Hopf bifurcation in a delayed reaction-diffusion-advection population model*

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Abstract

In this paper, we investigate a reaction-diffusion-advection model with time delay effect. The stability/instability of the spatially nonhomogeneous positive steady state and the associated Hopf bifurcation are investigated when the given parameter of the model is near the principle eigenvalue of an elliptic operator. Our result implies that time delay can make the spatially nonhomogeneous positive steady state unstable for a reaction-diffusion-advection model, and the model can exhibit oscillatory pattern through Hopf bifurcation.

Keywords: Reaction-diffusion; Advection; Delay; Hopf bifurcation; Spatial Heterogeneity.

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1 Introduction

During the past thirty years, delay induced instability has been investigated extensively for homogeneous reaction-diffusion equations with delay effect, and the spatial homogeneous and nonhomogeneous periodic solutions can occur through Hopf bifurcation. For models with the homogeneous Neumann boundary conditions, researchers were mainly concerned with the Hopf bifurcation near the constant positive equilibrium, see [9, 13, 15, 18, 19, 22, 26, 28, 31, 32] and the references therein. For models with the homogeneous Dirichlet boundary conditions, the positive equilibrium is always spatially nonhomogeneous. Busenberg and Huang [2] first studied the Hopf bifurcation near such spatially nonhomogeneous positive equilibrium, and they found that, for the following prototypical single population model,

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = d\Delta u(x,t) + \lambda u(x,t)\left(1 - u(x,t-\tau)\right), & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0, \end{cases} \tag{1.1}$$

time delay τ can make the unique spatially nonhomogeneous positive steady state unstable and induce Hopf bifurcation. Then, many authors investigated the Hopf bifurcation of models with the homogeneous Dirichlet boundary conditions, see [27, 33, 34, 36, 37]. Moreover, we refer to [8, 10, 20, 21] and the references therein for the Hopf bifurcation of models with the nonlocal delay effect and the homogenous Dirichlet boundary conditions.

In model (1.1), all the parameters are constant. However, due to the heterogeneity of the environment, the population may have a tendency to move up or down along the gradient of the habitats [1]. Therefore, it is more realistic to have the following model,

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = \nabla \cdot \left[d\nabla u - au\nabla m\right] + u(x,t)\left[m(x) - u(x,t-r)\right], & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0, \end{cases} \tag{1.2}$$

where $u(x,t)$ represents the population density at location x and time t, $d > 0$ is the diffusion coefficient, time delay $r > 0$ represents the maturation time, and Ω

is a bounded domain in $\mathbb{R}^n$ $(1 \leq n \leq 3)$ with a smooth boundary $\partial\Omega$. Moreover, the intrinsic growth rate $m(x)$ is spatially dependent and may change sign, which means that, the intrinsic growth rate of the population is positive on favorable habitats and negative on unfavorable ones, and a measures the tendency of the population to move up or down along the gradient of $m(x)$. For $r = 0$, Cantrell and Cosner [3, 4] investigated the effects of spatial heterogeneity on the dynamics of model (1.2) for the case of $a = 0$, and Belgacem and Cosner [1] considered the case of $a \neq 0$. We also refer to [5, 11, 12, 25, 29, 30] and the references therein for the effects of spatial heterogeneity on single population and two competing populations models.

In this paper, we mainly investigate whether time delay r can induce Hopf bifurcation for reaction-diffusion-advection model (1.2). As in [1], letting $v = e^{(-a/d)m(x)}u$, $t = \tilde{t}/d$, dropping the tilde sign, and denoting $\lambda = 1/d$, $\alpha = a/d$, $\tau = dr$, system (1.2) can be transformed as follows:

$$\begin{cases} \dfrac{\partial v}{\partial t} = e^{-\alpha m(x)}\nabla \cdot [e^{\alpha m(x)}\nabla v] + \lambda v\left[m(x) - e^{\alpha m(x)}v(x, t-\tau)\right], & x \in \Omega,\ t > 0, \\ v(x,t) = 0, & x \in \partial\Omega,\ t > 0. \end{cases} \tag{1.3}$$

Throughout the paper, unless otherwise specified, $m(x)$ satisfies the following assumption

$(\mathbf{A_1})$ $m(x) \in C^2(\overline{\Omega})$, and $\max_{x\in\overline{\Omega}} m(x) > 0$.

The following eigenvalue problem

$$\begin{cases} -e^{-\alpha m(x)}\nabla \cdot [e^{\alpha m(x)}\nabla v] = -\Delta v - \alpha\nabla m \cdot \nabla v = \lambda m(x) v, & x \in \Omega, \\ v(x) = 0, & x \in \partial\Omega, \end{cases} \tag{1.4}$$

is crucial to derive our main results. It follows from [1, 6, 30] that, under assumption $(\mathbf{A_1})$, (1.4) has a unique principal eigenvalue $\lambda_* > 0$ admitting a strictly positive eigenfunction $\phi \in C_0^{1+\delta}(\overline{\Omega})$ for some $\delta \in (0,1)$. Then, we can obtain the similar results as the case of spatial homogeneity [2, 33]: for $\lambda \in (\lambda_*, \lambda^*]$, where $0 < \lambda^* - \lambda_* \ll 1$, there exists a sequence of values $\{\tau_n(\lambda)\}_{n=0}^{\infty}$, such that, when $\tau = \tau_n(\lambda)$, Eq. (1.3) occurs Hopf bifurcation at the unique spatially nonhomogeneous positive steady state. Note that $\lambda = 1/d$, where d is the diffusion coefficient of model (1.2). Then, we see that there exists $d_* < 1/\lambda_*$, such that for $d \in [d_*, 1/\lambda_*)$, there exists a sequence of values $\{r_n(d)\}_{n=0}^{\infty}$, such that Eq. (1.2) occurs Hopf bifurcation when delay $r = r_n(d)$.

The rest of the paper is organized as follows. In Section 2, we study the stability and Hopf bifurcation of the spatially nonhomogeneous positive steady state for Eq. (1.3). In Section 3, we derive an explicit formula, which can be used to determine the direction of the Hopf bifurcation and the stability of the bifurcating periodic orbits. In Section 4, we give some remarks on the model with zero-flux boundary condition, and some numerical simulations are illustrated to support the obtained theoretical results. As in [8, 10], throughout the paper, we also denote the spaces $X = H^2(\Omega) \cap H_0^1(\Omega)$, $Y = L^2(\Omega)$, $C = C([-\tau, 0], Y)$, and $\mathcal{C} = C([-1, 0], Y)$. Moreover, we denote the complexification of a linear space Z to be $Z_{\mathbb{C}} := Z \oplus iZ = \{x_1 + ix_2 |\ x_1, x_2 \in Z\}$, the domain of a linear operator L by $\mathscr{D}(L)$, the kernel of L by $\mathscr{N}(L)$, and the range of L by $\mathscr{R}(L)$. For Hilbert space $Y_{\mathbb{C}}$, we use the standard inner product $\langle u, v\rangle = \displaystyle\int_{\Omega} \overline{u}(x)v(x)dx$.

2 Stability and Hopf bifurcation

In this section, we first consider the existence of positive steady states of Eq. (1.3), which satisfy:

$$\begin{cases} \nabla \cdot [e^{\alpha m(x)}\nabla v] + \lambda e^{\alpha m(x)} v \left[m(x) - e^{\alpha m(x)} v\right] = 0, & x \in \Omega, \\ v(x) = 0, & x \in \partial\Omega. \end{cases} \tag{2.1}$$

Actually, it follows from [1, 30] that, for $\tau = 0$, model (1.3) has a unique positive steady state which is global attractive among non-trivial nonnegative solutions if $\lambda > \lambda_*$, and the trivial steady state is global attractive if $\lambda \le \lambda_*$. Denote

$$L := \nabla \cdot [e^{\alpha m(x)}\nabla] + \lambda_* e^{\alpha m(x)} m(x), \tag{2.2}$$

where $\lambda_* > 0$ is the unique principal eigenvalue of problem (1.4) admitting a strictly positive eigenfunction ϕ. Note that

$$X = \mathscr{N}(L) \oplus X_1, \ \ Y = \mathscr{N}(L) \oplus Y_1,$$

where

$$\begin{aligned} \mathscr{N}(L) =& \text{span}\{\phi\}, \ \ X_1 = \left\{ y \in X : \int_\Omega \phi(x) y(x) dx = 0 \right\}, \\ Y_1 =& \mathscr{R}(L) = \left\{ y \in Y : \int_\Omega \phi(x) y(x) dx = 0 \right\}. \end{aligned} \tag{2.3}$$

Then we can give a profile of the unique positive steady state near λ_*.

Theorem 2.1. *There exist $\lambda^* > \lambda_*$ and a continuously differential mapping $\lambda \mapsto (\xi_\lambda, \beta_\lambda)$ from $[\lambda_*, \lambda^*]$ to $X_1 \times \mathbb{R}^+$ such that, for $\lambda \in (\lambda_*, \lambda^*]$, the unique positive steady state of Eq.* (1.3) *has the following form*

$$u_\lambda = \beta_\lambda(\lambda - \lambda_*)[\phi + (\lambda - \lambda_*)\xi_\lambda]. \tag{2.4}$$

Moreover, for $\lambda = \lambda_*$,

$$\beta_{\lambda_*} = \frac{\displaystyle\int_\Omega m(x) e^{\alpha m(x)} \phi^2(x) dx}{\displaystyle\lambda_* \int_\Omega e^{2\alpha m(x)} \phi^3(x) dx}, \tag{2.5}$$

and $\xi_{\lambda_*} \in X_1$ is the unique solution of the following equation

$$L\xi + \phi\left(m(x)e^{\alpha m(x)} - \lambda_* \beta_{\lambda_*} e^{2\alpha m(x)} \phi\right) = 0, \tag{2.6}$$

where L is defined as in Eq. (2.2).

