Nonexistence of Nonconstant Steady-State Solutions in a Triangular Cross-Diffusion Model

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Abstract

In this paper we study the Shigesada-Kawasaki-Teramoto model for two competing species with triangular cross-diffusion. We determine explicit parameter ranges within which the model exclusively possesses constant steady state solutions.

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1 Introduction

We consider the problem

$$\begin{cases} \Delta[(d_1 + a_{11}u + a_{12}v)u] + u(1-u-a_1 v) = 0, & x \in \Omega, \\ d_2 \Delta v + v(1-v-a_2 u) = 0, & x \in \Omega, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 1$, with smooth boundary, where $d_1 > 0, d_2 > 0, a_{12} \geq 0, a_{11} \geq 0, a_1 \geq 0$ and $a_2 \geq 0$ are parameters, and where $\nu = \nu_\Omega$ denotes the outward normal vector field on $\partial\Omega$. Our goal consists in identifying parameter ranges within which the system (1.1) exclusively possesses spatially homogeneous solutions. Thinking of u and v as representing population densities, we will focus here on nonnegative solutions, where throughout the sequel by a *nonnegative classical solution* of (1.1) we mean a pair $(u, v) \in (C^1(\overline{\Omega}) \cap C^2(\Omega))^2$ which is such that $u \geq 0$ and $v \geq 0$ in $\overline{\Omega}$, and that each identity in (1.1) is satisfied in the pointwise sense.

System (1.1) is a special case of the Shigesada-Kawasaki-Teramoto model (abbreviated as SKT henceforth), which was proposed in [13] to describe the spatial segregation of two competing species. There have been extensive studies on the existence of non-constant positive steady states of the SKT model; See [3, 4, 5, 6, 7, 8, 9, 12, 14, 15, 16]. Surveys of the SKT model can be found in [10, 11, 17]. Some of these works also investigate the non-existence of non-constant positive steady states of the SKT model. For instance, in the weak competition case (i.e., $a_1 < 1$ and $a_2 < 1$), it is shown in [5] that when $a_{11} = 0$, system (1.1) has no non-constant positive solution if one of the following three quantities is small: a_{12}/d_1, a_{12}/d_2, $a_{12}/\sqrt{d_1 d_2}$. The main goal of the paper is to provide *explicit* bounds for various parameters such that system (1.1) has no non-constant positive solution. These results will facilitate further understanding of the global dynamics of system (1.1).

Main results. The first class of our main results in this direction addresses the case when the quantity u is not influenced by any self-diffusion in the sense that the coefficient a_{11} in the first equation from (1.1) vanishes. In this situation, we shall establish two sufficient criteria for nonexistence of inhomogeneous solutions in cases when one of the competition parameters exceeds the critical value 1, whereas the other does not. According to the asymmetry in (1.1) induced by the assumption therein that only the first population is capable of cross-diffusive migration, our respective assumptions (1.3) and (1.6) on the further system parameters will be substantially different: When the second population has a competitive advantage in that (1.2) holds, we will only require the smallness condition (1.3) on its diffusion rate, without any restriction on the strength of cross-diffusion; in the opposite case (1.5), however, our hypothesis (1.6) will additionally involve the cross-diffusion parameter a_{12}. More precisely:

Theorem 1.1 *Suppose that $a_{11} = 0, a_{12} \geq 0, d_1 > 0, d_2 > 0, a_1 \geq 0$ and $a_2 \geq 0$, and that (u, v) is a nonnegative classical solution of (1.1).*

i) If

$$a_1 > 1 > a_2, \tag{1.2}$$

and if moreover

$$d_1 \geq d_2, \tag{1.3}$$

then

$$\text{either } (u,v) \equiv (0,0), \qquad \text{or } (u,v) \equiv (1,0), \qquad \text{or } (u,v) \equiv (0,1). \tag{1.4}$$

ii) In the case when

$$a_1 < 1 < a_2 \tag{1.5}$$

as well as

$$d_1 + 2a_{12} \le d_2, \tag{1.6}$$

again (1.4) is valid.

Next addressing the full system possibly containing self-diffusion in the first component, we shall detect an alternative set of conditions, essentially reflecting suitable smallness of cross-diffusion, as sufficient for nonexistence of inhomogeneous solutions. In contrast to Theorem 1.1, the following statement inter alia covers the case of weak competition when both $a_1 < 1$ and $a_2 < 1$ and hence coexistence of both populations is possible already in the associated spatially homogeneous counterpart of (1.1).

Theorem 1.2 *Let $d_1 > 0, d_2 > 0, a_{11} \ge 0, a_{12} \ge 0, a_1 > 0$ and $a_2 > 0$, and suppose that $(u,v) \in (C^1(\overline{\Omega}) \cap C^2(\Omega))^2$ is a classical solution of (1.1) such that $u \ge 0$ and $v \ge 0$ in Ω.*

i) If

$$a_1 < 1 < a_2 \tag{1.7}$$

and

$$\frac{a_{12}}{d_2} < 2 - a_1 + 2\sqrt{1-a_1}, \tag{1.8}$$

then (1.4) holds.

ii) In the case when

$$a_1 < 1 \qquad \text{and} \qquad a_2 < 1 \tag{1.9}$$

as well as

$$\frac{a_{12}}{d_2} < \min\left\{2 - a_1 + 2\sqrt{a_1}\,,\ \frac{a_1(1-a_1a_2)}{1-a_2}\right\}, \tag{1.10}$$

it follows that

$$\text{either } (u,v) \equiv (0,0), \quad \text{or} \quad (u,v) \equiv (1,0), \quad \text{or} \quad (u,v) \equiv (0,1), \quad \text{or} \quad (u,v) \equiv \Big(\frac{1-a_1}{1-a_1a_2}, \frac{1-a_2}{1-a_1a_2}\Big). \tag{1.11}$$

iii) If

$$a_1 > 1 > a_2 \tag{1.12}$$

and

$$\frac{a_{12}}{d_2} < \frac{1}{a_2}, \tag{1.13}$$

then again (1.4) holds.

