{2}\right| > \frac{l}{2} \end{cases}. \tag{7.4.3}$$

7.4.1.11 The function $\underline{\chi_c}$

For all $c \geq 2\sqrt{\frac{1-a}{2}}$, $\underline{\chi_c}$ denotes the unique (decreasing) profile of traveling wave solution of

$$\partial_t u - \partial_{xx} u = u\left(\frac{1-a}{2} - u\right) \tag{7.4.4}$$

connecting 0 to $\frac{1-a}{2}$ at speed c and satisfying $\underline{\chi_c}(0) = \frac{1-a}{4}$.

7.4.1.12 The functions $f_\delta(c)$ and $\Lambda_\delta(c,\tilde{c})$

For all $\delta \in \left[0, \frac{1}{2}\right)$, f_δ denotes the function

$$f_\delta : \begin{array}{ccc} \left[2\sqrt{1-a_\delta}, +\infty\right) & \to & \left(0, 2\left(\sqrt{1-a_\delta} + \sqrt{a_\delta}\right)\right] \\ c & \mapsto & c - \sqrt{c^2 - 4(1-a_\delta)} + 2\sqrt{a_\delta} \end{array}.$$

Notice right now that provided $\tilde{c} - f_\delta(c) > -4\sqrt{a_\delta}$, $\tilde{c} - f_\delta(c)$ has exactly the sign of $\tilde{c}^2 - 4(\lambda_\delta(c)(\tilde{c}-c)+1)$. Indeed, by the fact that $(\lambda_\delta(c))^2 - c\lambda_\delta(c) + 1 - a_\delta = 0$,

$$\begin{aligned} \tilde{c}^2 - 4(\lambda_\delta(c)(\tilde{c}-c)+1) &= (\tilde{c} - 2\lambda_\delta(c))^2 - 4\left(1 - \lambda_\delta(c)\,c + (\lambda_\delta(c))^2\right) \\ &= \left(\tilde{c} - c + \sqrt{c^2 - 4(1-a_\delta)}\right)^2 - 4a_\delta \\ &= \left(\tilde{c} - f_\delta(c) + 2\sqrt{a_\delta}\right)^2 - \left(2\sqrt{a_\delta}\right)^2 \\ &= \left(\tilde{c} - f_\delta(c)\right)\left(\tilde{c} - f_\delta(c) + 4\sqrt{a_\delta}\right). \end{aligned}$$

For all $\delta \in \left[0, \frac{1}{2}\right)$, Λ_δ denotes the function

$$\Lambda_\delta : (c,\tilde{c}) \mapsto \frac{1}{2}\left(\tilde{c} - \sqrt{\tilde{c}^2 - 4(\lambda_\delta(c)(\tilde{c}-c)+1)}\right).$$

Its domain is the set of all $(c,\tilde{c})$ such that $c \geq 2\sqrt{1-a_\delta}$ and $\tilde{c} \geq \max(c, f_\delta(c))$ and it is decreasing with respect to both c and $\tilde{c}$. As a function of c only, with a fixed $\tilde{c}$, it bijectively maps $\left[2\sqrt{1-a_\delta}, +\infty\right)$ onto

$$\left(\frac{1}{2}\left(\tilde{c} - \sqrt{\tilde{c}^2 - 4(a_\delta + 1)}\right), \frac{1}{2}\left(\tilde{c} - \sqrt{\tilde{c}^2 - 4\left(\tilde{c}\sqrt{1-a_\delta} + 2a_\delta - 1\right)}\right)\right].$$

The family $(\Lambda_\delta)_{\delta \in \left[0, \frac{1}{2}\right)}$ is increasing. Recalling the earlier definition of Λ, we find $\Lambda_0 = \Lambda$.

Finally, by construction, for all $(c,\tilde{c})$ such that $c \geq 2\sqrt{1-a_\delta}$ and $\tilde{c} \geq \max(c, f_\delta(c))$, $\Lambda_\delta(c,\tilde{c})$ satisfies

$$(\Lambda_\delta(c,\tilde{c}))^2 - \tilde{c}\Lambda_\delta(c,\tilde{c}) + \lambda_\delta(c)(\tilde{c} - c) + 1 = 0. \tag{7.4.5}$$

7.4.1.13 The function $\overline{w_{\delta,c,\tilde{c}}}$

For all $\delta \in \left[0, \frac{1}{2}\right)$, $c \geq c_{LLW}^{\delta}$ and $\tilde{c} \geq \max(c, f_\delta(c))$, $\overline{w_{\delta,c,\tilde{c}}}$ denotes the function

$$\overline{w_{\delta,c,\tilde{c}}} : (t,x) \mapsto \mathrm{e}^{-\lambda_\delta(c)(\tilde{c}-c)t}\mathrm{e}^{-\Lambda_\delta(c,\tilde{c})(x-\tilde{c}t)}.$$

In view of (7.4.5),

$$\overline{w_{\delta,c,\tilde{c}}}(t,x) = \mathrm{e}^{\left((\Lambda_\delta(c,\tilde{c}))^2+1\right)t}\mathrm{e}^{-\Lambda_\delta(c,\tilde{c})x} \text{ for all } (t,x) \in [0,+\infty) \times \mathbb{R}.$$

Recalling the earlier definition of $\overline{w_{c,\tilde{c}}}$, we find $\overline{w_{c,\tilde{c}}} = \overline{w_{0,c,\tilde{c}}}$.

7.4.1.14 The function $\underline{w_{c,\tilde{c},A,\eta}}$

For all $c \geq c_{LLW}$, $\tilde{c} \geq c$ such that $\tilde{c} > f(c)$, $\eta \in \left(0, \min\left(\Lambda(c,\tilde{c}), \sqrt{\tilde{c}^2 - 4(\lambda(c)(\tilde{c}-c)+1)}\right)\right)$ and $A > 0$, $\underline{w_{c,\tilde{c},A,\eta}}$ denotes

$$\underline{w_{c,\tilde{c},A,\eta}} : (t,x) \mapsto \mathrm{e}^{-\lambda(c)(\tilde{c}-c)t} \max\left(0, \mathrm{e}^{-\Lambda(c,\tilde{c})(x-\tilde{c}t+x_w)} - K_w \mathrm{e}^{-(\Lambda(c,\tilde{c})+\eta)(x-\tilde{c}t+x_w)}\right),$$

where

$$K_w = \max\left(1, \frac{1+aA}{\eta\left(\sqrt{\tilde{c}^2 - 4(\lambda(c)(\tilde{c}-c)+1)} - \eta\right)}\right) = \max\left(1, \frac{1+aA}{\eta(\tilde{c} - \eta - 2\Lambda_0(c,\tilde{c}))}\right)$$

and $x_w = \frac{\ln K_w}{\eta}$ is fixed so that, for all $t \geq 0$, $x \mapsto \underline{w_{c,\tilde{c},A,\eta}}(t,x)$ is positive in $(\tilde{c}t, +\infty)$, null elsewhere, increasing in $\left(\tilde{c}t, \frac{\ln(\Lambda(c,\tilde{c})+\eta)-\ln(\Lambda(c,\tilde{c}))}{\eta} + \tilde{c}t\right)$ and decreasing in $\left(\frac{\ln(\Lambda(c,\tilde{c})+\eta)-\ln(\Lambda(c,\tilde{c}))}{\eta} + \tilde{c}t, +\infty\right)$. Hereafter, the point where the global maximum is attained at $t=0$ is denoted X_w.

7.4.1.15 The function $\underline{z_{\delta,c,\tilde{c}}}$

For all $c \geq c_{LLW}$, $\tilde{c} \in (f(c) - 4\sqrt{a}, f(c))$ and $\delta \in \left[0, \frac{1}{4}\left(-\tilde{c}^2 + 4(\lambda(c)(\tilde{c}-c)+1)\right)\right)$, $\underline{z_{c,\tilde{c},\delta}}$ denotes the function defined by

$$\underline{z_{\delta,c,\tilde{c}}}(t,x) = \begin{cases} \mathrm{e}^{-\lambda(c)(\tilde{c}-c)t}\mathrm{e}^{-\frac{\tilde{c}}{2}(x-\tilde{c}t)} \sin\left(\frac{\pi}{2R_z}(x-\tilde{c}t)\right) & \text{if } x - \tilde{c}t \in [0, 2R_z] \\ 0 & \text{otherwise} \end{cases}.$$

where

$$R_z = \frac{\pi}{\sqrt{-\tilde{c}^2 + 4(\lambda(c)(\tilde{c}-c)+1-\delta)}}.$$

Hereafter, the point where the global maximum is attained at $t=0$ is denoted X_z.

7.4.2 Several useful lemmas

Lemma 7.15. *Let $c > c_{LLW}$ and (φ, ψ) be a profile of traveling wave solution of (7.1.1) with speed c.*

Then there exist $A > 0$ and $B > 0$ such that

$$\varphi(\xi) = Ae^{-\lambda(c)\xi} + h.o.t. \text{ as } \xi \to +\infty$$

and

$$\psi(\xi) = Be^{\lambda^{-\infty}(c)\xi} + h.o.t. \text{ as } \xi \to -\infty$$

where

$$\lambda^{-\infty}(c) = \frac{1}{2d}\left(\sqrt{c^2 + 4rd(b-1)} - c\right),$$

The proof of this lemma is quite lengthy. Therefore it is postponed to the appendix (see Corollary 7.35 and Corollary 7.38).

Lemma 7.16. *The function $\delta \mapsto c^{\delta}_{LLW}$ is continuous and nondecreasing in $\left[0, \frac{1}{2}\right)$.*

Proof. Recalling that

$$c^{\delta}_{LLW} = \sqrt{1+\delta}\,\hat{c}^{\delta}_{LLW},$$

where $\hat{c}^{\delta}_{LLW}$ is the minimal wave speed of the system (7.1.2) where (r, a, b) is replaced by $\left(\frac{(1-2\delta)r}{1+\delta}, a_\delta, \frac{(1+\delta)b}{1-2\delta}\right)$, the continuity of $\delta \mapsto c^{\delta}_{LLW}$ follows directly from the theorem due to Kan-on [101] establishing the continuity of the spreading speed of (7.1.2) with respect to the parameters (r, a, b).

The monotonicity follows from the comparison principle. □

Lemma 7.17. *Let $\delta \in [0, 1)$. Then for all $R \geq R^{\omega}_{\delta}$,*

$$\max_{[-R,R]} \underline{\omega_{\delta,R}} < 1 - \delta.$$

Furthermore, if $\delta > 0$, then there exists $R_\delta \geq R^{\omega}_{\delta}$ such that, for all $R \geq R_\delta$,

$$\max_{[-R,R]} \underline{\omega_{\delta,R}} \geq 1 - 2\delta,$$

and there exists a unique $x_{\delta,R} \in (-R, R)$ such that $\underline{\omega_{\delta,R}}$ is increasing in $[-R, x_{\delta,R}]$, decreasing in $[x_{\delta,R}, R]$ and maximal at $x_{\delta,R}$.

Proof. The first inequality follows very classically from the first and second order conditions at any local maximum and from the strong maximum principle.

The second inequality comes from the locally uniform convergence of $\underline{\omega_{\delta,R}}$ to $1-\delta$ as $R \to +\infty$. This fact is also well-known and its proof is not detailed here.

Finally, the piecewise strict monotonicity comes from the inequality

$$-d\underline{\omega_{\delta,R}}'' - \left(2\sqrt{r(1-\delta)d} - \delta\right)\underline{\omega_{\delta,R}}' > 0 \text{ in } (-R, R),$$

which implies the nonexistence of local minima. □

The function $\underline{\alpha_l}$ satisfies of course a similar property.

Lemma 7.18. *For all $l \geq L_\alpha$,*

$$\max_{[0,l]} \underline{\alpha_l} < 1 - a.$$

Furthermore, there exists $L \geq L_\alpha$ such that, for all $l \geq L$,

$$\max_{[0,l]} \underline{\alpha_l} \geq \frac{1-a}{2},$$

and there exists a unique $x_l \in (-l, l)$ such that $\underline{\alpha_l}$ is increasing in $[0, x_l]$, decreasing in $[x_l, l]$ and maximal at x_l.

Lemma 7.19. *For all $\delta \in \left[0, \frac{1}{2}\right)$, $c \geq c_{LLW}^{\delta}$ and $\tilde{c} \geq \max\left(c, f_\delta\left(c\right)\right)$, $\overline{w_{\delta,c,\tilde{c}}}$ satisfies*

$$\partial_t \overline{w_{\delta,c,\tilde{c}}} - \partial_{xx} \overline{w_{\delta,c,\tilde{c}}} = \overline{w_{\delta,c,\tilde{c}}} \text{ in } [0,+\infty) \times \mathbb{R}.$$

Proof. The following equality being straightforward,

$$\partial_t \overline{w_{\delta,c,\tilde{c}}} - \partial_{xx} \overline{w_{\delta,c,\tilde{c}}} - \overline{w_{\delta,c,\tilde{c}}} = \left(-\lambda_\delta\left(c\right)\left(\tilde{c}-c\right) + \tilde{c}\Lambda_\delta\left(c,\tilde{c}\right) - \left(\Lambda_\delta\left(c,\tilde{c}\right)\right)^2 - 1\right)\overline{w_{\delta,c}}, \tag{7.4.6}$$

the conclusion follows from (7.4.5). □

Quite similarly, we have the following lemma.

Lemma 7.20. *For all $c \geq c_{LLW}$, $\tilde{c} \in \left(f\left(c\right) - 4\sqrt{a}, f\left(c\right)\right)$ and $\delta \in \left[0, \frac{1}{4}\left(-\tilde{c}^2 + 4\left(\lambda\left(c\right)\left(\tilde{c}-c\right)+1\right)\right)\right)$, $\underline{z_{\delta,c,\tilde{c}}}$ satisfies*

$$\partial_t \underline{z_{\delta,c,\tilde{c}}} - \partial_{xx} \underline{z_{\delta,c,\tilde{c}}} = \left(1-\delta\right)\underline{z_{\delta,c,\tilde{c}}} \text{ in } [0,+\infty) \times \mathbb{R}. \tag{7.4.7}$$

Proof. It suffices to verify

$$-\lambda\left(c\right)\left(\tilde{c}-c\right) + \frac{\tilde{c}^2}{4} + \left(\frac{\pi}{2R_z}\right)^2 - \left(1-\delta\right) = 0,$$

which, in view of the definition of R_z, is equivalent to

$$-\lambda\left(c\right)\left(\tilde{c}-c\right) + \frac{\tilde{c}^2}{4} + \left(\frac{-\tilde{c}^2 + 4\lambda\left(c\right)\left(\tilde{c}-c\right) + 4\left(1-\delta\right)}{4}\right) - \left(1-\delta\right) = 0.$$

The last statement obviously holds. □

Lemma 7.21. *For all $c \geq c_{LLW}$, $\tilde{c} \geq c$ such that $\tilde{c} > f\left(c\right)$, $\eta \in \left(0, \min\left(\Lambda\left(c,\tilde{c}\right), \sqrt{\tilde{c}^2 - 4\left(\lambda\left(c\right)\left(\tilde{c}-c\right)+1\right)}\right)\right)$ and $A > 0$, the function $\underline{w_{c,\tilde{c},A,\eta}}$ satisfies, for all $\sigma \geq \eta$,*

$$\partial_t \underline{w_{c,\tilde{c},A,\eta}} - \partial_{xx} \underline{w_{c,\tilde{c},A,\eta}} \leq \underline{w_{c,\tilde{c},A,\eta}}\left(1 - \underline{w_{c,\tilde{c},A,\eta}} - aAe^{-\sigma(x-\tilde{c}t+x_w)}\right). \tag{7.4.8}$$

Remark. The above inequality is to be understood in the weak sense associated with generalized sub-solutions.

Proof. For $x - \tilde{c}t < 0$, $\underline{w_{c,\tilde{c},A,\eta}}$ is trivial and the inequality obviously holds. We focus on the case $x - \tilde{c}t > 0$, where $\underline{w_{c,\tilde{c},A,\eta}}$ reduces to

$$(t,x) \mapsto \mathrm{e}^{-\lambda(c)(\tilde{c}-c)t}\left(\mathrm{e}^{-\Lambda(c,\tilde{c})(x-\tilde{c}t+x_w)} - K_w \mathrm{e}^{-(\Lambda(c,\tilde{c})+\eta)(x-\tilde{c}t+x_w)}\right).$$

First, differentiating, we find:

$$\begin{aligned}\partial_t \underline{w_{c,\tilde{c},A,\eta}}(t,x) &= -\lambda(c)(\tilde{c}-c)\,\underline{w_{c,\tilde{c},A,\eta}}(t,x)\\ &\quad + \tilde{c}\Lambda(c,\tilde{c})\,\mathrm{e}^{-\lambda(c)(\tilde{c}-c)t}\mathrm{e}^{-\Lambda(c,\tilde{c})(x-\tilde{c}t+x_w)}\\ &\quad - K_w\left(\tilde{c}\left(\Lambda(c,\tilde{c})+\eta\right)\right)\mathrm{e}^{-\lambda(c)(\tilde{c}-c)t}\mathrm{e}^{-(\Lambda(c,\tilde{c})+\eta)(x-\tilde{c}t+x_w)},\end{aligned}$$

$$\partial_{xx}\underline{w_{c,\tilde{c},A,\eta}} = \mathrm{e}^{-\lambda(c)(\tilde{c}-c)t}\left(\left(\Lambda(c,\tilde{c})\right)^2\mathrm{e}^{-\Lambda(c,\tilde{c})(x-\tilde{c}t+x_w)} - K_w\left(\Lambda(c,\tilde{c})+\eta\right)^2\mathrm{e}^{-(\Lambda(c,\tilde{c})+\eta)(x-\tilde{c}t+x_w)}\right),$$

so that the auxiliary function

$$Q:(t,x)\mapsto \mathrm{e}^{\lambda(c)(\tilde{c}-c)t}\left(-\partial_t\underline{w_{c,\tilde{c},A,\eta}}+\partial_{xx}\underline{w_{c,\tilde{c},A,\eta}}+\underline{w_{c,\tilde{c},A,\eta}}\right)(t,x)$$

satisfies

$$\begin{aligned}Q(t,x) &= \left(\lambda(c)(\tilde{c}-c)-\tilde{c}\Lambda(c,\tilde{c})+\left(\Lambda(c,\tilde{c})\right)^2+1\right)\mathrm{e}^{-\Lambda(c,\tilde{c})(x-\tilde{c}t+x_w)}\\ &\quad - K_w\left(\lambda(c)(\tilde{c}-c)-\tilde{c}\left(\Lambda(c,\tilde{c})+\eta\right)+\left(\Lambda(c,\tilde{c})+\eta\right)^2+1\right)\mathrm{e}^{-(\Lambda(c,\tilde{c})+\eta)(x-\tilde{c}t+x_w)}.\end{aligned}$$

Using (7.4.5), it follows

$$Q(t,x) = K_w\eta\left(\tilde{c}-\eta-2\Lambda(c,\tilde{c})\right)\mathrm{e}^{-(\Lambda(c,\tilde{c})+\eta)(x-\tilde{c}t+x_w)},$$

that is, recalling the definition of $\Lambda_\delta(c,\tilde{c})$ as well as that of K_w,

$$Q(t,x)\geq(1+aA)\,\mathrm{e}^{-(\Lambda(c,\tilde{c})+\eta)(x-\tilde{c}t+x_w)}.$$

Next, getting rid of all the negative terms and using $\mathrm{e}^{-\lambda(c)(\tilde{c}-c)t}\leq 1$, we find

$$\mathrm{e}^{\lambda(c)(\tilde{c}-c)t}\underline{w_{c,\tilde{c},A,\eta}}\left(\underline{w_{c,\tilde{c},A,\eta}}+aA\mathrm{e}^{-\sigma(x-\tilde{c}t+x_w)}\right)\leq \mathrm{e}^{-\Lambda(c,\tilde{c})(x-\tilde{c}t+x_w)}\left(\mathrm{e}^{-\Lambda(c,\tilde{c})(x-\tilde{c}t+x_w)}+aA\mathrm{e}^{-\sigma(x-\tilde{c}t+x_w)}\right)$$

Finally, using $x>\tilde{c}t$, $x_w=\frac{1}{\eta}\ln K_w\geq 0$ as well as the assumption $0<\eta\leq\min\{\Lambda(c,\tilde{c}),\sigma\}$, we find

$$\begin{aligned}\mathrm{e}^{\eta(x-\tilde{c}t+x_w)}\left(\mathrm{e}^{-\Lambda(c,\tilde{c})(x-\tilde{c}t+x_w)}+aA\mathrm{e}^{-\sigma(x-\tilde{c}t+x_w)}\right) &\leq \mathrm{e}^{-(\Lambda(c,\tilde{c})-\eta)x_w}+aA\mathrm{e}^{-(\sigma-\eta)x_w}\\ &\leq 1+aA\end{aligned}$$

and the proof is therefore ended. □

With an analogous proof, we obtain directly the following lemma.

