PROPAGATION PHENOMENA FOR A NONLOCAL REACTION-DIFFUSION MODEL WITH BOUNDED PHENOTYPIC TRAITS

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Abstract. In this paper we study the stability of cylinder front waves and propagation of solution for a nonlocal Fisher type model describing the segregation of a population with nonlocal competition among bounded and continuous phenotypic traits. By applying spectral analysis and separation of variables we first prove the spectral and local exponential stability of cylinder waves with noncritical speeds in some exponentially weighted spaces. By applying detailed analysis with spectral expansion and special sub-supper solution construction, we further prove the uniform boundedness of solution and global asymptotic stability of waves for more general nonnegative initial value, and prove that the spreading speed and asymptotic behavior of solution are determined by the decay rate of initial value, which also extends some classical results on the stability of planar waves for Fisher-KPP equation to the nonlocal Fisher model in multi-dimensional cylinder case.

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1. Introduction and statement of main results

To investigate the intra-specific competition among multiple phenotypes within a single population, the following non-local reaction-diffusion model was proposed in [17],

$$\begin{cases} \partial_t u(t,x,y) - d_x \triangle_x u(t,x,y) - d_y \triangle_y u(t,x,y) \\ = [1-\alpha g(y-\theta) - \int_\Omega K(x,y,y')u(t,x,y')\mathrm{d}y']u(t,x,y), \ (t,x,y) \in \mathbb{R}^+ \times \mathbb{R} \times \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0, \quad (t,x,y) \in \mathbb{R}^+ \times \mathbb{R} \times \partial\Omega, \\ u(0,x,y) = u_0(x,y), \quad (x,y) \in \mathbb{R} \times \Omega. \end{cases} \tag{1.1}$$

Here $u(t,x,y)$ represents the density of a population that is structured by a continuous spatial variable $x \in \mathbb{R}$ and continuous bounded phenotypical traits $\vec{y} \in \Omega \subset \mathbb{R}^n$, with Ω the set of all possible traits. The traits could be, for instance, rate of food intake or age at maturity. The terms $d_x \triangle_x u$ and $d_y \triangle_y u$ measure the spatial diffusion

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and mutations respectively. The nonlocal term $\int_\Omega K(x,y,y')u(t,x,y')\mathrm{d}y'$ indicates that the intra-specific competition occurs among all the individuals at each location x. The birth rate of populations is given by the fitness function $1-\alpha g(y-\theta)$ where g is positive except $g(0)=0$, this assumption takes into account the impact of natural selection on population survival. Here α is a parameter that quantifies the intensity of selection towards the optimal value θ. More detailed information about the biological background of the nonlocal model (1.1) can be referred to [16, 23, 26].

Over the past decade, the propagation phenomena arising from the model (1.1) have attracted tremendous attention among mathematicians. For unbounded domains, wave propagation in the form of planar waves and cylinder waves are observed. A travelling front solution (or a cylinder front solution) of equation (1.1) is a solution $u(t,x,y)$ in form of $\phi(x-ct,y)$ which connects zero to a non-trivial state with a constant speed $c\in\mathbb{R}$ and $\phi(z,y)$ is monotone in z for each $y\in\bar{\Omega}$. For the nonlocal model (1.1) with the simplified kernel $K(x,y,y')=K(y')$ and $\theta=0$ in the whole space $(x,y)\in\mathbb{R}\times\mathbb{R}^n$ or with bounded trait $y\in\Omega$, by applying spectral expansion (or separation of variables method), H. Berestycki et al. [4] obtained the existence and uniqueness of cylinder front solutions $\phi_c(x-ct,y)$ of (1.1) for $c\geq c^*$, and $\phi_c(x-ct,y)$ must be in the form of $V_c(x-ct)\phi_0(y)$. Under some additional assumptions on $K(y)$ for $y\in\mathbb{R}^n$, in [4] it is also proved that the minimal speed c^* is the spreading speed of the solution with compact supported initial value.

For the nonlocal model (1.1) in whole space $(x,y)\in\mathbb{R}\times\mathbb{R}^n$ with more general kernel $K(x,y,y')$ and $\theta(x)=bx$, M. Alfaro et al. [1] proved the existence of cylinder waves $\phi_c(x-ct,y)$ by employing Harnack's inequality and topological fixed-point argument. Subsequently, accelerating invasions have been analysed in [24] if the initial value displays a heavy tail in the direction $y-bx=0$. M. Alfaro and G. Peltier [3] proved the existence of steady-state solutions and pulsating fronts for the case when θ is periodic in x. For the model (1.1) in moving environment with $\theta=b\cdot(x-c_mt)$, M. Alfaro et al. [2] investigated the existence of waves and the spreading speed of solution. For the nonlocal model (1.1) in bounded Ω with constant kernel $K\equiv 1$, by applying Hamilton-Jacobi approach, E. Bouin and S. Mirrahimi [9] investigated the asymptotic spreading speeds of the solution and the asymptotic behavior of $u(t,x,y)$ or $\int_\Omega u(t,x,y)dy$.

When the spatial diffusion rate of a population varies (see [27][31]) and is measured by the trait variable y such as the leg length of cane toads, O. Bénichou et al. [6] proposed the following biological diffusion model

$$\partial_t u(t,x,y)-y\triangle_x u(t,x,y)-d\triangle_y u(t,x,y)=r[1-\int_\Omega u(t,x,y')\mathrm{d}y']u(t,x,y), \tag{1.2}$$

where Ω is a bounded or unbounded set in $[0,+\infty)$, d and r are positive constants.

There are some deep and interesting theoretical works on the wave propagation and spreading speed of solution for the model (1.2) when the traits are bounded. The investigation on the spreading speed of the solutions to model (1.2) with bounded Ω started with a Hamilton-Jacobi framework that was formally developed in [7] and rigorously carried out in [32]. By applying the Leray-Schauder degree argument similar to [1], the existence of traveling waves with a minimal speed to model (1.2) with bounded Ω was obtained in [8]. Subsequently, E. Bouin et al. [11] proved that the spreading speed of solution with compact supported initial value is the minimal wave speed with the Bramson's logarithmic delay.

For the model (1.2) with unbounded set $\Omega = (0,\infty)$, N. Berestycki et al. [5] applied probabilistic techniques and E. Bouin et al. [10] used PDE method to prove that the spreading speed of the solution with compact supported initial value is unbounded, and the associated population front travels superlinearly in time (in order of $t^{3/2}$), see also [15] for more detailed estimates on the accelerated propagation.

Another related nonlocal Fisher model is in the form of $u_t = u_{xx} + (1 - \int_{\mathbb{R}} \phi(x-y)u(t,y)\mathrm{d}y)u$, where x is a spatial variable, and the nonlocal competition characterize the long range intra-specific competition. Some recent work on the existence of traveling waves and the spreading speed of solution for this type of nonlocal Fisher equations can be refered to [12, 19, 21, 25] and the references therein.

It is worth mentioning that different from the investigation of classical reaction-diffusion models, the comparison principle can't be applied directly to the aforementioned models with nonlinear coupled nonlocal reaction terms, thus the sub/super solution method or some techniques such as sliding method or monotone iteration schemes can't be applied directly to the nonlinear model, which leads to additional difficulties in establishing sharp estimates on the bound of solution in time and in determining the asymptotic behavior of solution in time with more general initial value, and as far as we know even for the simplest nonlocal model (1.1) with bounded Ω there are no theoretical results on the stability of waves or the asymptotic behavior of solution with more general initial value except the case when the initial value has compact support.

This paper focuses on the non-local reaction-diffusion model (1.1) in the cylinder domain $\mathbb{R}\times\Omega$, where Ω is bounded and $K(x,y,y') = K(y')$. Without loss of generality, we choose $d_x = d_y = 1$ (after re-scaling of x and y) and we recast (1.1) as follows

$$\begin{cases} \partial_t u(t,x,y) - \triangle_{x,y} u(t,x,y) & \\ = \left[1 - g(y) - \int_\Omega K(y')u(t,x,y')\mathrm{d}y'\right] u(t,x,y), & (t,x,y)\in\mathbb{R}^+\times\mathbb{R}\times\Omega, \\ \frac{\partial u}{\partial \nu} = 0, & (t,x,y)\in\mathbb{R}^+\times\mathbb{R}\times\partial\Omega, \\ u(0,x,y) = u_0(x,y), & (x,y)\in\mathbb{R}\times\Omega. \end{cases} \tag{1.3}$$

Next, we discuss the assumptions on K and g. The function $g(y)$ is bounded and measurable (and can be sign-changing). Furthermore, let $\{\lambda_j\}_{j=0}^{+\infty}$ denote all the eigenvalues of the operator $-\triangle_y + g(y)$ under homogeneous Neumann boundary condition on $\partial\Omega$, with $\lambda_0 < \lambda_1 \le \lambda_2 \le \cdots$. It is well known that the fist eigenvalue λ_0 is simple and corresponds to a positive eigenfunction $\psi_0(y)$, and denote $\{\psi_j(y)\}_{j=0}^{+\infty}$ be the sequence of the eigenfunctions which forms an orthonormal basis of $L^2(\Omega)$, i.e. $\int_\Omega \psi_j^2(y)\mathrm{d}y = 1$, and $\int_\Omega \psi_i(y)\psi_j(y)\mathrm{d}y = 0$ for $i,j \ge 0$ and $i \ne j$.

In this paper the assumptions of K and g can be summarized as follows

(H1): $g \in \mathrm{L}_\infty(\Omega), \quad \lambda_0 < 1, \quad K \in \mathrm{L}_2(\Omega), \quad K(y) \ge 0, \text{ and } K(y) \not\equiv 0.$

It is easy to check that for any $c \ge 2\sqrt{1-\lambda_0}$ the expression $V_c(x-ct)\psi_0(y)$ is a traveling front solution of (1.3), where $V_c(x-ct)$ is the planar front solution of following Fisher-KPP equation satisfying

$$\begin{cases} V_c''(\xi) + V_c'(\xi) + \left((1-\lambda_0) - V_c(\xi)\displaystyle\int_\Omega \psi_0(y)K(y)\mathrm{d}y\right) V_c(\xi) = 0, \ \ \xi\in\mathbb{R}, \\ V_c(-\infty) = \mu_0, \ V_c(+\infty) = 0, \end{cases} \tag{1.4}$$

with $\mu_0 = (1-\lambda_0)\left(\int_\Omega K(y)\psi_0(y)\mathrm{d}y\right)^{-1} > 0$.

By applying the argument based on separation of variables and detailed asymptotic estimates, it is also proved in [4] that, under the assumption of **(H1)**, the problem (1.3) has a positive and bounded cylinder front solution $\phi_c(z,y)$ $(z = x-ct)$ with ϕ_c decreasing in z if and only if $c \geq 2\sqrt{1-\lambda_0}$, and the cylinder front $\phi_c(z,y)$ is unique (neglecting the shift in z) and thus $\phi_c(x-ct,y) = V_c(x-ct)\psi_0(y)$.

In this paper, we study the local and global asymptotic stability of the cylinder waves $V(x-ct)\psi_0(y)$ to model (1.3) in various settings.

For the remainder of this paper, we further assume $\mu_0 = 1$ without loss of generality, i.e. $\int_\Omega \psi_0(y)K(y)\mathrm{d}y = 1-\lambda_0$. This is possible by replacing $u(t,x,y)$ by $\frac{1}{\mu_0}u(t,x,y)$ (and accordingly for the cylinder wave) for the original model (1.3). Then the re-scaled $V_c(\xi)$ satisfies

$$\begin{cases} V_c''(\xi) + V_c'(\xi) + (1-\lambda_0)(1-V_c(\xi))V_c(\xi) = 0, & \xi \in \mathbb{R}, \\ V_c(-\infty) = 1, \ V_c(+\infty) = 0. \end{cases} \tag{1.5}$$

By applying detailed spectral analysis and the classical stability theories of traveling waves based on analytic semigroup theories, in the following section we shall prove that all the cylinder waves $V_c(x-ct)\psi_0(y)$ with noncritical speeds are spectrally stable and nonlinearly exponentially stable in some appropriate spaces. Our results on nonlinear exponential stability of cylinder waves are stated as follows.