Proof. Noticing that

$$\lambda_* \int_\omega m(x) e^{\alpha m(x)} \phi^2(x) dx = \int_\Omega e^{\alpha m(x)} |\nabla \phi(x)|^2 dx > 0, \tag{2.7}$$

we see that β_{λ_*} is well defined and positive. It follows that

$$\phi\left(m(x)e^{\alpha m(x)} - \lambda_* \beta_{\lambda_*} e^{2\alpha m(x)} \phi\right) \in \mathscr{R}(L) = Y_1,$$

and hence ξ_{λ_*} is well defined. Substituting $u = \beta(\lambda-\lambda_*)\left[\phi + (\lambda-\lambda_*)\xi\right]$ into Eq. (2.1), we see that (β, ξ) satisfies

$$m(\xi, \beta, \lambda) = L\xi + m(x)e^{\alpha m(x)}\left[\phi + (\lambda-\lambda_*)\xi\right] - \lambda\beta e^{2\alpha m(x)}[\phi + (\lambda-\lambda_*)\xi]^2 = 0.$$

Noticing that Ω is a bounded domain in $\mathbb{R}^n (1 \le n \le 3)$ with a smooth boundary $\partial\Omega$, we see that X_1 is compactly imbedded into $C^\gamma(\overline{\Omega})$ for some $\gamma \in (0,1)$, and hence $m(\xi, \beta, \lambda)$ is a function from $X_1 \times \mathbb{R}^2$ to Y. It follows from Eqs. (2.5) and (2.6) that $m(\xi_{\lambda_*}, \beta_{\lambda_*}, \lambda_*) = 0$, and

$$D_{(\xi,\beta)}m(\xi_{\lambda_*}, \beta_{\lambda_*}, \lambda_*)[\eta, \epsilon] = L\eta - \lambda_*\epsilon e^{2\alpha m(x)}\phi^2,$$

where $D_{(\xi,\beta)}m(\xi_{\lambda_*}, \beta_{\lambda_*}, \lambda_*)[\eta, \epsilon]$ is the Fréchet derivative of m with respect to (ξ, β) at $(\xi_{\lambda_*}, \beta_{\lambda_*}, \lambda_*)$. One can easily check that $D_{(\xi,\beta)}m(\xi_{\lambda_*}, \beta_{\lambda_*}, \lambda_*)$ is a bijection from $X_1 \times \mathbb{R}$ to Y. Then, it follows from the implicit function theorem that there exist $\lambda^* > \lambda_*$ and a continuously differentiable mapping $\lambda \mapsto (\xi_\lambda, \beta_\lambda) \in X_1 \times \mathbb{R}^+$ such that

$$m(\xi_\lambda, \beta_\lambda, \lambda) = 0, \quad \lambda \in [\lambda_*, \lambda^*].$$

Therefore, $\beta_\lambda(\lambda-\lambda_*)[\phi + (\lambda-\lambda_*)\xi_\lambda]$ is a positive solution of Eq. (2.1). $\square$

Linearizing system (1.3) at u_λ, we have

$$\begin{cases} \dfrac{\partial v}{\partial t} = e^{-\alpha m(x)}\nabla \cdot [e^{\alpha m(x)}\nabla v] + \lambda\left[m(x) - e^{\alpha m(x)}u_\lambda\right]v \\ \quad - \lambda e^{\alpha m(x)}u_\lambda v(x, t-\tau), & x \in \Omega,\ t > 0, \\ v(x,t) = 0, & x \in \partial\Omega,\ t > 0. \end{cases} \tag{2.8}$$

It follows from [35] that the solution semigroup of Eq. (2.8) has the infinitesimal generator $A_\tau(\lambda)$ satisfying

$$A_\tau(\lambda)\Psi = \dot{\Psi}, \tag{2.9}$$

where

$$\mathscr{D}(A_\tau(\lambda)) = \{\Psi \in C_{\mathbb{C}} \cap C^1_{\mathbb{C}} :\ \Psi(0) \in X_{\mathbb{C}}, \dot{\Psi}(0) = e^{-\alpha m(x)}\nabla \cdot [e^{\alpha m(x)}\nabla\Psi(0)] + \lambda\left[m(x) - e^{\alpha m(x)}u_\lambda\right]\Psi(0) - \lambda e^{\alpha m(x)}u_\lambda\Psi(-\tau)\},$$

and $C^1_{\mathbb{C}} = C^1([-\tau, 0], Y_{\mathbb{C}})$. Moreover, $\mu \in \mathbb{C}$ is an eigenvalue of $A_\tau(\lambda)$, if and only if there exists $\psi(\neq 0) \in X_{\mathbb{C}}$ such that $\Delta(\lambda, \mu, \tau)\psi = 0$, where

$$\begin{aligned} &\Delta(\lambda, \mu, \tau)\psi : \\ =& e^{-\alpha m(x)}\nabla \cdot [e^{\alpha m(x)}\nabla\psi] + \lambda \left[m(x) - e^{\alpha m(x)}u_\lambda\right]\psi - \lambda e^{\alpha m(x)}u_\lambda \psi e^{-\mu\tau} - \mu\psi. \end{aligned} \tag{2.10}$$

We will show that the eigenvalues of $A_\tau(\lambda)$ could pass through the imaginary axis when time delay τ increases. Actually, one can easily check that $A_\tau(\lambda)$ has a purely imaginary eigenvalue $\mu = i\nu$ $(\nu > 0)$ for some $\tau \geq 0$, if and only if

$$e^{-\alpha m(x)}\nabla \cdot [e^{\alpha m(x)}\nabla\psi] + \lambda \left[m(x) - e^{\alpha m(x)}u_\lambda\right]\psi - \lambda e^{\alpha m(x)}u_\lambda \psi e^{-i\theta} - i\nu\psi = 0 \tag{2.11}$$

is solvable for some value of $\nu > 0$, $\theta \in [0, 2\pi)$, and $\psi(\neq 0) \in X_{\mathbb{C}}$. First, we give the following estimates for solutions of (2.11).

Lemma 2.2. *If $(\nu_\lambda, \theta_\lambda, \psi_\lambda)$ solves Eq.* (2.11) *with $\nu_\lambda > 0$, $\theta_\lambda \in [0, 2\pi)$, and $\psi_\lambda(\neq 0) \in X_{\mathbb{C}}$, then*

$$\nu_\lambda \int_\Omega e^{\alpha m(x)}|\psi_\lambda|^2 dx = \lambda \sin\theta_\lambda \int_\Omega e^{2\alpha m(x)}u_\lambda|\psi_\lambda|^2 dx, \tag{2.12}$$

and $\dfrac{\nu_\lambda}{\lambda - \lambda_*}$ *is bounded for $\lambda \in (\lambda_*, \lambda^*]$.*

Proof. Substituting $(\nu_\lambda, \theta_\lambda, \psi_\lambda)$ into Eq. (2.11), multiplying (2.11) by $e^{\alpha m(x)}\overline{\psi}_\lambda$, and integrating the result over Ω, we have

$$\begin{aligned} &\left\langle \psi_\lambda, \nabla \cdot [e^{\alpha m(x)}\nabla\psi_\lambda]\right\rangle + \lambda \int_\Omega \left[m(x)e^{\alpha m(x)} - e^{2\alpha m(x)}u_\lambda\right]|\psi_\lambda|^2 dx \\ &- \lambda \int_\Omega e^{2\alpha m(x)}u_\lambda|\psi_\lambda|^2 dx e^{-i\theta_\lambda} - i\nu_\lambda \int_\Omega e^{\alpha m(x)}|\psi_\lambda|^2 dx = 0. \end{aligned}$$

Noticing that

$$\left\langle \psi_\lambda, \nabla \cdot [e^{\alpha m(x)}\nabla\psi_\lambda]\right\rangle = -\int_\Omega e^{\alpha m(x)}|\nabla\psi_\lambda|^2 dx < 0,$$

we see that Eq. (2.12) holds. Therefore,

$$\frac{\nu_\lambda}{\lambda - \lambda_*} = \frac{\lambda \sin\theta_\lambda \int_\Omega e^{2\alpha m(x)}u_\lambda|\psi_\lambda|^2 dx}{(\lambda - \lambda_*)\int_\Omega e^{\alpha m(x)}|\psi_\lambda|^2 dx} \leq \lambda|\beta_\lambda|e^{\alpha \max_\Omega m(x)}\left[\|\phi\|_\infty + (\lambda - \lambda_*)\|\xi_\lambda\|_\infty\right].$$

It follows from the continuity of $\lambda \mapsto (\|\xi_\lambda\|_\infty, \beta_\lambda)$ that $\dfrac{\nu_\lambda}{\lambda - \lambda_*}$ is bounded for $\lambda \in (\lambda_*, \lambda^*]$. □

The following result is similar to Lemma 2.3 of [2] and we omit the proof here.

Lemma 2.3. *If* $z \in X_{\mathbb{C}}$ *and* $\langle \phi, z\rangle = 0$, *then* $|\langle Lz, z\rangle| \geq \lambda_2 \|z\|^2_{Y_{\mathbb{C}}}$, *where* λ_2 *is the second eigenvalue of operator* $-L$.

Now, for $\lambda \in (\lambda_*, \lambda^*]$, letting

$$\begin{aligned} &\psi = r\phi + (\lambda - \lambda_*)z, \quad z \in (X_1)_{\mathbb{C}}, \quad r \geq 0, \\ &\|\psi\|^2_{Y_{\mathbb{C}}} = r^2\|\phi\|^2_{Y_{\mathbb{C}}} + (\lambda-\lambda_*)^2\|z\|^2_{Y_{\mathbb{C}}} = \|\phi\|^2_{Y_{\mathbb{C}}}, \end{aligned} \tag{2.13}$$

and substituting (2.4), (2.13) and $\nu = (\lambda - \lambda_*)h$ into Eq. (2.11), we see that (ν, θ, ψ) solves Eq. (2.11), where $\nu > 0$, $\theta \in [0, 2\pi)$ and $\psi \in X_{\mathbb{C}}(\|\psi\|^2_{Y_{\mathbb{C}}} = \|\phi\|^2_{Y_{\mathbb{C}}})$, if and only if the following system:

$$\begin{cases} g_1(z, r, h, \theta, \lambda) := Lz - \lambda\beta_\lambda e^{2\alpha m(x)}\left[\phi + (\lambda-\lambda_*)\xi_\lambda\right]\left[r\phi + (\lambda-\lambda_*)z\right]e^{-i\theta} \\ +[r\phi + (\lambda-\lambda_*)z]\left\{m(x)e^{\alpha m(x)} - \lambda\beta_\lambda e^{2\alpha m(x)}\left[\phi + (\lambda-\lambda_*)\xi_\lambda\right] - ihe^{\alpha m(x)}\right\} = 0 \\ g_2(z, r, \lambda) := (r^2 - 1)\|\phi\|^2_{Y_{\mathbb{C}}} + (\lambda-\lambda_*)^2\|z\|^2_{Y_{\mathbb{C}}} = 0 \end{cases} \tag{2.14}$$

is solvable for some value of $z \in (X_1)_{\mathbb{C}}$, $h > 0$, $r \geq 0$, and $\theta \in [0, 2\pi)$. Define $G : (X_1)_{\mathbb{C}} \times \mathbb{R}^4 \to Y_{\mathbb{C}} \times \mathbb{R}$ by $G = (g_1, g_2)$, and we find that $G(z, r, h, \theta, \lambda) = 0$ is uniquely solvable for $\lambda = \lambda_*$.

Lemma 2.4. *The following equation*

$$\begin{cases} G(z, r, h, \theta, \lambda_*) = 0 \\ z \in (X_1)_{\mathbb{C}},\ h > 0\ r \geq 0,\ \theta \in [0, 2\pi) \end{cases} \tag{2.15}$$

has a unique solution $(z_{\lambda_*}, r_{\lambda_*}, h_{\lambda_*}, \theta_{\lambda_*})$. *Here*

$$r_{\lambda_*} = 1, \ \ \theta_{\lambda_*} = \pi/2, \ \ h_{\lambda_*} = \frac{\int_\Omega m(x)e^{\alpha m(x)}\phi^2 dx}{\int_\Omega e^{\alpha m(x)}\phi^2(x)dx}, \tag{2.16}$$

and $z_{\lambda_*} \in (X_1)_{\mathbb{C}}$ *is the unique solution of*

$$Lz = -i\lambda_*\beta_{\lambda_*}e^{2\alpha m(x)}\phi^2 + ih_{\lambda_*}e^{\alpha m(x)}\phi - \phi\left(m(x)e^{\alpha m(x)} - \lambda_*\beta_{\lambda_*}e^{2\alpha m(x)}\phi\right), \tag{2.17}$$

where L *is defined as in Eq.* (2.2).