Lemma 7.22. *For all $c>2\sqrt{rd}$, $\eta\in\left(0,\min\left(\lambda_v(c),\frac{1}{d}\sqrt{c^2-4rd}\right)\right)$ and $B>0$, $\underline{\beta_{c,B,\eta}}$ satisfies, for all $\sigma\geq\eta$,*

$$-d\underline{\beta_{c,B,\eta}}''-c\underline{\beta_{c,B,\eta}}'\leq r\underline{\beta_{c,B,\eta}}\left(1-\underline{\beta_{c,B,\eta}}-bBe^{-\sigma(\xi+\xi_\beta)}\right)\ \text{ in }\ (\mathbb{R}\backslash\{0\}).$$

Lemma 7.23. *For all $\delta\in(0,1)$, $c>2\sqrt{rd}$ and $h>0$, $\underline{\pi_{\delta,c,h}}$ satisfies*

$$-d\underline{\pi_{\delta,c,h}}''-c\underline{\pi_{\delta,c,h}}'\leq r\underline{\pi_{\delta,c,h}}\left(1-\delta-\underline{\pi_{\delta,c,h}}\right)\ \text{ in }\ \left[-\sqrt{\frac{c}{rh}},\sqrt{\frac{c}{rh}}\right].$$

Furthermore, there exists $h^\star>0$ such that, for all $h\in(0,h^\star]$,

$$\max_{\left[-\sqrt{\frac{c}{rh}},0\right]}\underline{\pi_{\delta,c,h}}\geq 1-2\delta,$$

$$\max_{\left[-\sqrt{\frac{c}{rh}},0\right]}\underline{\pi_{\delta,c,h}}>\max\left(\underline{\pi_{\delta,c,h}}(0),\underline{\pi_{\delta,c,h}}\left(-\sqrt{\frac{c}{rh}}\right)\right).$$

Remark. It should be achievable to prove that the global maximum of $\underline{\pi_{\delta,c,h}}$ in $\left(-\sqrt{\frac{c}{rh}},0\right)$ is actually unique and that $\underline{\pi_{\delta,c,h}}$ is increasing in $(-\infty,\xi^\star)$ and decreasing in $(\xi^\star,0)$ but this is really unnecessary for our purpose.

Proof. Recalling that $\underline{\pi_{\delta,c,h}}(\xi)=\underline{\pi_{\delta,c}}(\xi)+h\xi$ (see Subsection 7.4.1.8), we have

$$\begin{aligned}
&-d\underline{\pi_{\delta,c,h}}''(\xi)-c\underline{\pi_{\delta,c,h}}'(\xi)\\
&=r\underline{\pi_{\delta,c}}(\xi)\left(1-\delta-\underline{\pi_{\delta,c}}(\xi)\right)-ch\\
&=r\underline{\pi_{\delta,c,h}}(\xi)\left(1-\delta-\underline{\pi_{\delta,c,h}}(\xi)\right)-hr\left(\xi\left(1-\delta-\underline{\pi_{\delta,c}}(\xi)\right)+\frac{c}{r}-\underline{\pi_{\delta,c,h}}(\xi)\,\xi\right)\\
&=r\underline{\pi_{\delta,c,h}}(\xi)\left(1-\delta-\underline{\pi_{\delta,c,h}}(\xi)\right)-hr\left(-h\xi^2+\left(1-\delta-2\underline{\pi_{\delta,c}}(\xi)\right)\xi+\frac{c}{r}\right).
\end{aligned}$$

It is easily verified that, in $\left[-\sqrt{\frac{c}{rh}},\sqrt{\frac{c}{rh}}\right]$,

$$-h\xi^2+\left(1-\delta-2\underline{\pi_{\delta,c}}(\xi)\right)\xi+\frac{c}{r}>-h\xi^2+\frac{c}{r}\geq 0,$$

where we used the facts

$$\underline{\pi_{\delta,c,h}}>\frac{1-\delta}{2}\quad\text{for }\xi<0,\qquad\text{and}\qquad\underline{\pi_{\delta,c,h}}<\frac{1-\delta}{2}\quad\text{for }\xi>0.$$

And the stated differential inequality is established.

The maximum of $\underline{\pi_{\delta,c,h}}$ in $\left[-\sqrt{\frac{c}{rh}},0\right]$ is larger than or equal to

$$\underline{\pi_{\delta,c,h}}\left(-\sqrt{\frac{c}{rh}}\right)=\underline{\pi_{\delta,c}}\left(-\sqrt{\frac{c}{rh}}\right)-\sqrt{\frac{ch}{r}},$$

which is itself larger than or equal to $1-2\delta$ if h is small enough.

Finally, since $\underline{\pi'_{\delta,c}}\left(-\sqrt{\frac{c}{rh}}\right)$ vanishes exponentially as $h\to 0$,

$$\underline{\pi'_{\delta,c,h}}\left(-\sqrt{\frac{c}{rh}}\right)=\underline{\pi'_{\delta,c}}\left(-\sqrt{\frac{c}{rh}}\right)+h>0,\qquad\text{and}\qquad\underline{\pi_{\delta,c,h}}'(0)=\underline{\pi_{\delta,c}}'(0)+h<0,$$

for all sufficiently small h. This implies that the values at $\xi=0$ and $-\sqrt{\frac{c}{rh}}$ are smaller than the aforementioned maximum. □

Lemma 7.24. *For all $\delta\in\left[0,\frac{1}{2}\right)$, $c\geq c^{\delta}_{LLW}$ and $A>0$, $\underline{\theta_{\delta,c,A}}$ satisfies*

$$-d\underline{\theta_{\delta,c,A}}''-c\underline{\theta_{\delta,c,A}}'-r\underline{\theta_{\delta,c,A}}(1-\delta-b)=0\ \text{in }\mathbb{R}.\tag{7.4.9}$$

Proof. Note that $\underline{\theta_{\delta,c,A}}$ is a linear combination of

$$\xi\mapsto\exp\left(\frac{1}{2d}\left(\pm\sqrt{c^2+4rd(b-1+\delta)}-c\right)\xi\right),$$

where $\frac{1}{2d}\left(\pm\sqrt{c^2+4rd(b-1+\delta)}-c\right)$ are the two distinct roots of the characteristic polynomial associated with the above linear ODE (7.4.9). □

Lemma 7.25. *For all $c > 2\sqrt{1-a}$, $\tilde{c} \geq c$ such that*

$$\tilde{c} > f(c), \quad \eta \in \left(0, \min\left(\sqrt{\tilde{c}^2 - 4(\lambda(c)(\tilde{c}-c)+1)}\right), \lambda_v(\tilde{c})\right), \quad \text{and} \quad A > 0,$$

there exists $\zeta_0 \in \mathbb{R}$ such that the equation

$$\underline{\chi_c}(x - ct + \zeta_0) = \underline{w_{c,\tilde{c},A,\eta}}(t,x)$$

admits for all $t \geq 0$ an isolated solution $x_0(t) \in \mathbb{R}$ such that

1. *$\underline{\chi_c}(x - ct + \zeta_0) > \underline{w_{c,\tilde{c},A,\eta}}(t,x)$ in a left-sided neighborhood of $x_0(t)$;*
2. *$\underline{\chi_c}(x - ct + \zeta_0) < \underline{w_{c,\tilde{c},A,\eta}}(t,x)$ in a right-sided neighborhood of $x_0(t)$;*
3. *$\tilde{c}t < x_0(t) < X_w + \tilde{c}t$.*

Furthermore, $x_0 \in \mathscr{C}^1([0,+\infty),(0,+\infty))$.

Proof. Recall from standard results on the KPP equation that, since $c > 2\sqrt{1-a}$, there exists $\zeta_{0,1} \in \mathbb{R}$ such that

$$\underline{\chi_c}(x + \zeta_{0,1}) \sim \mathrm{e}^{-\lambda(c)x} \text{ as } x \to +\infty.$$

Hence there exists $\zeta_0 \in \mathbb{R}$ such that, for all $x \geq 0$,

$$\underline{\chi_c}(x + \zeta_0) \leq \frac{1}{2} e^{-\lambda(c)x} \max_{y \in \mathbb{R}} \underline{w_{c,\tilde{c},A,\eta}}(0,y) \leq \frac{1}{2} \max_{y \in \mathbb{R}} \underline{w_{c,\tilde{c},A,\eta}}(0,y)$$

with $\max_{y \in \mathbb{R}} \underline{w_{c,\tilde{c},A,\eta}}(0,y)$ uniquely attained at X_w.

From the intermediate value theorem and the respective strict monotonicities of $\underline{\chi_c}$ in $\mathbb{R}$ and $x \mapsto \underline{w_{c,\tilde{c},A,\eta}}(0,x)$ in $[0, X_w]$, it clearly follows that $\underline{\chi_c}(x+\zeta_0) = \underline{w_{c,\tilde{c},A,\eta}}(0,x)$ admits a unique solution $x_0(0)$ in $(0, X_w)$.

Next, to define in the same way $x_0(t)$, it suffices to verify that for all $t > 0$,

$$\underline{w_{c,\tilde{c},A,\eta}}(t, X_w + \tilde{c}t) > \underline{\chi_c}(X_w + (\tilde{c}-c)t + \zeta_0).$$

Since $X_w + \tilde{c}t \geq 0$, it is *a fortiori* sufficient to verify that for all $t \geq 0$,

$$\mathrm{e}^{-\lambda(c)(\tilde{c}-c)t} \max_{x \in \mathbb{R}} \underline{w_{c,\tilde{c},A,\eta}}(0,x) > \frac{1}{2} \max_{x \in \mathbb{R}} \underline{w_{c,\tilde{c},A,\eta}}(0,x)\, \mathrm{e}^{-\lambda(c)(X_w + (\tilde{c}-c)t)}.$$

This inequality reduces in fact to $2 > \mathrm{e}^{-\lambda(c)X_w}$, which holds as $\lambda(c)$ and X_w are both positive. The existence of $x_0(t)$ for all $t > 0$ follows.

Finally, the regularity of x_0 follows from the aforementioned monotonicities and the implicit function theorem. □

Lemma 7.26. *For all $\delta \in \left[0, \frac{1}{2}\right)$, $c \geq c^{\delta}_{LLW}$ and $\kappa \in (0, \delta]$, there exists $\zeta_{1,\kappa} \in \mathbb{R}$ and $A_\kappa > 0$ such that the equation*

$$\underline{\theta_{\delta,c,A_\kappa}}(\xi) = \underline{\psi_{\delta,c}}(\xi - \zeta_{1,\kappa})$$

admits an isolated solution $\xi_{1,\kappa} \in \mathbb{R}$ such that

1. *$\underline{\theta_{\delta,c,A_\kappa}}(\xi) > \underline{\psi_{\delta,c}}(\xi - \zeta_{1,\kappa})$ for ξ in a left-sided neighborhood of $\xi_{1,\kappa}$;*
2. *$\underline{\theta_{\delta,c,A_\kappa}}(\xi) < \underline{\psi_{\delta,c}}(\xi - \zeta_{1,\kappa})$ for ξ in a right-sided neighborhood of $\xi_{1,\kappa}$;*
3. *$\underline{\psi_{\delta,c}}(\xi_{1,\kappa} - \zeta_{1,\kappa}) \leq \kappa$;*
4. *$\zeta_{1,\kappa} - \xi_{1,\kappa} \to +\infty$ as $\kappa \to 0$.*

Proof. Let δ, c and κ be given as in the statement, define

$$\lambda^{-\infty} = \frac{1}{2d}\left(\sqrt{c^2+4rd\left(b-1+(b+2)\,\delta\right)}-c\right),$$

$$\lambda_\theta^+ = \frac{1}{2d}\left(\sqrt{c^2+4rd\left(b-1+\delta\right)}-c\right),$$

$$\lambda_\theta^- = \frac{1}{2d}\left(-\sqrt{c^2+4rd\left(b-1+\delta\right)}-c\right),$$

$$\xi_\theta = \frac{d\ln A}{\sqrt{c^2+4rd\left(b-1+\delta\right)}} = \frac{\ln A}{\lambda_\theta^+-\lambda_\theta^-},$$

and notice that

$$\lambda_\theta^- < 0 < \lambda_\theta^+ < \lambda^{-\infty}.$$

Let $\tilde{\kappa}\in(0,\kappa]$ such that $(1-\tilde{\kappa})\,\lambda^{-\infty} > \lambda_\theta^+$.

In view of Lemma 7.15,

$$\lim_{\xi\to-\infty}\left(\frac{\underline{\psi_{\delta,c}}'(\xi)}{\underline{\psi_{\delta,c}}(\xi)}\right) = \lambda^{-\infty}.$$

Therefore, by monotonicity of $\underline{\psi_{\delta,c}}$, there exists $\zeta_\kappa\in\mathbb{R}$ such that for all $\xi\leq 0$,

$$\underline{\psi_{\delta,c}}(\xi-\zeta_\kappa)\leq\kappa,$$

$$\left(1-\frac{\tilde{\kappa}}{2}\right)\lambda^{-\infty}\leq\frac{\underline{\psi_{\delta,c}}'(\xi-\zeta_\kappa)}{\underline{\psi_{\delta,c}}(\xi-\zeta_\kappa)}\leq\left(1+\frac{\tilde{\kappa}}{2}\right)\lambda^{-\infty}.$$

Note that $\zeta_\kappa\to+\infty$ as $\kappa\to 0$. It remains to find $A>0$, $\zeta_1>\zeta_\kappa$ and $\xi_1\in(0,\zeta_1-\zeta_\kappa]$ such that

$$\underline{\theta_{\delta,c,A}}(\xi_1) = \underline{\psi_{\delta,c}}(\xi_1-\zeta_1),$$

$$\frac{\underline{\theta_{\delta,c,A}}'(\xi_1)}{\underline{\theta_{\delta,c,A}}(\xi_1)}\leq(1-\tilde{\kappa})\,\lambda^{-\infty}.$$

For all $\xi\in\mathbb{R}$,

$$\underline{\theta_{\delta,c,A}}'(\xi) = A\lambda_\theta^+\mathrm{e}^{\lambda_\theta^+(\xi-\xi_\theta)}-\lambda_\theta^-\mathrm{e}^{\lambda_\theta^-(\xi-\xi_\theta)}>0,$$

whence for all $\xi>0$ the condition

$$\frac{\underline{\theta_{\delta,c,A}}'(\xi)}{\underline{\theta_{\delta,c,A}}(\xi)}<(1-\tilde{\kappa})\,\lambda^{-\infty}$$

holds true if and only if

$$(1-\tilde{\kappa})\,\lambda^{-\infty}>\lambda_\theta^+ + \frac{\lambda_\theta^+-\lambda_\theta^-}{A\mathrm{e}^{\left(\lambda_\theta^+-\lambda_\theta^-\right)(\xi-\xi_\theta)}-1},$$

that is if and only if

$$A\mathrm{e}^{\left(\lambda_\theta^+-\lambda_\theta^-\right)(\xi-\xi_\theta)}-1>\frac{\lambda_\theta^+-\lambda_\theta^-}{(1-\tilde{\kappa})\,\lambda^{-\infty}-\lambda_\theta^+},$$

that is if and only if $\xi > \xi_1$ where

$$\xi_1 = \max\left(0, \xi_\theta + \frac{1}{\lambda_\theta^+ - \lambda_\theta^-}\left(\ln\left(1 + \frac{\lambda_\theta^+ - \lambda_\theta^-}{(1-\tilde{\kappa})\,\lambda^{-\infty} - \lambda_\theta^+}\right) - \ln A\right)\right).$$

In view of the definition of ξ_θ,

$$\begin{aligned}\xi_1 &= \max\left(0, \frac{1}{\lambda_\theta^+ - \lambda_\theta^-}\ln\left(1 + \frac{\lambda_\theta^+ - \lambda_\theta^-}{(1-\tilde{\kappa})\,\lambda^{-\infty} - \lambda_\theta^+}\right)\right) \\ &= \frac{1}{\lambda_\theta^+ - \lambda_\theta^-}\ln\left(1 + \frac{\lambda_\theta^+ - \lambda_\theta^-}{(1-\tilde{\kappa})\,\lambda^{-\infty} - \lambda_\theta^+}\right).\end{aligned}$$

In particular, $\xi_1 > 0$ does not depend on A and, by construction, we have

$$\frac{\underline{\theta_{\delta,c,A}}'(\xi)}{\underline{\theta_{\delta,c,A}}(\xi)} < (1-\tilde{\kappa})\,\lambda^{-\infty} \text{ for all } \xi > \xi_1,$$

$$\frac{\underline{\theta_{\delta,c,A}}'(\xi_1)}{\underline{\theta_{\delta,c,A}}(\xi_1)} = (1-\tilde{\kappa})\,\lambda^{-\infty}.$$

Now, the function $\underline{\theta_{\delta,c,A}}$ is increasing with $\underline{\theta_{\delta,c,A}}(0) = 0$ and

$$\begin{aligned}\underline{\theta_{\delta,c,A}}(\xi_1) &= A\mathrm{e}^{\lambda_\theta^+(\xi_1 - \xi_\theta)} - \mathrm{e}^{\lambda_\theta^-(\xi_1 - \xi_\theta)} \\ &= A^{1 - \frac{\lambda_\theta^+}{\lambda_\theta^+ - \lambda_\theta^-}}\mathrm{e}^{\lambda_\theta^+\xi_1} - A^{-\frac{\lambda_\theta^-}{\lambda_\theta^+ - \lambda_\theta^-}}\mathrm{e}^{\lambda_\theta^-\xi_1} \\ &= A^{\frac{-\lambda_\theta^-}{\lambda_\theta^+ - \lambda_\theta^-}}\left(\mathrm{e}^{\lambda_\theta^+\xi_1} - \mathrm{e}^{\lambda_\theta^-\xi_1}\right).\end{aligned}$$

As a function of A, this quantity is increasing (recall $\lambda_\theta^- < 0$) and vanishes as $A \to 0$. We fix now A such that

$$\underline{\theta_{\delta,c,A}}(\xi_1) = \underline{\psi_{\delta,c}}(-\zeta_\kappa) \leq \kappa.$$

Defining $\zeta_1 = \xi_1 + \zeta_\kappa > \zeta_\kappa$, we obtain indeed

$$\underline{\theta_{\delta,c,A}}(\xi_1) = \underline{\psi_{\delta,c}}(\xi_1 - \zeta_1) \leq \kappa,$$

$$\begin{aligned}\underline{\theta_{\delta,c,A}}'(\xi_1) &= (1-\tilde{\kappa})\,\lambda^{-\infty}\underline{\theta_{\delta,c,A}}(\xi_1) \\ &= (1-\tilde{\kappa})\,\lambda^{-\infty}\underline{\psi_{\delta,c}}(\xi_1 - \zeta_1) \\ &< \underline{\psi_{\delta,c}}'(\xi_1 - \zeta_1),\end{aligned}$$

as well as the limit

$$\lim_{\kappa \to 0}(\zeta_1 - \xi_1) = \lim_{\kappa \to 0}\zeta_\kappa = +\infty.$$

This completes the proof. □

Lemma 7.27. *There exists $\delta_0 \in \left(0, \frac{1}{2}\right)$ such that, for all $\delta \in [0, \delta_0)$, $c > c_{LLW}^\delta$ and $\tilde{c} \geq \max(c, f_\delta(c))$, there exists $\zeta_2 \in \mathbb{R}$ such that the equation*

$$\overline{\varphi_{\delta,c}}(x - ct) = \overline{w_{\delta,c,\tilde{c}}}(t, x - \zeta_2)$$

admits for all $t \geq 0$ an isolated solution $x_2(t) \in \mathbb{R}$ such that

1. $\overline{\varphi_{\delta,c}}(x-ct) > \overline{w_{\delta,c,\tilde{c}}}(t,x-\zeta_2)$ *for all* $x\in(x_2(t),+\infty)$*;*
2. $\overline{\varphi_{\delta,c}}(x-ct) < \overline{w_{\delta,c,\tilde{c}}}(t,x-\zeta_2)$ *for all* $x\in(-\infty,x_2(t))$*;*
3. $\overline{\varphi_{\delta,c,\tilde{c}}}(x_2(t)-ct) \leq \frac{\delta}{b}$.

Furthermore,

1. $x_2 \in \mathscr{C}^1([0,+\infty),(\zeta_2,+\infty))$*;*
2. $x_2(t) = \tilde{c}t + O(1)$ *as* $t\to+\infty$*.*

Remark. As $\delta\to 0$, $f_\delta(c)\to f(c)$. It can be verified that $(f_\delta(c))_{\delta\in\left[0,\frac{1}{2}\right)}$ is increasing, so that the convergence occurs from above.

Proof. Recall from Lemma 7.15 that there exists $\zeta\in\mathbb{R}$ such that,

$$\overline{\varphi_{\delta,c}}(\xi-\zeta) \sim \mathrm{e}^{-\lambda_\delta(c)\xi} \text{ as } \xi\to+\infty.$$

Hence, by the intermediate value theorem, for each $t\geq 0$ and each $\zeta_2\in\mathbb{R}$, the equation

$$\begin{aligned}\overline{\varphi_{\delta,c}}(x-ct) &= \overline{w_{\delta,c,\tilde{c}}}(t,x-\zeta_2)\\ &= \mathrm{e}^{-\lambda_\delta(c)(\tilde{c}-c)t}\mathrm{e}^{-\Lambda_\delta(c,\tilde{c})(x-\zeta_2-\tilde{c}t)}\end{aligned}$$

admits at least one solution $x(t)$ provided $\Lambda_\delta(c,\tilde{c}) > \lambda_\delta(c)$. This inequality is true indeed, since it is equivalent to

$$\tilde{c}-\sqrt{\tilde{c}^2-4(\lambda_\delta(c)(\tilde{c}-c)+1)} > 2\lambda_\delta(c),$$

that is to

$$\tilde{c}^2-4\lambda_\delta(c)\tilde{c}+4(\lambda_\delta(c))^2 > \tilde{c}^2-4(\lambda_\delta(c)(\tilde{c}-c)+1),$$

that is to

$$(\lambda_\delta(c))^2-c\lambda_\delta(c)+1>0,$$

that is (recalling that $\lambda_\delta(c)$ is characterized by $(\lambda_\delta(c))^2-c\lambda_\delta(c)+1-a_\delta=0$) to the obviously true following inequality,

$$a_\delta>0.$$

Since $\overline{\varphi_{\delta,c}}(\xi)<1+\delta$ for all $\xi\in\mathbb{R}$, any such solution satisfies

$$-\ln(1+\delta) < \lambda_\delta(c)(\tilde{c}-c)t+\Lambda_\delta(c,\tilde{c})(x(t)-\zeta_2-\tilde{c}t),$$

that is

$$x(t) > \zeta_2+\left(\tilde{c}-\frac{\lambda_\delta(c)(\tilde{c}-c)+\ln(1+\delta)/t}{\Lambda_\delta(c,\tilde{c})}\right)t.$$

By

$$\lim_{\delta'\to 0}\left(\tilde{c}-\frac{\lambda_{\delta'}(c)(\tilde{c}-c)+\ln(1+\delta')/t}{\Lambda_{\delta'}(c,\tilde{c})}\right) = \tilde{c}-\frac{\lambda(c)(\tilde{c}-c)}{\Lambda(c,\tilde{c})}$$

uniformly for $t\geq 1$, and, due to the preceding observation,

$$\frac{\lambda(c)(\tilde{c}-c)}{\Lambda(c,\tilde{c})} < \tilde{c}-c,$$

we deduce that $x(t)>\zeta_2+ct$ provided δ is small enough. Therefore the set of solutions is bounded from below and admit an infimum $I(t)>\zeta_2+ct$. Back to the exponential estimates, it is also clear that the set of solutions is bounded from above and admits therefore a supremum $S(t)$.