Plan of the paper. Our analysis will follow two approaches which both are at their core based on arguments from the context of the maximum principle for scalar elliptic equations, but which substantially differ from each other in their overall view on (1.1): In order to establish Theorem 1.1, we shall interpret (1.1) as actually determining the unknown (w, v), where $w := (d_1 + a_{11}u + a_{12}v)u$ depends linearly on u precisely under the hypothesis $a_{11} = 0$ made in Theorem 1.1. In this situation, it will turn out in Section 2 that under the respective assumptions, for a certain $g : (0, \infty) \to (0, \infty)$ and any given nonconstant solution, the function $\nabla \cdot (g^2(v)\nabla \frac{w}{g(v)})$ cannot attain any zero in Ω which is impossible due to the boundary conditions in (1.1).
In the derivation of Theorem 1.2, the tridiagonal structure of (1.1) is used through the observation that the first two equations therein are equivalent to the elliptic system given by

$$\begin{cases} 0 = (d_1 + 2a_{11}u + a_{12}v)\Delta u + 2(a_{11}\nabla u + a_{12}\nabla v) \cdot \nabla u \\ \qquad\qquad +u \cdot \Big\{(a_2\delta v - 1)u + 1 - (a_1 + \delta)v + \delta v^2\Big\}, & x \in \Omega, \\ 0 = d_2\Delta v + v(1 - v - a_2 u), & x \in \Omega, \end{cases} \tag{1.14}$$

where

$$\delta := \frac{a_{12}}{d_2} \ge 0. \tag{1.15}$$

Thereby rewritten in non-divergence form, the system becomes accessible to arguments based on the use of the maximum principle in a rather elementary manner, followed by appropriate exploitation of accordingly obtained algebraic inequalities (Section 3).

Before going into details, let us first observe that nonnegative classical solutions of (1.1) are actually strictly positive in each of their components in which they are nontrivial:

Proposition 1.3 *Let $d_1 > 0, d_2 > 0, a_{11} \ge 0, a_{12} \ge 0, a_1 \ge 0$ and $a_2 \ge 0$, and suppose that (u, v) is a nonnegative classical solution of (1.1). Then if $u \not\equiv 0$, we have $u > 0$ in $\overline{\Omega}$, and if $v \not\equiv 0$, then $v > 0$ in $\overline{\Omega}$.*

Proof. Using (1.14) we see that $\widetilde{u}(x, t) := u(x),\ x \in \overline{\Omega},\ t \ge 0$, trivially solves $\widetilde{u}_t = A(x)\Delta\widetilde{u} + B(x) \cdot \nabla\widetilde{u} + H(x)\widetilde{u}$ in $\Omega \times (0, \infty)$ and $\frac{\partial \widetilde{u}}{\partial \nu} = 0$ on $\partial\Omega \times (0, \infty)$ with $A(x) := d_1 + 2a_{11}u(x) + a_{12}v(x), B(x) := 2a_{11}\nabla u(x) + 2a_{12}\nabla v(x)$ and $H(x) := (a_2\delta v(x) - 1)u(x) + 1 - (a_1 + \delta)v(x) + \delta v^2(x)$, $x \in \overline{\Omega}$. Assuming $u \not\equiv 0$, we moreover obtain that $\widetilde{u}(\cdot, 0) \equiv u$ is nonnegative but nontrivial, so that by means of the strong maximum principle for scalar parabolic equations ([2, Theorem 10.13]) we infer that $\widetilde{u} > 0$ in $\overline{\Omega} \times (0, \infty)$, and that hence $u > 0$ in $\overline{\Omega}$. The corresponding property of v can be seen in a similar manner. $\square$

2 Proof of Theorem 1.1

Proof of Theorem 1.1. i) The proof is based on a contradiction argument. Assuming on the contrary that (1.1) admits a nonnegative classical solution (u, v) such that $u \not\equiv 0$ and $v \not\equiv 0$, using that $a_{11} = 0$ and that u and v are positive due to Proposition 1.3 we can rewrite (1.1) according to

$$\begin{cases} \frac{\Delta[(d_1 + a_{12}v)u]}{u} = -1 + u + a_1 v, & x \in \Omega, \\ \frac{d_2\Delta v}{v} = -1 + v + a_2 u, & x \in \Omega, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{2.1}$$

Writing

$$w := (d_1 + a_{12}v)u,$$

we thus see that thanks to (1.2),

$$\begin{aligned}
\frac{\Delta w}{w} = \frac{\Delta[(d_1+a_{12}v)u]}{[(d_1+a_{12}v)u]} &= \frac{1}{d_1+a_{12}v}\cdot\frac{\Delta[(d_1+a_{12}v)u]}{u} \\
&= \frac{1}{d_1+a_{12}v}\cdot(-1+u+a_1v) \\
&> \frac{1}{d_1+a_{12}v}\cdot(-1+v+a_2u) \\
&= \frac{1}{d_1+a_{12}v}\cdot\frac{d_2\Delta v}{v} \\
&= \frac{\Delta v}{(d+\beta v)v}, \qquad x\in\Omega, \qquad (2.2)
\end{aligned}$$

where $d := \frac{d_1}{d_2}$ satisfies $d \ge 1$ due to (1.3), and where $\beta := \frac{a_{12}}{d_2} \ge 0$.
We now let

$$g(s) := \Big(\frac{s}{d+\beta s}\Big)^{\frac{1}{d}}, \qquad s>0,$$

and readily verify that

$$g(s) > 0 \qquad \text{for all } s>0 \qquad (2.3)$$

and

$$g'(s) = \frac{s^{\frac{1}{d}-1}}{(d+\beta s)^{\frac{1}{d}+1}} > 0 \qquad \text{for all } s>0 \qquad (2.4)$$

as well as

$$g''(s) = -\frac{[(d-1)+2\beta s]\cdot s^{\frac{1}{d}-2}}{(d+\beta s)^{\frac{1}{d}+2}} < 0 \qquad \text{for all } s>0, \qquad (2.5)$$

the latter relying on the inequalities $d \ge 1$ and $\beta \ge 0$. Thus, defining

$$\varphi := \frac{w}{g(v)},$$

upon a direct calculation using (2.2), (2.4) and (2.5), we find that

$$\begin{aligned}
\nabla\cdot(g^2(v)\nabla\varphi) &= \nabla\cdot(g(v)\nabla w - wg'(v)\nabla v) \\
&= g(v)\Delta w - wg'(v)\Delta v - wg''(v)|\nabla v|^2 \\
&> g(v)\cdot\frac{w}{(d+\beta v)v}\Delta v - wg'(v)\Delta v - wg''(v)|\nabla v|^2 \\
&= w\Delta v\cdot\Big(\frac{g(v)}{(d+\beta v)v} - g'(v)\Big) - wg''(v)|\nabla v|^2 \\
&= -wg''(v)|\nabla v|^2 \\
&\ge 0 \qquad \text{for all } x\in\Omega, \qquad (2.6)
\end{aligned}$$