Theorem 1. *Under the assumption of* **(H1)**, *for each* $c > c^* = 2\sqrt{1-\lambda_0}$ *and* $a > 0$ *satisfying*

$$0 < \frac{-c-\sqrt{c^2-4(1-\lambda_0)}}{2} < a < \frac{-c+\sqrt{c^2-4(1-\lambda_0)}}{2},$$

the cylinder traveling front $V_c(x-ct)\psi_0(y)$ *of* (1.3) *is locally exponentially stable in the following exponentially weighted space*

$$X_a = \{u(x,y): \ w_a(x)u(x,y) \in X, \ \ \|u\|_{X_a} = \|w_a u\|_X\}, \ \ X = \mathrm{C}_{\mathrm{unif}}(\mathbb{R}\times\overline{\Omega}),$$

where $w_a(z) = 1 + \mathrm{e}^{az}$. *In other words, if the initial perturbation* $\|u_0(x,y) - V_c(x)\psi_0(y)\|_{X_a}$ *is sufficiently small, then there exist positive constants* M *and* σ_c *such that the equation* (1.3) *admits a unique global solution* $u(t,z+ct,y)$ *satisfying*

$$\|u(t,z+ct,y) - V_c(z)\psi_0(y))\|_{X_a} \leqslant M\mathrm{e}^{-\sigma_c t}, \forall t > 0.$$

In this paper we also investigate the uniform boundedness and the long-time behavior of the solution for the nonlocal parabolic equation (1.3) with more general nonnegative initial value, where the nonnegative initial value need not to be a small perturbation of a cylinder wave. In Section 3, under the assumption of **(H1)**, by applying spectral expansion $u(t,x,y) = \sum\limits_{j=0}^{\infty} v_j(t,x)\psi_j(y)$, we investigate the related Cauchy problem of the coupled system of $v_j(t,x)$, and by detailed spectral analysis and applying comparison principle to some auxiliary linear evolutional models, we can prove that the boundedness and long time behavior of the solution $u(t,x,y)$ to nonlinear model (1.3) are determined by that of $v_0(t,x) = \int_\Omega u(t,x,y)\psi_0(y)\mathrm{d}y$, then by investigating the Cauchy problem of $v_0(t,x)$ in one dimensional space, it can be proved that $\|v_0(t,\cdot)\|_{\mathrm{L}_\infty(\mathbb{R})}$ and $\|u(t,\cdot)\|_{\mathrm{L}_\infty(\mathbb{R}\times\Omega)}$ are uniformly bounded in time for any nonnegative initial value, which can be stated as follows.

Theorem 2. *There exist positive constants* δ_0, M_0 *and* M, *such that for any given nonzero and nonnegative bounded initial value* $u_0(x,y) \in \mathrm{L}_\infty(\mathbb{R}\times\Omega)$, *there exist* *positive global solution $u(t,x,y)$ of (1.3), which is also uniformly bounded in time and satisfies*

$$\|u(t,x,y) - v_0(t,x)\psi_0(y)\|_{L_\infty(\mathbb{R}\times\Omega)} \leqslant M e^{-\delta_0 t}(\|u_0\|_{L_\infty(\mathbb{R}\times\Omega)} + 1), \quad \forall t > 0, x \in \mathbb{R}, \tag{1.6}$$

and

$$0 < v_0(t,x) \leqslant M_0(\|u_0\|_{L_\infty(\mathbb{R}\times\Omega)} + 1), \quad \forall t > 0, x \in \mathbb{R}, \tag{1.7}$$

where $v_0(t,x)$ satisfies the following initial value problem

$$\begin{cases} \frac{\partial}{\partial t} v_0 - \frac{\partial^2}{\partial^2 x} v_0 = (1-\lambda_0)[(1 - v_0 - b_0(t,x)]v_0, & t > 0,\ x \in \mathbb{R}, \\ v_0(0,x) = \langle u_0(x,\cdot), \psi_0(\cdot)\rangle, & x \in \mathbb{R}, \end{cases} \tag{1.8}$$

with $b_0(t,x) = \frac{1}{(1-\lambda_0)}\int_\Omega K(y)(u(t,x,y) - v_0(t,x)\psi_0(y))\,dy$ decaying exponentially in time:

$$\sup_{x\in\mathbb{R}} |b_0(t,x)| \leqslant M e^{-\delta_0 t}(\|u_0\|_{L_\infty(\mathbb{R}\times\Omega)} + 1), \quad \forall t > 0. \tag{1.9}$$

In Section 4, we further investigate the global asymptotic stability of cylinder waves and the asymptotic behavior of the solution $u(t,x,y)$ in the x direction as $t \to \infty$, for more general nonnegative initial value which decay exponentially at only one end or with compact support. By virtue of Theorem 2, we focus on investigating the long time behavior of $v_0(t,x) = \langle u(t,x,\cdot), \psi_0(\cdot)\rangle$ as $t \to +\infty$, where $v_0(t,x)$ satisfies a nonlinear equation in one dimensional space, a projected PDE equation of $u(t,x,\cdot)$ after spectral expansion, which can be treated as a Fisher-KPP equation in one dimensional space $v_t - v_{xx} = (1-\lambda_0)v(1 - b_0(t,x) - v)$, with a nonlocal heterogeneous perturbation term

$$b_0 = \frac{1}{(1-\lambda_0)}\int_\Omega K(y)(u - v_0\psi_0(y))\,dy$$

such that $|b_0(t,x)| \leqslant M e^{-\delta_0 t}$ for any $t > 0,\ x \in \mathbb{R}$.

For the Cauchy problem of the classical Fisher-KPP model $u_t = \triangle_{x,y}u + u(1-u)$ in higher dimensional cylinder space or in one dimensional space, there are many literatures (see [13, 20, 22, 29, 30, 33] for some classical results) which reveal that the spreading speed and the long time behavior of the solution is determined by the asymptotic behavior of the initial value at two ends, especially the decaying rate of the initial value $u_0(x,y)$ at $x = +\infty$ determines the spreading speed of the solution in positive x direction. Recently for some Fisher type equation with some special types of heterogeneous resource term depending only on t or x (periodic in t or x) or $x - ct$, there are many interesting works on some new types wave phenomena induced by heterogeneity and long time behavior of the solution. However for the Fisher-KPP equation with more general heterogenous resource term $b(t,x)$ or with nonlocal competition term, as far as we know, there are fewer works on the stability of waves or long time behavior of the solution with general initial value, and it is not clear whether the spreading speed of the solution can still be determined by the decay rate of the initial value.

Now we state our main results on the asymptotic behavior of solution as follows.

Theorem 3. *(Global asymptotic stability of cylinder waves with more general initial value) Under the assumption of* **(H1)** *and let $\int_\Omega K(y)\psi_0(y)\,dy = 1 - \lambda_0$. For any* *nonnegative initial value* $u_0(x,y) \in \mathrm{L}_\infty(\mathbb{R}\times\Omega)$ *satisfying*

$$\lim_{x\to-\infty}\inf\int_\Omega u_0(x,y)\psi_0(y)\mathrm{d}y > 0 \ \ and \ \lim_{x\to\infty}\mathrm{e}^{\sigma x}\int_\Omega u_0(x,y)\psi_0(y)\mathrm{d}y = r > 0, \tag{1.10}$$

with $0<\sigma<\sqrt{1-\lambda_0}$, (1.3) *has a unique global solution* $u(t,x,y)$, *which satisfies*

$$\lim_{t\to\infty}\|u(t,z+ct,y) - V_c\left(z-\tfrac{1}{\sigma}\ln r\right)\psi_0(y)\|_{\mathrm{L}_\infty(\mathbb{R}\times\Omega)} = 0, \tag{1.11}$$

where $c=\sigma+\frac{\sqrt{1-\lambda_0}}{\sigma}\in(2\sqrt{1-\lambda_0},\infty)$, *and* $V_c(z)$ *is the unique planar wave solution of* (1.5) *satisfying* $\lim\limits_{x\to+\infty}\mathrm{e}^{\sigma z}V_c(z)=1$.

Theorem 4. *(Global exponential stability of cylinder waves in exponentially weighted space) Under the assumption of Theorem 3, if the initial value* $u_0(x,y)$ *satisfies, for some* $a>\sigma$ *with* $0<a-\sigma\ll 1$,

$$\int_\Omega u_0(x,y)\psi_0(y)dy \sim r\mathrm{e}^{-\sigma x}+O(\mathrm{e}^{-ax}),\ x\to+\infty,\ r>0,\ \ 0<\sigma<\sqrt{1-\lambda_0},$$

then there exist positive constants M, δ_a *and* $z_0>0$ *such that the problem* (1.3) *admits a unique global solution* $u(t,z+ct,y)$ *satisfying*

$$\|u(t,z+ct,y)-V_c(z-\tfrac{1}{\sigma}\ln r)\psi_0(y))\|_{X_a}\leqslant M\mathrm{e}^{-\delta_a t},\forall t>0.$$

Theorem 5. *Under the assumption of* **(H1)**. *If the initial value* $u_0(x,y)$ *is nonnegative and has compact support in the cylinder, then there exists two functions* $\xi_-(t)$ *and* $\xi_+(t)$ *such that the solution* $u(t,x,y)$ *of* (1.3) *satisfies*

$$\begin{aligned}\lim_{t\to+\infty}\|u(t,x,y)-V_{c^*}(x-\xi_+(t))\psi_0(y)\|_{\mathrm{L}_\infty(\mathbb{R}^+\times\Omega)}=0,\\ \lim_{t\to+\infty}\|u(t,x,y)-V_{c^*}(-x-\xi_-(t))\psi_0(y)\|_{\mathrm{L}_\infty(\mathbb{R}^-\times\Omega)}=0,\end{aligned}\tag{1.12}$$

with $c^*=2\sqrt{1-\lambda_0}$. *In addition, there exists a positive constant* C *such that*

$$|\xi_\pm(t)-2\sqrt{1-\lambda_0}t+\frac{3}{2}\ln t|\leqslant C,\ \ \text{for}\ \ t>>1.$$

This paper is organised as follows. In section 2, by apply spectral analysis we prove the spectral and local exponential stability of the cylinder waves with noncritical speeds to model (1.3) in some weighted spaces. In section 3, by combining the spectral expansion method and detailed asymptotic analysis with the sub-supper solution method, we prove the uniform boundedness of the solution to model (1.3) in time for any nonnegative initial value. In Section 4, we investigate the long time behavior of the solution with more general initial value decaying with some exponential rates at one end or with compact support, and prove Theorems 3-5.