Recall that the asymptotic estimate for $\overline{\varphi_{\delta,c}}$ can be differentiated. Setting $g : (t,x) \mapsto \overline{\varphi_{\delta,c}}(x-ct) - \overline{w_{\delta,c,\tilde{c}}}(t, x-\zeta_2)$, we find that for any $t \geq 0$ and any solution $x(t) \in [I(t), S(t)]$,

$$\partial_x g(t, x(t)) = \overline{\varphi_{\delta,c}}(x(t)-ct)\left(\left(\frac{\overline{\varphi_{\delta,c}}'}{\overline{\varphi_{\delta,c}}}\right)(x(t)-ct) + \Lambda_\delta(c,\tilde{c})\right).$$

Since

$$\lim_{\xi\to+\infty}\left(\frac{\overline{\varphi_{\delta,c}}'}{\overline{\varphi_{\delta,c}}}\right)(\xi) + \Lambda_\delta(c,\tilde{c}) = -\lambda_\delta(c) + \Lambda_\delta(c,\tilde{c}) < 0,$$

we can choose ζ_2 large enough so that $\left(\frac{\overline{\varphi_{\delta,c}}'}{\overline{\varphi_{\delta,c}}}\right)(\xi) < 0$ for all $\xi \geq \zeta_2$. Since $x(t) - ct > \zeta_2$ for all $t \geq 0$, we deduce

$$\left(\frac{\overline{\varphi_{\delta,c}}'}{\overline{\varphi_{\delta,c}}}\right)(x(t)-ct) + \Lambda_\delta(c,\tilde{c}) < 0,$$

whence g is decreasing with respect to x in a neighborhood of $x(t)$. This implies directly the uniqueness of $x(t)$, namely $I(t) = S(t)$. From now on, we denote this unique solution $x_2(t)$. Of course, the regularity of x_2 follows directly from the implicit function theorem. The above yields that $x_2(t) - ct \geq \zeta_2$ for all $t \geq 0$.

Provided ζ_2 is large enough, for all $\xi \geq \zeta + \zeta_2$,

$$(1-\delta)\,\mathrm{e}^{-\lambda_\delta(c)\xi} \leq \overline{\varphi_{\delta,c}}(\xi-\zeta) \leq (1+\delta)\,\mathrm{e}^{-\lambda_\delta(c)\xi}.$$

At $\xi = x_2(t) - ct + \zeta \geq \zeta + \zeta_2$, this reads

$$(1-\delta)\,\mathrm{e}^{-\lambda_\delta(c)(x_2(t)-ct+\zeta)} \leq \overline{w_{\delta,c,\tilde{c}}}(t, x_2(t)-\zeta_2) \leq (1+\delta)\,\mathrm{e}^{-\lambda_\delta(c)(x_2(t)-ct+\zeta)},$$

that is

$$\begin{aligned}\ln(1-\delta) - \lambda_\delta(c)(x_2(t)-ct+\zeta) &\leq -\lambda_\delta(c)(\tilde{c}-c)t - \Lambda_\delta(c,\tilde{c})(x_2(t)-\zeta_2-\tilde{c}t)\\ &\leq \ln(1+\delta) - \lambda_\delta(c)(x_2(t)-ct+\zeta).\end{aligned}$$

The first inequality yields

$$x_2(t) \leq \frac{\tilde{c}(\Lambda_\delta(c,\tilde{c})-\lambda_\delta(c))t - \ln(1-\delta) + \lambda_\delta(c)\zeta + \Lambda_\delta(c,\tilde{c})\zeta_2}{\Lambda_\delta(c,\tilde{c})-\lambda_\delta(c)}$$

and the second inequality yields

$$x_2(t) \geq \frac{\tilde{c}(\Lambda_\delta(c,\tilde{c})-\lambda_\delta(c))t - \ln(1+\delta) + \lambda_\delta(c)\zeta + \Lambda_\delta(c,\tilde{c})\zeta_2}{\Lambda_\delta(c,\tilde{c})-\lambda_\delta(c)}.$$

Together these two estimates give that the asymptotic speed of x_2 is exactly $\tilde{c}$.

Finally, using once again $x_2(t) - ct \geq \zeta_2$, we find

$$\overline{\varphi_{\delta,c}}(x_2(t)-ct) \leq (1+\delta)\,\mathrm{e}^{-\lambda_\delta(c)(x_2(t)-ct+\zeta)} \leq (1+\delta)\,\mathrm{e}^{-\lambda_\delta(c)(\zeta+\zeta_2)},$$

and the inequality

$$\overline{\varphi_{\delta,c}}(x_2(t)-ct) \leq \frac{\delta}{b} \text{ for all } t \geq 0$$

is indeed satisfied provided ζ_2 is large enough. $\square$

Thanks again to the intermediate value theorem and the implicit function theorem, we can similarly establish the following lemmas. Since they involve the quantities L, x_L and $h^\star$, we recall that these are defined in Lemma 7.18 and Lemma 7.23 respectively.

Lemma 7.28. *There exists $\delta_1 \in \left(0, \frac{1}{2}\right)$ such that, for all $\delta \in (0, \delta_1)$, $c \in \left[c_{LLW}^{\delta}, 2\right)$ and $\zeta_3 \in \mathbb{R}$, the equation*

$$\underline{\psi_{\delta,c}}(x - ct) = \underline{\omega_{\delta,R_\delta}}\left(x - \left(2\sqrt{r(1-2\delta)d} - \delta\right)t - \zeta_3\right)$$

admits for all $t \geq 0$ an isolated solution $x_3(t) \in \mathbb{R}$ such that

1. *$\underline{\psi_{\delta,c}}(x - ct) > \underline{\omega_{\delta,R_\delta}}\left(x - \left(2\sqrt{r(1-2\delta)d} - \delta\right)t - \zeta_3\right)$ for x in a left-sided neighborhood of $x_3(t)$;*
2. *$\underline{\psi_{\delta,c}}(x - ct) < \underline{\omega_{\delta,R_\delta}}\left(x - \left(2\sqrt{r(1-2\delta)d} - \delta\right)t - \zeta_3\right)$ for x in a right-sided neighborhood of $x_3(t)$;*
3. *for all $t \geq 0$,*
$$-R_\delta < x_3(t) - \left(2\sqrt{r(1-2\delta)d} - \delta\right)t - \zeta_3 < x_{\delta,R_\delta}.$$

Furthermore, $x_3 \in \mathscr{C}^1([0,+\infty), \mathbb{R})$.

Lemma 7.29. *For all $\delta \in \left(0, \frac{1}{2}\right)$, $c \geq c_{LLW}^{\delta}$, $\tilde{c} > 2\sqrt{rd}$ such that $\tilde{c} \geq c$ and $h \in (0, h^\star)$, there exists $\zeta_3^0 \in \mathbb{R}$ such that, for all $\zeta_3 \geq \zeta_3^0$, the equation*

$$\underline{\psi_{\delta,c}}(x - ct) = \underline{\pi_{\delta,\tilde{c},h}}(x - \tilde{c}t - \zeta_3)$$

admits for all $t \geq 0$ an isolated solution $x_3(t) \in \mathbb{R}$ such that

1. *$\underline{\psi_{\delta,c}}(x - ct) > \underline{\pi_{\delta,\tilde{c},h}}(x - \tilde{c}t - \zeta_3)$ for x in a left-sided neighborhood of $x_3(t)$;*
2. *$\underline{\psi_{\delta,c}}(x - ct) < \underline{\pi_{\delta,\tilde{c},h}}(x - \tilde{c}t - \zeta_3)$ for x in a right-sided neighborhood of $x_3(t)$;*
3. *for all $t \geq 0$,*
$$-\sqrt{\frac{c}{rh}} < x_3(t) - \tilde{c}t - \zeta_3 < 0.$$

Furthermore, $x_3 \in \mathscr{C}^1([0,+\infty), \mathbb{R})$.

Remark. We have to point out here that the preceding two lemmas defining x_3 will never be used concurrently and no conflict of notation will occur. Lemma 7.28 will be used only in the proof of Proposition 7.5 whereas Lemma 7.29 will be used only in the proof of Proposition 7.6. In other words, going back to Theorem 7.1 and Theorem 7.3, they corresponds to different values of c_1: Lemma 7.28 corresponds to $c_1 = 2\sqrt{rd}$ whereas Lemma 7.29 corresponds to $c_1 > 2\sqrt{rd}$.

Lemma 7.30. *For all $\delta \in \left[0, \frac{1}{2}\right)$, $c > 2\sqrt{rd}$, $\eta \in \left(0, \frac{1}{d}\sqrt{c^2 - 4rd}\right)$, $B > 0$, there exists $\zeta_4 \in \mathbb{R}$ such that the equation*

$$\underline{\pi_{\delta,c,h}}(\xi) = \underline{\beta_{c,B,\eta}}(\xi - \zeta_4)$$

admits an isolated solution $\xi_4 \in \mathbb{R}$ such that

1. *$\underline{\pi_{\delta,c,h}}(\xi + \zeta_4) > \underline{\beta_{c,B,\eta}}(\xi)$ for ξ in a left-sided neighborhood of ξ_4;*
2. *$\underline{\pi_{\delta,c,h}}(\xi + \zeta_4) < \underline{\beta_{c,B,\eta}}(\xi)$ for ξ in a right-sided neighborhood of ξ_4;*
3. *$\xi_4 > 0$.*

Lemma 7.31. *For all $c > 2\sqrt{1-a}$, there exists $\zeta_0 \in \mathbb{R}$ such that, for all $\kappa \in \left(0, \frac{1-a}{2}\right)$, the equation*

$$\underline{\alpha_L}(x) = \underline{\chi_c}(x - ct - \zeta_0)$$

admits for all $t \geq 0$ a minimal solution $x_{0,\kappa}(t) \in \mathbb{R}$ such that

1. *$\underline{\alpha_L}(x) > \underline{\chi_c}(x - ct - \zeta_0)$ for x in a left-sided neighborhood of $x_{0,\kappa}(t)$;*
2. *$\underline{\chi_c}(x_{0,\kappa}(0) - \zeta_0) = \kappa$;*
3. *$x_L < x_{0,\kappa}(t) < L$.*

Furthermore, $x_{0,\kappa} \in \mathscr{C}^1([0,+\infty), (x_L, L))$.

Notice that in the above lemma, $x_{0,\kappa}(0) = \left(\underline{\chi_c}\right)^{-1}(\kappa) + \zeta_0$.

Lemma 7.32. *For all $c > 2\sqrt{1-a}$, $\tilde{c} \geq c$ such that $\tilde{c} \in (f(c) - 4\sqrt{a}, f(c))$, $\delta \in \left[0, \frac{1}{4}\left(-\tilde{c}^2 + 4(\lambda(c)(\tilde{c} - c) + 1)\right)\right)$ and $\zeta > \left(\underline{\chi_c}\right)^{-1}\left(\frac{\delta}{2}\right)$, the equation*

$$\underline{\chi_c}(x - ct) = \frac{\underline{\chi_c}(\zeta)}{\underline{z_{c,\tilde{c},\delta}}(0, X_z)} \underline{z_{c,\tilde{c},\delta}}(t, x - \zeta)$$

admits for all $t \geq 0$ an isolated solution $x_1(t) \in \mathbb{R}$ such that

1. *$\underline{\chi_c}(x - ct) > \frac{\underline{\chi_c}(\zeta)}{\underline{z_{c,\tilde{c},\delta}}(0,X_z)} \underline{z_{c,\tilde{c},\delta}}(t, x - \zeta)$ for x in a left-sided neighborhood of $x_1(t)$;*
2. *$\underline{\chi_c}(x - ct) < \frac{\underline{\chi_c}(\zeta)}{\underline{z_{c,\tilde{c},\delta}}(0,X_z)} \underline{z_{c,\tilde{c},\delta}}(t, x - \zeta)$ for x in a right-sided neighborhood of $x_1(t)$;*
3. *$\tilde{c}t + \zeta < x_1(t) < X_z + \tilde{c}t + \zeta$.*

Furthermore, $x_1 \in \mathscr{C}^1([0,+\infty), (\zeta, +\infty))$.

Remark. Similarly to the third interface x_3 which is defined in two separate lemmas, the zeroth interface is defined concurrently by Lemma 7.25 and Lemma 7.31 and the first interface is defined concurrently by Lemma 7.26 and Lemma 7.32. Lemma 7.25 will be used only in the proof of Proposition 7.6, Lemma 7.26 will be used only in the proof of Proposition 7.6 and in that of Proposition 7.5, Lemma 7.31 and Lemma 7.32 will be used only in the proof of Proposition 7.4.

There exists a small $\delta^\star \in \left(0, \frac{1}{2}\right)$ such that all the lemmas of this subsection involving a parameter δ can be applied in the range $\delta \in (0, \delta^\star)$. By construction, all the objects depending on δ defined in the preceding subsection are also well-defined in this range.

7.4.3 Construction of the super-solutions and sub-solutions for Theorem 7.3

In this subsection, we prove Proposition 7.6.

Let $c_1 > 2\sqrt{rd}$ and $c_2 > c_{LLW}$ such that $c_1 > c_2$ and $c_1 > f(c_2)$. In order to construct a satisfying approximated speed $c_2^\delta \simeq c_2$, we need to find c_2^δ such that:

1. $c_2^\delta > c_{LLW}^\delta$;
2. $c_2^\delta \to c_2$ as $\delta \to 0$;
3. $c_2 < c_2^\delta < c_1$;
4. $f_\delta\left(c_2^\delta\right) < c_1$
5. $\Lambda_\delta\left(c_2^\delta, c_1\right)$ is well-defined;
6. $\Lambda_\delta\left(c_2^\delta, c_1\right) \leq \Lambda(c_2, c_1)$.

The condition (6) above is equivalent to $\lambda_\delta\left(c_2^\delta\right)\left(c_1-c_2^\delta\right)\leq\lambda\left(c_2\right)\left(c_1-c_2\right)$, that is to

$$\frac{\lambda_\delta\left(c_2^\delta\right)}{\lambda\left(c_2\right)}\leq\frac{\left(c_1-c_2\right)}{\left(c_1-c_2^\delta\right)},$$

with a right-hand side necessarily larger than 1 provided (3) above is satisfied. Since the function $(\delta,c)\mapsto\lambda_\delta\left(c\right)$ is increasing with respect to c and decreasing with respect to δ, the sign of $\lambda_\delta\left(c_2^\delta\right)-\lambda\left(c_2\right)$ is unclear if we only assume $c_2^\delta>c_2$. Hence some care is needed and we cannot simply take a rough approximation like $c_2^\delta=c_2+\delta$.

In fact, since $a_\delta<a$ and $\lambda(c_2)<\sqrt{1-a}$, we can choose $\delta\in(0,\delta^\star)$ such that

$$\lambda\left(c_2\right)<\sqrt{1-a_\delta}.$$

Consequently, the following quantity is well-defined:

$$c_2^\delta=\left(\lambda_\delta^{-1}\circ\lambda\right)\left(c_2\right).$$

Since λ and λ_δ are both decreasing functions and $\lambda\left(c_2\right)<\lambda_\delta\left(c_2\right)$, it follows that $c_2^\delta>c_2$, whence

$$\begin{aligned}4\left(\lambda_\delta\left(c_2^\delta\right)\left(c_1-c_2^\delta\right)+1\right)&=4\left(\lambda\left(c_2\right)\left(c_1-c_2^\delta\right)+1\right)\\&<4\left(\lambda\left(c_2\right)\left(c_1-c_2\right)+1\right)\\&<c_1^2,\end{aligned}$$

where the last inequality is due to $c_1-f\left(c_2\right)>0$ (see also Subsection 7.4.1.12). By continuity, we can further assume that δ is so small that

$$\begin{cases}c_{LLW}\leq c_{LLW}^\delta<c_2^\delta\\ c_2<c_2^\delta<c_1\\ -4\sqrt{a_\delta}<c_1-f_\delta\left(c_2^\delta\right)\end{cases}.$$

It follows then, from Subsection 7.4.1.12, that

$$f_\delta\left(c_2^\delta\right)<f\left(c_2\right)<c_1,$$

whence the quantity $\Lambda_\delta\left(c_2^\delta,c_1\right)$ is well-defined. By definition, it satisfies

$$\begin{aligned}\Lambda_\delta\left(c_2^\delta,c_1\right)&=\frac{1}{2}\left(c_1-\sqrt{c_1^2-4\left(\lambda_\delta\left(c_2^\delta\right)\left(c_1-c_2^\delta\right)+1\right)}\right)\\&=\frac{1}{2}\left(c_1-\sqrt{c_1^2-4\left(\lambda\left(c_2\right)\left(c_1-c_2^\delta\right)+1\right)}\right)\\&<\frac{1}{2}\left(c_1-\sqrt{c_1^2-4\left(\lambda\left(c_2\right)\left(c_1-c_2\right)+1\right)}\right),\end{aligned}$$

so that $\Lambda_\delta\left(c_2^\delta,c_1\right)<\Lambda\left(c_2,c_1\right)$.

7.4.3.1 Super-solution

The pair $\left(\overline{u_\delta},\underline{v_\delta}\right)$ is defined by (see Figure 7.4.2)

$$\overline{u_\delta}\left(t,x\right)=\begin{cases}\min\left(1,\overline{\varphi_{\delta,c_2^\delta}}\left(x-c_2^\delta t-\zeta_{1,\kappa}\right)\right)&\text{if }x<x_2\left(t\right)+\zeta_{1,\kappa}\\ \overline{w_{\delta,c_2^\delta,c_1}}\left(t,x-\zeta_{1,\kappa}-\zeta_2\right)&\text{if }x\geq x_2\left(t\right)+\zeta_{1,\kappa}\end{cases},$$

$$\underline{v}_\delta(t,x) = \begin{cases} \max\left(0, \theta_{\delta,c_2^\delta,A_\kappa}\left(x - c_2^\delta t\right)\right) & \text{if } x < \xi_{1,\kappa} + c_2^\delta t \\ \underline{\psi_{\delta,c_2^\delta}}\left(x - c_2^\delta t - \zeta_{1,\kappa}\right) & \text{if } x \in \left[\xi_{1,\kappa} + c_2^\delta t, x_3(t) + \zeta_{1,\kappa}\right) \\ \underline{\pi_{\delta,c_1,h}}\left(x - c_1 t - \zeta_{1,\kappa} - \zeta_3\right) & \text{if } x \in \left[x_3(t) + \zeta_{1,\kappa}, \xi_4 + c_1 t + \zeta_{1,\kappa} + \zeta_3\right) \\ \underline{\beta_{c_1,B,\eta_\beta}}\left(x - c_1 t - \zeta_{1,\kappa} - \zeta_3 - \zeta_4\right) & \text{if } x \geq \xi_4 + c_1 t + \zeta_{1,\kappa} + \zeta_3 \end{cases},$$

where

- $\kappa \in (0, \delta]$ is fixed so small that $\zeta_{1,\kappa} - \xi_{1,\kappa} + x_2(0)$ is large enough so that for all $t \geq 0$, $\xi_{1,\kappa} + c_2^\delta t < x_2(t) + \zeta_{1,\kappa}$ (see Lemma 7.26(4) and Lemma 7.27 and use $x_2(t) \geq c_1 t + O(1) \geq c_2^\delta t + O(1)$);
- ζ_3 is fixed so large that, for all $t \geq 0$, $x_2(t) < x_3(t)$ (by Lemma 7.27 and Lemma 7.29, $x_2(t) - c_1 t$ and $x_3(t) - c_1 t$ are both bounded uniformly in $t \geq 0$, whence we can translate $x_3(t)$ to the right by increasing ζ_3);
- $h = \frac{h^\star}{2}$;
- $\eta_\beta = \frac{1}{2}\min\left(\frac{1}{d}\sqrt{c_1^2 - 4rd}, \Lambda_\delta\left(c_2^\delta, c_1\right)\right)$;
- $B = \mathrm{e}^{\Lambda_\delta\left(c_2^\delta, c_1\right)\xi_\beta} 2\overline{u_\delta}\left(0, \zeta_{1,\kappa} + \zeta_3 + \zeta_4\right)$.

The inequality

$$x_3(t) + \zeta_{1,\kappa} < \xi_4 + c_1 t + \zeta_{1,\kappa} + \zeta_3$$

is guaranteed by Lemma 7.29 and Lemma 7.30 which respectively show that $x_3(t) < c_1 t + \zeta_3$ and $\xi_4 > 0$. In conclusion, we have

$$\xi_{1,\kappa} + c_2^\delta t < x_2(t) + \zeta_{1,\kappa} < x_3(t) + \zeta_{1,\kappa} < \xi_4 + c_1 t + \zeta_{1,\kappa} + \zeta_3 \quad \text{for all } t \geq 0,$$

i.e. $\underline{v_\delta}$ is well-defined for all $t \geq 0$.