because $g(v) = v(d+\beta v)g'(v)$ by definition of g. On the other hand, for all $x \in \partial\Omega$ we have

$$\frac{\partial\varphi}{\partial\nu} = \frac{1}{g(v)}\frac{\partial w}{\partial\nu} - \frac{1}{g^2(v)}g'(v)\frac{\partial v}{\partial\nu} = \frac{1}{g(v)}\Big[a_{12}u\frac{\partial v}{\partial\nu} + (d_1 + a_{12}v)\frac{\partial u}{\partial\nu}\Big] - \frac{1}{g^2(v)}g'(v)\frac{\partial v}{\partial\nu} = 0$$

thanks to the zero Neumann boundary conditions in (2.1), and thus an integration by parts implies that

$$\int_\Omega \nabla\cdot(g^2(v)\nabla\varphi) = \int_{\partial\Omega} g^2(v)\frac{\partial\varphi}{\partial\nu} = 0.$$

This contradicts (2.6) and thereby readily establishes (1.4).

ii) The proof quite closely follows the above idea: If (1.1) had a nonnegative classical solution such that $u \not\equiv 0$ and $v \not\equiv 0$, then introducing the functions w, g and φ as in part i) we could proceed as in (2.2) to see that due to (1.5),

$$\frac{\Delta w}{w} < \frac{\Delta v}{(d+\beta v)v} \qquad \text{for all } x \in \Omega, \tag{2.7}$$

where $d := \frac{d_1}{d_2} > 0$ and $\beta := \frac{a_{12}}{d_2} \ge 0$. Accordingly,

$$\nabla\cdot(g^2(v)\nabla\varphi) < -wg''(v)|\nabla v|^2 \qquad \text{for all } x \in \Omega, \tag{2.8}$$

where now the function g satisfies

$$g(s) > 0 \quad \text{and} \quad g'(s) > 0 \qquad \text{for all } s > 0 \tag{2.9}$$

as well as

$$g''(s) = -\frac{[(d-1)+2\beta s]\cdot s^{\frac{1}{d}-2}}{(d+\beta s)^{\frac{1}{d}+2}} \qquad \text{for all } s > 0.$$

Here since $v \le 1$ by the maximum principle (cf. also Lemma 3.2 below), we see that

$$d - 1 + 2\beta v \le d - 1 + 2\beta = \frac{d_1 + 2a_{12} - d_2}{d_2} \le 0$$

thanks to the assumption (1.6), and thus

$$g''(v) \ge 0 \qquad \text{in } \Omega. \tag{2.10}$$

From this along with (2.8) we infer that

$$\nabla\cdot(g^2(v)\nabla\varphi) < 0 \qquad \text{for all } x \in \Omega, \tag{2.11}$$

and that hence

$$0 > \int_\Omega \nabla\cdot(g^2(v)\nabla\varphi) = \int_{\partial\Omega} g^2(v)\frac{\partial\varphi}{\partial\nu} = 0,$$

which is impossible. $\square$

3 Analysis of (1.14). Proof of Theorem 1.2

We next focus on the interpretation (1.14) of (1.1), where throughout the sequel we shall suppose without explicit further mentioning that $d_1 > 0, d_2 > 0, a_{11} \geq 0$ and $a_{12} \geq 0$, and where in most parts of our analysis we will consider the number δ appearing therein as an independent nonnegative parameter, not necessarily linked to a_{12} and d_2 in the style of (1.15). In this general setting, we shall separately obtain three results on nonexistence of inhomogeneous solutions to (1.14) which will finally imply the respective statements from Theorem 1.2 i), ii) and iii).

3.1 An application of the scalar maximum principle

Let us prepare our analysis of (1.14) by stating the following general consequence of the scalar maximum principle when combined with the Hopf boundary point lemma.

Lemma 3.1 *Let $A \in C^0(\overline{\Omega})$ be positive in $\overline{\Omega}$, and let $B \in C^0(\overline{\Omega};\mathbb{R}^n), F \in C^0(\overline{\Omega})$ and $z \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ be such that*

$$\begin{cases} A(x)\Delta z + B(x)\cdot\nabla z + F(x) = 0, & x \in \Omega, \\ \frac{\partial z}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{3.1}$$

Then there exists $x_0 \in \overline{\Omega}$ such that

$$z(x_0) = \max_{x\in\overline{\Omega}} z(x) \tag{3.2}$$

and

$$F(x_0) \geq 0. \tag{3.3}$$

Proof. The proof is similar to that of Proposition 2.2 in [5]. We write $M := \max_{x\in\overline{\Omega}} z(x)$ and first consider the case when $z < M$ inside Ω. Then there must exist $x_0 \in \partial\Omega$ such that $z(x_0) = M$, and since Ω satisfies an interior sphere condition, we may pick $y \in \Omega$ and $r > 0$ such that $\overline{B_r(y)} \cap \partial\Omega = \{x_0\}$. Now if (3.3) was false, then $F(x_0) < 0$, so that by continuity of F we could find $\varepsilon \in (0,r)$ such that $F < 0$ in $B_\varepsilon(x_0)\cap\Omega$. Since for $y_0 := \frac{x_0+y}{2}$ and $r_0 := \frac{\varepsilon}{2}$ one can easily verify that $B := B_{r_0}(y_0)$ satisfies $B \subset B_\varepsilon(x_0)\cap\Omega$ and $\overline{B}\cap\partial\Omega = \{x_0\}$, it then follows from (3.1) that $A(x)\Delta z + B(x)\cdot\nabla z = -F(x) > 0$ for all $x \in B$, whence the fact that $z|_B < M = z(x_0)$ allows for an application of the Hopf boundary point lemma ([1, Lemma 3.4]) which asserts that $\frac{\partial z}{\partial \nu_B}(x_0) > 0$ and hence contradicts the zero Neumann boundary condition from (3.1), because ν_B coincides with $\nu = \nu_\Omega$ at x_0.
Thus knowing that actually (3.3) must hold in this situation, we are left with the case when $M = \max_{x\in\Omega} z(x)$, in which $z(x_0) = M$ for some $x_0 \in \Omega$. But at this interior maximum point we necessarily have $\Delta z(x_0) \leq 0$ and $\nabla z(x_0) = 0$, whence (3.1) directly yields (3.3). □

This firstly implies in quite a straightforward manner the following pointwise upper and lower bounds for v in terms of the maximal and minimal values of u.