Proof. From Eq. (2.14), we see that $g_2(z,r,\lambda_*)=0$ if and only if $r=r_{\lambda_*}=1$. Note that

$$g_1(z,r_{\lambda_*},h,\theta,\lambda_*)=Lz-\lambda_*\beta_{\lambda_*}e^{2\alpha m(x)}\phi^2e^{-i\theta} \\ -ihe^{\alpha m(x)}\phi+\phi\left(m(x)e^{\alpha m(x)}-\lambda_*\beta_{\lambda_*}e^{2\alpha m(x)}\phi\right). \tag{2.18}$$

Then

$$\begin{cases} g_1(z,r_{\lambda_*},h,\theta,\lambda_*)=0 \\ z\in(X_1)_{\mathbb{C}},\ h>0\ r\geq 0,\ \theta\in[0,2\pi) \end{cases}$$

is solvable if and only if

$$\begin{cases} \lambda_*\beta_{\lambda_*}\int_\Omega e^{2\alpha m(x)}\phi^3dx\sin\theta=h\int_\Omega e^{\alpha m(x)}\phi^2dx \\ \lambda_*\beta_{\lambda_*}\int_\Omega e^{2\alpha m(x)}\phi^3dx\cos\theta=0 \end{cases} \tag{2.19}$$

is solvable for a pair (θ,h) with $h>0$ and $\theta\in[0,2\pi)$. This, combined with Eq. (2.5), leads to

$$\theta=\theta_{\lambda_*}=\pi/2,\quad h=h_{\lambda_*}=\frac{\lambda_*\beta_{\lambda_*}\int_\Omega e^{2\alpha m(x)}\phi^3dx}{\int_\Omega e^{\alpha m(x)}\phi^2dx}=\frac{\int_\Omega m(x)e^{\alpha m(x)}\phi^2dx}{\int_\Omega e^{\alpha m(x)}\phi^2dx}. \tag{2.20}$$

Consequently, $g_1(z,r_{\lambda_*},h_{\lambda_*},\theta_{\lambda_*},\lambda_*)=0$ has a unique solution z_{λ_*}, which satisfies Eq. (2.17). □

Then we solve $G=0$ for $\lambda\in(\lambda_*,\lambda^*]$.

Theorem 2.5. *There exist $\tilde{\lambda}^*>\lambda_*$ and a continuously differentiable mapping $\lambda\mapsto(z_\lambda,r_\lambda,h_\lambda,\theta_\lambda)$ from $[\lambda_*,\tilde{\lambda}^*]$ to $(X_1)_{\mathbb{C}}\times\mathbb{R}^3$ such that $G(z_\lambda,r_\lambda,r_\lambda,\theta_\lambda,\lambda)=0$. Moreover, for $\lambda\in[\lambda_*,\tilde{\lambda}^*]$,*

$$\begin{cases} G(z,r,h,\theta,\lambda)=0 \\ z\in(X_1)_{\mathbb{C}},\ h,\ r\geq 0,\ \theta\in[0,2\pi) \end{cases} \tag{2.21}$$

has a unique solution $(z_\lambda,r_\lambda,h_\lambda,\theta_\lambda)$.

Proof. Let $T=(T_1,T_2):(X_1)_{\mathbb{C}}\times\mathbb{R}^3\mapsto Y_{\mathbb{C}}\times\mathbb{R}$ be the Fréchet derivative of G with respect to (z,r,h,θ) at $(z_{\lambda_*},r_{\lambda_*},h_{\lambda_*},\theta_{\lambda_*},\lambda_*)$. Then,

$$T_1(\chi,\kappa,\epsilon,\vartheta)=L\chi-i\epsilon e^{\alpha m(x)}\phi+\vartheta\lambda_*\beta_{\lambda_*}e^{2\alpha m(x)}\phi^2 \\ +\kappa\phi\left[m(x)e^{\alpha m(x)}-\lambda_*\beta_{\lambda_*}e^{2\alpha m(x)}\phi-ih_{\lambda_*}e^{\alpha m(x)}+i\lambda_*\beta_{\lambda_*}e^{2\alpha m(x)}\phi\right],$$

$$T_2(\kappa)=2\kappa\|\phi\|^2_{Y_{\mathbb{C}}}.$$

One can easily check that T is a bijection from $(X_1)_{\mathbb{C}} \times \mathbb{R}^3$ to $Y_{\mathbb{C}} \times \mathbb{R}$. This, combined with the implicit function theorem, implies that there exist $\tilde{\lambda}^* > \lambda_*$ and a continuously differentiable mapping $\lambda \mapsto (z_\lambda, r_\lambda, h_\lambda, \theta_\lambda)$ from $[\lambda_*, \tilde{\lambda}^*]$ to $X_{\mathbb{C}} \times \mathbb{R}^3$ such that $G(z_\lambda, r_\lambda, h_\lambda, \theta_\lambda, \lambda) = 0$. To prove the uniqueness, we only need to verify that if $z \in (X_1)_{\mathbb{C}}$, r^λ, $h^\lambda > 0$, $\theta^\lambda \in [0, 2\pi)$, and $G(z^\lambda, r^\lambda, h^\lambda, \theta^\lambda, \lambda) = 0$, then

$$(z^\lambda, r^\lambda, h^\lambda, \theta^\lambda) \to (z_{\lambda_*}, r_{\lambda_*}, h_{\lambda_*}, \theta_{\lambda_*}) = \left(z_{\lambda_*}, 1, h_{\lambda_*}, \frac{\pi}{2}\right)$$

as $\lambda \to \lambda_*$ in the norm of $X_{\mathbb{C}} \times \mathbb{R}^3$. It follows from Lemma 2.2 and Eq. (2.14) that $\{h^\lambda\}, \{r^\lambda\}$ and $\{\theta^\lambda\}$ are bounded for $\lambda \in [\lambda_*, \tilde{\lambda}^*]$. Note that $\{\beta_\lambda\}$ and $\{\xi_\lambda\}$ are bounded for $\lambda \in [\lambda_*, \tilde{\lambda}^*]$. As in Theorem 2.4 of [2], we can obtain that there exist $M_1, M_2 > 0$ such that

$$\lambda_2 \|z^\lambda\|^2_{Y_{\mathbb{C}}} \leq |\langle Lz, z\rangle| \leq M_1 \|\phi\|_{Y_{\mathbb{C}}} \|z^\lambda\|_{Y_{\mathbb{C}}} + M_2(\lambda - \lambda_*)\|z^\lambda\|^2_{Y_{\mathbb{C}}},$$

where λ_2 is defined as in Lemma 2.3. Therefore, if $\tilde{\lambda}_*$ is sufficiently small, $\{z^\lambda\}$ is bounded in $Y_{\mathbb{C}}$ for $\lambda \in [\lambda_*, \tilde{\lambda}^*]$. Since the operator L^{-1} is bounded, we see that $\{z^\lambda\}$ is also bounded in $(X_1)_{\mathbb{C}}$, which implies that $\{(z^\lambda, r^\lambda, h^\lambda, \theta^\lambda) : \lambda \in (\lambda_*, \tilde{\lambda}^*]\}$ is precompact in $Y_{\mathbb{C}} \times \mathbb{R}^3$. Then, there exists a subsequence $\{(z^{\lambda^n}, r^{\lambda^n}, h^{\lambda^n}, \theta^{\lambda^n})\}$ such that

$$(z^{\lambda^n}, r^{\lambda^n}, h^{\lambda^n}, \theta^{\lambda^n}) \to (z^{\lambda_*}, r^{\lambda_*}, h^{\lambda_*}, \theta^{\lambda_*}) \text{ in } Y_{\mathbb{C}} \times \mathbb{R}^3, \quad \lambda^n \to \lambda_* \text{ as } n \to \infty.$$

Taking the limit of the equation $L^{-1} g_1(z^{\lambda^n}, r^{\lambda^n}, h^{\lambda^n}, \theta^{\lambda^n}, \lambda^n) = 0$ as $n \to \infty$, we see that $G(z^{\lambda_*}, r^{\lambda_*}, h^{\lambda_*}, \theta^{\lambda_*}, \lambda_*) = 0$. It follows from Lemma 2.4 that

$$(z^{\lambda_*}, r^{\lambda_*}, h^{\lambda_*}, \theta^{\lambda_*}) = (z_{\lambda_*}, r_{\lambda_*}, h_{\lambda_*}, \theta_{\lambda_*}).$$

This completes the proof. □

From Theorem 2.5, we derive the following result.

Theorem 2.6. *For each $\lambda \in (\lambda_*, \tilde{\lambda}^*]$, the following equation*

$$\begin{cases} \Delta(\lambda, i\nu, \tau)\psi = 0 \\ \nu \geq 0,\ \tau \geq 0,\ \psi(\neq 0) \in X_{\mathbb{C}} \end{cases}$$

has a solution (ν, τ, ψ)*, if and only if*

$$\nu = \nu_\lambda = (\lambda - \lambda_*)h_\lambda,\ \psi = c\psi_\lambda,\ \tau = \tau_n = \frac{\theta_\lambda + 2n\pi}{\nu_\lambda},\quad n = 0, 1, 2, \cdots, \tag{2.22}$$

where $\psi_\lambda = r_\lambda \phi + (\lambda - \lambda_*)z_\lambda$*,* c *is a nonzero constant, and* $z_\lambda, r_\lambda, h_\lambda, \theta_\lambda$ *are defined as in Theorem 2.5.*

In the following, we will always assume $\lambda \in (\lambda_*, \tilde{\lambda}^*]$ for simplicity, where $0 < \lambda^* - \lambda_* \ll 1$. Actually, the value of $\tilde{\lambda}^*$ may be chosen smaller than the one in Theorem 2.5, since further perturbation arguments are used. Now, we give some estimates to prove the simplicity of $i\nu_\lambda$.

Lemma 2.7. *Assume that* $\lambda \in (\lambda_*, \tilde{\lambda}^*]$*. Then, for* $n = 0, 1, 2, \cdots,$

$$S_n(\lambda) := \int_\Omega e^{\alpha m(x)} \psi_\lambda^2 dx - \lambda \tau_n e^{-i\theta_\lambda} \int_\Omega e^{2\alpha m(x)} u_\lambda \psi_\lambda^2 dx \neq 0, \tag{2.23}$$

where ψ_λ *is defined as in Theorem 2.6.*

Proof. It follows from Theorems 2.5 and 2.6 that $\theta_\lambda \to \pi/2$, $\tau_n(\lambda - \lambda_*) \to (\frac{\pi}{2} + 2n\pi)/h_{\lambda_*}$, $\psi_\lambda \to \phi$ in $X_\mathbb{C}$ as $\lambda \to \lambda_*$. This, combined with Eq. (2.20), yields

$$\begin{aligned}
&\lim_{\lambda \to \lambda_*} S_n(\lambda) \\
&= \int_\Omega e^{\alpha m(x)} \phi^2 dx + \frac{i\beta_{\lambda_*}\lambda_*}{h_{\lambda_*}} \left(\frac{\pi}{2} + 2n\pi\right) \int_\Omega e^{2\alpha m(x)} \phi^3 dx \\
&= \left[1 + i(\frac{\pi}{2} + 2n\pi)\right] \int_\Omega e^{\alpha m(x)} \phi^2 dx \neq 0.
\end{aligned} \tag{2.24}$$

This completes the proof. $\square$

Then, by virtue of Lemma 2.7, we obtain that $i\nu$ is simple as follows.