7.4.3.2 Sub-solution

First define the pair $(\underline{u}, \overline{v})$ by (see Figure 7.4.1)

$$\underline{u}(t,x) = \begin{cases} \underline{\chi_{c_2}}\left(x - c_2 t + \zeta_0\right) & \text{if } x < x_0(t) \\ \underline{w_{c_2,c_1,A,\eta_w}}(t,x) & \text{if } x \geq x_0(t) \end{cases},$$

$$\overline{v}(t,x) = \min\left(1, \mathrm{e}^{-\lambda_v(c_1)(x - c_1 t)}\right),$$

where $\eta_w = \frac{1}{2}\min\left(\sqrt{c_1^2 - 4\left(\lambda(c_2)(c_1 - c_2) + 1\right)}, \lambda_v(c_1)\right)$.

The function $\underline{u}$ depends on a constant $A > 0$ which will be fixed later on.

7.4.3.3 Up to some translations, the sub-solution $(\underline{u}, \overline{v})$ is initially smaller than the super-solution $(\overline{u_\delta}, \underline{v_\delta})$

First, let $\underline{V} : \mathbb{R} \to [0,1]$ be the smallest nonincreasing continuous function such that

$$\underline{v_\delta}(0,x) \leq \underline{V}(x) \text{ for all } x \in \mathbb{R}$$

and let $\zeta_5 \in \mathbb{R}$ such that, for all $t \geq 0$, $x \mapsto \underline{v_\delta}(t, x + c_1 t)$ is $\mathscr{C}^1$ and nonincreasing in $(\zeta_5, +\infty)$. The existence of ζ_5 follows from the fact that the last discontinuity of $\partial_x \underline{v_\delta}$ and the last local maximum of $\underline{v_\delta}$ move both at most at speed c_1. The limit of $\underline{V}$ at $-\infty$ is smaller than 1 and $\underline{V}(x) = \underline{v_\delta}(0,x)$ if $x > \zeta_5$. Therefore, since $\overline{v}$ and $\underline{v_\delta}$ have the same exponential decay at $+\infty$, there exists $\zeta_6 \geq \zeta_5$ such that:

1. for all $t \geq 0$, $x \mapsto \underline{v_\delta}(t, x + c_1 t)$ is $\mathscr{C}^1$ and nonincreasing in $(\zeta_6, +\infty)$;
2. for all $x \in \mathbb{R}$, $\underline{v_\delta}(0, x) \leq \underline{V}(x) \leq \overline{v}(0, x - \zeta_6)$.

Notice that with this definition of ζ_5 and ζ_6, the irregularity of $\overline{v}$ is initially on the right of the last irregularity of $\underline{v_\delta}$. Since the distance between these two points is nondecreasing with respect to t, it is bounded from below by the initial distance.

Next, quite similarly, we define $\zeta_7 \in \mathbb{R}$ such that

$$\underline{u}(0, x + \zeta_7) \leq \overline{u_\delta}(0, x) \text{ for all } x \in \mathbb{R}.$$

The irregularity of $\underline{u}$ moves faster than the first irregularity of $\overline{u_\delta}$ (as $c_1 > c_2^\delta$), whence it is impossible to guarantee that they stay ordered. This is not a major issue but some additional care will be required later on. Still, without loss of generality, we assume that ζ_7 is so large that the irregularity of $\underline{u}$ and the second (last) irregularity of $\overline{u_\delta}$, which both move at speed c_1, stay ordered.

7.4.3.4 Cleansing

Now that all required translations are done, we fix

$$A = 2\mathrm{e}^{\lambda_v(c_1)x_w}\mathrm{e}^{\lambda_v(c_1)(\zeta_6+\zeta_7)},$$

and thus there remains only one parameter: δ.

From now on, all the subscripts referring to fixed parameters are omitted. Furthermore, since all the properties of the functions $\underline{\chi}$, $\underline{w}$, $\overline{w_\delta}$, $\left(\overline{\varphi_\delta}, \underline{\psi_\delta}\right)$, $\underline{\theta_\delta}$, $\underline{\omega_\delta}$, $\underline{\pi_\delta}$, $\underline{\beta}$ we are interested in are invariant by translation, we assume that these functions were correctly normalized from the beginning, so that $\zeta_{1,\kappa} = \zeta_3 = \zeta_4 = \zeta_7 = 0$, and we fix $x_0(0) = 0$. Similarly, we define $C_\delta = \mathrm{e}^{\lambda_v(c_1)\zeta_6} > 0$ so that $\overline{v_\delta}(t, x) = \min\left(1, C_\delta \mathrm{e}^{-\lambda_v(c_1)(x - c_1 t)}\right)$ and x_w and ξ_β are redefined so that Lemma 7.21 and Lemma 7.22 stay true as stated. Notice that $\underline{\chi}$, $\underline{w}$, $\underline{\beta}$, $\underline{u}$ and $\overline{v}$ now depend on δ because of these various normalizations (and consequently these notations come with a subscript δ from now on).

To summarize, the super- and sub-solutions are now defined as follows:

$$\underline{u_\delta}(t, x) = \begin{cases} \underline{\chi_\delta}(x - c_2 t) & \text{if } x < x_0(t) \\ \underline{w_\delta}(t, x) & \text{if } x \geq x_0(t) \end{cases},$$

$$\overline{v_\delta}(t, x) = \min\left(1, C_\delta \mathrm{e}^{-\lambda_v(c_1)(x - c_1 t)}\right),$$

$$\overline{u_\delta}(t, x) = \begin{cases} \min\left(1, \overline{\varphi_\delta}\left(x - c_2^\delta t\right)\right) & \text{if } x < x_2(t) \\ \overline{w_\delta}(t, x) & \text{if } x \geq x_2(t) \end{cases},$$

$$\underline{v_\delta}(t, x) = \begin{cases} \max\left(0, \underline{\theta_\delta}\left(x - c_2^\delta t\right)\right) & \text{if } x < \xi_1 + c_2^\delta t \\ \underline{\psi_\delta}\left(x - c_2^\delta t\right) & \text{if } x \in \left[\xi_1 + c_2^\delta t, x_3(t)\right) \\ \underline{\pi_\delta}(x - c_1 t) & \text{if } x \in \left[x_3(t), \xi_4 + c_1 t\right) \\ \underline{\beta_\delta}(x - c_1 t) & \text{if } x \geq \xi_4 + c_1 t \end{cases}.$$

Furthermore, the interfaces satisfy, for all $t \geq 0$,

$$\begin{cases} x_0(t) < x_2(t) \\ \xi_1 + c_2^\delta t < x_2(t) \\ x_2(t) < x_3(t) \\ \xi_4 + c_1 t < \frac{\ln C_\delta}{\lambda_v(c_1)} + c_1 t \end{cases}.$$

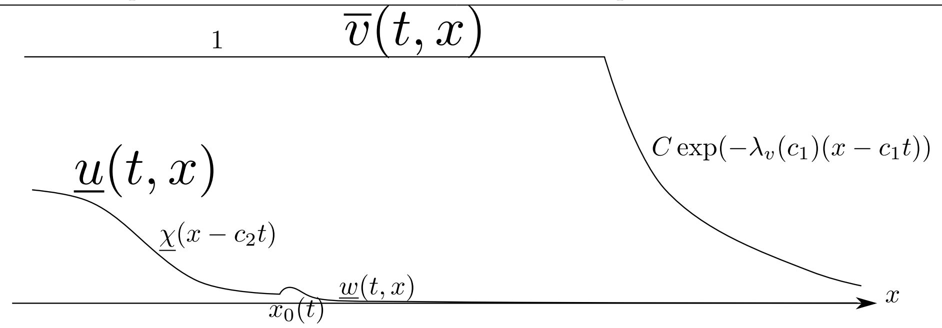


Figure 7.4.1 – Sub-solution $\left(\underline{u_\delta},\overline{v_\delta}\right)$ for Theorem 7.3

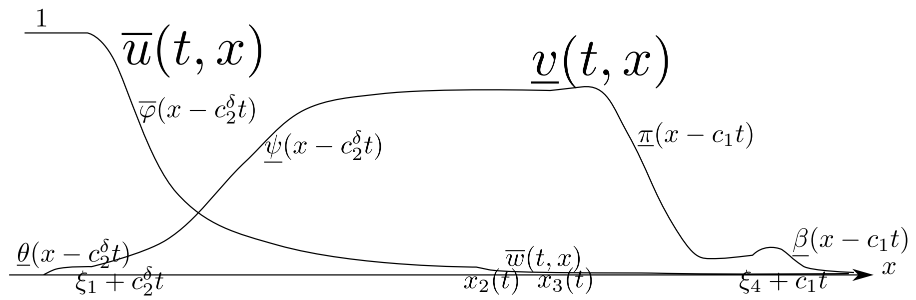


Figure 7.4.2 – Super-solution $\left(\overline{u_\delta},\underline{v_\delta}\right)$ for Theorem 7.3

7.4.3.5 Verification of the differential inequalities

Let us point out that by Theorem 7.9 and Theorem 7.10 and by construction of the pairs $\left(\overline{u_\delta},\underline{v_\delta}\right)$ and $\left(\underline{u_\delta},\overline{v_\delta}\right)$, it suffices to verify the differential inequalities

$$P\left(\overline{u_\delta},\underline{v_\delta}\right)\succeq F\left(\overline{u_\delta},\underline{v_\delta}\right) \tag{7.4.10}$$

and

$$P\left(\underline{u_\delta},\overline{v_\delta}\right)\preceq F\left(\underline{u_\delta},\overline{v_\delta}\right) \tag{7.4.11}$$

where the functions are regular in order to establish that $\left(\overline{u_\delta},\underline{v_\delta}\right)$ and $\left(\underline{u_\delta},\overline{v_\delta}\right)$ are indeed a super-solution and a sub-solution of (7.1.1) respectively. Also, the differential inequalities can also be verified before the translations are performed.

In what follows, for the sake of brevity, we voluntarily omit the mentions of the points (t,x), $x-c_1t$, $x-c_2t$ or $x-c_2^\delta t$ where the various functions are evaluated. In view of the construction, it should be unambiguous.

First, we consider (7.4.10). By Lemma 7.24 and Lemma 7.26, for all (t,x) such that

$$\left(\overline{u_\delta},\underline{v_\delta}\right)(t,x)=\left(1,\underline{\theta_\delta}\left(x-c_2^\delta t\right)\right),$$

we find $\underline{\theta_\delta} \leq \kappa \leq \delta$ and

$$\begin{aligned} P\left(\overline{u_\delta}, \underline{v_\delta}\right) - F\left(\overline{u_\delta}, \underline{v_\delta}\right) &= \begin{pmatrix} a\underline{\theta_\delta} \\ -c_2^\delta \underline{\theta_\delta}' - d\underline{\theta_\delta}'' - r\underline{\theta_\delta}\left(1 - \underline{\theta_\delta} - b\right) \end{pmatrix} \\ &= \begin{pmatrix} a\underline{\theta_\delta} \\ -r\underline{\theta_\delta}\left(\delta - \underline{\theta_\delta}\right) \end{pmatrix} \\ &\succeq (0,0). \end{aligned}$$

By definition of $\left(\overline{\varphi_\delta}, \underline{\psi_\delta}\right)$, for all (t,x) such that

$$\left(\overline{u_\delta}, \underline{v_\delta}\right)(t,x) = \left(1, \underline{\psi_\delta}\left(x - c_2^\delta t\right)\right),$$

we find, using $\underline{\psi_\delta} \leq 1 + \delta$,

$$\begin{aligned} P\left(\overline{u_\delta}, \underline{v_\delta}\right) - F\left(\overline{u_\delta}, \underline{v_\delta}\right) &= \begin{pmatrix} a\underline{\psi_\delta} \\ -c_2^\delta \underline{\psi_\delta}' - d\underline{\psi_\delta}'' - r\underline{\psi_\delta}\left(1 - \underline{\psi_\delta} - b\right) \end{pmatrix} \\ &= \begin{pmatrix} a\underline{\psi_\delta} \\ -r\underline{\psi_\delta}\left(2\delta + b\left(\overline{\varphi_\delta} - 1\right)\right) \end{pmatrix} \\ &\succeq \begin{pmatrix} 0 \\ -2r\delta\underline{\psi_\delta} \end{pmatrix} \\ &\succeq (0,0). \end{aligned}$$

Similarly, for all (t,x) such that

$$\left(\overline{u_\delta}, \underline{v_\delta}\right)(t,x) = \left(\overline{\varphi_\delta}\left(x - c_2^\delta t\right), \underline{\psi_\delta}\left(x - c_2^\delta t\right)\right),$$

we find

$$\begin{aligned} P\left(\overline{u_\delta}, \underline{v_\delta}\right) - F\left(\overline{u_\delta}, \underline{v_\delta}\right) &= \begin{pmatrix} -c_2^\delta \overline{\varphi_\delta}' - \overline{\varphi_\delta}'' - \overline{\varphi_\delta}\left(1 - \overline{\varphi_\delta} - a\underline{\psi_\delta}\right) \\ -c_2^\delta \underline{\psi_\delta}' - d\underline{\psi_\delta}'' - r\underline{\psi_\delta}\left(1 - \underline{\psi_\delta} - b\overline{\varphi_\delta}\right) \end{pmatrix} \\ &= \begin{pmatrix} \delta\overline{\varphi_\delta} \\ -2r\delta\underline{\psi_\delta} \end{pmatrix} \\ &\succeq (0,0). \end{aligned}$$

By Lemma 7.19 and Lemma 7.27, for all (t,x) such that

$$\left(\overline{u_\delta}, \underline{v_\delta}\right)(t,x) = \left(\overline{w_\delta}(t,x), \underline{\psi_\delta}\left(x - c_2^\delta t\right)\right),$$

we find, using $\overline{w_\delta} \leq \overline{\varphi_\delta}$ (Lemma 7.27(1)),

$$\begin{aligned} P\left(\overline{u_\delta}, \underline{v_\delta}\right) - F\left(\overline{u_\delta}, \underline{v_\delta}\right) &= \begin{pmatrix} \partial_t \overline{w_\delta} - \partial_{xx}\overline{w_\delta} - \overline{w_\delta}\left(1 - \overline{w_\delta} - a\underline{\psi_\delta}\right) \\ -c_2^\delta \underline{\psi_\delta}' - d\underline{\psi_\delta}'' - r\underline{\psi_\delta}\left(1 - \underline{\psi_\delta} - b\overline{w_\delta}\right) \end{pmatrix} \\ &= \begin{pmatrix} \overline{w_\delta}\left(\overline{w_\delta} + a\underline{\psi_\delta}\right) \\ -r\underline{\psi_\delta}\left(2\delta + b\left(\overline{\varphi_\delta} - \overline{w_\delta}\right)\right) \end{pmatrix} \\ &\succeq (0,0). \end{aligned}$$

By Lemma 7.23 and Lemma 7.27, for all (t,x) such that

$$\left(\overline{u_\delta}, \underline{v_\delta}\right)(t,x) = \left(\overline{w_\delta}(t,x), \underline{\pi_\delta}\left(x - c_1 t\right)\right),$$

we find, using $\overline{w_\delta} \leq \frac{\delta}{b}$ (Lemma 7.27(3)),

$$\begin{aligned} P\left(\overline{u_\delta}, \underline{v_\delta}\right) - F\left(\overline{u_\delta}, \underline{v_\delta}\right) &= \begin{pmatrix} \partial_t \overline{w_\delta} - \partial_{xx}\overline{w_\delta} - \overline{w_\delta}\left(1 - \overline{w_\delta} - a\underline{\pi_\delta}\right) \\ -c_1\underline{\pi_\delta}' - d\underline{\pi_\delta}'' - r\underline{\pi_\delta}\left(1 - \underline{\pi_\delta} - b\overline{w_\delta}\right) \end{pmatrix} \\ &\succeq \begin{pmatrix} \overline{w_\delta}\left(\overline{w_\delta} + a\underline{\pi_\delta}\right) \\ -r\underline{\pi_\delta}\left(\delta + \underline{\pi_\delta} - b\overline{w_\delta}\right) \end{pmatrix} \\ &\succeq (0,0). \end{aligned}$$

By Lemma 7.22, Lemma 7.27 and construction of B, for all (t,x) such that

$$\left(\overline{u_\delta}, \underline{v_\delta}\right)(t,x) = \left(\overline{w_\delta}(t,x), \underline{\beta_\delta}(x - c_1 t)\right),$$

we find, By definition of $\underline{\beta_\delta}$, and that of η_β in Subsection 7.4.3.1,

$$\begin{aligned} P\left(\overline{u_\delta}, \underline{v_\delta}\right) - F\left(\overline{u_\delta}, \underline{v_\delta}\right) &= \begin{pmatrix} \partial_t \overline{w_\delta} - \partial_{xx}\overline{w_\delta} - \overline{w_\delta}\left(1 - \overline{w_\delta} - a\underline{\beta_\delta}\right) \\ -c_1\underline{\beta_\delta}' - d\underline{\beta_\delta}'' - r\underline{\beta_\delta}\left(1 - \underline{\beta_\delta} - b\overline{w_\delta}\right) \end{pmatrix} \\ &\succeq \begin{pmatrix} \overline{w_\delta}\left(\overline{w_\delta} + a\underline{\beta_\delta}\right) \\ rb\underline{\beta_\delta}\left(\overline{w_\delta} - B\mathrm{e}^{-\Lambda_\delta\left(c_2^\delta, c_1\right)(x - c_1 t + \xi_\beta)}\right) \end{pmatrix} \\ &\succeq (0,0). \end{aligned}$$

Finally, we consider the differential inequalities associated with $(\underline{u_\delta}, \overline{v_\delta})$. By definition of $\underline{\chi_\delta}$, for all (t,x) such that

$$\left(\underline{u_\delta}, \overline{v_\delta}\right)(t,x) = \left(\underline{\chi_\delta}(x - c_2 t), 1\right),$$

we find

$$\begin{aligned} P\left(\underline{u_\delta}, \overline{v_\delta}\right) - F\left(\underline{u_\delta}, \overline{v_\delta}\right) &= \begin{pmatrix} -c_2\underline{\chi_\delta}' - \underline{\chi_\delta}'' - \underline{\chi_\delta}\left(1 - a - \underline{\chi_\delta}\right) \\ rb\underline{\chi_\delta} \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ rb\underline{\chi_\delta} \end{pmatrix} \\ &\preceq (0,0). \end{aligned}$$

By Lemma 7.21, Lemma 7.25 and by construction of $A = 2C_\delta \mathrm{e}^{\lambda_v(c_1)x_w}$, for all (t,x) such that

$$\left(\underline{u_\delta}, \overline{v_\delta}\right)(t,x) = \left(\underline{w_\delta}(t,x), 1\right),$$

we find, using $C_\delta \mathrm{e}^{-\lambda_v(c_1)(x - c_1 t)} \geq 1$,

$$\begin{aligned} P\left(\underline{u_\delta}, \overline{v_\delta}\right) - F\left(\underline{u_\delta}, \overline{v_\delta}\right) &= \begin{pmatrix} \partial_t \underline{w_\delta} - \partial_{xx}\underline{w_\delta} - \underline{w_\delta}\left(1 - \underline{w_\delta} - a\right) \\ rb\underline{w_\delta} \end{pmatrix} \\ &\preceq \begin{pmatrix} a\underline{w_\delta}\left(1 - A\mathrm{e}^{-\lambda_v(c_1)(x - c_1 t + x_w)}\right) \\ 0 \end{pmatrix} \\ &\preceq \begin{pmatrix} a\underline{w_\delta}\left(1 - 2C_\delta \mathrm{e}^{-\lambda_v(c_1)(x - c_1 t)}\right) \\ 0 \end{pmatrix} \\ &\preceq (0,0). \end{aligned}$$

Similarly, for all (t,x) such that

$$\left(\underline{u_\delta}, \overline{v_\delta}\right)(t,x) = \left(\underline{w_\delta}(t,x), C_\delta \mathrm{e}^{-\lambda_v(c_1)(x - c_1 t)}\right),$$

we find

$$\begin{aligned}
P\left(\underline{u_\delta},\overline{v_\delta}\right)-F\left(\underline{u_\delta},\overline{v_\delta}\right) &= \begin{pmatrix}\partial_t\underline{w_\delta}-\partial_{xx}\underline{w_\delta}-\underline{w_\delta}\left(1-\underline{w_\delta}-aC_\delta \mathrm{e}^{-\lambda_v(c_1)(x-c_1t)}\right)\\ rb\underline{w_\delta}\end{pmatrix}\\
&\preceq \begin{pmatrix}a\underline{w_\delta}\mathrm{e}^{-\lambda_v(c_1)(x-c_1t)}\left(C_\delta-A\mathrm{e}^{-\lambda_v(c_1)x_w}\right)\\ 0\end{pmatrix}\\
&\preceq (0,0)\,.
\end{aligned}$$

7.4.4 Construction of the super-solutions for Theorem 7.1

In this subsection, we prove Proposition 7.5.