Lemma 3.2 *Let $a_1 > 0, a_2 > 0$ and $\delta \geq 0$, and assume that (u,v) is a nonnegative classical solution of (1.14) for which $v \not\equiv 0$. Then*

$$1 - a_2 M_1 \leq v(x) \leq 1 - a_2 m_1 \qquad \textit{for all } x \in \overline{\Omega}, \tag{3.4}$$

where $M_1 := \max_{x\in\overline{\Omega}} u(x)$ and $m_1 := \min_{x\in\overline{\Omega}} u(x)$.

Proof. In Lemma 3.1 taking $A \equiv d_2, B \equiv 0$ and $F(x) := v(x) \cdot (1 - v(x) - a_2 u(x))$, $x \in \overline{\Omega}$, we infer the existence of $x_0 \in \overline{\Omega}$ such that $v(x_0) = M := \max_{x \in \overline{\Omega}} v(x)$ and $M \cdot (1 - M - a_2 u(x_0)) \geq 0$. Since $M > 0$ according to our assumption that $v \not\equiv 0$, by definition of m_1 this implies that

$$M \leq 1 - a_2 u(x_0) \leq 1 - a_2 m_1 \tag{3.5}$$

and hence proves the second inequality in (3.4), whereas the first can be derived in quite a similar manner by applying Lemma 3.1 to $-v$ and relying on the strict positivity of v in $\overline{\Omega}$, as guaranteed by Proposition 1.3. □

Due to the more complicated kinetic term in the first equation from (1.14) when compared to the second, our respective conclusions from Lemma 3.1 for u naturally become more involved:

Lemma 3.3 *Let $a_1 > 0, a_2 > 0$ and $\delta \geq 0$, and suppose that (u, v) is a nonnegative classical solution of (1.14) fulfilling $u \not\equiv 0$. Then there exists $x_0 \in \overline{\Omega}$ such that $u(x_0) = M_1$ and*

$$\Big(1 - a_2 \delta v(x_0)\Big) \cdot M_1 \leq 1 - (a_1 + \delta) v(x_0) + \delta v^2(x_0), \tag{3.6}$$

where $M_1 := \max_{x \in \overline{\Omega}} u(x)$.

Proof. On the basis of (1.14), an application of Lemma 3.1 to $z(x) := u(x), A(x) := d_1 + 2a_{11} u(x) + a_{12} v(x), B(x) := 2a_{11} \nabla u(x) + 2a_{12} \nabla v(x)$ and $F(x) := u(x) \cdot \Big\{(a_2 \delta v(x) - 1) u(x) + 1 - (a_1 + \delta) v(x) + \delta v^2(x)\Big\}$, $x \in \overline{\Omega}$, shows that there exists $x_0 \in \overline{\Omega}$ such that $u(x_0) = M_1$ and $F(x_0) \geq 0$. Since M_1 is assumed to be positive, the latter implies that $(a_2 \delta v(x_0) - 1) M_1 + 1 - (a_1 + \delta) v(x_0) + \delta v^2(x_0) \geq 0$, which is equivalent to (3.6). □

Lemma 3.4 *Let $a_1 > 0, a_2 > 0$ and $\delta \geq 0$, and let (u, v) be a nonnegative classical solution of (1.14) such that $u \not\equiv 0$. Then there exists $x_0 \in \overline{\Omega}$ such that writing $m_1 := \min_{x \in \overline{\Omega}} u(x)$ we have $u(x_0) = m_1$ and*

$$\Big(1 - a_2 \delta v(x_0)\Big) \cdot m_1 \geq 1 - (a_1 + \delta) v(x_0) + \delta v^2(x_0). \tag{3.7}$$

Proof. Applying Lemma 3.1 to $-u$ yields $x_0 \in \overline{\Omega}$ such that $u(x_0) = m_1$ and

$$m_1 \cdot \Big\{ \Big(a_2 \delta v(x_0) - 1\Big) \cdot m_1 + 1 - (a_1 + \delta) v(x_0) - \delta v^2(x_0) \Big\} \leq 0.$$

As m_1 must be positive by Proposition 1.3, this entails that in fact

$$\Big(a_2 \delta v(x_0) - 1\Big) \cdot m_1 + 1 - (a_1 + \delta) v(x_0) - \delta v^2(x_0) \leq 0,$$

and that hence (3.7) is valid. □

3.2 Basic properties associated with the inequalities (3.6) and (3.7)

In order to prepare a suitable further exploitation of the inequalities from Lemma 3.2, Lemma 3.3 and Lemma 3.4, let us state some useful properties of the terms in (3.6) and (3.7) containing v.

Firstly, the role of the condition appearing in (1.8) and also in (1.10) is predicated on the following elementary but important observation on the quadratic polynomial appearing on the right-hand sides of (3.6) and (3.7).

Lemma 3.5 *Suppose that $a_1 \in (0,1)$, and that $\delta \ge 0$ is such that*

$$\delta < 2 - a_1 + 2\sqrt{1-a_1}. \tag{3.8}$$

Then

$$\psi(s) := 1 - (a_1+\delta)s + \delta s^2, \qquad s \in \mathbb{R}, \tag{3.9}$$

satisfies

$$\psi(s) > 0 \qquad \text{for all } s \in [0,1]. \tag{3.10}$$

PROOF. We evidently may assume that δ is positive and that hence ψ possesses two complex roots s_+ and s_- which are given by

$$s_\pm := \frac{a_1 + \delta \pm \sqrt{\delta^2 - 2(2-a_1)\delta + a_1^2}}{2\delta}, \tag{3.11}$$

and which are both nonreal when $\delta^2 - 2(2-a_1)\delta + a_1^2 < 0$, that is, when $\delta \in (\delta_-, \delta_+)$ with

$$\delta_\pm := 2 - a_1 \pm 2\sqrt{1-a_1},$$

whence (3.10) is obvious for any such δ. We are thus left with the case when $\delta \in (0, \delta_-]$, in which we first observe that necessarily $\delta \le a_1$ due to the fact that

$$a_1 - \delta_- = 2\sqrt{1-a_1} - 2(1-a_1) = 2\sqrt{1-a_1}\cdot(1-\sqrt{1-a_1}) \ge 0.$$