Theorem 2.8. *Assume that* $\lambda \in (\lambda_*, \tilde{\lambda}^*]$*. Then* $\mu = i\nu_\lambda$ *is a simple eigenvalue of* A_{τ_n} *for* $n = 0, 1, 2, \cdots$*, where* $i\nu_\lambda$ *and* τ_n *are defined as in Theorem 2.6.*

Proof. It follows from Theorem 2.6 that $\mathscr{N}[A_{\tau_n}(\lambda) - i\nu_\lambda] = \text{Span}[e^{i\nu_\lambda \theta}\psi_\lambda]$, where $\theta \in [-\tau_n, 0]$ and ψ_λ is defined as in Theorem 2.6. If $\phi_1 \in \mathscr{N}[A_{\tau_n}(\lambda) - i\nu_\lambda]^2$, then

$$[A_{\tau_n}(\lambda) - i\nu_\lambda]\phi_1 \in \mathscr{N}[A_{\tau_n}(\lambda) - i\nu_\lambda] = \text{Span}[e^{i\nu_\lambda \theta}\psi_\lambda].$$

Therefore, there exists a constant a such that

$$[A_{\tau_n}(\lambda) - i\nu_\lambda]\phi_1 = ae^{i\nu_\lambda\theta}\psi_\lambda,$$

which yields

$$\begin{aligned}
&\dot{\phi_1}(\theta) = i\nu_\lambda\phi_1(\theta) + ae^{i\nu_\lambda\theta}\psi_\lambda, \quad \theta \in [-\tau_n, 0],\\
&\dot{\phi_1}(0) = e^{-\alpha m(x)}\nabla \cdot [e^{\alpha m(x)}\nabla\phi_1(0)]\\
&\qquad + \lambda\left[m(x) - e^{\alpha m(x)}u_\lambda\right]\phi_1(0) - \lambda e^{\alpha m(x)}u_\lambda\phi_1(-\tau_n).
\end{aligned} \tag{2.25}$$

From the first equation of Eq. (2.25), we see that

$$\begin{aligned}
&\phi_1(\theta) = \phi_1(0)e^{i\nu_\lambda\theta} + a\theta e^{i\nu_\lambda\theta}\psi_\lambda,\\
&\dot{\phi_1}(0) = i\nu_\lambda\phi_1(0) + a\psi_\lambda.
\end{aligned} \tag{2.26}$$

Eq. (2.25) and Eq. (2.26) imply that

$$\begin{aligned}
&e^{\alpha m(x)}\Delta(\lambda, i\nu_\lambda, \tau_n)\phi_1(0)\\
=&\nabla \cdot [e^{\alpha m(x)}\nabla\phi_1(0)] - i\nu_\lambda e^{\alpha m(x)}\psi_1(0)\\
&+\lambda\left[m(x)e^{\alpha m(x)} - e^{2\alpha m(x)}u_\lambda\right]\phi_1(0) - \lambda e^{2\alpha m(x)}u_\lambda\phi_1(0)e^{-i\theta_\lambda}\\
=&ae^{\alpha m(x)}\left(\psi_\lambda - \lambda\tau_n u_\lambda\psi_\lambda e^{\alpha m(x)}e^{-i\theta_\lambda}\right).
\end{aligned} \tag{2.27}$$

Since $\Delta(\lambda, i\nu_\lambda, \tau_n)\psi_\lambda = 0$, we have $\Delta(\lambda, -i\nu_\lambda, \tau_n)\overline{\psi}_\lambda = 0$. This, combined with Eq. (2.27), yields

$$\begin{aligned}
0 &= \left\langle e^{\alpha m(x)}\Delta(\lambda, -i\nu_\lambda, \tau_n)\overline{\psi}_\lambda, \phi_1(0)\right\rangle = \left\langle\overline{\psi}_\lambda, e^{\alpha m(x)}\Delta(\lambda, i\nu_\lambda, \tau_n)\phi_1(0)\right\rangle\\
&= a\left(\int_\Omega e^{\alpha m(x)}\psi_\lambda^2 dx - \lambda\tau_n e^{-i\theta_\lambda}\int_\Omega\int_\Omega u_\lambda\psi_\lambda^2 e^{2\alpha m(x)}dx\right),
\end{aligned}$$

which implies that $a = 0$ from Lemma 2.7. Therefore,

$$\mathscr{N}[A_{\tau_n}(\lambda) - i\nu_\lambda]^j = \mathscr{N}[A_{\tau_n}(\lambda) - i\nu_\lambda], \;\; j = 2, 3, \cdots, \;\; n = 0, 1, 2, \cdots,$$

and $\lambda = i\nu_\lambda$ is a simple eigenvalue of A_{τ_n} for $n = 0, 1, 2, \cdots$. □

Note that $\mu = i\nu_\lambda$ is a simple eigenvalue of A_{τ_n}. It follows from the implicit function theorem that there are a neighborhood $O_n \times D_n \times H_n \subset \mathbb{R} \times \mathbb{C} \times X_{\mathbb{C}}$ of $(\tau_n, i\nu_\lambda, \psi_\lambda)$

and a continuously differential function $(\mu(\tau), \psi(\tau)) : O_n \to D_n \times H_n$ such that for each $\tau \in O_n$, the only eigenvalue of $A_\tau(\lambda)$ in D_n is $\mu(\tau)$, and

$$\begin{aligned} & e^{\alpha m(x)}\Delta(\lambda, \mu(\tau), \tau)\psi(\tau) = \nabla \cdot [e^{\alpha m(x)}\nabla\psi(\tau)] - e^{\alpha m(x)}\mu(\tau)\psi(\tau) \\ & + \lambda \left[m(x)e^{\alpha m(x)} - e^{2\alpha m(x)}u_\lambda\right] \psi(\tau) - \lambda e^{2\alpha m(x)}u_\lambda\psi(\tau)e^{-\mu(\tau)\tau} = 0. \end{aligned} \tag{2.28}$$

Moreover, $\mu(\tau_n) = i\nu_\lambda$, and $\psi(\tau_n) = \psi_\lambda$. Then we have the following transversality condition.

Theorem 2.9. *Assume that* $\lambda \in (\lambda_*, \tilde{\lambda}^*]$. *Then*

$$\frac{d\mathcal{R}e[\mu(\tau_n)]}{d\tau} > 0, \qquad n = 0, 1, 2, \cdots.$$

Proof. Differentiating Eq.(2.28) with respect to τ at $\tau = \tau_n$ yields

$$\begin{aligned} & \frac{d\mu(\tau_n)}{d\tau}\left[-e^{\alpha m(x)}\psi_\lambda + \lambda\tau_n e^{2\alpha m(x)}u_\lambda\psi_\lambda e^{-i\theta_\lambda}\right] \\ & + e^{\alpha m(x)}\Delta(\lambda, i\nu_\lambda, \tau_n)\frac{d\psi(\tau_n)}{d\tau} + i\nu_\lambda\lambda e^{2\alpha m(x)}u_\lambda\psi_\lambda e^{-i\theta_\lambda} = 0. \end{aligned} \tag{2.29}$$

Note that

$$\left\langle \overline{\psi}_\lambda, e^{\alpha m(x)}\Delta(\lambda, i\nu_\lambda, \tau_n)\frac{\psi(\tau_n)}{d\tau} \right\rangle = \left\langle e^{\alpha m(x)}\Delta(\lambda, -i\nu_\lambda, \tau_n)\overline{\psi}_\lambda, \frac{\psi(\tau_n)}{d\tau} \right\rangle = 0. \tag{2.30}$$

Then, multiplying Eq. (2.29) by ψ_λ and integrating the result over Ω, we have

$$\begin{aligned} \frac{d\mu(\tau_n)}{d\tau} = & \frac{i\nu_\lambda\lambda e^{-i\theta_\lambda}\displaystyle\int_\Omega e^{2\alpha m(x)}u_\lambda\psi_\lambda^2 dx}{\int_\Omega e^{\alpha m(x)}\psi_\lambda^2 dx - \lambda\tau_n e^{-i\theta_\lambda}\int_\Omega e^{2\alpha m(x)}u_\lambda\psi_\lambda^2 dx} \\ = & \frac{1}{|S_n(\lambda)|^2}\left(i\nu_\lambda\lambda e^{-i\theta_\lambda}\int_\Omega e^{\alpha m(x)}\psi_\lambda^2 dx \int_\Omega e^{2\alpha m(x)}u_\lambda\psi_\lambda^2 dx \right. \\ & \left. -i\nu_\lambda\lambda^2\tau_n\left[\int_\Omega e^{2\alpha m(x)}u_\lambda\psi_\lambda^2 dx\right]^2\right). \end{aligned} \tag{2.31}$$

It follows from Eq. (2.20) and the expression of u_λ, θ_λ, ν_λ and ψ_λ that

$$\lim_{\lambda\to\lambda_*}\frac{1}{(\lambda-\lambda_*)^2}\frac{d\mathcal{R}e[\mu(\tau_n)]}{d\tau} = \frac{h_{\lambda_*}^2}{\lim_{\lambda\to\lambda_*}|S_n(\lambda)|^2}\left(\int_\Omega e^{\alpha m(x)}\phi^2 dx\right)^2 > 0.$$

□

From Theorems 2.6, 2.8 and 2.9, we have the result on the distribution of eigenvalues of $A_\tau(\lambda)$.

Theorem 2.10. *For $\lambda \in (\lambda_*, \tilde{\lambda}^*]$, the infinitesimal generator $A_\tau(\lambda)$ has exactly $2(n+1)$ eigenvalues with positive real parts when $\tau \in (\tau_n, \tau_{n+1}]$, $n = 0, 1, 2, \cdots$.*

Then we obtain the stability and associated Hopf bifurcations of the positive steady state solution u_λ. We remark that the local Hopf bifurcation theorem for partial functional differential equations was proved in [35] (see Theorem 4.5 on page 208).