Let $c_2>\max\left(c_{LLW},f^{-1}\left(2\sqrt{rd}\right)\right)$ and let $\delta\in(0,\delta^\star)$ such that $c_{LLW}^\delta<c_2$. Define

$$c_1^\delta=2\sqrt{r\left(1-2\delta\right)d}-\delta.$$

Recall from Subsection 7.4.1.12 that, given a fixed $\tilde{c}$, the function $c\mapsto\Lambda_\delta\left(c,\tilde{c}\right)$ is decreasing and bijectively maps $\left[2\sqrt{1-a_\delta},+\infty\right)$ onto

$$\left(\frac{1}{2}\left(\tilde{c}-\sqrt{\tilde{c}^2-4\left(a_\delta+1\right)}\right),\frac{1}{2}\left(\tilde{c}-\sqrt{\tilde{c}^2-4\left(\tilde{c}\sqrt{1-a_\delta}+2a_\delta-1\right)}\right)\right].$$

Thus the equation

$$\Lambda_\delta\left(c_2^\delta,c_1^\delta\right)=\Lambda\left(c_2,2\sqrt{rd}\right).$$

admits a unique solution c_2^δ if and only if

$$c_1^\delta-\sqrt{\left(c_1^\delta\right)^2-4\left(a_\delta+1\right)}<2\Lambda\left(c_2,2\sqrt{rd}\right)\leq c_1^\delta-\sqrt{\left(c_1^\delta\right)^2-4\left(c_1^\delta\sqrt{1-a_\delta}+2a_\delta-1\right)}.\quad(7.4.12)$$

Since $c_2\in\left(c_{LLW},+\infty\right)\subset\left(2\sqrt{1-a},+\infty\right)$, we have by the above discussion

$$2\sqrt{rd}-\sqrt{\left(2\sqrt{rd}\right)^2-4\left(a+1\right)}<2\Lambda\left(c_2,2\sqrt{rd}\right)<2\sqrt{rd}-\sqrt{\left(2\sqrt{rd}\right)^2-4\left(2\sqrt{rd}\sqrt{1-a}+2a-1\right)}.$$

By the facts that $c_1^\delta\to2\sqrt{rd}$ and $a_\delta\to a$ as $\delta\to0$, we deduce that we can in fact assume that δ is so small that (7.4.12) holds. Hence c_2^δ is well-defined.

Furthermore, by continuity, c_2^δ converges to c_2 as $\delta\to0$, and thus $c_2^\delta>c_{LLW}^\delta$. In summary, we can assume that δ is so small that c_1^δ and c_2^δ are well-defined, respectively close to c_1 and c_2, and satisfy the following:

$$c_{LLW}^\delta<c_2^\delta\quad\text{and}\quad\Lambda_\delta\left(c_2^\delta,c_1^\delta\right)=\Lambda\left(c_2,2\sqrt{rd}\right)\qquad(7.4.13)$$

7.4.4.1 Super-solution

The pair $\left(\overline{u_\delta},\underline{v_\delta}\right)$ is defined by

$$\overline{u_\delta}\left(t,x\right)=\begin{cases}\min\left(1,\overline{\varphi_{\delta,c_2^\delta}}\left(x-c_2^\delta t-\zeta_{1,\kappa}\right)\right) & \text{if } x<x_2\left(t\right)+\zeta_{1,\kappa}\\ \overline{w_{\delta,c_2^\delta,c_1^\delta}}\left(t,x-\zeta_{1,\kappa}-\zeta_2\right) & \text{if } x\geq x_2\left(t\right)+\zeta_{1,\kappa}\end{cases},$$

$$\underline{v_\delta}(t,x) = \begin{cases} \max\left(0, \underline{\theta_{\delta,c_2^\delta,A_\kappa}}\left(x - c_2^\delta t\right)\right) & \text{if } x < \xi_{1,\kappa} + c_2^\delta t \\ \underline{\psi_{\delta,c_2^\delta}}\left(x - c_2^\delta t - \zeta_{1,\kappa}\right) & \text{if } x \in \left[\xi_{1,\kappa} + c_2^\delta t, x_3(t) + \zeta_{1,\kappa}\right) \\ \underline{\omega_{\delta,R_\delta}}\left(x - c_1^\delta t - \zeta_{1,\kappa} - \zeta_3\right) & \text{if } x \geq x_3(t) + \zeta_{1,\kappa} \end{cases},$$

where

— $\kappa \in (0,\delta]$ is fixed so small that, for all $t \geq 0$, $\xi_{1,\kappa} + c_2^\delta t < x_2(t) + \zeta_{1,\kappa}$ (see Lemma 7.26);

— ζ_3 is fixed so large that, for all $t \geq 0$, $x_2(t) < x_3(t)$ (see Lemma 7.28).

Thus, we have

$$\xi_{1,\kappa} + c_2^\delta t < x_2(t) + \zeta_{1,\kappa} < x_3(t) + \zeta_{1,\kappa} \quad \text{for all } t \geq 0.$$

7.4.4.2 Cleansing

Just as in the previous case, we normalize and simplify the notations so that $x_2(0) = 0$ and the super-solution is defined as follows:

$$\overline{u_\delta}(t,x) = \begin{cases} \min\left(1, \overline{\varphi_\delta}\left(x - c_2^\delta t\right)\right) & \text{if } x < x_2(t) \\ \overline{w_\delta}(t,x) & \text{if } x \geq x_2(t) \end{cases},$$

$$\underline{v_\delta}(t,x) = \begin{cases} \max\left(0, \underline{\theta_\delta}\left(x - c_2^\delta t\right)\right) & \text{if } x < \xi_1 + c_2^\delta t \\ \underline{\psi_\delta}\left(x - c_2^\delta t\right) & \text{if } x \in \left[\xi_1 + c_2^\delta t, x_3(t)\right) \\ \underline{\omega_\delta}\left(x - c_1^\delta t\right) & \text{if } x \geq x_3(t) \end{cases}.$$

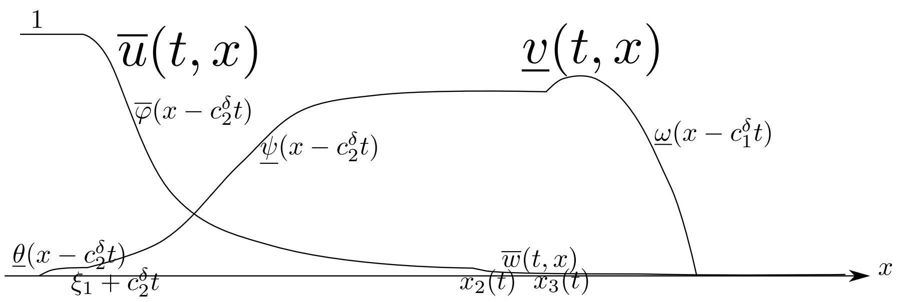


Figure 7.4.3 – Super-solution $\left(\overline{u_\delta}, \underline{v_\delta}\right)$ for Theorem 7.1

7.4.4.3 Verification of the differential inequalities

Just as in the previous case, we verify that $\left(\overline{u_\delta}, \underline{v_\delta}\right)$ is indeed a super-solution. The only new component to account for is $\underline{\omega_\delta}$, which can be handled easily in view of its definition.

7.4.5 Construction of the sub-solutions for Theorem 7.2

In this subsection, we prove Proposition 7.4.

Let $c_1 > 2\sqrt{rd}$ and $c_2 > c_{LLW}$ such that $c_1 \geq c_2$ and $c_1 < f(c_2)$. Let $c > c_2$ so close to c_2 that $c_1 < f(c)$ and let $\delta \in (0,\delta^\star)$ and

$$\tilde{c} \in \left(\max\left(c_1, f(c) - 4\sqrt{a}\right), f(c)\right).$$

7.4.5.1 Sub-solution

The pair $\left(\underline{u_{\delta,\zeta,\kappa}},\overline{v_{\delta,\zeta}}\right)$ is defined by

$$\underline{u_{\delta,\zeta,\kappa}}(t,x)=\begin{cases}\underline{\alpha_L}(x) & \text{if } x<x_{0,\kappa}(t)\\ \underline{\chi_c}(x-ct-\zeta_0) & \text{if } x\in[x_{0,\kappa}(t),x_1(t)+\zeta_0)\\ \frac{\underline{\chi_c}(\zeta-\zeta_0)}{\underline{z_{c,\tilde{c},\delta}}(0,X_z-\zeta_0)}\underline{z_{c,\tilde{c},\delta}}(t,x-\zeta_0-\zeta) & \text{if } x\geq x_1(t)+\zeta_0\end{cases},$$

$$\overline{v_{\delta,\zeta}}(t,x)=\min\left(1,\mathrm{e}^{-\lambda_v(\tilde{c})(x-y_{\delta,\zeta}-\tilde{c}t)}\right),$$

where

$$y_{\delta,\zeta}=\frac{\ln\delta-\ln(2a)}{\lambda_v(\tilde{c})}+\zeta_0+\zeta$$

and $\kappa\in\left(0,\min\left(\frac{1-a}{2},\frac{\delta}{2}\right)\right)$ and $\zeta>L$ are parameters.

Note that $\overline{v_{\delta,\zeta}}(t,x)\leq\frac{\delta}{2a}$ for all $x\geq\zeta_0+\zeta+\tilde{c}t$. By Lemma 7.32(3), we have $x_1(t)>\tilde{c}t+\zeta$ and thus $\overline{v_{\delta,\zeta}}(t,x)\leq\frac{\delta}{2a}$ for all $x\geq x_1(t)+\zeta_0$. Notice also that the support of $x\mapsto\underline{u_{\delta,\zeta,\kappa}}(0,x)$ is included in $[0,L+\zeta+2R_z]$.

7.4.5.2 Cleansing

Again, we normalize and simplify:

$$\underline{u_{\delta,\zeta,\kappa}}(t,x)=\begin{cases}\underline{\alpha}(x) & \text{if } x<x_0(t)\\ \underline{\chi}(x-ct) & \text{if } x\in[x_0(t),x_1(t))\\ \underline{z_\delta}(t,x-\zeta) & \text{if } x\geq x_1(t)\end{cases},$$

$$\overline{v_{\delta,\zeta}}(t,x)=\min\left(1,C_\delta\mathrm{e}^{-\lambda_v(\tilde{c})(x-\zeta-\tilde{c}t)}\right).$$

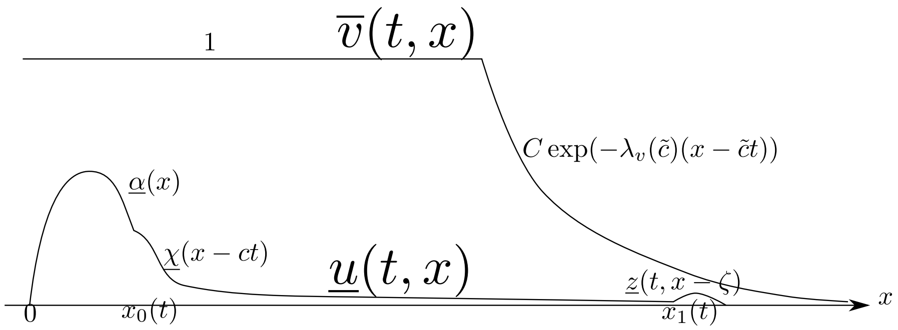


Figure 7.4.4 – Sub-solution $\left(\underline{u_{\delta,\zeta,\kappa}},\overline{v_{\delta,\zeta}}\right)$ for Theorem 7.2

7.4.5.3 Verification of the differential inequalities

Again, we verify that $\left(\underline{u_{\delta,\zeta,\kappa}},\overline{v_{\delta,\zeta}}\right)$ is a sub-solution. The only new components are $\underline{\alpha}$ and $\underline{z_\delta}$, the latter being handled with Lemma 7.20.

7.5 Discussion

As a preliminary remark, let us point out that analogous results can be obtained with the exact same method for the coexistence case $a < 1$, $b < 1$. In that case the solutions are characterized by a profile connecting $(0,0)$ to $(0,1)$ to $\left(\frac{1-a}{1-ab}, \frac{1-b}{1-ab}\right)$.

7.5.1 On the consequences of Theorem 7.1, Theorem 7.2 and Theorem 7.3

Consider here the Cauchy problem associated with Theorem 7.1, namely the initial condition u_0 of the slower and stronger species has a support included in $(-\infty, 0]$ while the initial condition v_0 of the faster and weaker species has compact support. Treating $2\sqrt{rd}$ as a parameter, Theorem 7.1 says that, while the species v always spreads at speed $2\sqrt{rd}$ if it persists, the species u:

— lags behind v and spreads at speed c_{LLW} if $2\sqrt{rd} \geq f(c_{LLW})$;

— lags behind v and spreads at speed $f^{-1}(2\sqrt{rd}) > c_{LLW}$ if $2 < 2\sqrt{rd} < f(c_{LLW})$;

— drives v to extinction and spreads at speed 2 if $2\sqrt{rd} < 2$.

In general, it is unclear whether $c_{LLW} = 2\sqrt{1-a}$ or not. Hence the condition $2\sqrt{rd} \geq f(c_{LLW})$ might be difficult to check in practice. However, since

$$\max_{c\in\left[2\sqrt{1-a},2\right]} f(c) = f(2\sqrt{1-a}) = 2\left(\sqrt{1-a} + \sqrt{a}\right),$$

the condition $\sqrt{rd} > \sqrt{1-a} + \sqrt{a}$ always implies $2\sqrt{rd} > f(c_{LLW})$ and consequently always implies that u invades at speed c_{LLW}. In particular, the maximum of $a \mapsto \sqrt{1-a} + \sqrt{a}$ in $(0,1)$ being $\sqrt{2}$, if $rd > 2$, then u invades at speed c_{LLW} independently of the value of a and b. In ecological terms, if v is a sufficiently fast invader, then it decelerates optimally any stronger and slower competitor.

Applied to a pair $(u_0, 0)$, the nonexistence result reduces to a well-known property of the KPP equation satisfied by u in isolation: all solutions spread at least at speed 2.

In view of Figure 7.1.1 and Figure 7.1.2, it is tempting to refer to the pair of speeds

$$(c_2^\star, c_1^\star) = \left(\max\left(c_{LLW}, f^{-1}\left(2\sqrt{rd}\right)\right), 2\sqrt{rd}\right)$$

as a "minimal pair". But in our opinion, such a terminology would be misleading. Indeed, a very natural conjecture in view of the KPP literature is that the propagating terraces attract initial data with appropriate exponential decays ($\lambda_v(c_1)$ for v_0 and $\Lambda(c_2, c_1)$ for u_0). Assume this conjecture is true indeed, assume $2 < 2\sqrt{rd} < f(c_{LLW})$ and fix a compactly supported or Heaviside-like u_0. Then decreasing the decay of v_0 will accelerate the invasion of v but decelerate that of u (with the obvious convention that a compactly supported v_0 has an infinite decay).

More generally, this paper presents several results that are complementary to that of Lewis, Li and Weinberger, with several surprising consequences. It shows that c_{LLW} is not always the relevant speed when predicting the speed of the invasion of u in the territory of v. The initial spatial distribution of v has to be taken into account and in particular, it can be inappropriate to approximate a very large territory by an unbounded territory. Also, even if c_{LLW} is linearly determined and therefore only depends on a, the speed of u might still depend on rd.

Our acceleration result can be heuristically understood as a pulled property, in the sense that very small densities of u on the right of the territory of v are still sufficiently large to increase the speed of u on the left of the territory of v. Of course, it would be interesting to verify the existence of such pulled accelerations in real biological invasions. Indeed, at first glance, our result might very well be described by ecological modelers as a strong case against diffusion equations: dispersal operators preserving compact supports, like the nonlinear diffusion of the porous form, $\partial_t u - \Delta(u^m)$, or the diffusion with free boundary studied in the last decade by Du and his collaborators (see for instance [59, 145]), will never lead to such a result.

7.5.2 On the boundary of the set of admissible pairs of speeds

In the present paper, the question of existence at the boundary of the set of admissible pairs is not settled. It is in fact more subtle than expected.

Assuming only $2\sqrt{rd} > 2$, this boundary is naturally partitioned as $V \cup G \cup H \cup D$, where

$$V = \{c_{LLW}\} \times \left(\max\left(2\sqrt{rd}, f(c_{LLW})\right), +\infty\right),$$

$$G = \left\{(c, f(c)) \mid c \in \left[c_{LLW}, \max\left(c_{LLW}, f^{-1}\left(2\sqrt{rd}\right)\right)\right)\right\},$$

$$H = \left[\max\left(c_{LLW}, f^{-1}\left(2\sqrt{rd}\right)\right), 2\sqrt{rd}\right) \times \left\{2\sqrt{rd}\right\},$$

$$D = \left\{(c, c) \mid c \geq 2\sqrt{rd}\right\},$$

and where G is possibly empty whereas V, H and D are always nonempty.

Points on $V \cup G$ should correspond to pairs (u_0, v_0) with u_0 supported in a left half-line and v_0 exponentially decaying. Using both Theorem 7.3 and Theorem 7.2 as well as a limiting argument and the comparison principle, it is possible to obtain the existence of such a terrace with a pair (u_0, v_0) of this form.

However, on H, which corresponds naively to pairs (u_0, v_0) with compactly supported v_0 and exponentially decaying u_0, such a construction seems to be impossible. A different, likely more delicate, argument is needed to deal with H. Still, we believe existence holds there.

On the contrary, on D, the question remains completely open. Indeed, on D, propagating terraces reduce to non-monotonic traveling waves connecting $(0,0)$ to $(1,0)$ with an intermediate bump of v. To the best of our knowledge, such traveling waves have never been studied. Even though it might be tempting to conjecture their nonexistence, we prefer to remain cautious here.

7.5.3 On the proofs

In the proof of Theorem 7.1, the approximated speed c_1^δ is necessary in the following sense: it is impossible to construct another $\underline{v}$ spreading this time exactly at speed $2\sqrt{rd}$. This is an immediate consequence of the Bramson shift for the KPP equation [29]: the level sets of the solution of the KPP equation satisfied by v in isolation with compactly supported initial data are asymptotically located at $2\sqrt{rd}t + s_{Bramson}(t)$, with $s_{Bramson}(t) = -\frac{3}{2}\log t + o(\log t)$. By comparison, it is then easily verified that for the solution (u, v) of our competitive system, there exists a time shift $s(t) \leq s_{Bramson}(t)$ such that the level sets of v in our problem are located at $2\sqrt{rd}t + s(t)$.

Similarly, in the proofs of Theorem 7.1 and of Theorem 7.3, we believe that the approximated speed c_2^δ are needed to account for a time shift $\tilde{s}(t) \neq 0$ describing the position of the level sets of u. The characterization of this shift is completely open; the only hint provided by our approach is that $\tilde{s}(t)$ is asymptotically nonnegative (contrarily to $s(t)$ and $s_{Bramson}(t)$).

Acknowledgments

This research was conducted at the Ohio State University and co-funded by the Mathematics Research Institute of the Ohio State University and the ANR NONLOCAL project. The authors thank deeply both institutions.

7.A On competition–diffusion traveling waves connecting $(1,0)$ to $(0,1)$

In this appendix, the parameters (d,r,a,b) are not fixed anymore and can vary. We define

$$\Pi = (0,+\infty)^2 \times (0,1) \times (1,+\infty).$$

For all $(d,r,a,b) \in \Pi$, $c_{LLW}^{d,r,a,b}$ denotes the associated spreading speed of the system (7.1.2). Subsequently, we define

$$E = \left\{ (c,d,r,a,b) \in (0,+\infty) \times \Pi \mid c \geq c_{LLW}^{d,r,a,b} \right\}.$$

7.A.1 Exact exponential decays

For all $P = (c,d,r,a,b) \in E$, we define

$$\lambda_{1,P}^{-\infty} = \frac{\sqrt{c^2+4}-c}{2},$$

$$\lambda_{2,P}^{-\infty} = \frac{\sqrt{c^2+4rd\,(b-1)}-c}{2d},$$

$$\lambda_{1,P}^{+\infty} = \frac{c+\sqrt{c^2+4rd}}{2d},$$

$$\lambda_{2,P}^{+\infty} = \frac{c+\sqrt{c^2-4\,(1-a)}}{2},$$

$$\lambda_{3,P}^{+\infty} = \frac{c-\sqrt{c^2-4\,(1-a)}}{2}.$$

Lemma 7.33. *Let $P = (c,d,r,a,b) \in E$ and (φ,ψ) be a profile of traveling wave solution of (7.1.1) with speed c. Define $R_P^{-\infty} : \lambda \mapsto \lambda^2 + c\lambda - 1$ and $R_P^{+\infty} : \lambda \mapsto d\lambda^2 - c\lambda - r$.*

Then the asymptotic behaviors of (φ,ψ) are as follows.