Consequently, for such choices of δ the number s_- from (3.11) satisfies

$$2\delta\cdot(s_- - 1) = a_1 - \delta - \sqrt{\delta^2 - 2(2-a_1)\delta + a_1^2} > 0,$$

because then

$$(a_1-\delta)^2 - \sqrt{\delta^2 - 2(2-a_1)\delta + a_1^2}^{\,2} = (a_1^2 - 2a_1\delta + \delta^2) - (\delta^2 - 4\delta + 2a_1\delta + a_1^2) = 4(1-a_1)\delta > 0$$

according to our assumption that $a_1 < 1$. Whenever $\delta \in (0, \delta_-]$, we thus obtain that $s_+ \ge s_- > 1$, which in view of (3.9) implies (3.10) also in this case. $\square$

Next, in view of (3.6) and (3.7) it is not surprising that a crucial role in our subsequent analysis will be played by the function ϕ defined on $(-\infty, \frac{1}{a_2\delta})$ by

$$\phi(s) := \frac{1-(a_1+\delta)s + \delta s^2}{1 - a_2\delta s} \equiv \frac{\psi(s)}{1-a_2\delta s}, \qquad s \in \Big(-\infty, \frac{1}{a_2\delta}\Big), \tag{3.12}$$

which has its derivative ϕ' given by

$$\phi'(s) = \frac{a_2\delta - a_1 - \delta + 2\delta s - a_2\delta^2 s^2}{(1-a_2\delta s)^2}, \qquad s \in \Big(-\infty, \frac{1}{a_2\delta}\Big). \tag{3.13}$$

Two elementary properties thereof are stated in the following.

Lemma 3.6 *Let $a_1 > 0, a_2 > 0$ and $\delta \ge 0$, and let ϕ be as defined in (3.12).*

i) If $1 - a_1a_2 - a_2\delta + a_2^2\delta \le 0$ or $\delta = 0$, then ϕ' has no zero in $(-\infty, \frac{1}{a_2\delta})$.

ii) If $1 - a_1a_2 - a_2\delta + a_2^2\delta > 0$ and $\delta > 0$, then ϕ' possesses precisely one zero in $(-\infty, \frac{1}{a_2\delta})$, which is attained at

$$s_\star := \frac{1 - \sqrt{1 - a_1a_2 - a_2\delta + a_2^2\delta}}{a_2\delta}. \tag{3.14}$$

Proof. Both statements can be verified on the basis of (3.13) in a straightforward manner. $\Box$

3.3 The case $a_1 < 1 < a_2$

Now in the situation addressed in Theorem 1.2 i), the function ϕ defined in (3.12) enjoys a further favorable property, asserting that its graph remains a certain decreasing line within a conveniently large interval.

Lemma 3.7 *Assume that $a_1 < 1$, $a_2 > 1$ and $\delta \ge 0$. Then the function ϕ from (3.12) satisfies*

$$\phi(s) > \frac{1-s}{a_2} \qquad \text{for all } s \in \Big[0, \frac{a_2 - 1}{(a_1a_2-1)_+}\Big). \tag{3.15}$$

Proof. The elementary derivation of this may be left to the reader. $\Box$

By means of a contradictory argument based on Lemma 3.2 and Lemma 3.4, we can thereby exclude the existence of inhomogeneous solutions in the framework described in Theorem 1.2 i).

Lemma 3.8 *Let $a_1 < 1$, $a_2 > 1$ and $\delta \ge 0$ be such that $\delta < 2 - a_1 + 2\sqrt{1-a_1}$, and suppose that (u, v) is a nonnegative classical solution of (1.14) satisfying $v \not\equiv 0$. Then $u \equiv 0$ and $v \equiv 1$.*

Proof. Let us assume on the contrary that $u \not\equiv 0$. Then Lemma 3.4 provides $x_0 \in \overline{\Omega}$ such that $u(x_0) = m_1 := \min_{x\in\overline{\Omega}} u(x)$ and that

$$\Big(1 - a_2\delta v(x_0)\Big) \cdot m_1 \ge 1 - (a_1 + \delta)v(x_0) + \delta v^2(x_0), \tag{3.16}$$

where in view of Lemma 3.2 we know that

$$0 \le v(x_0) \le 1 - a_2 m_1 \tag{3.17}$$

and that thus, in particular, $v(x_0) \le 1$. Therefore Lemma 3.5 applies so as to warrant that the right-hand side in (3.16) is positive, whence (3.16) necessarily requires that

$$v(x_0) < \frac{1}{a_2\delta}, \tag{3.18}$$

and that consequently

$$m_1 \geq \phi(v(x_0)) \tag{3.19}$$

by definition of ϕ. Now if $1 - a_1a_2 - a_2\delta + a_2^2\delta \leq 0$ and hence

$$(a_2 - 1)\delta \leq \frac{a_1a_2 - 1}{a_2}, \tag{3.20}$$

then Lemma 3.6 says that ϕ' has no zero in $(-\infty, \frac{1}{a_2\delta})$, so that since

$$\phi'(0) = (a_2 - 1)\delta - a_1 \leq \frac{a_1a_2 - 1}{a_2} - a_1 = -\frac{1}{a_2} < 0$$

by (3.20), it follows that ϕ decreases throughout $(-\infty, \frac{1}{a_2\delta})$. As (3.20) additionally entails that

$$\delta \leq \frac{a_1a_2 - 1}{a_2(a_2 - 1)} < \frac{a_2 - 1}{a_2(a_2 - 1)} = \frac{1}{a_2}$$

due to our assumption that $a_1 < 1$, we moreover see that $[0, 1 - a_2m_1] \subset [0, 1] \subset (-\infty, \frac{1}{a_2\delta})$, which means that ϕ is decreasing on $[0, 1 - a_2m_1]$ and that thus (3.19) together with (3.17) implies that

$$m_1 \geq \phi(1 - a_2m_1).$$

Again by definition of ϕ, this is equivalent to

$$\Big\{1 - a_2\delta \cdot (1 - a_2m_1)\Big\} \cdot m_1 \geq 1 - (a_1 + \delta) \cdot (1 - a_2m_1) + \delta(1 - a_2m_1)^2,$$

which in turn shows that

$$(1 - a_1a_2)m_1 \geq 1 - a_1 > 0,$$

which is impossible because for (3.20) we necessarily must have $1 - a_1a_2 < 0$.