Theorem 2.11. *For $\lambda \in (\lambda_*, \tilde{\lambda}^*]$, the positive steady state u_λ of Eq. (1.3) is locally asymptotically stable when $\tau \in [0, \tau_0)$, and unstable when $\tau \in (\tau_0, \infty)$. Moreover, when $\tau = \tau_n$, $(n = 0, 1, 2, \cdots)$, system (1.3) occurs Hopf bifurcation at the positive steady state u_λ.*

3 The direction of the Hopf bifurcation

In this section, we combine the methods in [14, 16, 17, 24] to analyze the direction of the Hopf bifurcation of Eq. (1.3). Letting $U(t) = u(\cdot, t) - u_\lambda$, $t = \tau\tilde{t}$, $\tau = \tau_n + \gamma$, and dropping the tilde sign, system (1.3) can be transformed as follows:

$$\frac{dU(t)}{dt} = \tau_n e^{-\alpha m(x)} \nabla \cdot [e^{\alpha m(x)} \nabla U(t)] + \tau_n L_0(U_t) + J(U_t, \gamma), \tag{3.1}$$

where $U_t \in \mathcal{C} = C([-1, 0], Y)$, and

$$\begin{aligned} L_0(U_t) &= \lambda \left[m(x) - e^{\alpha m(x)} u_\lambda\right] U(t) - \lambda e^{\alpha m(x)} u_\lambda U(t-1), \\ J(U_t, \gamma) &= \gamma \tau_n e^{-\alpha m(x)} \nabla \cdot [e^{\alpha m(x)} \nabla U(t)] + \gamma L_0(U_t) - (\gamma + \tau_n)\lambda e^{\alpha m(x)} U(t) U(t-1). \end{aligned}$$

Then Eq. (3.1) occurs Hopf bifurcation near the zero equilibrium when $\gamma = 0$. Let $\mathcal{A}_{\tau_n}$ be the infinitesimal generator of the linearized equation

$$\frac{dU(t)}{dt} = \tau_n e^{-\alpha m(x)} \nabla \cdot [e^{\alpha m(x)} \nabla U(t)] + \tau_n L_0(U_t). \tag{3.2}$$

It follows from [35] that

$$\begin{aligned} \mathcal{A}_{\tau_n} \Psi =& \dot{\Psi}, \\ \mathscr{D}(\mathcal{A}_{\tau_n}) =& \Big\{ \Psi \in \mathcal{C}_{\mathbb{C}} \cap \mathcal{C}^1_{\mathbb{C}} : \ \Psi(0) \in X_{\mathbb{C}}, \dot{\Psi}(0) = \tau_n e^{-\alpha m(x)} \nabla \cdot [e^{\alpha m(x)} \nabla \Psi(0)] \\ & + \lambda \tau_n \left[m(x) - e^{\alpha m(x)} u_\lambda\right] \Psi(0) - \lambda \tau_n e^{\alpha m(x)} u_\lambda \Psi(-1) \Big\}, \end{aligned}$$

where $\mathcal{C}^1_{\mathbb{C}} = C^1([-1,0], Y_{\mathbb{C}})$, and Eq. (3.1) can be written in the following abstract form

$$\frac{dU_t}{dt} = \mathcal{A}_{\tau_n} U_t + X_0 J(U_t, \gamma), \tag{3.3}$$

where

$$X_0(\theta) = \begin{cases} 0, & \theta \in [-1,0), \\ I, & \theta = 0. \end{cases}$$

It follows from Theorem 2.10 that $\mathcal{A}_{\tau_n}$ has only one pair of purely imaginary eigenvalues $\pm i\nu_\lambda \tau_n$, which are simple, and the corresponding eigenfunction with respect to $i\nu_\lambda\tau_n$ (respectively, $-i\nu_\lambda\tau_n$) is $\psi_\lambda e^{i\nu_\lambda\tau_n\theta}$ (respectively, $\overline{\psi_\lambda} e^{-i\nu_\lambda\tau_n\theta}$) for $\theta \in [-1,0]$, where ψ_λ is defined as in Theorem 2.6.

Following [16, 34], we introduce the formal duality $\langle\langle \cdot,\cdot \rangle\rangle$ in $\mathcal{C}$ by

$$\langle\langle \tilde{\Psi}, \Psi \rangle\rangle = \langle \tilde{\Psi}(0), \Psi(0) \rangle_1 - \lambda\tau_n \int_{-1}^0 \left\langle \tilde{\Psi}(s+1), u_\lambda e^{\alpha m(x)} \Psi(s) \right\rangle_1 ds, \tag{3.4}$$

for $\Psi \in \mathcal{C}_{\mathbb{C}}$ and $\tilde{\Psi} \in \mathcal{C}^*_{\mathbb{C}} := C([0,1], Y_{\mathbb{C}})$, where

$$\langle u, v \rangle_1 = \int_\Omega e^{\alpha m(x)} \overline{u}(x) v(x) dx.$$

Since $m(x)$ is bounded and $e^{\alpha m(x)}$ is positive, we see that $Y_{\mathbb{C}}$ is also a Hilbert space with this product, and

$$e^{\alpha \min_\Omega m(x)} \langle v, v \rangle \le \langle v, v \rangle_1 \le e^{\alpha \max_\Omega m(x)} \langle v, v \rangle.$$

As in [23], we can compute the formal adjoint operator $\mathcal{A}^*_{\tau_n}$ of $\mathcal{A}_{\tau_n}$ with respect to the formal duality.

Lemma 3.1. *The formal adjoint operator $\mathcal{A}^*_{\tau_n}$ of $\mathcal{A}_{\tau_n}$ is defined by*

$$\mathcal{A}^*_{\tau_n} \tilde{\Psi}(s) = -\dot{\tilde{\Psi}}(s),$$

and the domain

$$\mathscr{D}(\mathcal{A}^*_{\tau_n}) = \Big\{ \tilde{\Psi} \in \mathcal{C}^*_{\mathbb{C}} \cap (\mathcal{C}^*_{\mathbb{C}})^1 : \tilde{\Psi}(0) \in X_{\mathbb{C}}, -\dot{\tilde{\Psi}}(0) = \tau_n e^{-\alpha m(x)} \nabla \cdot [e^{\alpha m(x)} \nabla \tilde{\Psi}(0)] + \lambda\tau_n \left[m(x) - e^{\alpha m(x)} u_\lambda \right] \tilde{\Psi}(0) - \lambda\tau_n e^{\alpha m(x)} u_\lambda \tilde{\Psi}(1) \Big\},$$

where $(\mathcal{C}^*_{\mathbb{C}})^1 = C^1([0,1], Y_{\mathbb{C}})$. Moreover, $\mathcal{A}^*_{\tau_n}$ and $\mathcal{A}_{\tau_n}$ satisfy

$$\langle\langle \mathcal{A}^*_{\tau_n} \tilde{\Psi}, \Psi \rangle\rangle = \langle\langle \tilde{\Psi}, \mathcal{A}_{\tau_n} \Psi \rangle\rangle \ \textit{for } \Psi \in \mathscr{D}(\mathcal{A}_{\tau_n}) \textit{ and } \tilde{\Psi} \in \mathscr{D}(\mathcal{A}^*_{\tau_n}). \tag{3.5}$$

Proof. For $\Psi \in \mathscr{D}(\mathcal{A}_{\tau_n})$ and $\tilde{\Psi} \in \mathscr{D}(\mathcal{A}^*_{\tau_n})$,

$$\begin{aligned}
\langle\langle \tilde{\Psi}, \mathcal{A}_{\tau_n}\Psi\rangle\rangle =& \left\langle \tilde{\Psi}(0), (\mathcal{A}_{\tau_n}\Psi)(0)\right\rangle_1 - \lambda\tau_n \int_{-1}^{0} \left\langle \tilde{\Psi}(s+1), u_\lambda e^{\alpha m(x)}\dot{\Psi}(s)\right\rangle_1 ds \\
=& \left\langle \tilde{\Psi}(0), \tau_n e^{-\alpha m(x)}\nabla\cdot\left[e^{\alpha m(x)}\nabla\Psi(0)\right]\right\rangle_1 - \lambda\tau_n \left[\left\langle \tilde{\Psi}(s+1), u_\lambda e^{\alpha m(x)}\Psi(s)\right\rangle_1\right]_{-1}^{0} \\
&+ \left\langle \tilde{\Psi}(0), \lambda\tau_n\left[m(x) - e^{\alpha m(x)}u_\lambda\right]\Psi(0) - \lambda\tau_n e^{\alpha m(x)}u_\lambda\Psi(-1)\right\rangle_1 \\
&+\lambda\tau_n \int_{-1}^{0} \left\langle \dot{\tilde{\Psi}}(s+1), u_\lambda e^{\alpha m(x)}\Psi(s)\right\rangle_1 ds \\
=& \left\langle (\mathcal{A}^*_{\tau_n}\tilde{\Psi})(0), \Psi(0)\right\rangle_1 - \lambda\tau_n \int_{-1}^{0} \left\langle -\dot{\tilde{\Psi}}(s+1), u_\lambda e^{\alpha m(x)}\Psi(s)\right\rangle_1 ds \\
=& \langle\langle \mathcal{A}^*_{\tau_n}\tilde{\Psi}, \Psi\rangle\rangle.
\end{aligned}$$

□

Similarly, it follows from Theorem 2.10 that the operator $\mathcal{A}^*_{\tau_n}$ has only one pair of purely imaginary eigenvalues $\pm i\nu_\lambda\tau_n$, which are simple, and the associated eigenfunction with respect to $-i\nu_\lambda\tau_n$ (respectively, $i\nu_\lambda\tau_n$) is $\overline{\psi}_\lambda e^{i\nu_\lambda\tau_n s}$ (respectively, $\psi_\lambda e^{-i\nu_\lambda\tau_n s}$) for $s \in [0,1]$, where ψ_λ is defined as in Theorem 2.6. From [35], we see that the center subspace of Eq. (3.1) is $P = \operatorname{span}\{p(\theta), \overline{p}(\theta)\}$, where $p(\theta) = \psi_\lambda e^{i\nu_\lambda\tau_n\theta}$ is the eigenfunction of $\mathcal{A}_{\tau_n}$ with respect to $i\nu_\lambda\tau_n$. The formal adjoint subspace of P is $P^* = \operatorname{span}\{q(s), \overline{q}(s)\}$, where $q(s) = \overline{\psi}_\lambda e^{i\nu_\lambda\tau_n s}$ is the eigenfunction of $\mathcal{A}^*_{\tau_n}$ with respect to $-i\nu_\lambda\tau_n$. Let $\Phi_p = (p(\theta), \overline{p}(\theta))$, $\Psi_P = \dfrac{1}{\overline{S_n}(\lambda)}(q(s), \overline{q}(s))^T$, where $S_n(\lambda)$ is defined in Lemma 2.7, and one can easily check that $\langle\langle \Psi_p, \Phi_p\rangle\rangle = I$, where I is the identity matrix in $\mathbb{R}^{2\times 2}$. Moreover, $\mathcal{C}_\mathbb{C}$ can be decomposed as $\mathcal{C}_\mathbb{C} = P \oplus Q$, where

$$Q = \{\Psi \in \mathcal{C}_\mathbb{C} : \langle\langle \tilde{\Psi}, \Psi\rangle\rangle = 0 \text{ for all } \tilde{\Psi} \in P^*\}.$$