1. *There exist $A > 0$ and $B > 0$ such that, as $\xi \to -\infty$:*
   a) *if $\lambda_{2,P}^{-\infty} > \lambda_{1,P}^{-\infty}$, then*

$$\begin{cases} \varphi(\xi) = 1 - Ae^{\lambda_{1,P}^{-\infty}\xi} + h.o.t. \\ \psi(\xi) = Be^{\lambda_{2,P}^{-\infty}\xi} + h.o.t. \end{cases};$$

   b) *if $\lambda_{2,P}^{-\infty} < \lambda_{1,P}^{-\infty}$, then $R_P^{-\infty}\left(\lambda_{2,P}^{-\infty}\right) < 0$ and*

$$\begin{cases} \varphi(\xi) = 1 + \frac{a}{R_P^{-\infty}\left(\lambda_{2,P}^{-\infty}\right)} Be^{\lambda_{2,P}^{-\infty}\xi} + h.o.t. \\ \psi(\xi) = Be^{\lambda_{2,P}^{-\infty}\xi} + h.o.t. \end{cases};$$

c) if $\lambda_{2,P}^{-\infty}=\lambda_{1,P}^{-\infty}$, then $c+2\lambda_{2,P}^{-\infty}=\sqrt{c^2+4}>0$ and

$$\begin{cases}\varphi(\xi)=1-B|\xi|e^{\lambda_{2,P}^{-\infty}\xi}+h.o.t.\\ \psi(\xi)=\frac{c+2\lambda_{2,P}^{-\infty}}{a}Be^{\lambda_{2,P}^{-\infty}\xi}+h.o.t.\end{cases}.$$

2. There exist $A\in\mathbb{R}$, $B\in\mathbb{R}$ and $C\geq 0$ such that $B>0$ if $C=0$ and, as $\xi\to+\infty$:
   a) if $c>2\sqrt{1-a}$,
      i. if $\lambda_{1,P}^{+\infty}<\lambda_{3,P}^{+\infty}$, then $A>0$ and

$$\begin{cases}\varphi(\xi)=Be^{-\lambda_{2,P}^{+\infty}\xi}+Ce^{-\lambda_{3,P}^{+\infty}\xi}+h.o.t.\\ \psi(\xi)=1-Ae^{-\lambda_{1,P}^{+\infty}\xi}+h.o.t.\end{cases};$$

      ii. if $\lambda_{1,P}^{+\infty}=\lambda_{3,P}^{+\infty}$, then $A>0$ if $C=0$ and

$$\begin{cases}\varphi(\xi)=\frac{2d\lambda_{1,P}^{+\infty}-c}{a}Ce^{-\lambda_{1,P}^{+\infty}\xi}+Be^{-\lambda_{2,P}^{+\infty}\xi}+h.o.t.\\ \psi(\xi)=1-(A+C\xi)e^{-\lambda_{1,P}^{+\infty}\xi}+h.o.t.\end{cases};$$

      iii. if $\lambda_{1,P}^{+\infty}\in\left(\lambda_{3,P}^{+\infty},\lambda_{2,P}^{+\infty}\right)$, then $R_P^{+\infty}\left(\lambda_{3,P}^{+\infty}\right)<0$, $A>0$ if $C=0$ and

$$\begin{cases}\varphi(\xi)=Be^{-\lambda_{2,P}^{+\infty}\xi}+Ce^{-\lambda_{3,P}^{+\infty}\xi}+h.o.t.\\ \psi(\xi)=1-Ae^{-\lambda_{1,P}^{+\infty}\xi}+\frac{rb}{R_P^{+\infty}\left(\lambda_{3,P}^{+\infty}\right)}Ce^{-\lambda_{3,P}^{+\infty}\xi}+h.o.t.\end{cases};$$

      iv. if $\lambda_{1,P}^{+\infty}=\lambda_{2,P}^{+\infty}$, then $R_P^{+\infty}\left(\lambda_{3,P}^{+\infty}\right)<0$ and

$$\begin{cases}\varphi(\xi)=\frac{2d\lambda_{1,P}^{+\infty}-c}{a}Be^{-\lambda_{1,P}^{+\infty}\xi}+Ce^{-\lambda_{3,P}^{+\infty}\xi}+h.o.t.\\ \psi(\xi)=1-B\xi e^{-\lambda_{1,P}^{+\infty}\xi}+\frac{rb}{R_P^{+\infty}\left(\lambda_{3,P}^{+\infty}\right)}Ce^{-\lambda_{3,P}^{+\infty}\xi}+h.o.t.\end{cases};$$

      v. if $\lambda_{1,P}^{+\infty}>\lambda_{2,P}^{+\infty}$, then $R_P^{+\infty}\left(\lambda_{2,P}^{+\infty}\right)<0$, $R_P^{+\infty}\left(\lambda_{3,P}^{+\infty}\right)<0$ and

$$\begin{cases}\varphi(\xi)=Be^{-\lambda_{2,P}^{+\infty}\xi}+Ce^{-\lambda_{3,P}^{+\infty}\xi}+h.o.t.\\ \psi(\xi)=1+\frac{rb}{R_P^{+\infty}\left(\lambda_{2,P}^{+\infty}\right)}Be^{-\lambda_{2,P}^{+\infty}\xi}+\frac{rb}{R_P^{+\infty}\left(\lambda_{3,P}^{+\infty}\right)}Ce^{-\lambda_{3,P}^{+\infty}\xi}+h.o.t.\end{cases};$$

   b) if $c=2\sqrt{1-a}$,
      i. if $\lambda_{1,P}^{+\infty}<\lambda_{2,P}^{+\infty}$, then $A>0$ and

$$\begin{cases}\varphi(\xi)=(B+C\xi)e^{-\lambda_{2,P}^{+\infty}\xi}+h.o.t.\\ \psi(\xi)=1-Ae^{-\lambda_{1,P}^{+\infty}\xi}+h.o.t.\end{cases}.$$

      ii. if $\lambda_{1,P}^{+\infty}=\lambda_{2,P}^{+\infty}$, then $2d\lambda_{1,P}^{+\infty}-c=\sqrt{c^2+4rd}>0$ and

$$\begin{cases}\varphi(\xi)=\frac{2d\lambda_{1,P}^{+\infty}-c}{a}(B+C\xi)e^{-\lambda_{1,P}^{+\infty}\xi}+h.o.t.\\ \psi(\xi)=1-\left(B+\frac{1}{2}C\xi\right)\xi e^{-\lambda_{1,P}^{+\infty}\xi}+h.o.t.\end{cases};$$

iii. if $\lambda_{1,P}^{+\infty} > \lambda_{2,P}^{+\infty}$*, then* $R_P^{+\infty}\left(\lambda_{2,P}^{+\infty}\right) < 0$ *and*

$$\begin{cases} \varphi(\xi) = (B + C\xi)\, e^{-\lambda_{2,P}^{+\infty}\xi} + h.o.t. \\ \psi(\xi) = 1 + \frac{rb}{R_P^{+\infty}\left(\lambda_{2,P}^{+\infty}\right)}(B + C\xi)\, e^{-\lambda_{2,P}^{+\infty}\xi} + h.o.t. \end{cases};$$

Proof. This result follows from a standard yet lengthy phase-plane analysis. The detailed proof can be found for instance in Kan-on [101] or in Morita–Tachibana [114]. □

Compiling these estimates, we obtain the following two corollaries.

Corollary 7.34. *Let* $P = (c, d, r, a, b) \in E$ *and* (φ, ψ) *be a profile of traveling wave solution of the corresponding system with speed c. Then there exist* $i \in \{2, 3\}$*,* $C > 0$*,* $D > 0$ *and* $(i_+, j_+) \in \{0, 1\} \times \{0, 1, 2\}$ *such that, as* $\xi \to +\infty$*,*

$$\begin{cases} \varphi(\xi) = C\xi^{i_+} e^{-\lambda_{i,P}^{+\infty}\xi} + h.o.t. \\ \psi(\xi) = 1 - D\xi^{j_+} e^{-\min\left(\lambda_{1,P}^{+\infty}, \lambda_{i,P}^{+\infty}\right)\xi} + h.o.t. \end{cases}.$$

Corollary 7.35. *Let* $P = (c, d, r, a, b) \in E$ *and* (φ, ψ) *be a profile of traveling wave solution of the corresponding system with speed c. Let* $i_- = 2 - \#\left\{\lambda_{1,P}^{-\infty}, \lambda_{2,P}^{-\infty}\right\}$*.*

Then there exist $A > 0$ *and* $B > 0$ *such that, as* $\xi \to -\infty$*,*

$$\begin{cases} \varphi(\xi) = 1 - A|\xi|^{i_-} e^{\min\left(\lambda_{1,P}^{-\infty}, \lambda_{2,P}^{-\infty}\right)\xi} + h.o.t. \\ \psi(\xi) = Be^{\lambda_{2,P}^{-\infty}\xi} + h.o.t. \end{cases}.$$

7.A.2 Component-wise monotonicity of the profiles

Thanks to Corollary 7.34 and a sliding argument, we can show the component-wise monotonicity. We point out that Roques–Hosono–Bonnefon–Boivin [131] showed that the slow or fast decay problem is related to the pulled or pushed front problem.

Proposition 7.36. *Let* $P = (c, d, r, a, b) \in E$*. Let* $(\varphi, \psi) \in \mathscr{C}^2\left(\mathbb{R}, [0, 1]^2\right)$ *be a profile of traveling wave solution of (7.1.1) with speed c connecting* $(1, 0)$ *to* $(0, 1)$*.*

Then (φ, ψ) *is component-wise strictly monotonic, i.e.*

$$(\varphi, \psi)(\xi_1) \succ (\varphi, \psi)(\xi_2) \quad \textit{whenever} \quad \xi_1 < \xi_2.$$

Proof. The proof relies upon a sliding argument.

The sliding argument for monostable problems has three main steps: first, showing that if two profiles are correctly ordered at some point far on the left, then they remain correctly ordered everywhere on the left of this point; next, showing thanks to the first step and the exponential estimates at $+\infty$ that, up to some translation, the two profiles are globally ordered; finally, showing by optimizing the aforementioned translation that the two profiles actually coincide.

Notice that since the exponential estimates of Lemma 7.33 can be differentiated, they imply the component-wise strict monotonicity of (φ, ψ) near $\pm\infty$. Thus we can define $R > 0$ such that (φ, ψ) is component-wise strictly monotonic in $\mathbb{R}\backslash[-R, R]$. In particular, we can assume that

$$(\varphi, \psi)(-R) \succ (\varphi, \psi)(\xi) \succ (\varphi, \psi)(R) \text{ for all } \xi \in (-R, R). \tag{7.A.1}$$

Step 1: We claim that there is $\tau_1 > 0$ such that for all $\tau \geq \tau_1$,

$$(\varphi,\psi)(\xi-\tau) \succ (\varphi,\psi)(\xi) \quad \text{for all } \xi \in \mathbb{R}. \tag{7.A.2}$$

In view of the monotonicity of (φ,ψ) in $\mathbb{R}\setminus(-R,R)$, and (7.A.1), the claim clearly holds once we take $\tau_1 = 2R$.

Step 2: Define $\tau^\star$ to be the infimum of all $\tau \in (0,2R]$ such that (7.A.2) holds true. It remains to show that $\tau^\star = 0$. Suppose to the contrary that $\tau^\star > 0$. By construction,

$$(\varphi,\psi)(\xi-\tau^\star) \succeq (\varphi,\psi)(\xi) \text{ for all } \xi \in \mathbb{R}.$$

Moreover, by (7.A.1) and monotonicity of (φ,ψ) in $\mathbb{R}\setminus(-R,R)$, we see that for each $\tau \in \left[\frac{\tau^\star}{2}, 2\tau^\star\right]$,

$$(\varphi,\psi)(\xi-\tau) \succ (\varphi,\psi)(\xi) \text{ for all } \xi \in \mathbb{R}\setminus(-R+\tau,R),$$

and in particular for all $\xi \in \mathbb{R}\setminus(-R+\tau^\star/2,R)$. By the minimality of $\tau^\star > 0$, there exists $\xi^\star \in [-R+\tau^\star/2,R]$ such that equality holds for at least one of the components. The strong comparison principle yields

$$(\varphi,\psi)(\xi-\tau^\star) = (\varphi,\psi)(\xi) \quad \text{for all } \xi \in \mathbb{R}.$$

This implies (φ,ψ) is periodic with period $\tau^\star$, and contradicts $(\varphi,\psi)(-\infty) = (1,0)$ and $(\varphi,\psi)(+\infty) = (0,1)$.

Hence $\tau^\star = 0$ and, subsequently, for all $\tau > 0$, we have

$$(\varphi,\psi)(\xi-\tau) \succ (\varphi,\psi)(\xi) \text{ for all } \xi \in \mathbb{R},$$

which exactly means that (φ,ψ) is component-wise strictly monotonic. ☐

7.A.3 Ordering of the decays

By a similar proof, we can characterize more precisely the decays. We point out that Roques–Hosono–Bonnefon–Boivin [131] showed that the slow or fast decay problem is related to the pulled or pushed front problem.

Lemma 7.37. *Let $p = (d,r,a,b) \in \Pi$, $c \geq c^p_{LLW}$ and $\hat{c} \geq c$. Define $P = (c,p) \in E$ and $\hat{P} = (\hat{c},p) \in E$.*

Let $(\varphi,\psi) \in \mathscr{C}^2\left(\mathbb{R},[0,1]^2\right)$ and $\left(\hat{\varphi},\hat{\psi}\right) \in \mathscr{C}^2\left(\mathbb{R},[0,1]^2\right)$ be two profiles of traveling wave solution of (7.1.1) with speed c and $\hat{c}$ respectively. Denote (i,C,D,i_+,j_+) and $\left(\hat{i},\hat{C},\hat{D},\hat{i_+},\hat{j_+}\right)$ the quantities given by Corollary 7.34 when applied to (φ,ψ) and $\left(\hat{\varphi},\hat{\psi}\right)$ respectively.

Then at least one of the following estimates fails:

$$\hat{C}\xi^{\hat{i_+}}e^{-\lambda^{+\infty}_{\hat{i},\hat{P}}\xi} = o\left(C\xi^{i_+}e^{-\lambda^{+\infty}_{i,P}\xi}\right) \text{ as } \xi \to +\infty,$$

$$\hat{D}\xi^{\hat{j_+}}e^{-\min\left(\lambda^{+\infty}_{1,\hat{P}},\lambda^{+\infty}_{\hat{i},\hat{P}}\right)\xi} = o\left(D\xi^{j_+}e^{-\min\left(\lambda^{+\infty}_{1,P},\lambda^{+\infty}_{i,P}\right)\xi}\right) \text{ as } \xi \to +\infty.$$

Proof. The proof is by contradiction: we assume from now on that, on the contrary, the above two asymptotic estimates are satisfied. This means that, near $+\infty$, any translation of (φ,ψ) dominates $\left(\hat{\varphi},\hat{\psi}\right)$ (in the sense of the competitive ordering).

Here are the three steps of the sliding argument of this proof.

Step 1: choose $\xi_0 \in \mathbb{R}$ sufficiently close to $-\infty$ and such that for all $\xi \leq \xi_0$,

$$\left(\hat{\varphi}, \hat{\psi}\right)(\xi) \succeq \left(\frac{3}{4}, \frac{1}{4}\right) \text{ and } \hat{\varphi}(\xi) \geq \max\left(\frac{5-a}{8-4a}, \frac{3+b}{4b} + \hat{\psi}(\xi)\right). \tag{7.A.3}$$

Notice that such a ξ_0 exists indeed, since $(\hat{\varphi}, \hat{\psi})(-\infty) = (1, 0)$, and $\max\left(\frac{5-a}{8-4a}, \frac{3+b}{4b}\right) < 1$ with $a < 1$ and $b > 1$. We claim that if there exists $\tau \in \mathbb{R}$ such that

$$(\varphi, \psi)(\xi_0 - \tau) \succ \left(\hat{\varphi}, \hat{\psi}\right)(\xi_0),$$

then

$$(\varphi, \psi)(\xi - \tau) \succ \left(\hat{\varphi}, \hat{\psi}\right)(\xi) \text{ for all } \xi \leq \xi_0.$$

Clearly, there exists $\varepsilon \in \left(0, \frac{1}{4}\right]$ such that

$$(\varphi, \psi)(\xi - \tau) \succeq \left(\hat{\varphi}, \hat{\psi}\right)(\xi) + \varepsilon(-1, 1) \text{ for all } \xi \leq \xi_0.$$

Now, let $\varepsilon^\star \in \left[0, \frac{1}{4}\right]$ be the infimum of all these ε and assume by contradiction that $\varepsilon^\star > 0$. In view of the limiting values at $-\infty$ and of the inequality at ξ_0, there exists $\xi^\star \in (-\infty, \xi_0)$ such that

$$(\varphi, \psi)(\xi^\star - \tau) \succeq \left(\hat{\varphi}, \hat{\psi}\right)(\xi^\star) + \varepsilon^\star(-1, 1)$$

with, most importantly, equality for at least one of the components. Let us verify that $(\varphi_{\varepsilon^\star}, \psi_{\varepsilon^\star}) = \left(\hat{\varphi}, \hat{\psi}\right) + \varepsilon^\star(-1, 1)$ is a sub-solution. Since (φ, ψ) satisfies by definition

$$\begin{cases} -\hat{\varphi}'' - c\hat{\varphi}' = (\hat{c} - c)\hat{\varphi}' + \hat{\varphi}\left(1 - \hat{\varphi} - a\hat{\psi}\right) \\ -d\hat{\psi}'' - c\hat{\psi}' = (\hat{c} - c)\hat{\psi}' + r\hat{\psi}\left(1 - \hat{\psi} - b\hat{\varphi}\right) \end{cases},$$

we find (note that, by Proposition 7.36, $(\hat{\varphi}', \hat{\psi}') \preceq (0, 0)$)

$$\begin{cases} -\varphi''_{\varepsilon^\star} - c\varphi'_{\varepsilon^\star} - \varphi_{\varepsilon^\star}(1 - \varphi_{\varepsilon^\star} - a\psi_{\varepsilon^\star}) < \varepsilon^\star(1 - (2-a)\varphi_{\varepsilon^\star} - (1-a)\varepsilon^\star - a\psi_{\varepsilon^\star}) \\ -d\psi''_{\varepsilon^\star} - c\psi'_{\varepsilon^\star} - r\psi_{\varepsilon^\star}(1 - \psi_{\varepsilon^\star} - b\varphi_{\varepsilon^\star}) > -r\varepsilon^\star(1 - (2-b)\psi_{\varepsilon^\star} - (b-1)\varepsilon^\star - b\varphi_{\varepsilon^\star}) \end{cases}.$$

From

$$\begin{pmatrix} 1 - (2-a)\varphi_{\varepsilon^\star} - (1-a)\varepsilon^\star - a\psi_{\varepsilon^\star} \\ -(1 - (2-b)\psi_{\varepsilon^\star} - (b-1)\varepsilon^\star - b\varphi_{\varepsilon^\star}) \end{pmatrix} \preceq \begin{pmatrix} 1 - (2-a)\hat{\varphi} + \frac{1-a}{4} - a\hat{\psi} \\ -\left(1 - (2-b)\hat{\psi} + \frac{b-1}{4} - b\hat{\varphi}\right) \end{pmatrix},$$

we deduce by (7.A.3) that

$$\begin{pmatrix} 1 - (2-a)\varphi_{\varepsilon^\star} - (1-a)\varepsilon^\star - a\psi_{\varepsilon^\star} \\ -(1 - (2-b)\psi_{\varepsilon^\star} - (b-1)\varepsilon^\star - b\varphi_{\varepsilon^\star}) \end{pmatrix} \preceq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ for all } \xi \leq \xi_0.$$

We are now in position to apply the strong comparison principle of Theorem 7.11 and deduce from the existence of $\xi^\star$ a contradiction. Hence $\varepsilon^\star = 0$, that is

$$(\varphi, \psi)(\xi - \tau) \succeq \left(\hat{\varphi}, \hat{\psi}\right)(\xi) \text{ for all } \xi \leq \xi_0.$$

Finally, by strong comparison principle the strict inequality must hold for any $\xi \leq \xi_0$.

Step 2: in this step, we show the existence of τ_1 such that, for all $\tau \geq \tau_1$,

$$(\varphi,\psi)(\xi-\tau) \succ \left(\hat{\varphi},\hat{\psi}\right)(\xi) \text{ for all } \xi \in \mathbb{R}.$$

To this end, we fix ξ_0 as in (7.A.3) and choose $\tau_0 > 0$ large so that

$$(\varphi,\psi)(\xi_0-\tau) \succ \left(\hat{\varphi},\hat{\psi}\right)(\xi_0) \text{ for all } \tau \geq \tau_0.$$

By Step 1, we deduce that for all $\tau \geq \tau_0$,

$$(\varphi,\psi)(\xi-\tau) \succeq \left(\hat{\varphi},\hat{\psi}\right)(\xi) \text{ for all } \xi \leq \xi_0.$$

Next, we use the asymptotic behavior of (φ,ψ) and $(\hat{\varphi},\hat{\psi})$ at $+\infty$ to choose $\tau_1 \geq \tau_0$ such that for all $\tau \geq \tau_1$,

$$(\varphi,\psi)(\xi-\tau) \succeq \left(\hat{\varphi},\hat{\psi}\right)(\xi) \text{ for all } \xi \geq \xi_0.$$

The above two inequalities complete Step 2.

Step 3: define $\tau^\star$ as the infimum of all τ such that the preceding inequality holds true. By construction,

$$(\varphi,\psi)(\xi-\tau^\star) \succeq \left(\hat{\varphi},\hat{\psi}\right)(\xi) \text{ for all } \xi \in \mathbb{R}.$$

It suffices to show the existence of $\xi^\star \in \mathbb{R}$ such that $(\varphi,\psi)(\xi^\star-\tau^\star) \succeq (\varphi,\psi)(\xi^\star)$ with equality for at least one component. Granted, then the strong comparison principle yields

$$(\varphi,\psi)(\xi-\tau^\star) = \left(\hat{\varphi},\hat{\psi}\right)(\xi) \text{ for all } \xi \in \mathbb{R},$$

and the proof is ended. Suppose by contradiction that such a $\xi^\star$ does not exist, that is

$$(\varphi,\psi)(\xi-\tau^\star) \succ \left(\hat{\varphi},\hat{\psi}\right)(\xi) \text{ for all } \xi \in \mathbb{R}.$$

Now, the asymptotic behavior assumed at the beginning of this proof implies

$$\lim_{\xi\to+\infty} \frac{\varphi(\xi-\tau^\star)}{\hat{\varphi}(\xi)} = +\infty, \quad \text{and} \quad \lim_{\xi\to+\infty} \frac{1-\psi(\xi-\tau^\star)}{1-\hat{\psi}(\xi)} = +\infty.$$

Hence, there exists $\xi_1 > 0$ large and $\delta > 0$ small such that for all $\tau \in (\tau^\star-\delta,\tau^\star)$,

$$(\varphi,\psi)(\xi-\tau) \succ \left(\hat{\varphi},\hat{\psi}\right)(\xi) \text{ for all } \xi \geq \xi_1.$$

By taking $\delta > 0$ small, we have also that, for all $\tau \in (\tau^\star-\delta,\tau^\star)$,

$$(\varphi,\psi)(\xi-\tau) \succ \left(\hat{\varphi},\hat{\psi}\right)(\xi) \quad \text{for all } \xi \in [\xi_0,\xi_1].$$

Finally, the result in Step 1 implies that for all $\tau \in (\tau^\star-\delta,\tau^\star)$,

$$(\varphi,\psi)(\xi-\tau) \succ \left(\hat{\varphi},\hat{\psi}\right)(\xi) \text{ for all } \xi \in \mathbb{R}.$$

This contradicts the minimality of $\tau^\star$. □

From the preceding lemma, Lemma 7.33 and the respective monotonicities of $c \mapsto \lambda_{1,P}^{+\infty}$, $c \mapsto \lambda_{2,P}^{+\infty}$ and $c \mapsto \lambda_{3,P}^{+\infty}$, we deduce the following corollary which is a refinement of Lemma 7.33. Basically, it discards the possibility of solutions having a fast decay and a super-critical speed.