2. Local exponential stability of cylinder waves in some weighted spaces

In this section, we investigate the spectral and local exponential stability of cylinder wave solution $\phi_c(x-ct,y)$ with $c>c^*$ for the model (1.3) in some appropriate spaces, where the cylinder wave solution $\phi_c(\xi,y)$ satisfies the following boundary value problem

$$\begin{cases} 0 = c\phi_c(\xi, y) + \triangle_{\xi,y}\phi_c(\xi, y) - g(y)\phi_c(\xi, y) + \left[1 - \int_\Omega K(y')\phi_c(\xi, y')\mathrm{d}y'\right]\phi_c(\xi, y), \\ \frac{\partial \phi_c}{\partial \nu} = 0, (\xi, y) \in \mathbb{R} \times \partial\Omega, \\ \lim_{\xi\to+\infty} \phi_c(\xi, \cdot) = 0, \ \liminf_{\xi\to-\infty} \phi_c(\xi, \cdot) > 0. \end{cases} \tag{2.1}$$

Let λ_0 be the principal eigenvalue of $-\Delta_y + g(y)$ in Ω under homogeneous Neumann boundary condition on $\partial\Omega$. It is proved in [4] that if $\lambda_0 < 1$ then for any $c \geq 2\sqrt{1-\lambda_0}$, (2.1) has a unique positive bounded solution with separate variable expression $\phi_c(\xi, y) = V_c(\xi)\psi_0(y)$, where under the assumption $\int_\Omega \psi_0(y)K(y)\mathrm{d}y = 1 - \lambda_0$ (after the re-scaling of $\phi_c(\xi, y)$), $V(\xi)$ satisfies

$$\begin{cases} V_c''(\xi) + cV_c'(\xi) + (1-\lambda_0)(1 - V_c(\xi))V_c(\xi) = 0, \ \ \xi \in \mathbb{R}, \\ V_c(-\infty) = 1, \ V_c(+\infty) = 0. \end{cases} \tag{2.2}$$

It is well known that for $c \geqslant 2\sqrt{1-\lambda_0}$, the planar wave solution $V_c(\xi)$ of (2.2) decays exponentially at both ends uniformly in $y \in \Omega$ and satisfies

$$\begin{cases} \text{if } c \geqslant c^*, \ V_c(\xi) - 1 \sim \mathrm{e}^{\mu^+\xi}, \text{ as } \xi \to -\infty, \\ \text{if } c > c^*, \ V_c(\xi) \sim \mathrm{e}^{-\sigma^-\xi}, \text{ as } \xi \to +\infty, \\ \text{if } c = c^*, \ V_c(\xi) \sim \xi\mathrm{e}^{-\sigma^+\xi}, \text{ as } \xi \to +\infty, \end{cases} \tag{2.3}$$

where

$$\mu^+ = \frac{-c + \sqrt{c^2 + 4(1-\lambda_0)}}{2} > 0, \ \ \sigma^\pm = \frac{c \pm \sqrt{c^2 - 4(1-\lambda_0)}}{2} > 0.$$

In moving coordinate (ξ, y, t) (with $\xi = x - ct$) the initial boundary value problem (1.3) can be rewritten as follows

$$\begin{cases} \partial_t u = \triangle_{\xi,y}u + c\partial_\xi u - g(y)u + \left[1 - \int_\Omega K(y')u(t, \xi, y')\mathrm{d}y'\right]u, & t > 0, (\xi, y) \in \mathbb{R} \times \Omega, \\ \frac{\partial u}{\partial \nu} = 0, & t > 0, \ \ (\xi, y) \in \partial\Sigma, \\ u(0, \xi, y) = u_0(\xi, y), & (\xi, y) \in \Sigma. \end{cases} \tag{2.4}$$

To prove the local asymptotic stability of the cylinder waves in some appropriate space, we first investigate the following linearized evolutional equation of (2.4) around the cylinder wave $\phi_c(\xi, y)$

$$\begin{aligned} \partial_t v = \triangle_{\xi,y}v + c\partial_\xi v - g(y)v + \left(1 - \int_\Omega K(y')\phi_c(\xi, y')\mathrm{d}y'\right)v - \phi_c\int_\Omega K(y')v(t, \xi, y')\mathrm{d}y' \\ \triangleq \mathcal{L}_c v. \end{aligned} \tag{2.5}$$

It is easy to check that the operator $\mathcal{L}_c$ generates an analytic semigroup in the Banach space $L_2(\Sigma)$, $\Sigma = \mathbb{R} \times \Omega$ with the domain $D(\mathcal{L}_c) = \mathrm{H}_v^2(\Sigma)$, and respectively in the Banach space $\mathrm{C}_{\mathrm{unif}}(\overline{\Sigma})$, with domain $D(\mathcal{L}_c) = X^2$ given by

$$X^2 = \left\{ u \in \mathrm{C}_{\mathrm{unif}}(\overline{\Sigma}) \bigcap (\bigcap_{q\geq 1} W^{2,q}_{\mathrm{loc}}(\Sigma)), \ \triangle_{x,y}u \in \mathrm{C}_{\mathrm{unif}}(\overline{\Sigma}), \text{ and } \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Sigma \right\}.$$

By applying the analytic semigroup theories and stability theories of traveling waves, to prove the local exponential stability/instability of cylinder waves in space $X = \mathrm{C_{unif}}(\overline{\Sigma})$ or $\mathrm{H}^k(\Sigma)$, it suffices to investigate the spectral distribution of the linear operator $\mathcal{L}_c$ in X or $\mathrm{H}^k(\Sigma)$.

For convenience of our investigation on the nonlinear local stability of the waves, in the following of this paper we choose the working space of $\mathcal{L}_c$ as $X = C_{\mathrm{unif}}(\overline{\Sigma})$, with domain $D(\mathcal{L}_c) = X^2$.

Let $\sigma(\mathcal{L}_c)$ be the spectral set of $\mathcal{L}_c$ in X, $\sigma_n(\mathcal{L}_c)$ the set consisting of the isolated eigenvalues of $\mathcal{L}_c$ with finite algebraic multiplicity and $\sigma_{ess}(\mathcal{L}_c) = \sigma(\mathcal{L}_c)\backslash\sigma_n(\mathcal{L}_c)$ the essential spectral set of $\mathcal{L}_c$.

2.1. **Location of $\sigma_{ess}(\mathcal{L}_c)$.** By applying the essential spectral theories to the elliptic operator $\mathcal{L}_c$ in $\mathrm{C_{unif}}(\overline{\Sigma})$ (see [28]) and $\mathrm{H}^k(\Sigma)$ (see [34]), it is known that the boundaries of the essential spectra of $\mathcal{L}_c$ are determined by the location of the spectra of the limiting operators $\mathcal{L}_c^{\pm}$ of $\mathcal{L}_c$ as $\xi \to \pm\infty$, with $\mathcal{L}_c^{\pm}$ defined by

$$\begin{aligned}&\mathcal{L}_c^{+}u \triangleq \triangle_{\xi,y}u + c\partial_\xi u - g(y)u + u, \quad u \in X^2,\\ &\mathcal{L}_c^{-}u \triangleq \triangle_{\xi,y}u + c\partial_\xi u - g(y)u - \psi_0(y)\textstyle\int_\Omega K(y')u(\xi,y')\mathrm{d}y' + \lambda_0 u, \quad u \in X^2,\end{aligned} \tag{2.6}$$

where $(\lambda_0, \psi_0(y))$ is defined in Section 1 and $\int_\Omega K(y')\psi_0(y')\mathrm{d}y' = 1 - \lambda_0$.

Without loss of generality, we investigate the essential spectral set of $\mathcal{L}_c$ in $X \cap L_2(\Sigma)$, after applying Fourier transform to $\mathcal{L}_c^-$ and $\mathcal{L}_c^+$ with respect to ξ, in the following we first investigate the location of eigenvalues of the corresponding operators $\widehat{\mathcal{L}_c^-}$ and $\widehat{\mathcal{L}_c^+}$ on bounded region Ω with parameter τ, i.e. the following eigenvalue problems

$$\begin{aligned}&\lambda^-(\tau)v(y) = \widehat{\mathcal{L}_c^-}v(y)\\ &\triangleq \triangle_y v(y) - g(y)v(y) + (-\tau^2 + ic\tau + \lambda_0)v(y) - (\textstyle\int_\Omega K(y')v(y')\mathrm{d}y')\psi_0(y),\end{aligned} \tag{2.7}$$

and

$$\lambda^+(\tau)v(y) = \widehat{\mathcal{L}_c^+}v(y) \triangleq \triangle_y v(y) - g(y)v(y) + (-\tau^2 + ic\tau + 1)v(y), \tag{2.8}$$

with eigenfunction $v(y) \in X^2 \cap \mathrm{H}^2_\nu(\Sigma)$.

For any given parameter $\tau \in \mathbb{R}$, let $\lambda^-(\tau)$ be an eigenvalue of (2.7) with an eigenfunction $v(y)$, note that we can represent the nonzero function $v(y)$ as

$$v(y) = \sum_{k=0}^{\infty} c_k\psi_k(y), \quad \text{with constant } c_{k_0} \neq 0, \text{ for some } k_0 \geq 0. \tag{2.9}$$

Substituting (2.9) into (2.7), we have

$$\begin{aligned}\lambda^-(\tau)\sum_{k=0}^{\infty} c_k\psi_k(y) = &\sum_{k=0}^{\infty} c_k(\triangle_y - g(y))\psi_k(y) + (-\tau^2 + ic\tau + \lambda_0)\sum_{k=0}^{\infty} c_k\psi_k(y)\\ &- \sum_{k=0}^{\infty} c_k\psi_0(y)\int_\Omega K(y')\psi_k(y')\mathrm{d}y'.\end{aligned} \tag{2.10}$$

For the case when there exists $k_0 \geq 1$ such that $c_{k_0} \neq 0$ in (2.9), multiplying (2.10) by $\psi_{k_0}(y)$ and integrating on Ω, it yields

$$\lambda^-(\tau) = -\lambda_{k_0} - \tau^2 + ic\tau + \lambda_0, \text{ for some } k_0 \geq 1. \tag{2.11}$$

For the remaining case when the eigenfunction $v(y) = \psi_0(y)$, multiplying (2.10) by $\psi_0(y)$ and integrating on Ω, we have

$$\lambda^-(\tau) = -\tau^2 + ic\tau - 1 + \lambda_0. \tag{2.12}$$

(2.11) and (2.12) imply that there exists $\delta_0 \geq \min\{1 - \lambda_0, \lambda_1 - \lambda_0\} > 0$ such that for any given $\tau \in \mathbb{R}$ all the eigenvalues of (2.7) denoted by $\lambda^-(\tau)$ satisfy $\lambda^-(\tau) \leq -\delta_0 < 0$. Thus

$$\sigma(\mathcal{L}_c^-) \subset \{\operatorname{Re}\lambda \leq -\delta_0 < 0\}. \tag{2.13}$$

It is easy to see that $\lambda^+(\tau)$ is an eigenvalue of the eigenvalue problem (2.8) with parameter τ, if and only if

$$-\tau^2 + ic\tau + 1 - \lambda^+(\tau) = \lambda_k, \text{ for some } k \geq 0,$$

thus

$$\sup\{\operatorname{Re}\lambda^+(\tau), \tau \in \mathbb{R}\} = 1 - \lambda_0 > 0, \tag{2.14}$$

if we choose $\tau = 0$ and eigenvalue $\lambda^+(0) = 1 - \lambda_0$ with eigenfunction $\psi_0(y)$.

The fact $\sigma(\mathcal{L}_c^+) \bigcap \{\operatorname{Re}\lambda > 0\} \neq \emptyset$ further means

$$\sigma_{\text{ess}}(\mathcal{L}_c) \bigcap \{\operatorname{Re}\lambda > 0\} \neq \emptyset,$$

which is also true when the working space of $\mathcal{L}_c$ is $\mathrm{C}_{\text{unif}}(\overline{\Sigma})$ or $L_2(\Sigma)$, thus for any $c \geq c^* = 2\sqrt{1 - \lambda_0}$ the cylinder waves $\phi_c(x - ct, y)$ are spectrally unstable and nonlinearly unstable in $\mathrm{C}_{\text{unif}}(\overline{\Sigma})$ and in $\mathrm{H}^k(\Sigma)$.