We consequently see that (3.20) cannot hold, so that actually $\delta > \frac{a_1a_2-1}{a_2(a_2-1)}$. Then however, in the case $a_1a_2 > 1$ we have

$$\frac{1}{a_2\delta} < \frac{a_2 - 1}{a_1a_2 - 1} = \frac{a_2 - 1}{(a_1a_2 - 1)_+},$$

which trivially extends so as to remain valid in fact regardless of the sign of $a_1a_2 - 1$. Therefore, Lemma 3.7 may be applied and ensures that $\phi(s) > \frac{1-s}{a_2}$ for all $s \in [0, \frac{1}{a_2\delta})$, so that again relying on (3.17) and (3.18) we infer from (3.19) that

$$m_1 \geq \phi(v(x_0)) > \frac{1 - v(x_0)}{a_2},$$

which is absurd, for once more due to (3.17) we know that

$$m_1 \leq \frac{1 - v(x_0)}{a_2}.$$

In conclusion, we obtain that indeed $u \equiv 0$, whereupon Lemma 3.2 guarantees that according to our hypothesis $v \not\equiv 0$ we have $v \equiv 1$. $\square$

3.4 The case $a_1 < 1$ and $a_2 < 1$

Next, in the case of weak competition we will rely on the following counterpart of Lemma 3.7, the proof of which is again straightforward and hence may be omitted here.

Lemma 3.9 *Assume that $a_1 < 1$, $a_2 < 1$ and $\delta \ge 0$. Then the function ϕ from (3.12) satisfies*

$$\phi(s) > \frac{1-s}{a_2} \qquad \text{for all } s > \frac{a_2-1}{a_1a_2-1} \tag{3.21}$$

and

$$\phi(s) < \frac{1-s}{a_2} \qquad \text{for all } s \in \Big[0, \frac{a_2-1}{a_1a_2-1}\Big). \tag{3.22}$$

The following additional observation will enable us to avoid inconvenient case distinctions when applying Lemma 3.6 in the proof of Lemma 3.11 below.

Lemma 3.10 *Let $a_1 < 1, a_2 < 1$ and $\delta \ge 0$ be such that $\delta < 2 - a_1 + 2\sqrt{1-a_1}$. Then $1 - a_1a_2 - a_2\delta + a_2^2\delta > 0$.*

PROOF. Writing $\chi(\xi) := \frac{1-a_1\xi}{\xi(1-\xi)}$, $\xi \in (0,1)$, we compute $\chi'(\xi) = \frac{-1+2\xi-a_1\xi^2}{\xi^2(1-\xi)^2}$ for $\xi \in (0,1)$, and thus see that $\chi'(\xi) = 0$ if and only if $\xi = \xi_- := \frac{1-\sqrt{1-a_1}}{a_1}$. Since

$$\begin{aligned}
\chi(\xi_-) &= \frac{\sqrt{1-a_1}}{\frac{1-\sqrt{1-a_1}}{a_1} - \frac{1-2\sqrt{1-a_1}+1-a_1}{a_1^2}} \\
&= \frac{a_1^2\sqrt{1-a_1}}{(2-a_1)\sqrt{1-a_1}+2a_1-2} \\
&= \frac{a_1^2\sqrt{1-a_1}\cdot\Big\{(2-a_1)\sqrt{1-a_1}-2a_1+2\Big\}}{(2-a_1)^2(1-a_1)-(2a_1-2)^2} \\
&= \frac{2-3a_1+a_1^2+2(1-a_1)\sqrt{1-a_1}}{1-a_1} \\
&= 2-a_1+2\sqrt{1-a_1},
\end{aligned}$$

and since $\chi = \chi(\xi) \to +\infty$ as $\xi \to 0$ and as $\xi \to 1$, it thus follows that $\chi(\xi) \ge 2 - a_1 + 2\sqrt{1-a_1}$ for all $\xi \in (0,1)$. Therefore, our assumption on δ ensures that

$$1 - a_1a_2 - a_2\delta + a_2^2\delta = a_2(1-a_2)\cdot\Big(\chi(a_2)-\delta\Big) > 0,$$

as claimed. □

By deriving mutually identical upper and lower pointwise bounds for both components of a given nontrivial solution on the basis of Lemma 3.2, Lemma 3.4, Lemma 3.3 and, again, Lemma 3.5, we can proceed to establish the second statement from Theorem 1.2.

Lemma 3.11 *Let $a_1 < 1$, $a_2 < 1$ and $\delta \ge 0$ be such that $\delta < 2 - a_1 + 2\sqrt{1-a_1}$ and $\delta < \frac{a_1(1-a_1a_2)}{1-a_2}$, and assume that (u,v) is a nonnegative classical solution of (1.14) fulfilling $u \not\equiv 0$ and $v \not\equiv 0$. Then*

$$u \equiv \frac{1-a_1}{1-a_1a_2} \qquad \text{and} \qquad v \equiv \frac{1-a_2}{1-a_1a_2}. \tag{3.23}$$

PROOF. We once more write $m_1 := \min_{x\in\overline{\Omega}} u(x)$ and $M_1 := \max_{x\in\overline{\Omega}} u(x)$ and first claim that

$$m_1 \ge \frac{1-a_1}{1-a_1a_2}. \tag{3.24}$$

To see this, we recall that Lemma 3.4 provides $x_0 \in \overline{\Omega}$ such that $u(x_0) = m_1$ and

$$\Big(1 - a_2\delta v(x_0)\Big) \cdot m_1 \ge \psi(v(x_0)) = 1 - (a_1+\delta)v(x_0) + \delta v^2(x_0), \tag{3.25}$$

whereas Lemma 3.2 guarantees that

$$1 - a_2 M_1 \le v(x) \le 1 - a_2 m_1 \qquad \text{for all } x \in \overline{\Omega}. \tag{3.26}$$