Corollary 7.38. *Let $P = (c, d, r, a, b) \in E$ with $c > c_{LLW}^{d,r,a,b}$ and (φ, ψ) be a profile of traveling wave solution of (7.1.1) with speed c.*

Then there exist $A > 0$ and $C > 0$ such that, as $\xi \to +\infty$:

1. *if $\lambda_{1,P}^{+\infty} < \lambda_{3,P}^{+\infty}$, then*
$$\begin{cases} \varphi(\xi) = Ce^{-\lambda_{3,P}^{+\infty}\xi} + h.o.t. \\ \psi(\xi) = 1 - Ae^{-\lambda_{1,P}^{+\infty}\xi} + h.o.t. \end{cases};$$
2. *if $\lambda_{1,P}^{+\infty} = \lambda_{3,P}^{+\infty}$, then*
$$\begin{cases} \varphi(\xi) = \frac{2d\lambda_{1,P}^{+\infty} - c}{a} Ce^{-\lambda_{1,P}^{+\infty}\xi} + h.o.t. \\ \psi(\xi) = 1 - C\xi e^{-\lambda_{1,P}^{+\infty}\xi} + h.o.t. \end{cases};$$
3. *if $\lambda_{1,P}^{+\infty} \in \left(\lambda_{3,P}^{+\infty}, \lambda_{2,P}^{+\infty}\right)$, then $R_P^{+\infty}\left(\lambda_{3,P}^{+\infty}\right) < 0$ and*
$$\begin{cases} \varphi(\xi) = Ce^{-\lambda_{3,P}^{+\infty}\xi} + h.o.t. \\ \psi(\xi) = 1 + \frac{rb}{R_P^{+\infty}\left(\lambda_{3,P}^{+\infty}\right)} Ce^{-\lambda_{3,P}^{+\infty}\xi} + h.o.t. \end{cases};$$
4. *if $\lambda_{1,P}^{+\infty} = \lambda_{2,P}^{+\infty}$, then $R_P^{+\infty}\left(\lambda_{3,P}^{+\infty}\right) < 0$ and*
$$\begin{cases} \varphi(\xi) = Ce^{-\lambda_{3,P}^{+\infty}\xi} + h.o.t. \\ \psi(\xi) = 1 + \frac{rb}{R_P^{+\infty}\left(\lambda_{3,P}^{+\infty}\right)} Ce^{-\lambda_{3,P}^{+\infty}\xi} + h.o.t. \end{cases};$$
5. *if $\lambda_{1,P}^{+\infty} > \lambda_{2,P}^{+\infty}$, then $R_P^{+\infty}\left(\lambda_{3,P}^{+\infty}\right) < 0$ and*
$$\begin{cases} \varphi(\xi) = Ce^{-\lambda_{3,P}^{+\infty}\xi} + h.o.t. \\ \psi(\xi) = 1 + \frac{rb}{R_P^{+\infty}\left(\lambda_{3,P}^{+\infty}\right)} Ce^{-\lambda_{3,P}^{+\infty}\xi} + h.o.t. \end{cases};$$

Remark. We emphasize that there exists a unique translation of the profile such that the normalization $C = 1$ holds. The remaining degree of freedom in the first case above (A can still take any positive value *a priori*) is the main difficulty regarding uniqueness.

7.A.4 Uniqueness and continuity

We are now in position to establish the following uniqueness result.

Proposition 7.39. *Let $P = (c, d, r, a, b) \in E$ such that $\lambda_{1,P}^{+\infty} \geq \lambda_{3,P}^{+\infty}$.*

Let $(\varphi, \psi) \in \mathscr{C}^2\left(\mathbb{R}, [0,1]^2\right)$ and $\left(\hat{\varphi}, \hat{\psi}\right) \in \mathscr{C}^2\left(\mathbb{R}, [0,1]^2\right)$ be two profiles of traveling wave solution of (7.1.1) with speed c.

Then (φ, ψ) and $\left(\hat{\varphi}, \hat{\psi}\right)$ coincide up to translation.

Proof. The proof relies upon a sliding argument again.

In view of Corollary 7.38, if $c > c_{LLW}^{d,r,a,b}$, the assumption $\lambda_{1,P}^{+\infty} \geq \lambda_{3,P}^{+\infty}$ immediately yields that the two profiles can be normalized so that they have the same decay at $+\infty$. Similarly, in view of Lemma 7.33, if $c = c_{LLW}^{d,r,a,b}$, then the two profiles can be normalized so that their decays either coincide or are well-ordered. In all cases, we can fix *a priori* the roles of the two profiles so that (φ, ψ) dominates $\left(\hat{\varphi}, \hat{\psi}\right)$ near $+\infty$. By following the first two steps of the proof of Lemma 7.37, we can assume without loss of generality the existence of $\tau_0 \in \mathbb{R}$ such that, for all $\tau \geq \tau_0$,

$$(\varphi, \psi)(\xi - \tau) \succeq \left(\hat{\varphi}, \hat{\psi}\right)(\xi) \text{ for all } \xi \in \mathbb{R}.$$

Next, define $\tau^\star \in \mathbb{R}$ as the infimum of all τ such that the preceding inequality holds true. It again suffices to show that there exists $\xi^\star \in \mathbb{R}$ where equality holds for one of the components. Assume on the contrary that no such $\xi^\star$ exists. Thus the preceding inequality is strict for both components for all $\xi \in \mathbb{R}$. Now, note that

$$\lim_{\xi \to +\infty} \frac{\varphi(\xi - \tau^\star)}{\hat{\varphi}(\xi)} \geq 1 \quad \text{and} \quad \lim_{\xi \to +\infty} \frac{1 - \psi(\xi - \tau^\star)}{1 - \hat{\psi}(\xi)} \geq 1.$$

Next, we claim that

$$\lim_{\xi \to +\infty} \frac{\varphi(\xi - \tau^\star)}{\hat{\varphi}(\xi)} = 1 \quad \text{or} \quad \lim_{\xi \to +\infty} \frac{1 - \psi(\xi - \tau^\star)}{1 - \hat{\psi}(\xi)} = 1.$$

Otherwise we may further reduce $\tau^\star$, just as in the proof of Lemma 7.37. Notice that this equality directly yields $\tau^\star = 0$, that is

$$(\varphi, \psi)(\xi) \succeq \left(\hat{\varphi}, \hat{\psi}\right)(\xi) \text{ for all } \xi \in \mathbb{R}.$$

Next, from the fact that $\lambda_{1,P}^{+\infty} \geq \lambda_{3,P}^{+\infty}$ and, depending on c, Corollary 7.38 or Lemma 7.33, both of the above limits are equal to 1.

Since the decay rate at $+\infty$ of both profiles coincide, we can reverse the profiles and repeat the proof. This leads to

$$\left(\hat{\varphi}, \hat{\psi}\right)(\xi) \succeq (\varphi, \psi)(\xi) \text{ for all } \xi \in \mathbb{R}.$$

Hence the two profiles actually coincide, which directly contradicts the assumption of nonexistence of $\xi^\star$.

In the end, $\xi^\star$ exists indeed and, by virtue of the strong comparison principle, the two normalized profiles coincide. In other words, the two profiles coincide up to translation. □

Corollary 7.40. *Let* $(d, r, a, b) \in \Pi$ *such that* $d \leq 2 + \frac{r}{1-a}$. *Then each speed* $c \geq c_{LLW}^{d,r,a,b}$ *is associated with a unique profile (up to translation).*

Proof. It suffices to prove that, for all $c \geq c_{LLW}^{d,r,a,b}$, $\lambda_{1,(c,d,r,a,b)}^{+\infty} \geq \lambda_{3,(c,d,r,a,b)}^{+\infty}$.

Noticing that this inequality is equivalent to $R_{(c,d,r,a,b)}^{+\infty}\left(\lambda_{3,(c,d,r,a,b)}^{+\infty}\right) \leq 0$, we find that we just have to prove that, for all $c \geq c_{LLW}^{d,r,a,b}$,

$$d \leq \frac{c\lambda_{3,(c,d,r,a,b)}^{+\infty} + r}{\left(\lambda_{3,(c,d,r,a,b)}^{+\infty}\right)^2}$$

and, using the polynomial equation satisfied by $\lambda_{3,(c,d,r,a,b)}^{+\infty} = \frac{c-\sqrt{c^2-4(1-a)}}{2}$, this reads

$$d \leq \frac{\left(\lambda_{3,(c,d,r,a,b)}^{+\infty}\right)^2 + 1 - a + r}{\left(\lambda_{3,(c,d,r,a,b)}^{+\infty}\right)^2} = 1 + \frac{1-a+r}{\left(\lambda_{3,(c,d,r,a,b)}^{+\infty}\right)^2}$$

It only remains to show that

$$\inf_{c \geq c_{LLW}^{d,r,a,b}} \frac{1-a+r}{\left(\lambda_{3,(c,d,r,a,b)}^{+\infty}\right)^2} \geq 1 + \frac{r}{1-a}.$$

The above inequality follows actually quite easily:

$$\inf_{c \geq c_{LLW}^{d,r,a,b}} \frac{1-a+r}{\left(\lambda_{3,(c,d,r,a,b)}^{+\infty}\right)^2} \geq \inf_{c \geq 2\sqrt{1-a}} \frac{1-a+r}{\left(\lambda_{3,(c,d,r,a,b)}^{+\infty}\right)^2} = \frac{1-a+r}{\sup\limits_{c \geq 2\sqrt{1-a}} \left(\lambda_{3,(c,d,r,a,b)}^{+\infty}\right)^2}$$

and, by monotonicity,

$$\sup_{c \geq 2\sqrt{1-a}} \left(\lambda_{3,(c,d,r,a,b)}^{+\infty}\right)^2 = \left(\lambda_{3,\left(2\sqrt{1-a},d,r,a,b\right)}^{+\infty}\right)^2 = 1 - a.$$

□

Finally, as a consequence of the uniqueness, we also have the continuity of the profiles with respect to the parameters.

Proposition 7.41. *Let*

$$E_u = \left\{ P \in E \mid \lambda_{1,P}^{+\infty} \geq \lambda_{3,P}^{+\infty} \right\}.$$

For all $P \in E_u$, let $\left(\Phi^P, \Psi^P\right)$ be the unique profile of traveling wave solution of (7.1.1) with speed c satisfying $\Psi^P(0) = \frac{1}{2}$.

Then $P \mapsto \left(\Phi^P, \Psi^P\right)$ is in $\mathscr{C}\left(\mathrm{int}E_u, \mathscr{C}_b\left(\mathbb{R}, \mathbb{R}^2\right)\right)$.

Proof. Let $P_\infty \in \overline{\mathrm{int}E_u}$ and $(P_n)_{n\in\mathbb{N}} \in (\mathrm{int}E_u)^{\mathbb{N}}$ such that $\lim\limits_{n\to+\infty} P_n = P_\infty$. By standard elliptic estimates (see Gilbarg–Trudinger [80]), the sequence $\left(\left(\Phi^{P_n}, \Psi^{P_n}\right)\right)_{n\in\mathbb{N}}$ converges, up to a diagonal extraction, in $\mathscr{C}_{loc}^2$. *A fortiori* it converges pointwise in $\mathbb{R}$. The limit $(\Phi_\infty, \Psi_\infty)$ is continuous, monotonic, and satisfies $\Psi_\infty(0) = \frac{1}{2}$. Using standard elliptic estimates to study the asymptotic behaviors, we find easily

$$\lim_{-\infty} (\Phi_\infty, \Psi_\infty) \in \{(1,0), (0,0)\} \text{ and } \lim_{+\infty} (\Phi_\infty, \Psi_\infty) = (0,1).$$

If Φ_∞ is null in $\mathbb{R}$, then $\xi \mapsto \Psi_\infty(-\xi)$ is a KPP traveling wave with negative speed, which is impossible. Therefore the limit at $-\infty$ of $(\Phi_\infty, \Psi_\infty)$ is $(1,0)$. This shows that the sequence of monotonic functions $\left(\left(\Phi^{P_n}, \Psi^{P_n}\right)\right)_{n\in\mathbb{N}}$ converges pointwise in $[-\infty, +\infty]$, whence by a variant of the Dini theorem it converges uniformly in $\mathbb{R}$. In view of the preceding uniqueness result, the limit is exactly $\left(\Phi^{P_\infty}, \Psi^{P_\infty}\right)$. Finally, a classical uniqueness and compactness argument shows that the previous diagonal extraction was not necessary and the sequence $\left(\left(\Phi^{P_n}, \Psi^{P_n}\right)\right)_{n\in\mathbb{N}}$ converges indeed in $\mathscr{C}_b\left(\mathbb{R}, \mathbb{R}^2\right)$ to $\left(\Phi^{P_\infty}, \Psi^{P_\infty}\right)$. □

Articles de l'auteur

[Gir17] Léo Girardin, *Competition in periodic media: I – Existence of pulsating fronts*, Discrete and Continuous Dynamical Systems - Series B **22** (2017), no. 4, 1341–1360.

[Gir18a] Léo Girardin, *Non-cooperative Fisher–KPP systems: Asymptotic behavior of traveling waves*, Mathematical Models and Methods in Applied Sciences **28** (2018), no. 06, 1067–1104.

[Gir18b] Léo Girardin, *Non-cooperative Fisher–KPP systems: traveling waves and long-time behavior*, Nonlinearity **31** (2018), no. 1, 108.

[GL18] Léo Girardin and King-Yeung Lam, *Invasion of an empty habitat by two competitors: spreading properties of monostable two-species competition–diffusion systems*, ArXiv e-prints (2018).

[GN15] Léo Girardin and Grégoire Nadin, *Travelling waves for diffusive and strongly competitive systems: relative motility and invasion speed*, European J. Appl. Math. **26** (2015), no. 4, 521–534.

[GN18] Léo Girardin and Grégoire Nadin, *Competition in periodic media: II – Segregative limit of pulsating fronts and "Unity is not Strength"-type result*, Journal of Differential Equations **265** (2018), no. 1, 98–156.

[GZ18] Léo Girardin and Alessandro Zilio, *Competition in periodic media: III – Existence & stability of segregated periodic coexistence states*, ArXiv e-prints (2018).

Bibliographie

[1] Ameera Al-Kiffai and Elaine C. M. Crooks, *Lack of symmetry in linear determinacy due to convective effects in reaction-diffusion-convection problems*, Tamkang J. Math. **47** (2016), no. 1, 51–70.

[2] Matthieu Alfaro and Quentin Griette, *Pulsating fronts for Fisher-KPP systems with mutations as models in evolutionary epidemiology*, ArXiv e-prints (2016).

[3] Antonio Ambrosetti and David Arcoya, *An introduction to nonlinear functional analysis and elliptic problems*, Progress in Nonlinear Differential Equations and their Applications, vol. 82, Birkhäuser Boston, Inc., Boston, MA, 2011.

[4] Sigurd Angenent, *The zero set of a solution of a parabolic equation*, Journal für die Reine und Angewandte Mathematik **390** (1988), 79–96.

[5] Ari Arapostathis, Mrinal K. Ghosh, and Steven I. Marcus, *Harnack's inequality for cooperative weakly coupled elliptic systems*, Comm. Partial Differential Equations **24** (1999), no. 9-10, 1555–1571.

[6] Anton Arnold, Laurent Desvillettes, and Céline Prévost, *Existence of nontrivial steady states for populations structured with respect to space and a continuous trait*, Commun. Pure Appl. Anal. **11** (2012), no. 1, 83–96.

[7] Donald G. Aronson, *Regularity properties of flows through porous media: The interface.*, Arch. Rational Mech. Anal. **37** (1970), 1–10.

[8] Donald G. Aronson and Hans F. Weinberger, *Nonlinear diffusion in population genetics, combustion, and nerve pulse propagation*, Partial differential equations and related topics, Springer, 1975, pp. 5–49.

[9] Nicolas Bacaër, *A short history of mathematical population dynamics*, Springer-Verlag London, Ltd., London, 2011.

[10] Guy Barles, Lawrence C. Evans, and Panagiotis E. Souganidis, *Wavefront propagation for reaction-diffusion systems of PDE*, Duke Math. J. **61** (1990), no. 3, 835–858.

[11] Olivier Bénichou, Vincent Calvez, Nicolas Meunier, and Raphaël Voituriez, *Front acceleration by dynamic selection in Fisher population waves*, Physical Review E **86** (2012), no. 4, 041908.

[12] Henri Berestycki, *Le nombre de solutions de certains problemes semi-linéaires elliptiques*, Journal of Functional Analysis **40** (1981), no. 1, 1–29.

[13] Henri Berestycki, Juliette Bouhours, and Guillemette Chapuisat, *Front blocking and propagation in cylinders with varying cross section*, Calc. Var. Partial Differential Equations **55** (2016), no. 3, –44.

[14] Henri Berestycki and François Hamel, *Front propagation in periodic excitable media*, Comm. Pure Appl. Math. **55** (2002), no. 8, 949–1032.

[15] Henri Berestycki, François Hamel, and Grégoire Nadin, *Asymptotic spreading in heterogeneous diffusive excitable media*, J. Funct. Anal. **255** (2008), no. 9, 2146–2189.

[16] Henri Berestycki, François Hamel, and Lionel Roques, *Analysis of the periodically fragmented environment model. I. Species persistence*, J. Math. Biol. **51** (2005), no. 1, 75–113.

[17] Henri Berestycki, François Hamel, and Lionel Roques, *Analysis of the periodically fragmented environment model. II. Biological invasions and pulsating travelling fronts*, J. Math. Pures Appl. (9) **84** (2005), no. 8, 1101–1146.

[18] Henri Berestycki, François Hamel, and Luca Rossi, *Liouville-type results for semilinear elliptic equations in unbounded domains*, Ann. Mat. Pura Appl. (4) **186** (2007), no. 3, 469–507.

[19] Henri Berestycki, Grégoire Nadin, Benoit Perthame, and Lenya Ryzhik, *The non-local Fisher–KPP equation: travelling waves and steady states*, Nonlinearity **22** (2009), no. 12, 2813.

[20] Henri Berestycki, Louis Nirenberg, and SR Srinivasa Varadhan, *The principal eigenvalue and maximum principle for second-order elliptic operators in general domains*, Communications on Pure and Applied Mathematics **47** (1994), no. 1, 47–92.

[21] Henri Berestycki and Luca Rossi, *On the principal eigenvalue of elliptic operators in* $\mathbb{R}^N$ *and applications*, Journal of the European Mathematical Society **8** (2006), no. 2, 195–215.

[22] Henri Berestycki and Luca Rossi, *Generalizations and properties of the principal eigenvalue of elliptic operators in unbounded domains*, Comm. Pure Appl. Math. **68** (2015), no. 6, 1014–1065.

[23] Joris Bierkens and André Ran, *A singular M-matrix perturbed by a nonnegative rank one matrix has positive principal minors; is it D-stable?*, Linear Algebra Appl. **457** (2014), 191–208.

[24] Isabeau Birindelli, Enzo Mitidieri, and Guido Sweers, *Existence of the principal eigenvalue for cooperative elliptic systems in a general domain*, Differ. Uravn. **35** (1999), no. 3, 325–333.

[25] Olivier Bonnefon, Jérome Coville, Jimmy Garnier, François Hamel, and Lionel Roques, *The spatio-temporal dynamics of neutral genetic diversity*, Ecological Complexity **20** (2014), 282–292.

[26] Emeric Bouin and Vincent Calvez, *Travelling waves for the cane toads equation with bounded traits*, Nonlinearity **27** (2014), no. 9, 2233–2253.

[27] Emeric Bouin, Vincent Calvez, Nicolas Meunier, Sepideh Mirrahimi, Benoît Perthame, Gael Raoul, and Raphaël Voituriez, *Invasion fronts with variable motility: phenotype selection, spatial sorting and wave acceleration*, Comptes Rendus Mathematique **350** (2012), no. 15, 761–766.

[28] Emeric Bouin, Christopher Henderson, and Lenya Ryzhik, *Super-linear spreading in local and non-local cane toads equations*, ArXiv e-prints (2015).

[29] Maury Bramson, *Convergence of solutions of the Kolmogorov equation to travelling waves*, Mem. Amer. Math. Soc. **44** (1983), no. 285, –190.

[30] Maury D. Bramson, *Maximal displacement of branching brownian motion*, Comm. Pure Appl. Math. **31** (1978), no. 5, 531–581.

[31] L. A. Caffarelli, A. L. Karakhanyan, and Fang-Hua Lin, *The geometry of solutions to a segregation problem for nondivergence systems*, Journal of Fixed Point Theory and Applications **5** (2009), no. 2, 319–351.

[32] Luis A. Caffarelli, *The regularity of free boundaries in higher dimensions*, Acta Math. **139** (1977), no. 3-4, 155–184.

[33] John R. Cannon and Hong-Ming Yin, *A periodic free boundary problem arising in some diffusion chemical reaction processes*, Nonlinear Anal. **15** (1990), no. 7, 639–648.

[34] Jack Carr and Adam Chmaj, *Uniqueness of travelling waves for nonlocal monostable equations*, Proc. Amer. Math. Soc. **132** (2004), no. 8, 2433–2439.

[35] Cécile Carrère, *Spreading speeds for a two-species competition-diffusion system*, J. Differential Equations **264** (2018), no. 3, 2133–2156.

[36] Nicolas Champagnat and Sylvie Méléard, *Invasion and adaptive evolution for individual-based spatially structured populations*, J. Math. Biol. **55** (2007), no. 2, 147–188.