In the following we try to prove that the cylinder waves $V_c(x - ct)\psi_0(y)$ with noncritical speed $c > 2\sqrt{1 - \lambda_0}$ are spectrally stable and nonlinearly exponentially stable in some exponentially weighted spaces of X with an exponential weight near $\xi = +\infty$. Let $w_a(\xi) = 1 + \mathrm{e}^{a\xi}$, define the exponentially weighted space X_a by

$$X_a = \{u(\xi, y) : \ w_a(\xi)u(\xi, y) \in X, \ \ \|u\|_{X_a} = \|w_a u\|_X\}, \tag{2.15}$$

and we can define the related exponentially weighted space of X^2 similarly and denoted by X_a^2.

Define the operator $\mathcal{L}_{c,a} : X_a^2 \to X_a$ as the restriction of $\mathcal{L}_c$ on X_a^2, and defined $\tilde{\mathcal{L}}_{c,a} : X^2 \to X$ as $\tilde{\mathcal{L}}_{c,a}v(\xi, y) = w_a(\xi)\mathcal{L}_c(w_a^{-1}(\xi)v(\xi, y)$ for $v(\xi, y) \in X^2$, obviously

$$\sigma_{\text{ess}}(\tilde{\mathcal{L}}_{c,a}) = \sigma_{\text{ess}}(\mathcal{L}_{c,a}), \sigma_n(\tilde{\mathcal{L}}_{c,a}) = \sigma_n(\mathcal{L}_{c,a}),$$

and $\|(\lambda I - \tilde{\mathcal{L}}_{c,a})^{-1}\|_{X \to X} = \|(\lambda I - \mathcal{L}_{c,a})^{-1}\|_{X_a \to X_a}$.

For $a > 0$ it is easy to check that the limiting operator of $\tilde{\mathcal{L}}_{c,a}$ as $\xi \to -\infty$ is still $\mathcal{L}_c^-$, while the limiting operator of $\tilde{\mathcal{L}}_{c,a}$ as $\xi \to +\infty$ denoted by $\tilde{\mathcal{L}}_{c,a}^+$ has the following expression

$$\tilde{\mathcal{L}}_{c,a}^+ v = \triangle_{\xi,y} v + 2av_\xi + a^2 v + cv_\xi + cav - g(y)v + v, \ \ v \in X^2.$$

To obtain the location of $\sigma_{\text{ess}}(\mathcal{L}_{c,a})$, it remains to investigate the location of $\sigma(\tilde{\mathcal{L}}_{c,a}^+)$, by applying Fourier transform to $\tilde{\mathcal{L}}_{c,a}^+$ with respect to ξ, we investigate the following eigenvalue problem with parameter $\tau \in \mathbb{R}$

$$\lambda^+(\tau)v(y) = \widehat{\tilde{\mathcal{L}}_{c,a}^+} v \triangleq \Delta_y v(y) - g(y)v(y) + (-\tau^2 + ic\tau + 2ia\tau + a^2 - ca + 1)v(y), \tag{2.16}$$

with zero Neumann boundary condition on $\partial\Omega$.

Obviously for any given $\tau \in \mathbb{R}$, $\lambda^+(\tau)$ is an eigenvalue of (2.16) if and only if

$$\lambda^+(\tau) = -\tau^2 + ic\tau + 2ia\tau + a^2 - ca + 1 - \lambda_k, \ \text{ for some } k \geq 0. \tag{2.17}$$

For any given $c > 2\sqrt{1-\lambda_0}$, if we choose $a > 0$ satisfying

$$\frac{c-\sqrt{c^2-4(1-\lambda_0)}}{2} < a < \frac{c+\sqrt{c^2-4(1-\lambda_0)}}{2}, \tag{2.18}$$

then by (2.17) it follows that there exists a positive constant δ_+ depending only on a and c such that for any given $\tau \in \mathbb{R}$ it holds that

$$\operatorname{Re}\lambda^+(\tau) < -\delta_+ < 0,$$

which with the location of $\sigma(\mathcal{L}_c^-)$ in (2.13) guarantees that

$$\sup\{\text{Re }\sigma_{ess}(\mathcal{L}_{c,a})\} \leqslant -\delta < 0, \;\; \delta = \{\delta_+, \lambda_1 - \lambda_0, 1-\lambda_0\} > 0.$$

Thus we have the following spectral estimates.

Lemma 2.1. *For any given $c > c^*$ and $a > 0$ satisfying* (2.18)*, let $\mathcal{L}_{c,a}$ be the restriction of $\mathcal{L}_c$ on the weighted space X_a, with weight function defined by $w_a(x) = 1 + \mathrm{e}^{ax}$, there exists small enough $\delta > 0$ such that*

$$\sup\{\text{Re }\sigma_{ess}(\mathcal{L}_{c,a})\} \leqslant -\delta < 0. \tag{2.19}$$

2.2. **Location of isolated eigenvalues of $\mathcal{L}_{c,a}$.** By Lemma 2.1, to prove the spectral stability and the nonlinear exponential stability of cylinder wave $V_c(x-ct)\psi_0(y)$ with $c > c^*$ in the weighted space X_a, it remains to prove the non-existence of unstable eigenvalues of $\mathcal{L}_{c,a}$. For this purpose, in this subsection we investigate the location of eigenvalues of $\mathcal{L}_{c,a}$ in the range $\Omega_\delta = \{\lambda \in \mathbf{C} : \operatorname{Re}\lambda \geqslant -\delta/2\}$ with small enough $\delta > 0$ satisfying (2.19).

Consider the eigenvalue problem

$$\begin{aligned}\lambda u(\xi,y) &= \mathcal{L}_{c,a}u(\xi,y)\\ &= \triangle_{\xi,y}u + c\partial_\xi u - g(y)u + (1-V_c(\xi)(1-\lambda_0))\,u - \phi_0(y)V_c(\xi)\int_\Omega K(y')u(\xi,y')\mathrm{d}y'\end{aligned} \tag{2.20}$$

with the eigenvalue λ satisfying $\operatorname{Re}\lambda \geqslant -\delta/2$ and having an eigenfunction $u(\xi,y) \in X_a^2$.

We express the eigenfunction $u(\xi,y)$ of $\mathcal{L}_{c,a}$ in X_a^2 by spectral expansion

$$u(\xi,y) = \sum_{i=0}^{\infty} v_i(\xi)\psi_i(y) \tag{2.21}$$

with $\psi_i(y)$ defined as in Section 1.

Substituting (2.21) into (2.20), it is easy to check that if λ is an eigenvalue of (2.20) with $\operatorname{Re}\lambda \geqslant -\delta_c/2$, then there exists some $k \geq 0$ such that $v_k(\xi) \not\equiv 0$ in (2.21) and $(\lambda, v_k(\xi))$ must be an eigenpair of the following eigenvalue problem

$$v_k''(\xi) + cv_k'(\xi) + [1-\lambda_k - (1-\lambda_0)V_c(\xi)]v_k(\xi) = \lambda v_k(\xi), \;\text{ if } v_k \neq 0, \;\text{ for some } k \geq 1; \tag{2.22}$$

or

$$v_0''(\xi) + cv_0'(\xi) + [1-\lambda_0 - 2(1-\lambda_0)V_c(\xi)]v_0(\xi) = \lambda v_0(\xi), \tag{2.23}$$

if $v_0(\xi)\psi_0(y)$ is an eigenfunction of (2.20).

Theorem 2.1. *For any given $c > 2\sqrt{1-\lambda_0}$ and a satisfying* (2.18)*, let $\delta > 0$ be small enough chosen as in Lemma 2.1.*
(i) If λ is an eigenvalue of $\mathcal{L}_{c,a}$ with $\operatorname{Re}\lambda \geq -\delta/2$, then λ must be real and the eigenfunction must be in the form of $v_0(\xi)\psi_0(y)$.

(ii) There exists small enough $\delta_{c,a} > 0$ such that there is no eigenvalue of $\mathcal{L}_{c,a}$ with $\operatorname{Re}\lambda \geqslant -\delta_{c,a}$.

Proof. Let λ be an isolated eigenvalue of $\mathcal{L}_{c,a}$ with $\operatorname{Re}\lambda \geq -\delta/2$ with $\delta > 0$ small enough chosen as in Lemma 2.1 and the eigenfunction $u(\xi, y) \in X_a^2$ expressed by (2.21).

We first assume that there exists $k \geq 1$ such that $v_k(\xi) \neq 0$ in (2.21), i.e. $(\lambda, v_k(\xi))$ is an eigenpair of (2.22) with $v_k(\xi)(1 + \mathrm{e}^{a\xi}) \in \mathrm{C}_{\mathrm{unif}}(\mathbb{R})$. Using the fact that

$$\operatorname{Re}\left(\frac{c - \sqrt{c^2 + 4(\lambda + \lambda_k - 1)}}{2}\right) < a < \operatorname{Re}\left(\frac{c + \sqrt{c^2 + 4(\lambda + \lambda_k - 1)}}{2}\right), \quad \forall \operatorname{Re}\lambda \geq -\delta/2,$$

then by applying the classical asymptotic analysis to (2.22) it holds that

$$v_k(\xi) \sim \exp\left\{\frac{-c - \sqrt{c^2 + 4(\lambda + \lambda_k - 1)}}{2}\right\}, \quad \text{as } \xi \to +\infty, \text{ if } \operatorname{Re}\lambda \geq -\delta/2. \tag{2.24}$$

Let $\widetilde{v}_k(\xi) = \mathrm{e}^{\frac{c}{2}\xi} v_k(\xi) \in \mathrm{H}^2(R)$, by (2.22) and (2.24), it is easy to check that $\widetilde{v}_k(\xi) \in \mathrm{H}^2(\mathbb{R})$ and satisfies the differential equation

$$\widetilde{v}_k''(\xi) + [-\frac{c^2}{4} + 1 - \lambda_k - (1 - \lambda_0)V_c(\xi)]\widetilde{v}_k(\xi) = \lambda\widetilde{v}_k(\xi), \text{ for some } k \geq 1, \tag{2.25}$$

which means that λ must be a real eigenvalue of the differential operator $L_k = \frac{\partial^2}{\partial\xi^2} + b_k(\xi)$ with an eigenfunction $\widetilde{v}_k(\xi) = \mathrm{e}^{\frac{c}{2}\xi} v_k(\xi) \in \mathrm{H}^2(\mathbb{R})$ and note that

$$b_k(\xi) \triangleq -\frac{c^2}{4} + 1 - \lambda_k - (1 - \lambda_0)V_c(\xi) < \lambda_0 - \lambda_1 \leq -\delta < 0, \quad \forall k \geq 1, \forall\, c > 2\sqrt{1 - \lambda_0}. \tag{2.26}$$

(2.26) further implies that

$$\sigma\left(\frac{\partial^2}{\partial\xi^2} + b_k(\xi)\right) \subset (-\infty, -\delta],\ \forall k \geq 1,$$

which contradicts with the assumptions $\operatorname{Re}\lambda \geq -\delta/2$ and $v_k(\xi) \neq 0$ for some $k \geq 1$, this proves that if λ is an eigenvalue of $\mathcal{L}_{c,a}$ with $\operatorname{Re}\lambda \geqslant -\delta/2$, then the eigenfunction in X_a^2 must be in the form of $v_0(\xi)\phi_0(y)$ and $(\lambda, v_0(\xi))$ is an eigenpair of (2.23).