Since the formulas of Hopf bifurcation are all relative to $\gamma = 0$ only, we set $\gamma = 0$ in Eq. (3.1). Let

$$w(z,\overline{z}) = w_{20}(\theta)\frac{z^2}{2} + w_{11}(\theta)z\overline{z} + w_{02}(\theta)\frac{\overline{z}^2}{2} + \cdots \tag{3.6}$$

be the center manifold with the range in Q, and then the flow of Eq. (3.1) on the center manifold can be written as:

$$U_t = \Phi_p \cdot (z(t), \overline{z}(t))^T + w(z(t), \overline{z}(t)),$$

where

$$
\begin{aligned}
\dot{z}(t) =& \frac{d}{dt}\langle\langle q(s), U_t\rangle\rangle \\
=& \langle\langle q(s), \mathcal{A}_{\tau_n} U_t\rangle\rangle + \frac{1}{S_n(\lambda)}\langle\langle q(s), X_0 J(U_t, 0)\rangle\rangle \\
=& i\nu_\lambda \tau_n z(t) + \frac{1}{S_n(\lambda)}\left\langle q(0), J\left(\Phi_p(z(t), \overline{z}(t))^T + w(z(t), \overline{z}(t)), 0\right)\right\rangle_1 \\
=& i\nu_\lambda \tau_n z(t) + g(z, \overline{z}).
\end{aligned}
\tag{3.7}
$$

Then,

$$
\begin{aligned}
g(z, \overline{z}) =& \frac{1}{S_n(\lambda)}\left\langle q(0), J\left(\Phi_p(z(t), \overline{z}(t))^T + w(z(t), \overline{z}(t)), 0\right)\right\rangle_1 \\
=& g_{20}\frac{z^2}{2} + g_{11} z\overline{z} + g_{02}\frac{\overline{z}^2}{2} + g_{21}\frac{z^2\overline{z}}{2} + \cdots,
\end{aligned}
\tag{3.8}
$$

and an easy calculation implies that

$$
\begin{aligned}
g_{20} =& -\frac{2\lambda\tau_n}{S_n(\lambda)} e^{-i\nu_\lambda\tau_n}\int_\Omega e^{2\alpha m(x)}\psi_\lambda^3 dx, \\
g_{11} =& -\left[\frac{\lambda\tau_n}{S_n(\lambda)}(e^{i\nu_\lambda\tau_n} + e^{-i\nu_\lambda\tau_n})\right]\int_\Omega e^{2\alpha m(x)}\psi_\lambda|\psi_\lambda|^2 dx, \\
g_{02} =& -\frac{2\lambda\tau_n}{S_n(\lambda)} e^{i\nu_\lambda\tau_n}\int_\Omega e^{2\alpha m(x)}\psi_\lambda\overline{\psi}_\lambda^2 dx, \\
g_{21} =& -\frac{2\lambda\tau_n}{S_n(\lambda)}\int_\Omega e^{2\alpha m(x)}\psi_\lambda^2 w_{11}(-1)dx - \frac{\lambda\tau_n}{S_n(\lambda)}\int_\Omega e^{2\alpha m(x)}|\psi_\lambda|^2 w_{20}(-1)dx \\
& -\frac{\lambda\tau_n}{S_n(\lambda)} e^{i\nu_\lambda\tau_n}\int_\Omega e^{2\alpha m(x)}|\psi_\lambda|^2 w_{20}(0)dx - \frac{2\lambda\tau_n}{S_n(\lambda)} e^{-i\nu_\lambda\tau_n}\int_\Omega e^{2\alpha m(x)}\psi_\lambda^2 w_{11}(0)dx.
\end{aligned}
\tag{3.9}
$$

To compute g_{21}, we need to compute $w_{20}(\theta)$ and $w_{11}(\theta)$ in the following. As in [8, 24], we see that $w_{20}(\theta)$ and $w_{11}(\theta)$ satisfy

$$
\begin{cases}
(2i\nu_\lambda\tau_n - \mathcal{A}_{\tau_n})w_{20} = H_{20}, \\
-\mathcal{A}_{\tau_n} w_{11} = H_{11}.
\end{cases}
\tag{3.10}
$$

Here, for $-1 \le \theta < 0$,

$$H_{20}(\theta) = -(g_{20}p(\theta) + \overline{g}_{02}\overline{p}(\theta)), \tag{3.11}$$

$$H_{11}(\theta) = -(g_{11}p(\theta) + \overline{g}_{11}\overline{p}(\theta)), \tag{3.12}$$

and, for $\theta = 0$,

$$H_{20}(0) = -\left(g_{20}p(0) + \overline{g}_{02}\overline{p}(0)\right) - 2\lambda\tau_n e^{-i\nu_\lambda\tau_n} e^{\alpha m(x)}\psi_\lambda^2, \tag{3.13}$$

$$H_{11}(0) = -\left(g_{11}p(0) + \overline{g}_{11}\overline{p}(0)\right) - \lambda\tau_n\left(e^{-i\nu_\lambda\tau_n} + e^{i\nu_\lambda\tau_n}\right)e^{\alpha m(x)}|\psi_\lambda|^2. \tag{3.14}$$

It follows from Eqs. (3.10)-(3.12) that $w_{20}(\theta)$ and $w_{11}(\theta)$ can be solved as follows:

$$w_{20}(\theta) = \frac{ig_{20}}{\nu_\lambda\tau_n}p(\theta) + \frac{i\overline{g}_{02}}{3\nu_\lambda\tau_n}\overline{p}(\theta) + Ee^{2i\nu_\lambda\tau_n\theta}, \tag{3.15}$$

and

$$w_{11}(\theta) = -\frac{ig_{11}}{\nu_\lambda\tau_n}p(\theta) + \frac{i\overline{g}_{11}}{\nu_\lambda\tau_n}\overline{p}(\theta) + F. \tag{3.16}$$

From Eq. (3.10) with $\theta = 0$, the definition of $\mathcal{A}_{\tau_n}$ and we see that E satisfies

$$(2i\nu_\lambda\tau_n - \mathcal{A}_{\tau_n})Ee^{2i\nu_\lambda\tau_n\theta}\Big|_{\theta=0} = -2\lambda\tau_n e^{-i\nu_\lambda\tau_n}e^{\alpha m(x)}\psi_\lambda^2,$$

or equivalently,

$$\Delta(\lambda, 2i\nu_\lambda, \tau_n)E = 2\lambda e^{-i\nu_\lambda\tau_n}e^{\alpha m(x)}\psi_\lambda^2. \tag{3.17}$$

Note that $2i\nu_\lambda$ is not the eigenvalue of $A_{\tau_n}(\lambda)$ for $\lambda \in (\lambda_*, \tilde{\lambda}^*]$, and hence

$$E = 2\lambda e^{-i\nu_\lambda\tau_n}\Delta(\lambda, 2i\nu_\lambda, \tau_n)^{-1}\left(e^{\alpha m(x)}\psi_\lambda^2\right).$$

Similarly, from Eqs. (3.10), (3.14), and (3.16), we have

$$F = \lambda\left(e^{-i\nu_\lambda\tau_n} + e^{i\nu_\lambda\tau_n}\right)\Delta(\lambda, 0, \tau_n)^{-1}\left(e^{\alpha m(x)}|\psi_\lambda|^2\right). \tag{3.18}$$

In the following, we obtain the similar result as in [8] for the expression of E and F.

Lemma 3.2. *Assume that E and F satisfy* (3.17) *and* (3.18)*, respectively. Then*

$$E = \frac{1}{\lambda - \lambda_*}(c_\lambda u_\lambda + \eta_\lambda), \quad F = \frac{\tilde{\eta}_\lambda}{\lambda - \lambda_*}, \tag{3.19}$$

where u_λ is defined as in (2.4)*, η_λ and $\tilde{\eta}_\lambda$ satisfy*

$$\langle u_\lambda, \eta_\lambda\rangle = 0, \quad \lim_{\lambda\to\lambda_*}\|\eta_\lambda\|_{Y_\mathbb{C}} = 0, \quad \lim_{\lambda\to\lambda_*}\|\tilde{\eta}_\lambda\|_{Y_\mathbb{C}} = 0,$$

and the constant c_λ satisfies $\lim_{\lambda\to\lambda_*}(\lambda - \lambda_*)c_\lambda = \dfrac{2i}{\alpha_{\lambda_*}^2(2i-1)}$.

Proof. We just prove the estimate for E, and that for F can be derived similarly. Denote the operator

$$L_\lambda := \nabla \cdot \left[e^{\alpha m(x)}\nabla\right] + \lambda e^{\alpha m(x)}[m(x) - e^{\alpha m(x)}u_\lambda], \tag{3.20}$$

and consequently $L_\lambda u_\lambda = 0$. Substituting E, defined as in Eq. (3.19), into Eq. (3.17), one can easily have

$$\begin{aligned} &L_\lambda \eta_\lambda - \lambda e^{-2i\nu_\lambda\tau_n} e^{2\alpha m(x)} u_\lambda (c_\lambda u_\lambda + \eta_\lambda) - 2i\nu_\lambda e^{\alpha m(x)}(c_\lambda u_\lambda + \eta_\lambda) \\ =&2(\lambda - \lambda_*)\lambda e^{-i\nu_\lambda\tau_n} e^{2\alpha m(x)} \psi_\lambda^2. \end{aligned} \tag{3.21}$$

Multiplying Eq. (3.21) by u_λ, and integrating the result over Ω, we have

$$\begin{aligned} &c_\lambda \left(\lambda e^{-2i\nu_\lambda\tau_n} \int_\Omega e^{2\alpha m(x)} u_\lambda^3 dx + 2i\nu_\lambda \int_\Omega e^{\alpha m(x)} u_\lambda^2 dx\right) \\ =& -\lambda e^{-2i\nu_\lambda\tau_n} \int_\Omega e^{2\alpha m(x)} u_\lambda^2 \eta_\lambda dx - 2i\nu_\lambda \int_\Omega e^{\alpha m(x)} u_\lambda \eta_\lambda dx \\ &-2\lambda e^{-i\nu_\lambda\tau_n}(\lambda - \lambda_*) \int_\Omega e^{2\alpha m(x)} u_\lambda \psi_\lambda^2 dx. \end{aligned} \tag{3.22}$$

Multiplying Eq. (3.21) by $\overline{\eta}_\lambda$, and integrating the result over Ω, we obtain

$$\begin{aligned} &\langle \eta_\lambda, L_\lambda \eta_\lambda\rangle - \lambda c_\lambda \int_\Omega e^{2\alpha m(x)} \overline{\eta}_\lambda u_\lambda^2 dx e^{-2i\nu_\lambda\tau_n} - 2i\nu_\lambda c_\lambda \int_\Omega e^{\alpha m(x)} u_\lambda \overline{\eta}_\lambda dx \\ =&\lambda \int_\Omega e^{2\alpha m(x)} u_\lambda |\eta_\lambda|^2 dx e^{-2i\nu_\lambda\tau_n} + 2i\nu_\lambda \int_\Omega e^{\alpha m(x)} |\eta_\lambda|^2 dx \\ &+2\lambda e^{-i\nu_\lambda\tau_n}(\lambda - \lambda_*) \int_\Omega e^{2\alpha m(x)} \overline{\eta}_\lambda \psi_\lambda^2 dx. \end{aligned} \tag{3.23}$$

It follows from the expression of ν_λ, u_λ, ψ_λ and τ_n that

$$\begin{aligned} &\psi_\lambda \to \phi, \;\; u_\lambda/(\lambda - \lambda_*) \to \beta_{\lambda_*}\phi \;\text{ in }\; C(\overline{\Omega}), \\ &\nu_\lambda/(\lambda - \lambda_*) \to h_{\lambda_*}, \;\; \nu_\lambda \tau_n \to \frac{\pi}{2} + 2n\pi. \end{aligned} \tag{3.24}$$