[37] Zhen-Qing Chen and Zhong X. Zhao, *Harnack principle for weakly coupled elliptic systems*, J. Differential Equations **139** (1997), no. 2, 261–282.

[38] Monica Conti, Susanna Terracini, and Gianmaria Verzini, *Asymptotic estimates for the spatial segregation of competitive systems*, Adv. Math. **195** (2005), no. 2, 524–560.

[39] Chris Cosner, *Challenges in modeling biological invasions and population distributions in a changing climate*, Ecological Complexity **20** (2014), 258–263.

[40] Jérome Coville and Frédéric Fabre, *Convergence to the equilibrium in a Lotka-Volterra ODE competition system with mutations*, ArXiv e-prints (2013).

[41] Elaine C. M. Crooks, Edward N. Dancer, and Danielle Hilhorst, *On long-time dynamics for competition-diffusion systems with inhomogeneous Dirichlet boundary conditions*, Topol. Methods Nonlinear Anal. **30** (2007), no. 1, 1–36.

[42] Elaine C. M. Crooks, Edward N. Dancer, Danielle Hilhorst, Masayasu Mimura, and Hirokazu Ninomiya, *Spatial segregation limit of a competition-diffusion system with Dirichlet boundary conditions*, Nonlinear Anal. Real World Appl. **5** (2004), no. 4, 645–665.

[43] Edward N. Dancer, *On the existence and uniqueness of positive solutions for competing species models with diffusion*, Trans. Amer. Math. Soc. **326** (1991), no. 2, 829–859.

[44] Edward N. Dancer, *On the principal eigenvalue of linear cooperating elliptic systems with small diffusion*, J. Evol. Equ. **9** (2009), no. 3, 419–428.

[45] Edward N. Dancer and Yi H. Du, *Competing species equations with diffusion, large interactions, and jumping nonlinearities*, J. Differential Equations **114** (1994), no. 2, 434–475.

[46] Edward N. Dancer and Zong M. Guo, *Some remarks on the stability of sign changing solutions*, Tohoku Math. J. (2) **47** (1995), no. 2, 199–225.

[47] Edward N. Dancer, Danielle Hilhorst, Masayasu Mimura, and Lambertus A. Peletier, *Spatial segregation limit of a competition-diffusion system*, European J. Appl. Math. **10** (1999), no. 2, 97–115.

[48] Edward N. Dancer, Kelei Wang, and Zhitao Zhang, *Uniform Hölder estimate for singularly perturbed parabolic systems of Bose–Einstein condensates and competing species*, Journal of Differential Equations **251** (2011), no. 10, 2737–2769.

[49] Edward N. Dancer, Kelei Wang, and Zhitao Zhang, *Dynamics of strongly competing systems with many species*, Trans. Amer. Math. Soc. **364** (2012), no. 2, 961–1005.

[50] Edward N. Dancer, Kelei Wang, and Zhitao Zhang, *The limit equation for the Gross–Pitaevskii equations and S. Terracini's conjecture*, Journal of Functional Analysis **262** (2012), no. 3, 1087–1131.

[51] Edward N. Dancer, Kelei Wang, and Zhitao Zhang, *Addendum to "The limit equation for the Gross–Pitaevskii equations and S. Terracini's conjecture" [J. Funct. Anal. 262 (3) (2012) 1087–1131] [MR2863857]*, J. Funct. Anal. **264** (2013), no. 4, 1125–1129.

[52] Edward N. Dancer and Zhitao Zhang, *Dynamics of Lotka–Volterra competition systems with large interaction*, J. Differential Equations **182** (2002), no. 2, 470–489.

[53] Djairo G. de Figueiredo, *Monotonicity and symmetry of solutions of elliptic systems in general domains*, NoDEA Nonlinear Differential Equations Appl. **1** (1994), no. 2, 119–123.

[54] Djairo G. de Figueiredo and Enzo Mitidieri, *Maximum principles for linear elliptic systems*, Rend. Istit. Mat. Univ. Trieste **22** (1990), no. 1-2, 36–66.

[55] Emmanuele DiBenedetto, *Degenerate parabolic equations*, Springer-Verlag, New York, 1993.

[56] Weiwei Ding, François Hamel, and Xiao-Qiang Zhao, *Transition fronts for periodic bistable reaction-diffusion equations*, Calc. Var. Partial Differential Equations **54** (2015), no. 3, 2517–2551.

[57] Weiwei Ding, François Hamel, and Xiao-Qiang Zhao, *Bistable pulsating fronts for reaction-diffusion equations in a periodic habitat*, Indiana Univ. Math. J. **66** (2017), no. 4, 1189–1265.

[58] Jack Dockery, Vivian Hutson, Konstantin Mischaikow, and Mark Pernarowski, *The evolution of slow dispersal rates: a reaction diffusion model*, J. Math. Biol. **37** (1998), no. 1, 61–83.

[59] Yihong Du and Zhigui Lin, *Spreading-vanishing dichotomy in the diffusive logistic model with a free boundary*, SIAM Journal on Mathematical Analysis **42** (2010), no. 1, 377–405.

[60] Yihong Du and Zhigui Lin, *Erratum: Spreading-vanishing dichotomy in the diffusive logistic model with a free boundary*, SIAM Journal on Mathematical Analysis **45** (2013), no. 3, 1995–1996.

[61] Arnaud Ducrot, Thomas Giletti, and Hiroshi Matano, *Spreading speeds for multidimensional reaction-diffusion systems of the prey-predator type*, in preparation.

[62] Arnaud Ducrot, Thomas Giletti, and Hiroshi Matano, *Existence and convergence to a propagating terrace in one-dimensional reaction-diffusion equations*, Trans. Amer. Math. Soc. **366** (2014), no. 10, 5541–5566.

[63] Steven R. Dunbar, *Travelling wave solutions of diffusive Lotka–Volterra equations*, J. Math. Biol. **17** (1983), no. 1, 11–32.

[64] Steven R. Dunbar, *Traveling wave solutions of diffusive Lotka–Volterra equations: a heteroclinic connection in* $\mathbb{R}^4$, Trans. Amer. Math. Soc. **286** (1984), no. 2, 557–594.

[65] Elizabeth C. Elliott and Stephen J. Cornell, *Dispersal polymorphism and the speed of biological invasions*, PLOS ONE **7** (2012), no. 7, 1–10.

[66] Lawrence C. Evans, *A chemical diffusion-reaction free boundary problem*, Nonlinear Anal. **6** (1982), no. 5, 455–466.

[67] Lawrence C. Evans, *Partial differential equations*, second ed., Graduate Studies in Mathematics, vol. 19, American Mathematical Society, Providence, RI, 2010.

[68] Jian Fang, Xiao Yu, and Xiao-Qiang Zhao, *Traveling waves and spreading speeds for time-space periodic monotone systems*, ArXiv e-prints (2015).

[69] Jian Fang and Xiao-Qiang Zhao, *Bistable traveling waves for monotone semiflows with applications*, J. Eur. Math. Soc. (JEMS) **17** (2015), no. 9, 2243–2288.

[70] Grégory Faye and Matt Holzer, *Bifurcation to locked fronts in two component reaction-diffusion systems*, ArXiv e-prints (2017).

[71] Paul C. Fife and J. B. McLeod, *The approach of solutions of nonlinear diffusion equations to travelling front solutions*, Archive for Rational Mechanics and Analysis **65** (1977), no. 4, 335–361.

[72] Ronald Aylmer Fisher, *The wave of advance of advantageous genes*, Annals of eugenics **7** (1937), no. 4, 355–369.

[73] Juraj Földes and Peter Poláčik, *On cooperative parabolic systems: Harnack inequalities and asymptotic symmetry*, Discrete Contin. Dyn. Syst. **25** (2009), no. 1, 133–157.

[74] Mark I. Freidlin, *Functional integration and partial differential equations*, Annals of Mathematics Studies, vol. 109, Princeton University Press, Princeton, NJ, 1985.

[75] Sheng-Chen Fu and Je-Chiang Tsai, *Wave propagation in predator-prey systems*, Nonlinearity **28** (2015), no. 12, 4389–4423.

[76] Jacques E. Furter and Julián López-Gómez, *On the existence and uniqueness of coexistence states for the Lotka–Volterra competition model with diffusion and spatially dependent coefficients*, Nonlinear Anal. **25** (1995), no. 4, 363–398.

[77] Robert A. Gardner, *Existence and stability of travelling wave solutions of competition models: a degree theoretic approach*, J. Differential Equations **44** (1982), no. 3, 343–364.

[78] Jimmy Garnier, Thomas Giletti, François Hamel, and Lionel Roques, *Inside dynamics of pulled and pushed fronts*, Journal de Mathématiques Pures et Appliquées **98** (2012), no. 4, 428–449.

[79] Jürgen Gärtner and Mark I. Freidlin, *The propagation of concentration waves in periodic and random media*, Dokl. Akad. Nauk SSSR **249** (1979), no. 3, 521–525.

[80] David Gilbarg and Neil S. Trudinger, *Elliptic partial differential equations of second order*, Classics in Mathematics, Springer-Verlag, 2001.

[81] Michael E. Gilpin and Francisco J. Ayala, *Global models of growth and competition*, Proceedings of the National Academy of Sciences **70** (1973), no. 12, 3590–3593.

[82] Quentin Griette and Gaël Raoul, *Existence and qualitative properties of travelling waves for an epidemiological model with mutations*, J. Differential Equations **260** (2016), no. 10, 7115–7151.

[83] Jong-Shenq Guo and Ying-Chih Lin, *The sign of the wave speed for the Lotka–Volterra competition-diffusion system*, Commun. Pure Appl. Anal. **12** (2013), no. 5, 2083–2090.

[84] Jong-Shenq Guo and Chang-Hong Wu, *Recent developments on wave propagation in 2-species competition systems*, Discrete Contin. Dyn. Syst. Ser. B **17** (2012), no. 8, 2713–2724.

[85] Jong-Shenq Guo and Chang-Hong Wu, *Traveling wave front for a two-component lattice dynamical system arising in competition models*, J. Differential Equations **252** (2012), no. 8, 4357–4391.

[86] Morton E. Gurtin and Richard C. MacCamy, *On the diffusion of biological populations*, Math. Biosci. **33** (1977), no. 1-2, 35–49.

[87] Morton E. Gurtin and Richard C. MacCamy, *Diffusion models for age-structured populations*, Math. Biosci. **54** (1981), no. 1-2, 49–59.

[88] François Hamel, *Qualitative properties of monostable pulsating fronts: exponential decay and monotonicity*, J. Math. Pures Appl. (9) **89** (2008), no. 4, 355–399.

[89] François Hamel and Grégoire Nadin, *Spreading properties and complex dynamics for monostable reaction–diffusion equations*, Communications in Partial Differential Equations **37** (2012), 511–537.

[90] François Hamel, James Nolen, Jean-Michel Roquejoffre, and Lenya Ryzhik, *A short proof of the logarithmic Bramson correction in Fisher-KPP equations*, Netw. Heterog. Media **8** (2013), no. 1, 275–289.

[91] Li Jun Hei and Jian Hua Wu, *Existence and stability of positive solutions for an elliptic cooperative system*, Acta Math. Sin. (Engl. Ser.) **21** (2005), no. 5, 1113–1120.

[92] Yuzo Hosono, *Singular perturbation analysis of travelling waves for diffusive Lotka–Volterra competition models*, Numerical and applied mathematics, Part II (Paris, 1988), vol. 1, Baltzer, Basel, 1989, pp. 687–692.

[93] Wenzhang Huang, *Problem on minimum wave speed for a Lotka–Volterra reaction-diffusion competition model*, J. Dynam. Differential Equations **22** (2010), no. 2, 285–297.

[94] Wenzhang Huang and Maoan Han, *Non-linear determinacy of minimum wave speed for a Lotka–Volterra competition model*, J. Differential Equations **251** (2011), no. 6, 1549–1561.

[95] Li-Chang Hung and Chiun-Chuan Chen, *A maximum principle for diffusive Lotka-Volterra systems of two competing species*, ArXiv e-prints (2015).

[96] Vivian Hutson, Yuan Lou, and Konstantin Mischaikow, *Spatial heterogeneity of resources versus Lotka–Volterra dynamics*, J. Differential Equations **185** (2002), no. 1, 97–136.

[97] Masato Iida, Roger Lui, and Hirokazu Ninomiya, *Stacked fronts for cooperative systems with equal diffusion coefficients*, SIAM J. Math. Anal. **43** (2011), no. 3, 1369–1389.

[98] Giorgio Israel, *La mathématisation du réel: essai sur la modélisation mathématique*, Éd. du Seuil, 1996.

[99] Robert Jensen, *Smoothness of the free boundary in the Stephan problem with supercooled water*, Illinois J. Math. **22** (1978), no. 4, 623–629.

[100] Yukio Kan-on, *Parameter dependence of propagation speed of travelling waves for competition-diffusion equations*, SIAM J. Math. Anal. **26** (1995), no. 2, 340–363.

[101] Yukio Kan-on, *Fisher wave fronts for the Lotka–Volterra competition model with diffusion*, Nonlinear Anal. **28** (1997), no. 1, 145–164.

[102] Kazuo Kishimoto, *The diffusive Lotka–Volterra system with three species can have a stable non-constant equilibrium solution*, Journal of Mathematical Biology **16** (1982), no. 1, 103–112.

[103] Kazuo Kishimoto and Hans F. Weinberger, *The spatial homogeneity of stable equilibria of some reaction-diffusion systems on convex domains*, J. Differential Equations **58** (1985), no. 1, 15–21.

[104] Andrei N. Kolmogorov, I. G. Petrovsky, and N. S. Piskunov, *Étude de l'équation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique*, Bulletin Université d'État à Moscou **1** (1937), 1–25.

[105] King-Yeung Lam and Yuan Lou, *Asymptotic behavior of the principal eigenvalue for cooperative elliptic systems and applications*, J. Dynam. Differential Equations **28** (2016), no. 1, 29–48.

[106] Hélène Leman, Sylvie Méléard, and Sepideh Mirrahimi, *Influence of a spatial structure on the long time behavior of a competitive Lotka–Volterra type system*, Discrete Contin. Dyn. Syst. Ser. B **20** (2015), no. 2, 469–493.

[107] Jean Leray and Jules Schauder, *Topologie et équations fonctionnelles*, Ann. Sci. École Norm. Sup. (3) **51** (1934), 45–78.

[108] Mark A. Lewis, Bingtuan Li, and Hans F. Weinberger, *Spreading speed and linear determinacy for two-species competition models*, J. Math. Biol. **45** (2002), no. 3, 219–233.

[109] Bingtuan Li, *Multiple invasion speeds in a two-species integro-difference competition model*, J. Math. Biol. **76** (2018), no. 7, 1975–2009.

[110] Bingtuan Li, Hans F. Weinberger, and Mark A. Lewis, *Spreading speeds as slowest wave speeds for cooperative systems*, Math. Biosci. **196** (2005), no. 1, 82–98.

[111] Gary M. Lieberman, *Second order parabolic differential equations*, World Scientific Publishing Co., Inc., River Edge, NJ, 1996.

[112] Guo Lin and Wan-Tong Li, *Asymptotic spreading of competition diffusion systems: the role of interspecific competitions*, European J. Appl. Math. **23** (2012), no. 6, 669–689.

[113] Xavier Mora, *Semilinear parabolic problems define semiflows on C^k spaces*, Trans. Amer. Math. Soc. **278** (1983), no. 1, 21–55.

[114] Yoshihisa Morita and Koichi Tachibana, *An entire solution to the Lotka-Volterra competition-diffusion equations*, SIAM J. Math. Anal. **40** (2009), no. 6, 2217–2240.

[115] Aled Morris, Luca Börger, and Elaine C. M. Crooks, *Individual variability in dispersal and invasion speed*, ArXiv e-prints (2016).

[116] Grégoire Nadin, *The principal eigenvalue of a space–time periodic parabolic operator*, Annali di Matematica Pura ed Applicata **188** (2009), no. 2, 269–295.

[117] Grégoire Nadin, *Traveling fronts in space-time periodic media*, J. Math. Pures Appl. (9) **92** (2009), no. 3, 232–262.

[118] Grégoire Nadin, *Some dependence results between the spreading speed and the coefficients of the space-time periodic Fisher-KPP equation*, European J. Appl. Math. **22** (2011), no. 2, 169–185.

[119] Grégoire Nadin, Benoît Perthame, and Min Tang, *Can a traveling wave connect two unstable states?: The case of the nonlocal Fisher equation*, C. R. Math. Acad. Sci. Paris **349** (2011), no. 9-10, 553–557.

[120] Kimie Nakashima and Tohru Wakasa, *Generation of interfaces for Lotka–Volterra competition–diffusion system with large interaction rates*, Journal of Differential Equations **235** (2007), no. 2, 586–608.

[121] James Nolen, Matthew Rudd, and Jack Xin, *Existence of KPP fronts in spatially-temporally periodic advection and variational principle for propagation speeds*, Dyn. Partial Differ. Equ. **2** (2005), no. 1, 1–24.

[122] Roger D. Nussbaum, *Convexity and log convexity for the spectral radius*, Linear Algebra Appl. **73** (1986), 59–122.

[123] Akira Okubo, Philip K. Maini, Mark H. Williamson, and James D. Murray, *On the spatial spread of the grey squirrel in Britain*, Proceedings of the Royal Society of London B: Biological Sciences **238** (1989), no. 1291, 113–125.

[124] Chia-Ven Pao, *Coexistence and stability of a competition-diffusion system in population dynamics*, J. Math. Anal. Appl. **83** (1981), no. 1, 54–76.

[125] Chia-Ven Pao, *Nonlinear parabolic and elliptic equations*, Plenum Press, New York, 1992.

[126] Peter Poláčik, *Propagating terraces and the dynamics of front-like solutions of reaction-diffusion equations on* $\mathbb{R}$, Mem. Amer. Math. Soc., to appear (2016).

[127] Jonathan R. Potts and Sergei V. Petrovskii, *Fortune favours the brave: Movement responses shape demographic dynamics in strongly competing populations*, Journal of Theoretical Biology **420** (2017), 190–199.

[128] Céline Prévost, *Applications of partial differential equations and their numerical simulations of population dynamics*, Ph.D. thesis, PhD Thesis, University of Orleans, 2004.

[129] Murray H. Protter and Hans F. Weinberger, *Maximum principles in differential equations*, Springer-Verlag, 1984.

[130] Emmanuel Risler, *Competition between stable equilibria in reaction-diffusion systems: the influence of mobility on dominance*, ArXiv e-prints (2017).

[131] Lionel Roques, Yuzo Hosono, Olivier Bonnefon, and Thomas Boivin, *The effect of competition on the neutral intraspecific diversity of invasive species*, J. Math. Biol. **71** (2015), no. 2, 465–489.

[132] David H. Sattinger, *Monotone methods in nonlinear elliptic and parabolic boundary value problems*, Indiana Univ. Math. J. **21** (1971/72), 979–1000.

[133] Nanako Shigesada and Kohkichi Kawasaki, *Biological invasions: theory and practice*, Oxford University Press, UK, 1997.

[134] John Gordon Skellam, *Random dispersal in theoretical populations*, Biometrika (1951), 196–218.

[135] Nicola Soave and Alessandro Zilio, *Uniform bounds for strongly competing systems: the optimal Lipschitz case*, Arch. Ration. Mech. Anal. **218** (2015), no. 2, 647–697.

[136] Guido Sweers, *Strong positivity in $C(\overline{\Omega})$ for elliptic systems*, Math. Z. **209** (1992), no. 2, 251–271.

[137] Min Ming Tang and Paul C. Fife, *Propagating fronts for competing species equations with diffusion*, Arch. Rational Mech. Anal. **73** (1980), no. 1, 69–77.

[138] Alan M. Turing, *The chemical basis of morphogenesis*, Philos. Trans. Roy. Soc. London Ser. B **237** (1952), no. 641, 37–72.

[139] Aizik I. Volpert, Vitaly A. Volpert, and Vladimir A. Volpert, *Traveling wave solutions of parabolic systems*, Translations of Mathematical Monographs, vol. 140, American Mathematical Society, Providence, RI, 1994, Translated from the Russian manuscript by James F. Heyda.

[140] Haiyan Wang, *Spreading speeds and traveling waves for non-cooperative reaction-diffusion systems*, J. Nonlinear Sci. **21** (2011), no. 5, 747–783.

[141] Hans F. Weinberger, *On spreading speeds and traveling waves for growth and migration models in a periodic habitat*, J. Math. Biol. **45** (2002), no. 6, 511–548.

[142] Hans F. Weinberger, Mark A. Lewis, and Bingtuan Li, *Analysis of linear determinacy for spread in cooperative models*, J. Math. Biol. **45** (2002), no. 3, 183–218.

[143] Jack X. Xin, *Existence and nonexistence of traveling waves and reaction-diffusion front propagation in periodic media*, J. Statist. Phys. **73** (1993), no. 5-6, 893–926.

[144] Xue Xin, *Existence and stability of traveling waves in periodic media governed by a bistable nonlinearity*, J. Dynam. Differential Equations **3** (1991), no. 4, 541–573.

[145] Jian Yang and Bendong Lou, *Traveling wave solutions of competitive models with free boundaries*, Discrete Contin. Dyn. Syst. Ser. B **19** (2014), no. 3, 817–826.

[146] Xiao Yu and Xiao-Qiang Zhao, *Propagation phenomena for a reaction–advection–diffusion competition model in a periodic habitat*, Journal of Dynamics and Differential Equations (2015), 1–26.

[147] Sergey V. Zelik, *Attractors of reaction-diffusion systems in unbounded domains and their spatial complexity*, Comm. Pure Appl. Math. **56** (2003), no. 5, 584–637.

[148] Andrej Zlatoš, *Existence and non-existence of transition fronts for bistable and ignition reactions*, Ann. Inst. H. Poincaré Anal. Non Linéaire **34** (2017), no. 7, 1687–1705.