By applying nearly the same argument as above, it can be proved that the eigenvalue λ must be real and λ is an eigenvalue of L_0 with an eigenfunction $v_0 \in \mathrm{H}^2(\mathbb{R})$ and L_0 defined by

$$L_0 = \frac{\partial^2}{\partial\xi^2} - \frac{c^2}{4} + 1 - \lambda_0 - 2(1 - \lambda_0)V_c(\xi).$$

Using the fact that

$$-\frac{c^2}{4} + 1 - \lambda_0 - 2(1 - \lambda_0)V_c(\xi) \leq -\frac{c^2}{4} + 1 - \lambda_0 = -\delta_c < 0, \text{ for } c > 2\sqrt{1 - \lambda_0},$$

which means that $\sigma(L_0) \subset (-\infty, -\delta_c]$, this completes the proof of Theorem 2.1 and Theorem 1 □

Remark 2.1. Note that the estimates (2.13) and (2.17) are still valid for the critical speed case $c = c^* = 2\sqrt{1 - \lambda_0}$, thus if we choose $a = \sqrt{1 - \lambda_0}$, then $\sigma_{\mathrm{ess}}(\mathcal{L}_{c^*,a}) \subset \{\operatorname{Re}\lambda < 0\} \cup \{0\}$, and it can be further proved that $\sigma(\mathcal{L}_{c^*,a}) \setminus \{0\} \subset \{\operatorname{Re}\lambda < 0\}$

and zero is not an eigenvalue of $\mathcal{L}_{c^*,a}$, but $0 \in \sigma_{\text{ess}}(\mathcal{L}_{c^*,a})$, the above stated spectral results of $\mathcal{L}_{c^*,a}$ are nearly the same as that for the planar wave front with the critical speed for Fisher equation $u_t = u_{xx} = (1-\lambda_0)u(1-u)$. By applying Green function method with detailed point-wise semigroup estimate, it was proved in [18] that for the Fisher equation $u_t = u_{xx} + (1-\lambda_0)u(1-u)$ if the small initial perturbation of $V_{c^*}(x)$ in X_a $(a = \sqrt{1-\lambda_0})$ decays faster than $x^{-2}\mathrm{e}^{-ax}$ at $x = +\infty$, then the solution tends to the planar wave $V_{c^*}(x - c^*t)$ in X_a and the perturbation of the wave decays algebraically in time. However in the multi-dimensional cylinder case even for the classical nonlinear parabolic equation it is still an open theoretical problem that whether the above mentioned weak spectral stability of the cylinder wave with critical speed can still guarantee some types of asymptotically stability of the wave.

3. Uniform boundedness of solutions with more general initial value

In this section under the assumption of **(H1)**, we investigate the initial boundary value problem

$$\begin{cases} u_t(t,x,y) - \triangle_{x,y}u(t,x,y) + g(y)u(t,x,y) & \\ = [1-m(t,x)]\,u(t,x,y), & (t,x,y) \in \mathbb{R}^+ \times \mathbb{R} \times \Omega, \\ m(t,x) = \int_\Omega K(y')u(t,x,y')\mathrm{d}y', & (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ \frac{\partial u}{\partial \nu} = 0, & (t,x,y) \in \mathbb{R}^+ \times \mathbb{R} \times \partial\Omega, \\ u(0,x,y) = u_0(x,y), & (x,y) \in \mathbb{R} \times \Omega. \end{cases} \tag{3.1}$$

with more general nonnegative initial value $u_0(x,y)$.

Lemma 3.1. *For any nonzero and nonnegative initial value $u_0 \in L_\infty(\mathbb{R}\times\Omega)$, problem (3.1) admits a unique global positive classical solution $u(t,x,y) \in C^\infty(\mathbb{R}^+ \times \mathbb{R}\times\Omega)$, which satisfies*

$$0 < u(t,x,y) \leqslant \mathrm{e}^t||u_0||_{\mathrm{L}_\infty(\mathbb{R}\times\Omega)}, \quad \forall (x,y) \in (\mathbb{R}\times\Omega), t > 0. \tag{3.2}$$

Proof. By applying comparison principle to (3.1) in the linear form, obviously $u(t,x,y) > 0$ for any $t > 0$ and $(x,y) \in \mathbb{R}\times\Omega$. It is easy to see that $\mathrm{e}^t||u_0||_{\mathrm{L}_\infty(\mathbb{R}\times\Omega)}$ is a super-solution of the following linear initial boundary value problem

$$\begin{cases} w_t - \triangle_{x,y}w + g(y)w = w, \ (t,x,y) \in \mathbb{R}^+ \times \mathbb{R} \times \Omega, \\ w(t,x,y) = u_0(x,y), \ (x,y) \in \mathbb{R}\times\Omega, \end{cases} \tag{3.3}$$

and $u(t,x,y)$ is a sub-solution of (3.3), then by comparison principle we have

$$u(t,x,y) \le \mathrm{e}^t||u_0||_{\mathrm{L}_\infty(\mathbb{R}\times\Omega)}, \quad \forall t > 0, \forall (x,y) \in \mathbb{R}\times\Omega.$$

□

By Lemma 3.1, denote $m(t,x) = \int_\Omega K(y')u(t,x,y')\mathrm{d}y'$ and $v_j(x,t) = \langle u(t,x,y), \psi_j(y)\rangle = \int_\Omega u(x,y,t)\psi_j(y)\mathrm{d}y$, then $u(t,x,y) = \sum_{j=0}^{\infty} v_j(t,x)\psi_j(y)$ and $v_j(t,x)$ $(j \ge 0)$ is the unique global solution of the following nonlinear initial value problem

$$\begin{cases} \frac{\partial}{\partial t}v_j - \frac{\partial^2}{\partial x^2}v_j + (\lambda_j - 1 + m(t,x))v_j = 0, \ (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ v_j(0,x) = \int_\Omega u_0(x,y)\psi_j(y)\mathrm{d}y, \ x \in \mathbb{R}. \end{cases} \tag{3.4}$$

Lemma 3.2. *Let $(\lambda_j, \psi_j(y))$, $(\ j = 0, 1, \cdot)$ be the eigenpair stated as in Section 1, there exists $J \geq 1$ such that $\lambda_j \geq 2$ for all $j \geq J+1$, and denote $u^{\perp}(t,x,y) = \sum_{j=J+1}^{\infty} v_j(x,t)\psi_j(y)$. Then for any given nonnegative bounded initial value $u_0(x,y)$, it holds that*

$$\sup_{x\in\mathbb{R}} ||u^{\perp}(t,x,y)||_{\mathrm{L}^2(\Omega)} \leqslant \mathrm{e}^{-t}||u_0||_{\mathrm{L}_\infty(\mathbb{R})\times \mathrm{L}_2(\Omega)}, \quad \forall t \geq 0. \tag{3.5}$$

Proof. Using the fact that $\lambda_j \geq 2$ for any $j \geq J+1$ and $m(t,x) \geq 0$ for $x \in \mathbb{R}$, it is easy to check that $\mathrm{e}^{-t}\|v_j(0,x)\|_{L_\infty(\mathbb{R})}$ and $-\mathrm{e}^{-t}\|v_j(0,x)\|_{L_\infty(\mathbb{R})}$ are supper and sub-solutions of (3.4) respectively for any $j \geq J+1$, thus

$$\sup_{x\in\mathbb{R}} |v_j(t,x)| \leq \mathrm{e}^{-t}\|v_j(0,x)\|_{L_\infty(\mathbb{R})}, \quad \forall t \geq 0, \forall j \geq J+1,$$

and

$$\begin{aligned}\sup_{x\in\mathbb{R}} ||u^{\perp}(t,x,\cdot)||_{\mathrm{L}_2(\Omega)} &= \sup_{x\in\mathbb{R}} \sum_{j\geq J+1}^{+\infty} |v_j(t,x)| \leq \mathrm{e}^{-t} \sum_{j\geq J+1}^{+\infty} \|v_j(0,x)\|_{L_\infty(\mathbb{R})} \\ &\leq \mathrm{e}^{-t}||u_0||_{\mathrm{L}_\infty(\mathbb{R})\times\mathrm{L}_2(\Omega)}, \quad \forall t \geq 0, \forall j \geq J+1.\end{aligned}$$

□

Next, we consider the finite sum $u(t,x,y) - u^{\perp}(t,x,y) = \sum_{j=0}^{J} v_j(x,t)\psi_j(y)$, we only need to deal with the functions $v_j(t,x)$ for $j = 0, 1, \cdot\cdot\cdot, J$. Denote by $c_j = ||\frac{\psi_j}{\psi_0}||_{\mathrm{L}_\infty(\Omega)}$, then

$$|v_j(0,x)| \leqslant \int_\Omega u_0(x,y)|\psi_j(y)|\mathrm{d}y \leqslant c_j \int_\Omega u_0(x,y)\psi_0(y)\mathrm{d}y = c_j v_0(0,x). \tag{3.6}$$

By (3.6) and the fact $\lambda_j - \lambda_0 \geq \delta_0 > 0$ for $j \geq 1$, applying sub-supper solution method to the linear problem (3.4), it can be proved that

$$|v_j(t,x)| \leqslant c_j \mathrm{e}^{-(\lambda_j-\lambda_0)t} v_0(t,x) \leq c_j \mathrm{e}^{-\delta_0 t} v_0(t,x), \ \ \forall\, x \in \mathbb{R},\ t > 0,\ j \geq 1. \tag{3.7}$$

Define

$$\begin{aligned} m_0(t,x) &= v_0(t,x) \int_\Omega K(y)\psi_0(y)\mathrm{d}y = (1-\lambda_0)v_0(t,x), \\ m_j(t,x) &= v_j(t,x) \int_\Omega K(y)\psi_j(y)\mathrm{d}y, \quad j = 1, 2, \cdot\cdot\cdot, J, \end{aligned} \tag{3.8}$$

then

$$|m_j(t,x)| \leqslant \int_\Omega K(y)|\psi_j(y)|\mathrm{d}y \cdot |v_j(t,x)| \leqslant (1-\lambda_0)c_j^2 \mathrm{e}^{-(\lambda_j-\lambda_0)t} v_0(t,x). \tag{3.9}$$

By Lemma 3.2 and (3.7)-(3.9), now we are ready to complete the proof of Theorem 2.

Proof of Theorem 2: Define

$$m^{\perp}(t,x) = \int_\Omega K(y)u^{\perp}(t,x,y)\mathrm{d}y,$$

then by Lemma 3.2 and (3.7), we have

$$\begin{aligned}|m^{\perp}(t,x)| &\leqslant ||K||_{\mathrm{L}_2(\Omega)}||u^{\perp}(t,x,\cdot)||_{\mathrm{L}_2(\Omega)} \\ &\leqslant \mathrm{e}^{-t}||K||_{\mathrm{L}_2(\Omega)}||u_0||_{\mathrm{L}_\infty(\mathbb{R})\times\mathrm{L}_2(\Omega)}.\end{aligned} \tag{3.10}$$

Denote $b_0(t,x) = \frac{1}{(1-\lambda_0)}\big(m(t,x) - m_0(t,x)\big)$, then the equation of $v_0(t,x)$ in the system of (3.4) can be written as

$$\frac{\partial}{\partial t}v_0 - \frac{\partial^2}{\partial x^2}v_0 = (1-\lambda_0)(1-b_0(t,x)-v_0)v_0,\ t>0, x\in\mathbb{R}. \tag{3.11}$$

(3.9) and (3.17) imply that

$$\begin{aligned}|b_0(t,x)| =& \frac{1}{(1-\lambda_0)}\left|\sum_{j=1}^{\infty} v_j(t,x)\int_\Omega K(y)\psi_i(y)\mathrm{d}y\right| \\ =& \frac{1}{(1-\lambda_0)}\left|\sum_{j=1}^{J} m_j(t,x) + m^\perp(t,x)\right| \\ \leqslant& \sum_{j=1}^{J} c_j^2 \mathrm{e}^{-(\lambda_j-\lambda_0)t}|v_0(t,x)| + \frac{\|K\|_{\mathrm{L}_2(\Omega)}}{(1-\lambda_0)}\mathrm{e}^{-t}\|u_0(x,\cdot)\|_{\mathrm{L}_2(\Omega)}.\end{aligned} \tag{3.12}$$

In particular, there exists a positive constant C_0 independent of u_0 such that

$$|b_0(t,x)| \le C_0\mathrm{e}^{-\delta_0 t}\left(|v_0(t,x)| + \|u_0(x,\cdot)\|_{L_2(\Omega)}\right) \quad \forall t\ge 0, x\in\mathbb{R}, \tag{3.13}$$

with $\delta_0 = \min\{1, \lambda_1-\lambda_0\} > 0$.