From Eqs. (3.22) and (3.24), we see that there exist constants $\tilde{\lambda} > \lambda_*$ and $M_0, M_1 > 0$ such that for, any $\lambda \in (\lambda_*, \tilde{\lambda})$,

$$|(\lambda - \lambda_*)c_\lambda| \le M_0 \|\eta_\lambda\|_{Y_\mathbb{C}} + M_1. \tag{3.25}$$

This, combined with Eqs. (3.23) and (3.24), implies that there exist constants $M_2,\ M_3 > 0$ such that for any $\lambda \in (\lambda_*, \tilde{\lambda})$,

$$|\lambda_2(\lambda)| \cdot \|\eta_\lambda\|_{Y_\mathbb{C}}^2 \le (\lambda - \lambda_*)M_2 \|\eta_\lambda\|_{Y_\mathbb{C}}^2 + M_3(\lambda - \lambda_*)\|\eta_\lambda\|_{Y_\mathbb{C}},$$

where $\lambda_2(\lambda)$ is the second eigenvalue of $-L_\lambda$. Since $\lim_{\lambda\to\lambda_*}\lambda_2(\lambda)=\lambda_2>0$, where λ_2, defined as in Lemma 2.3, is the second eigenvalue of $-L$, we have $\lim_{\lambda\to\lambda_*}\|\eta_\lambda\|_{Y_{\mathbb{C}}}=0$. This, together with (3.22), implies

$$\lim_{\lambda\to\lambda_*}(\lambda-\lambda_*)c_\lambda=\frac{2i}{\beta_{\lambda_*}^2(2i-1)}.$$

□

Therefore, by similar arguments to [8], one can easily check

$$\begin{aligned}&\lim_{\lambda\to\lambda_*}(\lambda-\lambda_*)g_{11}=0,\\&\lim_{\lambda\to\lambda_*}\mathcal{R}e[(\lambda-\lambda_*)^2g_{21}]<0.\end{aligned}\tag{3.26}$$

It is well-known that the real part of the following quantity determines the direction and stability of bifurcating periodic orbits (see [24, 35]):

$$C_1(0)=\frac{i}{2\nu_\lambda\tau_n}\left(g_{11}g_{20}-2|g_{11}|^2-\frac{|g_{02}|^2}{3}\right)+\frac{g_{21}}{2}.$$

It follows from Eq. (3.26) that $\lim_{\lambda\to\lambda_*}\mathcal{R}e[(\lambda-\lambda_*)^2C_1(0)]<0$. Hence we have the following result.

Theorem 3.3. *For $\lambda\in(\lambda_*,\lambda^*]$, where $\lambda^*-\lambda_*\ll 1$, let $\tau_n(\lambda)$ be the Hopf bifurcation points of Eq.* (1.3) *obtained in Theorem 2.6. Then for each $n\in\mathbb{N}\cup\{0\}$, the direction of the Hopf bifurcation at $\tau=\tau_n$ is forward and the bifurcating periodic solution from $\tau=\tau_0$ is orbitally asymptotically stable.*

4 No-flux boundary condition and simulation

In this section, we discussion model (1.2) with no-flux boundary condition, that is,

$$\begin{cases}\dfrac{\partial u(x,t)}{\partial t}=\nabla\cdot[d\nabla u-au\nabla m]+u(x,t)\left[m(x)-u(x,t-r)\right], & x\in\Omega,\ t>0,\\ d\partial_n u-au\partial_n m=0 & x\in\partial\Omega,\ t>0,\end{cases}\tag{4.1}$$

where n is the outward unit normal vector on $\partial\Omega$, and $\partial_n u = \nabla u \cdot n$. As in Eq. (1.2), we also derive an equivalent model of Eq. (4.1) as follows:

$$\begin{cases} \dfrac{\partial v}{\partial t} = e^{-\alpha m(x)} \nabla \cdot [e^{\alpha m(x)} \nabla v] + \lambda v \left[m(x) - e^{\alpha m(x)} v(x, t-\tau) \right], & x \in \Omega, \ t > 0, \\ \partial_n v = 0, & x \in \partial\Omega, \ t > 0. \end{cases} \tag{4.2}$$

Here $m(x)$ satisfies the following assumption:

$(\mathbf{A_2})$ $m(x) \in C^2(\overline{\Omega})$, $\max_{x\in\overline{\Omega}} m(x) > 0$, and $\int_\Omega m(x) e^{\alpha m(x)} dx < 0$; or

$(\mathbf{A_3})$ $m(x) \in C^2(\overline{\Omega})$, and $\int_\Omega m(x) e^{\alpha m(x)} dx > 0$.

Then the following discussion is divided into two cases.

4.1 Case I

In this case, $m(x)$ satisfies assumption (A_2). The method used for this case is similar to that for Dirichlet problem (1.3). In fact, it follows from [1] that the following problem

$$\begin{cases} -e^{-\alpha m(x)} \nabla \cdot [e^{\alpha m(x)} \nabla v] = -\Delta v - \alpha \nabla m \cdot \nabla v = \lambda m(x) v, & x \in \Omega, \\ \partial_n v = 0, & x \in \partial\Omega, \end{cases} \tag{4.3}$$

has a unique positive principal eigenvalue λ_*, and model (4.2) admits a unique positive steady state u_λ for $\lambda > \lambda_*$, if $m(x)$ satisfies assumption $(\mathbf{A_2})$. Moreover, we comment that the relation between λ_* and α was also investigated in [12]: if $\int_\Omega m(x) dx \geq 0$, then $\lambda_*(\alpha) = 0$ for all $\alpha \geq 0$; and if $m(x)$ change sign and $\int_\Omega m(x) dx < 0$, then there is a unique $\alpha_* > 0$ such that $\lambda_*(\alpha) > 0$ for $0 < \alpha < \alpha_*$, and $\lambda_*(\alpha) = 0$ for $\alpha > \alpha_*$.

Then, by similar arguments to Sections 2 and 3, we have the following results on model (4.2).

Theorem 4.1. *Assume that $m(x)$ satisfies assumption $(\mathbf{A_2})$. Then, for $\lambda \in (\lambda_*, \lambda^*]$, where $\lambda^* - \lambda_* \ll 1$, there exists a sequence $\{\tau_n\}_{n=0}^{\infty}$ such that the positive steady state u_λ of Eq. (4.2) is locally asymptotically stable when $\tau \in [0, \tau_0)$, unstable when $\tau \in (\tau_0, \infty)$, and system (4.2) occurs Hopf bifurcation at the positive steady state u_λ when $\tau = \tau_n$,*

$(n = 0, 1, 2, \cdots)$. *Moreover, the direction of the Hopf bifurcation at $\tau = \tau_n$ is forward and the bifurcating periodic solution from $\tau = \tau_0$ is orbitally asymptotically stable.*

4.2 Case II

Note that assumption $(\mathbf{A_2})$ is equivalent to $m(x)$ changing sign, $\int_\Omega m(x)dx < 0$ and $\alpha < \alpha_*$. Thus $\lambda_*(\alpha) > 0$ under assumption $(\mathbf{A_2})$. It will be of interest to study the dynamics of system (4.2) for $\alpha > \alpha_*$, i.e. to understand the joint effect of strong advection and time delay. Therefore, in this subsection, we consider the case that $m(x)$ satisfies assumption $(\mathbf{A_3})$. It follows from [7, 12] that, under assumption $(\mathbf{A_3})$, the unique positive principal eigenvalue $\lambda_*(\alpha)$ of problem (4.3) is zero, and the corresponding eigenfunction ϕ is constant. Moreover, for any $\lambda > 0$, system (1.3) has a unique positive steady state u_λ, which is globally asymptotically stable, and u_λ satisfies

$$\lim_{\lambda \to 0} u_\lambda(x) = \overline{m} := \frac{\int_\Omega m(x)e^{\alpha m(x)}dx}{\int_\Omega e^{2\alpha m(x)}dx} \quad \text{in} \quad C^{1+\delta}(\overline{\Omega}) \tag{4.4}$$

for some $\delta \in (0, 1)$. Let $u_0(x) = \overline{m}$, and then $\lambda \to u_\lambda$ is continuous from $[0, \infty)$ to $C^{1+\delta}(\overline{\Omega})$. For simplicity, we choose $\phi \equiv \overline{m}$, and then L, X_1 and Y_1 (defined in Eqs. (2.2) and (2.3)) have the following forms:

$$\begin{aligned} L =& \nabla \cdot [e^{\alpha m(x)} \nabla], \\ X_1 =& \left\{ y \in X : \int_\Omega y(x)dx = 0 \right\}, \\ Y_1 =& \mathscr{R}(L) = \left\{ y \in Y : \int_\Omega y(x)dx = 0 \right\}. \end{aligned}$$

In order to analyze eigenvalue problem (2.11), we first give the following estimates for solutions of (2.11).

Lemma 4.2. *Assume that $\lambda \in (0, \lambda^*]$. If $(\nu_\lambda, \theta_\lambda, \psi_\lambda)$ solves Eq. (2.11) with $\nu_\lambda > 0$, $\theta_\lambda \in [0, 2\pi)$, and $\psi_\lambda (\neq 0) \in X_\mathbb{C}$, then ν_λ/λ is bounded for $\lambda \in (0, \lambda^*]$.*

Proof. It follows from Eq. (2.12) that

$$\nu_\lambda/\lambda = \frac{\sin\theta_\lambda \int_\Omega e^{2\alpha m(x)} u_\lambda |\psi_\lambda|^2 dx}{\int_\Omega e^{\alpha m(x)} |\psi_\lambda|^2 dx} \leq e^{\alpha \max_\Omega m(x)} \|u_\lambda\|_\infty.$$

Then, from the continuity of $\lambda \mapsto \|u_\lambda\|_\infty$, we see that ν_λ/λ is bounded for $\lambda \in (0, \lambda^*]$. $\square$

We remark that Lemma 2.3 still holds for the case that $L = \nabla \cdot [e^{\alpha m(x)} \nabla]$. Now, for $\lambda \in (0, \lambda^*]$, letting

$$\begin{aligned} &\psi = r\overline{m} + \lambda z, \quad z \in (X_1)_{\mathbb{C}}, \quad r \geq 0, \\ &\|\psi\|_{Y_{\mathbb{C}}}^2 = r^2 \overline{m}^2 |\Omega| + \lambda^2 \|z\|_{Y_{\mathbb{C}}}^2 = \overline{m}^2 |\Omega|, \end{aligned} \tag{4.5}$$

and substituting (4.5) and $\nu = \lambda h$ into Eq. (2.11), we see that (ν, θ, ψ) solves Eq. (2.11), where $\nu > 0$, $\theta \in [0, 2\pi)$ and $\psi \in X_{\mathbb{C}}(\|\psi\|_{Y_{\mathbb{C}}}^2 = \|\phi\|_{Y_{\mathbb{C}}}^2)$, if and only if the following system:

$$\begin{cases} \tilde{g}_1(z, r, h, \theta, \lambda) := \nabla \cdot [e^{\alpha m(x)} \nabla z] + e^{\alpha m(x)} \left[m(x) - e^{\alpha m(x)} u_\lambda \right] (r\overline{m} + \lambda z) \\ -e^{2\alpha m(x)} u_\lambda (r\overline{m} + \lambda z) e^{-i\theta} - ihe^{\alpha m(x)} (r\overline{m} + \lambda z) = 0 \\ \tilde{g}_2(z, r, \lambda) := (r^2 - 1)\overline{m}^2 |\Omega| + \lambda^2 \|z\|_{Y_{\mathbb{C}}}^2 = 0 \end{cases} \tag{4.6}$$

Define $\tilde{G} : (X_1)_{\mathbb{C}} \times \mathbb{R}^4 \to Y_{\mathbb{C}} \times \mathbb{R}$ by $\tilde{G} = (g_1, g_2)$, and we see that $\tilde{G}(z, r, h, \theta, \lambda) = 0$ is also uniquely solvable for $\lambda = 0$.