In particular, we have $v(x_0) \le 1$ and hence $\psi(v(x_0)) > 0$ due to Lemma 3.5 and our assumption that $\delta < 2 - a_1 + 2\sqrt{1-a_1}$, whence (3.25) entails that

$$v(x_0) < \frac{1}{a_2\delta} \tag{3.27}$$

and

$$m_1 \ge \phi(v(x_0)). \tag{3.28}$$

We next note that according to Lemma 3.10 the inequality $\delta < 2 - a_1 + 2\sqrt{1-a_1}$ moreover implies that $1 - a_1a_2 - a_2\delta + a_2^2\delta > 0$, so that Lemma 3.6 says that in the case $\delta > 0$, the derivative ϕ' attains a zero at $s_\star := \frac{1-\sqrt{1-a_1a_2-a_2\delta+a_2^2\delta}}{a_2\delta}$. We now make use of our additional assumption that $\delta < \frac{a_1(1-a_1a_2)}{1-a_2}$, which namely warrants that

$$\sqrt{1-a_1a_2-a_2\delta+a_2^2\delta}^{\,2} - (1-a_1a_2)^2 = a_1a_2 - a_1^2a_2^2 - a_2\delta + a_2^2\delta = a_2 \cdot \Big\{a_1(1-a_1a_2) - (1-a_2)\delta\Big\} > 0.$$

Thus, $\sqrt{1-a_1a_2-a_2\delta+a_2^2\delta}^{\,2} > (1-a_1a_2)^2$ and hence

$$\sqrt{1-a_1a_2-a_2\delta+a_2^2\delta} < \frac{\sqrt{1-a_1a_2-a_2\delta+a_2^2\delta}^{\,2}}{1-a_1a_2} = 1 - \frac{a_2(1-a_2)\delta}{1-a_1a_2},$$

so that

$$s_\star > \frac{1 - \Big(1 - \frac{a_2(1-a_2)\delta}{1-a_1a_2}\Big)}{a_2\delta} = \frac{1-a_2}{1-a_1a_2}. \tag{3.29}$$

Therefore, the number $v(x_0)$ must satisfy

$$v(x_0) \le s_\star, \tag{3.30}$$

for otherwise $v(x_0) > \frac{1-a_2}{1-a_1a_2}$, whence Lemma 3.9 would imply that $\phi(v(x_0)) > \frac{1-v(x_0)}{a_2}$, which together with (3.26) and (3.28) would yield the absurd conclusion that

$$\frac{1-v(x_0)}{a_2} \ge m_1 \ge \phi(v(x_0)) > \frac{1-v(x_0)}{a_2}.$$

But now since $\phi'(0) = a_2\delta - a_1 - \delta < 0$ and hence ϕ decreases on $[0, s_\star]$ according to Lemma 3.6, a combination of (3.28) and (3.30) shows that

$$1 - a_2 m_1 \le 1 - a_2\phi(v(x_0)) \le 1 - a_2\phi(s_\star). \tag{3.31}$$

By (3.29) we can directly check that $1 - a_2\phi(s_\star) \le s_\star$ holds true, which together with (3.31) implies

$$1 - a_2 m_1 \le s_\star. \tag{3.32}$$

As ϕ decreases on $[0, s_\star]$, a combination of (3.30) with (3.32), (3.26) and (3.28) shows that in the considered case $\delta > 0$ we have

$$m_1 \ge \phi(v(x_0)) \ge \phi(1 - a_2 m_1), \tag{3.33}$$

which clearly continues to be valid also when $\delta = 0$ due to the evident fact that then $\phi' \equiv -a_1 < 0$. In both cases $\delta = 0$ and $\delta > 0$ we thus obtain that $m_1 \ge \phi(1 - a_2 m_1)$, which is equivalent to the inequality

$$\Big\{1 - a_2\delta \cdot (1 - a_2 m_1)\Big\} \cdot m_1 \ge 1 - (a_1 + \delta) \cdot (1 - a_2 m_1) + \delta(1 - a_2 m_1)^2$$

and thus to (3.24), because $a_1 a_2 < 1$.

We next improve our knowledge on v by using (3.24) along with (3.26) to infer that since $1 - a_2 \cdot \frac{1-a_1}{1-a_1 a_2} = \frac{1-a_2}{1-a_1 a_2}$, we actually have

$$v(x) \le \frac{1 - a_2}{1 - a_1 a_2} \qquad \text{for all } x \in \overline{\Omega}. \tag{3.34}$$

Therefore, if now we invoke Lemma 3.3 to find $x_1 \in \overline{\Omega}$ such that $u(x_1) = M_1$ and

$$\Big(1 - a_2\delta v(x_1)\Big) \cdot M_1 \le 1 - (a_1 + \delta)v(x_1) + \delta v^2(x_1), \tag{3.35}$$

we particularly know that again since $\delta < \frac{a_1(1-a_1a_2)}{1-a_2}$,

$$1 - a_2\delta v(x_1) \ge 1 - a_2\delta \cdot \frac{1-a_2}{1-a_1a_2} > 1 - a_2 \cdot \frac{a_1(1-a_1a_2)}{1-a_2} \cdot \frac{1-a_2}{1-a_1a_2} = 1 - a_1a_2 > 0.$$

Consequently, (3.35) in conjunction with the left inequality in (3.26) shows that

$$\frac{1 - v(x_1)}{a_2} \le M_1 \le \phi(v(x_1)) \tag{3.36}$$

and that hence necessarily

$$v(x_1) = \frac{1 - a_2}{1 - a_1 a_2}, \tag{3.37}$$

because if this was false then (3.34) would require that $v(x_1) < \frac{1-a_2}{1-a_1a_2}$, so that Lemma 3.9 would say that $\phi(v(x_1)) < \frac{1-v(x_1)}{a_2}$ and thereby contradict (3.36).

Knowing that (3.37) holds, however, we may conclude from Lemma 3.9 that $\phi(v(x_1)) = \frac{1-v(x_1)}{a_2}$, whence both inequalities in (3.36) must actually be identities and thus, in particular,

$$M_1 = \frac{1 - a_1}{1 - a_1 a_2}. \tag{3.38}$$

In view of (3.26), this guarantees that also

$$v(x) \geq 1 - a_2 M_1 = \frac{1-a_2}{1-a_1a_2} \qquad \text{for all } x \in \overline{\Omega},$$

so that collecting (3.24), (3.37), (3.34) and (3.38) establishes that both $u \equiv \frac{1-a_1}{1-a_1a_2}$ and $v \equiv \frac{1-a_2}{1-a_1a_2}$. □

3.5 The case $a_1 > 1 > a_2$

We finally concentrate on the case when the second population possesses the substantially stronger ability for competition formulated in Theorem 1.2 iii). Then the positivity properties of the right-hand sides in (3.6) and (3.7), as expressed in Lemma 3.5, evidently can no longer be expected. In fact, the following holds.