Next, we claim that

$$\|v_0(t,\cdot)\|_{\mathrm{L}_\infty(\mathbb{R})} \le C \quad \text{uniformly in } t\ge 0. \tag{3.14}$$

By Lemma 3.1 the assertion holds for finite t. For t large enough, we set $T >> 1$ such that

$$C_0\mathrm{e}^{-\delta_0 T} \leqslant \frac{1}{2}, \tag{3.15}$$

then (3.11) and (3.12) yield that

$$\begin{aligned}\frac{\partial}{\partial t}v_0 - \frac{\partial^2}{\partial x^2}v_0 =& (1-\lambda_0)(1-b_0(t,x)-v_0)v_0 \\ \le& (1-\lambda_0)\left[1 - v_0 + \frac{1}{2}\left(v_0(t,x) + \|u_0(x,\cdot)\|_{\mathrm{L}_2(\Omega)}\right)\right]v_0(t,x),\ \forall t\ge T.\end{aligned} \tag{3.16}$$

It follows from the maximum principle that

$$\|v_0(t,\cdot)\|_{\mathrm{L}_\infty(\mathbb{R})} \leqslant \max\{\|v_0(0,\cdot)\|_{\mathrm{L}_\infty(\mathbb{R})}, 2 + \|u_0(\cdot,\cdot)\|_{\mathrm{L}_\infty(\mathbb{R})\times\mathrm{L}_2(\Omega)}\},\ \ \forall t\geqslant T. \tag{3.17}$$

This proves (3.14). Lemma 3.2 and (3.7) also imply that

$$\|u(t,x,\cdot) - v_0(t,x)\psi_0(\cdot)\|_{\mathrm{L}_2(\Omega)} \leqslant \sum_{j=1}^{J} c_j^2\mathrm{e}^{-\delta_0 t}|v_0(t,x)| + \mathrm{e}^{-t}\|u_0(x,\cdot)\|_{\mathrm{L}_2(\Omega)} \le C\mathrm{e}^{-\delta_0 t}, \tag{3.18}$$

which with (3.14) further implies

$$\sup_{t>0}\|u(t,\cdot,\cdot)\|_{\mathrm{L}_\infty(\mathbb{R})\times\mathrm{L}_2(\Omega)} \le M_0. \tag{3.19}$$

By virtue of (3.19), the nonlinear equation (3.1) can be written in a form of linear heterogeneous parabolic equation $u_t = \triangle_{x,y}u - u + f(t,x,y)$ with $f(t,x,y) = u(2-m(t,x))$ satisfying $\|f(t,x,y)\|_{\mathrm{L}_\infty(0,+\infty)\times\mathrm{L}_\infty(\mathbb{R})\times\mathrm{L}_2(\Omega)} \le M_1$, and note that $\sigma(L_0) \subset \{\mathrm{Re}\ \lambda \le -1\}$ with $L_0 = \triangle_{x,y} - I$, and

$$\|\mathrm{e}^{L_0 t}\|_{L_p(\mathbb{R}\times\Omega)\to W_p^1(\mathbb{R}\times\Omega)} \le C_p t^{-1/2}e^{-1/2t},\ \ \forall t>0,\ 1<p<+\infty, \tag{3.20}$$

then by the decay estimate (3.20) and applying a recursive argument to (3.1) it is easy to show that there exist positive constants θ and C_θ such that for any $u_0 \in \mathrm{L}_\infty(\mathbb{R}\times\Omega)$ the unique classical solution of u of (3.1) also satisfies

$$\|u(t,x,y)\|_{C^\theta(\mathbb{R}\times\overline{\Omega})} \leq C_\theta(\|u_0\|_{\mathrm{L}_\infty(\mathbb{R}\times\Omega)} + M_0), \ \ \forall t \geq 1. \tag{3.21}$$

Estimate (3.21) can be similarly proved by applying interior $W^{1,2}_{p,p}$ estimates and bootstrap argument. By interpolation, (3.18) can be improved to

$$\sup_{(x,y)\in\mathbb{R}\times\Omega} |u(t,x,y) - v_0(t,x)\psi_0(y)| \leqslant C'\mathrm{e}^{-\delta_0' t} \quad \text{for } t \geq 1, \tag{3.22}$$

for some $C', \delta_0' > 0$. This proves Theorem 2.

4. Asymptotic behavior of solution with more general initial value

By virtue of the uniform boundedness of the solution and the estimates (1.6)-(1.7) proved in Theorem 2, to investigate the spreading speed and asymptotic behavior of the solution $u(t,x,y)$ in higher dimensional cylinder to the problem (3.1) with more general initial value, it suffices to investigate the long time behavior of $v_0(t,x) = \langle u(t,x,\cdot), \psi_0(\cdot)\rangle$ as $t \to +\infty$, where $v_0(t,x)$ satisfies the nonlinear equation (3.11) in one dimensional space, i.e. $v_t - v_{xx} = (1-\lambda_0)v(1-b_0(t,x)-v)$, with $b_0(t,x) = \frac{1}{1-\lambda_0}\int_\Omega K(y)(u(t,x,y) - v_0(t,x)\psi_0(y))\mathrm{d}y$. Due to the exponential decay in time of the coupled term $b(t,x)$ obtained in (3.13), the equation (3.11) of $v_0(t,x)$ can be treated as a Fisher-KPP equation with a heterogenous term $b(t,x)$.

In this section we shall focus on the investigation of the long time behavior of the solution of Fisher-KPP equation (3.11) with more general heterogenous decaying resource term $b_0(t,x)$, using the decaying estimate (3.13) of $b_0(t,x)$ in time, we shall prove that for more general initial value the spreading speed of the solution to problem (3.1) or problem (3.11) is still determined by the decay rate of the initial value and the solution may still tend to the wave with some noncritical speed or the critical speed in some appropriate sense.

4.1. **Global asymptotic stability of waves with noncritical speeds.**

In this subsection we investigate the Cauchy problem of (3.11), i.e.

$$\begin{cases} v_t - v_{xx} = (1-\lambda_0)v(1-b_0(t,x)-v), & x\in\mathbb{R},\ t>0,\\ v(0,x) = v_0^*(x), & x\in\mathbb{R}. \end{cases} \tag{4.1}$$

For any given $c > 2\sqrt{1-\lambda_0}$, let $V_c(x-ct)$ be the traveling front solution connecting 1 and 0 of (the limiting problem of) (4.1) with $b_0(t,x)\equiv 0$ (as $t\to+\infty$), and without loss of generality, we choose $V_c(z)$ be the unique wave solution satisfying the following boundary value problem

$$\begin{cases} cV_c' + V_c'' + (1-\lambda_0)V_c(1-V_c) = 0, & z\in\mathbb{R},\\ V_c(-\infty) = 1, \quad V_c(+\infty) = 0, & \\ \lim\limits_{z\to+\infty} \mathrm{e}^{\sigma z}V_c(z) = 1, & \sigma = \frac{c-\sqrt{c^2-4(1-\lambda_0)}}{2}. \end{cases} \tag{4.2}$$

Observe that $\sigma \mapsto c = \sigma + \frac{1-\lambda_0}{\sigma}$ is a bijection from $(0,\sqrt{1-\lambda_0})$ to $(2\sqrt{1-\lambda_0},\infty)$.

For any given $c > 2\sqrt{1-\lambda_0}$ and $r > 0$, let $\psi_c(x - ct; r)$ be the unique planar wave solution connecting r and 0 to the following boundary value problem:

$$\begin{cases} c\psi_c' + \psi_c'' + (1-\lambda_0)\psi_c(1 - \frac{\psi_c}{r}) = 0, & z \in \mathbb{R}, \\ \psi_c(-\infty, r) = r, \quad \psi_c(+\infty, r) = 0, & \\ \lim_{z\to+\infty} \mathrm{e}^{\sigma z}\psi_c(z) = 1, & \sigma = \frac{c - \sqrt{c^2 - 4(1-\lambda_0)}}{2}. \end{cases} \tag{4.3}$$

Obviously $\psi_c(z; 1) = V_c(z)$ and $\psi_c(z; r) = rV_c(z - z_r)$ and $r\mathrm{e}^{\sigma z_r} = 1$.

In this subsection we always assume that the initial value $v_0^*(x)$ is nonnegative, bounded and stays away from zero at $x = -\infty$, i.e.

$$0 < \underline{q}_0 < \liminf_{x\to-\infty} v_0^*(x) \le \limsup_{x\to-\infty} v_0^*(x) < \overline{q}_0, \text{ for } x \le 0; \tag{4.4}$$

and assume that the nonnegative bounded initial value $v_0^*(x)$ decays exponentially at $x = +\infty$ with the same decay rate of a wave for (4.2) or (4.3) with a noncritical speed, which means for some $c > 2\sqrt{1-\lambda_0}$

$$\lim_{x\to+\infty} \frac{v_0^*(x)}{V_c(x + x_0)} = 1, \text{ as } x \to +\infty, \text{ for some } x_0 \in \mathbb{R}, \tag{4.5}$$

or equivalently and without loss of generality, we assume the initial value of $v_0^*(x)$ satisfies the decay estimate

$$\lim_{x\to+\infty} v_0^*(x)\mathrm{e}^{\sigma x} = 1, \text{ for some } 0 < \sigma < 1\sqrt{1-\lambda_0}. \tag{4.6}$$

For the heterogeneous term $b_0(t, x)$, we assume that

$$|b_0(t, x)| \le C_0\mathrm{e}^{-\delta t}(v(t, x) + \mathrm{e}^{-\sigma(x-ct)} \wedge 1) \quad \text{for some } C_0, \delta, \sigma > 0. \tag{4.7}$$

Note that the decay estimate (3.13) implies (4.7).