Lemma 3.12 *Let $a_1 > 1$, $a_2 < 1$ and $\delta \geq 0$, and let ϕ be as given by (3.12).*
i) If $1 - a_1a_2 - a_2\delta + a_2^2\delta \leq 0$, then $\phi' < 0$ on $[0, \frac{1}{a_2\delta})$.
ii) If $1 - a_1a_2 - a_2\delta + a_2^2\delta > 0$, then with $s_\star$ given by (3.14) we have

$$\phi(s_\star) < 0. \tag{3.39}$$

Proof. i) Since ϕ' has no zero in $(-\infty, \frac{1}{a_2\delta})$ by Lemma 3.6, the claim results upon the observation that $\phi'(0) = a_2\delta - a_1 - \delta < 0$ thanks to our assumption that $a_2 < 1$.

ii) Using that $a_1 > 1$, we first note that $0 < (\delta + a_1 - 2)^2 + 4(a_1 - 1) = \delta^2 + (2a_1 - 4)\delta + a_1^2$ and hence $4\delta < \delta^2 + 2a_1\delta + a_1^2$. Multiplying both sides by a_2^2 and then adding $4(1 - a_1a_2 - a_2\delta)$, we see that this is equivalent to

$$4(1 - a_1a_2 - a_2\delta + a_2^2\delta) < a_2^2\delta^2 + 2a_1a_2^2\delta + a_1^2a_2^2 + 4(1 - a_1a_2 - a_2\delta) = (2 - a_1a_2 - a_2\delta)^2. \tag{3.40}$$

Now since $a_1 > 1$ moreover entails that

$$0 < (1-a_2)^2 < 1 - 2a_2 + a_1a_2^2 = (2 - a_1a_2)(1 - a_2) - (1 - a_1a_2)$$

and thus

$$\frac{1 - a_1a_2}{a_2(1-a_2)} < \frac{2 - a_1a_2}{a_2},$$

from our hypothesis on δ it follows that $\delta < \frac{2-a_1a_2}{a_2}$ and that therefore $2 - a_1a_2 - a_2\delta > 0$. Consequently, (3.40) is equivalent to the inequality

$$2\sqrt{1 - a_1a_2 - a_2\delta + a_2^2\delta} < 2 - a_1a_2 - a_2\delta$$

and hence to

$$2(1 - a_1a_2 - a_2\delta + a_2^2\delta) < (2 - a_1a_2 - a_2\delta)\sqrt{1 - a_1a_2 - a_2\delta + a_2^2\delta},$$

so that (3.39) is a result of (3.12). □

In consequence of this and, again, of Lemma 3.2 and Lemma 3.3, we obtain the nonexistence feature claimed in Theorem 1.2 iii).

Lemma 3.13 *Let $a_1 > 1$, $a_2 < 1$ and $\delta \ge 0$ be such that $\delta < \frac{1}{a_2}$, and let (u,v) be a nonnegative classical solution of (1.14) satisfying $v \not\equiv 0$. Then $u \equiv 0$ and $v \equiv 1$.*

Proof. Assuming for contradiction that $M_1 := \max_{x\in\overline{\Omega}} u(x)$ be positive, from Lemma 3.2 we know that

$$(1-a_2M_1)_+ \le v(x) \le 1 \qquad \text{for all } x \in \overline{\Omega}, \tag{3.41}$$

which together with our assumption $\delta < \frac{1}{a_2}$ in particular ensures that $1 - a_2\delta v(x) \ge 1 - a_2\delta > 0$ for all $x \in \overline{\Omega}$. Therefore, Lemma 3.3 shows that

$$M_1 \le \max_{s\in[(1-a_2M_1)_+,1]} \phi(s), \tag{3.42}$$

from which we derive a contradiction as follows:

Firstly, if $M_1 \ge \frac{1}{a_2}$ than (3.42) reduces to the inequality

$$M_1 \le \max_{s\in[0,1]} \phi(s). \tag{3.43}$$

Since Lemma 3.6 and Lemma 3.12 warrant that regardless of the sign of $1 - a_1a_2 - a_2\delta + a_2^2\delta$,

$$\phi \text{ does not attain a maximum in } (0,1), \tag{3.44}$$

and since

$$\phi(1) = \frac{1-a_1}{1-a_2\delta} < 0, \tag{3.45}$$

it follows that $\max_{s\in[0,1]} \phi(s) = \phi(0) = 1$, whence (3.43) entails that $M_1 \le 1$, contrary to our assumption that $M_1 \ge \frac{1}{a_2} > 1$.

We therefore must have

$$M_1 < \frac{1}{a_2} \tag{3.46}$$

and thus

$$M_1 \le \max_{s\in[1-a_2M_1,1]} \phi(s),$$

which again by (3.44) and (3.45) implies that

$$M_1 \le \phi(1-a_2M_1),$$

that is,

$$\Big\{1 - a_2\delta\cdot(1-a_2M_1)\Big\}\cdot M_1 \le 1 - (a_1+\delta)\cdot(1-a_2M_1) + \delta\cdot(1-a_2M_1)^2 \tag{3.47}$$

according to the definition of ϕ. As (3.47) is equivalent to the inequality

$$(1-a_1a_2)M_1 \le 1 - a_1 < 0,$$

in the case $a_1a_2 \le 1$ this is evidently absurd, whereas if $a_1a_2 > 1$ we obtain from the assumption $a_2 < 1$ that

$$M_1 \ge \frac{a_1-1}{a_1a_2-1} > \frac{a_1-1}{a_1a_2-a_2} = \frac{1}{a_2}$$

which is impossible due to (3.46).

In conclusion, we infer that $M_1 = 0$ and hence $u \equiv 0$ and, by (3.41), $v \equiv 1$. $\square$

3.6 Proof of Theorem 1.2

PROOF of Theorem 1.2. All three statements readily result on collecting the respective outcomes of Lemma 3.8, Lemma 3.11 and Lemma 3.13. □

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