Lemma 4.1. *Let $v(t, x)$ be a solution to (4.1) with initial value satisfying (4.6) for some $\sigma \in (0, \sqrt{1-\lambda_0})$. Assume in addition that $b_0(t, x)$ satisfies (4.7) for some positive C_0 and $\delta > 0$. Then the following statements hold true.*

(a) *For each $t > 0$, we have $\lim_{x\to+\infty} \mathrm{e}^{\sigma x}v(t, x) = \mathrm{e}^{\sigma ct}$, where $c = \sigma + \frac{1-\lambda_0}{\sigma}$.*

(b) *There exist positive constants t_1 and r_1 such that*

$$v(t, x) \ge r_1(\mathrm{e}^{-\sigma(x-ct)} \wedge 1), \quad t \ge t_1, \ x \in \mathbb{R}.$$

(c) *For each $\varepsilon > 0$, there exists $t_2 > 0$ such that*

$$|b_0(t, x)| \le \frac{\varepsilon}{1 + \varepsilon} v(t, x), \quad t \ge t_2, \ x \in \mathbb{R}.$$

Proof. To prove (a), we first observe that $v_t - v_{xx} \le \kappa v$, where $\kappa = (1 - \lambda_0)(1 + \|b\|_{L_\infty(\mathbb{R}^+\times\mathbb{R})})$. Hence, the comparison principle yields, for each $t > 0$,

$$0 \le v(s, x) \le \sup_{y\in\mathbb{R}} (\mathrm{e}^{\sigma y}v_0^*(y))\mathrm{e}^{-\sigma x + (\sigma^2+\kappa)s} \le C_t\mathrm{e}^{-\sigma x}, \quad (s, x) \in [0, t] \times \mathbb{R}. \tag{4.8}$$

Next, observe that by Duhamel's principle:

$$v(t, x) = \mathrm{e}^{(1-\lambda_0)t}(p_t * v_0^*)(x) + E(t, x), \tag{4.9}$$

where $p_t(x) = p(t, x) = \frac{1}{\sqrt{4\pi t}}\mathrm{e}^{-x^2/(4t)}$ and

$$E(t, x) = \int_0^t \mathrm{e}^{(1-\lambda_0)(t-s)} \int_{\mathbb{R}} p(t - s, x - x')(-b_0(s, x') - v(s, x'))v(s, x')\, dx'ds.$$

Thanks to (4.7) and (4.8), it follows that for each fixed $t > 0$,

$$|E(t,x)| \le C_t \int_0^t \mathrm{e}^{(1-\lambda_0)(t-s)} \int_{\mathbb{R}} p(t-s, x-x')(\mathrm{e}^{-\sigma x'})^2 \, dx' ds \le C_t' \mathrm{e}^{-2\sigma x}, \quad t \ge 0,$$

so that $\lim_{x\to\infty} \mathrm{e}^{\sigma x}|E(t,x)| = 0$ for any $t > 0$. Therefore, using (4.6) and (4.9) again, we have

$$\lim_{x\to\infty} \mathrm{e}^{\sigma x} v(t,x) = \lim_{x\to\infty} \mathrm{e}^{\sigma x}\mathrm{e}^{(1-\lambda_0)t}(p_t * v_0^*)(x) = \mathrm{e}^{\sigma ct}, \quad \forall t > 0. \tag{4.10}$$

To see the last equality, we note that $v_0^*(x) = \mathrm{e}^{-\sigma x}(1 + g(x))$ with $g(+\infty) = 0$, so that

$$\begin{aligned}
\mathrm{e}^{\sigma x}\mathrm{e}^{(1-\lambda_0)t}(p_t * v_0^*)(x) &= \frac{1}{\sqrt{4\pi t}}\mathrm{e}^{\sigma x + (1-\lambda_0)t} \int_{\mathbb{R}} \mathrm{e}^{-\frac{(x-x')^2}{4t}} \mathrm{e}^{-\sigma x'}(1 + g(x'))\, dx' \\
&= \frac{1}{\sqrt{4\pi t}}\mathrm{e}^{(1-\lambda_0)t} \int_{\mathbb{R}} \mathrm{e}^{-\frac{(x-x'-2\sigma t)^2}{4t} + \sigma^2 t}(1 + g(x'))\, dx' \\
&= \mathrm{e}^{(1-\lambda_0)t + \sigma^2 t} \int_{\mathbb{R}} p_t(x - x' - 2\sigma t)(1 + g(x'))\, dx' \\
&= \mathrm{e}^{\sigma ct} \int_{\mathbb{R}} p_t(\tilde{x})(1 + g(x - \tilde{x} - 2\sigma t))\, d\tilde{x}, \quad \forall t > 0.
\end{aligned}$$

Then one can take $x \to +\infty$ in the above by the dominant convergence theorem to obtain the last equality in (4.10). This completes the proof of (a).

For (b), note that $v(t,x)$ satisfies

$$v_t - v_{xx} \ge (1-\lambda_0)v(1-|b|-v) \ge (1-\lambda_0)v[1 - C_b\mathrm{e}^{-\delta t}(v + (\mathrm{e}^{-\sigma(x-ct)} \wedge 1) - v]. \tag{4.11}$$

By choosing $t_1 > 1$ large enough, we see that $v(t,x)$ is a supersolution of

$$w_t - w_{xx} = (1-\lambda_0)w[1 - \tfrac{\varepsilon}{2}\psi(x - ct; 1) - (1 + \tfrac{\varepsilon}{2})w], \quad t \ge t_1,\ x \in \mathbb{R}, \tag{4.12}$$

where we used

$$(\mathrm{e}^{-\sigma x} \wedge 1) \le B\psi(x;1) \quad \text{ for some } B > 1. \tag{4.13}$$

Next, observe that $\underline{w}(t,x) = r\psi(x-ct;1)$ is a subsolution of (4.12) for any $r \in (0,1)$. Finally, we can choose $r = r(\varepsilon)$ small enough so that

$$v(t_1, x) \ge r\psi(x - ct_2; 1), \quad x \in \mathbb{R}.$$

(This is thanks to (a) and $\lim_{x\to\infty} \mathrm{e}^{\sigma x}\psi(x - ct_1; 1) = \mathrm{e}^{\sigma ct_1}$.) We can then conclude by the comparison principle that

$$v(t,x) \ge r\psi(x - ct; 1) \ge \frac{r}{B}(\mathrm{e}^{-\sigma(x-ct)} \wedge 1), \quad t \ge t_1,\ x \in \mathbb{R}. \tag{4.14}$$

This proves (b). Assertion (c) follows from (4.7) and (b). □

Theorem 4.1. *Let $v(t,x)$ be a solution to* (4.1) *with initial value $v_0^*(x)$ satisfying* (4.4) *and* (4.6) *for some $\sigma \in (0, \sqrt{1-\lambda_0})$. Suppose, in addition, that* (4.7) *holds, then*

$$\lim_{t\to\infty}\left[\sup_{z\in\mathbb{R}} |v(t, z + ct) - V_c(z)|\right] = 0. \tag{4.15}$$

Proof. Fix $\varepsilon > 0$, then by Lemma 4.1 (a) and (c), there exists $t_\varepsilon \geq 1$, such that

$$(1-\lambda_0)v\left(1-\frac{v}{1-\varepsilon}\right) \leq v_t - v_{xx} \leq (1-\lambda_0)v\left(1-\frac{v}{1+\varepsilon}\right), \quad t \geq t_\varepsilon,\ x \in \mathbb{R}, \tag{4.16}$$

and

$$\lim_{x\to+\infty} \mathrm{e}^{\sigma x} v(t,x) = \mathrm{e}^{\sigma ct},\ \ \forall t \geq 0. \tag{4.17}$$

By comparison principle, we have

$$\tilde{v}^-(t,x) \leq v(t,x) \leq \tilde{v}^+(t,x), \quad t \geq t_\varepsilon,\ x \in \mathbb{R},$$

where $\tilde{v}^\pm(t,x)$ are, respectively, the solutions of the Cauchy problem of the following classical Fisher equation

$$\begin{cases} \tilde{v}_t - \tilde{v}_{xx} = (1-\lambda_0)\left(1-\frac{\tilde{v}}{1\pm\varepsilon}\right), & t \geq t_\varepsilon,\ x \in \mathbb{R}, \\ \tilde{v}(t_\varepsilon, x) = v(t_\varepsilon, x), & x \in \mathbb{R}. \end{cases} \tag{4.18}$$

Notice that by (4.17) and (4.18), the initial value $\tilde{v}(t_\varepsilon, x)$ satisfies

$$\lim_{z\to+\infty} \mathrm{e}^{\sigma z} \tilde{v}(t_\varepsilon, z + ct_\varepsilon) = 1,$$

then by [33, Theorem 9.3] it is follows that the solution $\tilde{v}^\pm(t,x)$ converges to the planar wave solution $\psi(x-ct; 1\pm\varepsilon)$ of (4.18) uniformly in the moving coordinate $z = x - ct$ as $t \to +\infty$; precisely speaking, we have

$$\lim_{t\to+\infty} \sup_{z\in\mathbb{R}} |\tilde{v}^\pm(t, z+ct) - \psi(z; 1\pm\varepsilon)| = 0,$$

where $\psi(z;r)$ is given in (4.3). Thus

$$\psi(z;1-\varepsilon) \leq \liminf_{t\to+\infty} v(t,z+ct) \leq \limsup_{t\to+\infty} v(t,z+ct) \leq \psi(z;1-\varepsilon), \forall z \in \mathbb{R}. \tag{4.19}$$

The proof is completed by letting $\varepsilon \searrow 0$ in (4.19). □

Obviously Theorem 3 follows from Theorem 2 and Theorem 4.1.

Remark 4.1. If the decay assumption (4.6) on the initial value is weakened to

$$v_0^*(x) = \mathrm{e}^{-(\sigma+o(1))x},\ \ x \to +\infty,$$

we conjecture that

$$\lim_{t\to\infty}\left[\sup_{x\in\mathbb{R}} |v(t,x) - \psi\left(x - ct + \xi(t)\right)|\right] = 0, \tag{4.20}$$

where $\xi(t)$ is in general a bounded function.

Proof of Theorem 4: Under the assumption that the initial value $u_0(x,y)$ satisfies the assumption (1.10) and

$$\int_\Omega u_0(x,y)\psi_0(y)\mathrm{d}y \sim r\mathrm{e}^{-\sigma x} + O(\mathrm{e}^{-ax}),\ \ \text{as } x \to +\infty, \tag{4.21}$$

for some $r > 0$, $\sigma \in (0, \sqrt{1-\lambda_0})$ and $a > \sigma$, which means $u_0(x,y) - V_c(x - \frac{1}{\sigma}\ln r)\psi_0(y) \in X_a$, for $c = \sigma + \frac{1-\lambda_0}{\sigma} > 2\sqrt{1-\lambda_0}$ and $a > \sigma$. By virtue of the local exponential stability of the wave $V_c(x+x_0)\psi_0(y)$ in some weighted space X_a (see Theorem 1), it suffices to consider the case $r = 1$ in (4.21) and prove that $\|u(t, z+ct, y) - V_c(z)\psi_0(y)\|_{X_a} \to 0$ as $t \to +\infty$ if $a - \sigma$ is small enough.

By Theorem 3, it is known that under the assumption (1.10),

$$\|u(t, z+ct, y) - V_c(z)\psi_0(y)\|_{\mathrm{L}_\infty(\mathbb{R}\times\Omega)} \to 0 \quad \text{as } t \to +\infty.$$

Denote $v(t,x) = \langle u(t,x,\cdot), \psi_0(\cdot)\rangle$, and let $\tilde{v}(t,z) = v(t,z+ct)$, then in the moving coordinate $z = x - ct$, $\tilde{v}(t,z)$ satisfies the following heterogeneous Fisher type equation:

$$\tilde{v}_t = \tilde{v}_{zz} + c\tilde{v}_z + (1-\lambda_0)\tilde{v}(1 - b_0(t,z+ct) - \tilde{v}),$$

then $\hat{v}(t,z) = \tilde{v}(t,z) - V_c(z)$, satisfies the nonlinear equation

$$\hat{v}_t = \hat{v}_{zz} + c\hat{v}_z + (1-\lambda_0)\hat{v} + F(t,z,\hat{v}),$$

with the initial value $\hat{v}_0(z) = \int_\Omega u_0(z,y)\psi_0(y) - V_c(z) \in \mathrm{L}_\infty(\mathbb{R})$, and $\hat{v}_0(z) = O(\mathrm{e}^{-az})$ for $z >> 1$ and $a > \sigma$, where

$$F(t,z,\hat{v}) = -b_0(t,z+ct)(\hat{v} + V_c(z)) - \hat{v}^2(t,z) - 2V_c(z)\hat{v}.$$

Under the assumption (1.10), by (4.7) and Theorem 3 we know that

$$|F(t,z,\hat{v})| \le C_t\mathrm{e}^{-2\sigma z} \wedge \eta(t),\ z \in \mathbb{R}, t > 0,$$

where $\eta(t) \to 0^+$ as $t \to +\infty$.

Note that

$$\hat{v}(t,z) = \mathrm{e}^{(1-\lambda_0)t}\int_{\mathbb{R}} p(t, z+ct-z')\hat{v}_0(z')dz' + \hat{F}(t,z,\hat{v}),$$

where $p(t,z) = \frac{1}{\sqrt{4\pi t}}\mathrm{e}^{-\frac{z^2}{4t}}$ and

$$\hat{F}(t,z) = \int_0^t \mathrm{e}^{(1-\lambda_0)(t-s)}\int_{\mathbb{R}} p(t-s, z+ct-z')F(s,z',\hat{v}(s,z'))\,dz'ds.$$

Choosing $a \in (\sigma, \sigma_+)$, with $\sigma_+ = \frac{c+\sqrt{c^2-4(1-\lambda_0)}}{2} > \sqrt{1-\lambda_0} > \sigma$, by detailed computation it can be verified that

$$\mathrm{e}^{(1-\lambda_0)t}\int_{\mathbb{R}} p(t,z+ct-z')\mathrm{e}^{-az'}dz' = \mathrm{e}^{-\delta_a t}\mathrm{e}^{-az},$$

with $-\delta_a = a^2 + ca + 1 - \lambda_0 < 0$, if $a \in (\sigma, \sigma_+)$, and it can be proved that

$$|\hat{F}(t,z)| \le C\mathrm{e}^{-2\sigma z}, \quad t \ge 0, z \ge 0,$$

thus for any given $\hat{v}_0 \in \mathrm{L}_\infty(\mathbb{R})$ satisfying $\hat{v}_0(z) = O(\mathrm{e}^{-az})$ for $z >> 1$ with $a \in (\sigma, \sigma_+)$ and $a < 2\sigma$, we have

$$\lim_{z\to+\infty} |\mathrm{e}^{az}\hat{v}(t,z)| \le C_0\mathrm{e}^{-\delta_a t}\|\mathrm{e}^{az}\hat{v}_0(z)\|_{\mathrm{L}_\infty(\mathbb{R})},\ t > 0,$$

which with Theorem 3 further implies that

$$\|u(t,z+ct,y) - V_c(z)\psi_0(y)\|_{X_a} \to 0 \text{ as } t \to +\infty,$$

then for large enough t we can apply Theorem 1 to yield the exponential decay in time of $\|u(t,z+ct,y) - V_c(z)\psi_0(y)\|_{X_a}$, which completes the proof of Theorem 4.

4.2. **Spreading speed of the solution with the Bramson logarithmic delay when the initial value has compact support.** In this subsection we investigate the spreading speed and asymptotic behavior of the solution of (1.3) with nonnegative compact supported initial value, in [4] it has been proved that the spreading speed of the solution must be the minimal speed $2\sqrt{1-\lambda_0}$, in this paper we try to prove that the propagation of the solution for (1.3) with bounded Ω still has Bramson's type of delay estimate, which also extends some classical results for the scalar Fisher-KPP equation to the nonlocal model (1.3). By Theorem 2 and estimate (3.13), to prove Theorem 5 it suffices to investigate the asymptotic behavior of solution $v(t,x)$ to the heterogenous Fisher type equation (4.1) with compact supported initial value. After re-scaling of the coordinates: $x \mapsto \sqrt{1-\lambda_0}x$ and $t \mapsto (1-\lambda_0)t$, it is easy to see that in the new coordinates $v(t,x)$ satisfies the equation (4.1) with $\lambda_0 = 0$, thus in the following of this subsection we just investigate the Cauchy problem of (4.1) with $\lambda_0 = 0$, i.e.

$$\begin{cases} v_t - v_{xx} = v(1 - b_0(t,x) - v), & t > 0,\ x \in \mathbb{R}, \\ v(0,x) = v_0^*(x), & x \in \mathbb{R}, \end{cases} \tag{4.22}$$

where $b_0(t,x)$ satisfies (3.13), which with Theorem 2 implies that for any given nonnegative bounded initial value $u_0(x,y)$ there exist positive constants C_0 and δ_0, such that

$$\|b_0(t,\cdot)\|_{L_\infty(\mathbb{R})} \le C_0 e^{-\delta_0 t},\ \ t > 0,$$

thus

$$\tilde{b}(t) = \sup_{x\in\mathbb{R}} |b_0(t,x)| \in L_1([0,\infty)). \tag{4.23}$$

Denote $b_1(t) = \tilde{b}(t)$, $b_2(t) = -\tilde{b}(t)$, and $v_i(t,x) = e^{\int_0^t b_i(s)ds} v(t,x)$ $(i = 1,2)$, it is easy to see that $v_i(t,x)$ satisfies

$$\frac{\partial}{\partial t} v_1 - \frac{\partial^2}{\partial x^2} v_1 \ge v_1(1 - v_1),\ \ t > 0, x \in \mathbb{R},$$

and

$$\frac{\partial}{\partial t} v_2 - \frac{\partial^2}{\partial x^2} v_2 \le v_2(1 - v_2),\ \ t > 0, x \in \mathbb{R},$$

Let $\Phi(x-2t)$ be the traveling wave solution with the minimal speed of Fisher equation $u_t = u_{xx} + u(1-u)$ satisfying

$$\begin{cases} \Phi''(z) + 2\Phi'(z) + \Phi(z)(1 - \Phi(z)) = 0, & z \in \mathbb{R}, \\ \Phi(-\infty) = 1, \quad \Phi(+\infty) = 0, \\ \Phi(0) = \frac{1}{2}. \end{cases} \tag{4.24}$$

Let $\tilde{v}(t,x)$ be the unique solution of

$$\begin{cases} \tilde{v}_t - \tilde{v}_{xx} = \tilde{v}(1 - \tilde{v}), & t > 0,\ x \in \mathbb{R}, \\ \tilde{v}(0,x) = v_0^*(x), & x \in \mathbb{R}, \end{cases}$$

Then by comparison principle it yields that

$$e^{-\int_0^t \tilde{b}(s)\,ds} \tilde{v}(t,x) \le v(t,x) \le e^{\int_0^t \tilde{b}(s)\,ds} \tilde{v}(t,x). \tag{4.25}$$

Theorem 5 is a consequence of Theorem 2 and the following theorem.

Theorem 4.2. *Let $b_0(t,x)$ satisfy (3.13) and let v be a solution of (4.22). There exists a constant $C \ge 0$ and two functions $\xi_\pm(0,\infty) \to \mathbb{R}$ such that $|\xi(t)| \le C$ for all $t > 0$, and*

$$\lim_{t\to+\infty} \sup_{z\in\mathbb{R}^+} \left| v(z + 2t - \tfrac{3}{2}\log t, t) - \Phi(z + \xi_+(t)) \right| = 0, \tag{4.26}$$

and

$$\lim_{t\to+\infty} \sup_{z\in\mathbb{R}^-} \left| v(z - (2t - \tfrac{3}{2}\log t), t) - \Phi(-z + \xi_-(t)) \right| = 0. \tag{4.27}$$

Proof. By Theorem 1.1 of [20] and (4.25), we have

$$\liminf_{t\to+\infty}\left(\min_{0\le x\le 2t-\frac{3}{2}\log t-C} v(t,x)\right)\ge e^{-\|\tilde{b}\|_{L^1}}\liminf_{t\to+\infty}\left(\min_{0\le x\le 2t-\frac{3}{2}\log t-C}\tilde{v}_1(t,x)\right)>0,\quad \text{as}\quad C\to+\infty,$$

and

$$\limsup_{t\to+\infty}\left(\max_{x\ge 2t-\frac{3}{2}\log t+C} v(t,x)\right)\le e^{\|\tilde{b}\|_{L^1}},\ \text{for } C>>1,$$
$$\limsup_{t\to+\infty}\left(\max_{x\ge 2t-\frac{3}{2}\log t+C}\tilde{v}_2(t,x)\right)\to 0,\ \ \text{for } C>>1.$$

Furthermore, by Propositions 2.3 and 3.1 of [20], there exist positive constants κ and ρ such that

$$\kappa z e^{-z}\le v(t,2t-\tfrac{3}{2}\log t+z)\le \rho(z+1)e^{-z},\ \ t\ge 1, 0\le z\le\sqrt{t}. \tag{4.28}$$

We can then repeat the proof of [20, Theorem 1.2] to prove (4.26). (4.27) can be proved by the same argument. □

Remark 4.2. If the nonnegative initial value $v_0^*(x)$ satisfies the assumption

$$0<\underline{q}_0<\liminf_{x\to-\infty} v_0^*(x)\le\limsup_{x\to-\infty} v_0^*(x)<\overline{q}_0,\ \text{for } x\le 0; \tag{4.29}$$

and

$$v_0^*(x)\equiv 0,\ \ \text{for}\ \ x>>1,$$

then estimate (4.26) is still valid and it can be proved that the solution $v(t,x)$ of (4.22) tends to $V_{c^*}(x-c^*t)$ in the following weak way

$$\lim_{t\to\infty}\sup_{x\in\mathbb{R}}|v(t,x)-V_{c^*}(x-c^*t-\eta(t))|=0, \tag{4.30}$$

with $\eta(t)=-\frac{3}{2}\ln t+O(1)$ for all $t>0$, which with Theorem 2 also means that the solution $u(t,x,y)$ of the nonlocal equation (1.3) tends to the cylinder wave $V_{c^*}(x-ct)\psi_0(y)$ in the similar weak sense.

Remark 4.3. If the initial value $v_0^*(x)$ satisfies (4.29) and decays with the same exponential rate as that of $V_{c^*}(x)$ at $x=+\infty$, i.e.

$$v_0^*(x)\sim xe^{-2x}\sim V_{c^*}(x),\ \ x\to+\infty,\ c^*=2,$$

due to the exponential decay $b_0(t,x)$ in time and the initial assumption, by constructing appropriate sub and supper solution to heterogeneous Fisher type equation $v_t=v_{xx}+v(1-b_0(t,x)-v)$, it is naturally expected that the shift $\eta(t)$ in (4.30) can be uniformly bounded for all $t>0$, and we conjecture that the shift $\eta(t)$ has a limit as $t\to+\infty$ if $v_0^*(x)\equiv V_{c^*}(x)$, for $x>>1$.

If $v_0^*(x)$ decays faster than $V_{c^*}(x)$ at $x=+\infty$, such as

$$v_0^*(x)=O(\mathrm{e}^{-2x}),\ \ x\to+\infty, \tag{4.31}$$

by virtue of (4.25) and Theorems 5 and 4.2, by applying comparison argument it can be proved that the spreading speed of the solution is still the critical speed. We conjecture that the estimate (4.30) is still valid with $\eta(t)$ satisfying $\frac{\eta(t)}{t}\to 0$ as $t\to+\infty$.

It is well known that for classical Fisher equation in one dimensional space, in [13, 22] the authors give more detailed description on the asymptotic behavior of solution and the spreading of the level set of solution, which are classified by the decay rate of the initial value $v_0^*(x)$ near $z=+\infty$. However due to the fact that the comparison principle can't be applied directly to the nonlocal model (1.3) or to the nonlinear heterogenous equation (4.22), some powerful techniques applied in [13, 22], which are based on the comparison principle for nonlinear homogeneous parabolic equation, can't be applied directly to the nonlocal model (1.3) or to equation (4.22) with a heterogeneous term. For the typical case when the initial value is compact supported (or a heaviside function), it is unknown that whether the bounded shifts $\xi_\pm(t)$ in Theorem 4.2 have limits, which may be not true for the nonlocal model (1.3) and the above mentioned conjectures are also open problems.

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