Lemma 4.3. *The following equation*

$$\begin{cases} \tilde{G}(z, r, h, \theta, 0) = 0 \\ z \in (X_1)_{\mathbb{C}}, \ h > 0 \ r \geq 0, \ \theta \in [0, 2\pi) \end{cases} \tag{4.7}$$

has a unique solution $(z_0, r_0, h_0, \theta_0)$*. Here*

$$r_0 = 1, \ \ \theta_0 = \pi/2, \ \ h_0 = \frac{\int_\Omega m(x) e^{\alpha m(x)} dx}{\int_\Omega e^{\alpha m(x)} dx}, \tag{4.8}$$

and $z_0 \in (X_1)_{\mathbb{C}}$ *is the unique solution of*

$$-\nabla \cdot [e^{\alpha m(x)} \nabla z] = e^{\alpha m(x)} \left[m(x) - e^{\alpha m(x)} \overline{m} \right] \overline{m} - e^{2\alpha m(x)} \overline{m}^2 e^{-i\theta_0} - ih_0 e^{\alpha m(x)} \overline{m}. \tag{4.9}$$

Proof. From Eq. (4.6), we see that $\tilde{g}_2(z, r, 0) = 0$ if and only if $r = r_0 = 1$. Note that

$$\begin{aligned} \tilde{g}_1(z, r_0, h, \theta, 0) = &\nabla \cdot [e^{\alpha m(x)} \nabla z] + e^{\alpha m(x)} \left[m(x) - e^{\alpha m(x)} \overline{m} \right] \overline{m} \\ &-e^{2\alpha m(x)} \overline{m}^2 e^{-i\theta} - ihe^{\alpha m(x)} \overline{m} = 0 \end{aligned} \tag{4.10}$$

Then

$$\begin{cases} \tilde{g}_1(z, r_0, h, \theta, 0) = 0 \\ z \in (X_1)_{\mathbb{C}},\ h > 0\ r \geq 0,\ \theta \in [0, 2\pi) \end{cases}$$

is solvable if and only if

$$\begin{cases} \overline{m}^2 \int_\Omega e^{2\alpha m(x)} dx \sin\theta = h\overline{m} \int_\Omega e^{\alpha m(x)} dx \\ \overline{m}^2 \int_\Omega e^{2\alpha m(x)} dx \cos\theta = 0 \end{cases} \tag{4.11}$$

is solvable for a pair (θ, h) with $h > 0$ and $\theta \in [0, 2\pi)$. Noticing that

$$\overline{m} = \frac{\int_\Omega m(x) e^{\alpha m(x)} dx}{\int_\Omega e^{2\alpha m(x)} dx},$$

we have

$$\theta = \theta_0 = \pi/2, \quad h = h_0 = \frac{\int_\Omega m(x) e^{\alpha m(x)} dx}{\int_\Omega e^{\alpha m(x)} dx}. \tag{4.12}$$

Consequently, $\tilde{g}_1(z, r_0, h_0, \theta_0, 0) = 0$ has a unique solution z_0, which satisfies Eq. (4.9). □

Then, we also have the following result on the solvability of $\tilde{G} = 0$ for $\lambda \in (0, \lambda^*]$.

Theorem 4.4. *There exist $\tilde{\lambda}^* > 0$ and a continuously differentiable mapping $\lambda \mapsto (z_\lambda, r_\lambda, h_\lambda, \theta_\lambda)$ from $[0, \tilde{\lambda}^*]$ to $(X_1)_{\mathbb{C}} \times \mathbb{R}^3$ such that $\tilde{G}(z_\lambda, r_\lambda, r_\lambda, \theta_\lambda, \lambda) = 0$. Moreover, for $\lambda \in [0, \tilde{\lambda}^*]$,*

$$\begin{cases} \tilde{G}(z, r, h, \theta, \lambda) = 0 \\ z \in (X_1)_{\mathbb{C}},\ h,\ r \geq 0,\ \theta \in [0, 2\pi) \end{cases} \tag{4.13}$$

has a unique solution $(z_\lambda, r_\lambda, h_\lambda, \theta_\lambda)$.

Proof. Let $\tilde{T} = (\tilde{T}_1, \tilde{T}_2) : (X_1)_{\mathbb{C}} \times \mathbb{R}^3 \mapsto Y_{\mathbb{C}} \times \mathbb{R}$ be the Fréchet derivative of $\tilde{G}$ with respect to (z, r, h, θ) at $(z_0, r_0, h_0, \theta_0, 0)$. An easy calculation yields

$$\begin{aligned} \tilde{T}_1(\chi, \kappa, \epsilon, \vartheta) =& \nabla \cdot [e^{\alpha m(x)} \nabla z] + \kappa e^{\alpha m(x)} \left[m(x) - e^{\alpha m(x)} \overline{m} \right] \overline{m} - \kappa e^{2\alpha m(x)} \overline{m}^2 e^{-i\theta_0} \\ & - i\kappa h_0 e^{\alpha m(x)} \overline{m} - i\epsilon e^{\alpha m(x)} \overline{m} + \vartheta e^{2\alpha m(x)} \overline{m}^2, \\ \tilde{T}_2(\kappa) =& 2\kappa \overline{m}^2 |\Omega|. \end{aligned}$$

Then, we check that $\tilde{T}$ is a bijection from $(X_1)_{\mathbb{C}} \times \mathbb{R}^3$ to $Y_{\mathbb{C}} \times \mathbb{R}$, and we only need to verify that T is an injective mapping. If $\tilde{T}_2(\kappa) = 0$, then $\kappa = 0$, and substituting $\kappa = 0$ into $\tilde{T}_1(\chi, \kappa, \epsilon, \vartheta) = 0$, we obtain $\vartheta = \epsilon = 0$. Therefore, T is an an injection. It follows from the implicit function theorem that there exist $\tilde{\lambda}^* > \lambda_*$ and a continuously differentiable mapping $\lambda \mapsto (z_\lambda, r_\lambda, h_\lambda, \theta_\lambda)$ from $[\lambda_*, \tilde{\lambda}^*]$ to $X_{\mathbb{C}} \times \mathbb{R}^3$ such that $\tilde{G}(z_\lambda, r_\lambda, h_\lambda, \theta_\lambda, \lambda) = 0$. By the arguments similar to Lemma 2.5, the uniqueness can be proved, and here we omit the proof. □

Summarizing the above result, we have the following result.

Theorem 4.5. *For each $\lambda \in (0, \tilde{\lambda}^*]$, the following equation*

$$\begin{cases} \Delta(\lambda, i\nu, \tau)\psi = 0 \\ \nu \geq 0,\ \tau \geq 0,\ \psi(\neq 0) \in X_{\mathbb{C}} \end{cases}$$

has a solution (ν, τ, ψ), if and only if

$$\nu = \nu_\lambda = \lambda h_\lambda,\ \psi = c\psi_\lambda,\ \tau = \tau_n = \frac{\theta_\lambda + 2n\pi}{\nu_\lambda},\ \ n = 0, 1, 2, \cdots, \tag{4.14}$$

where $\psi_\lambda = r_\lambda \overline{m} + \lambda z_\lambda$, c is a nonzero constant, and $z_\lambda, r_\lambda, h_\lambda, \theta_\lambda$ are defined as in Theorem 4.4.

The simplicity of $i\nu$ and the transversality condition can also be derived as in Lemma 2.7, Theorems 2.8 and 2.9, and we also omit the proof here. Therefore, for case II, we also derive the existence of Hopf bifurcation.

Theorem 4.6. *Assume that $m(x)$ satisfies assumption $(\mathbf{A_3})$. Then, for $\lambda \in (0, \lambda^*]$, where $0 < \lambda^* \ll 1$, there exists a sequence $\{\tau_n\}_{n=0}^{\infty}$ such that the positive steady state u_λ of Eq. (4.2) is locally asymptotically stable when $\tau \in [0, \tau_0)$, unstable when $\tau \in (\tau_0, \infty)$, and system (4.2) occurs Hopf bifurcation at the positive steady state u_λ when $\tau = \tau_n$, $(n = 0, 1, 2, \cdots)$.*

At the end of this section, some numerical simulations are given to support our theoretical results. We will show that large delay τ can make the spatial nonhomogeneous positive steady state unstable and induce Hopf bifurcation for models (1.3) and (4.2), see Fig. 1 and Fig. 2, respectively.

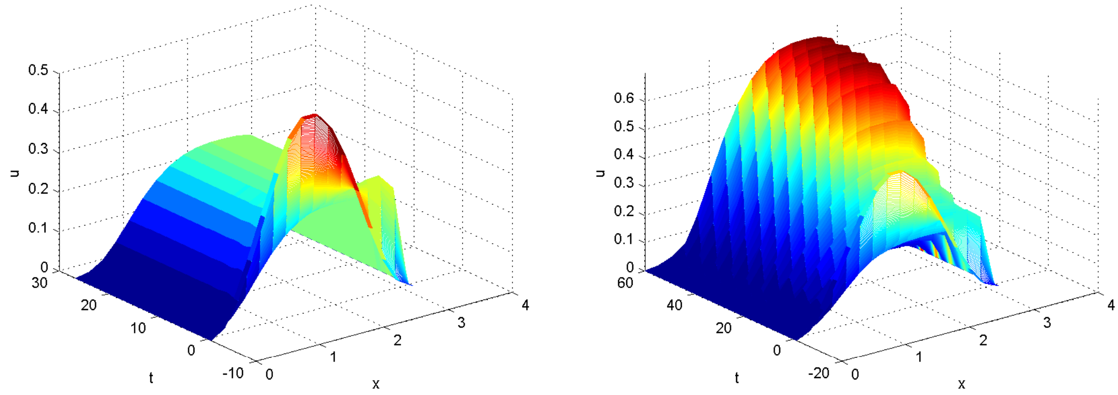

Figure 1: Eq. (1.3) occurs Hopf bifurcation with the homogeneous Dirichlet boundary condition. Here we choose $m(x) = -2\sin 2x$, $\Omega = (0, \pi)$, $\lambda = 1$ and $\alpha = 1$. (Left): $\tau = 0.8$; (Right): $\tau = 1.5$.

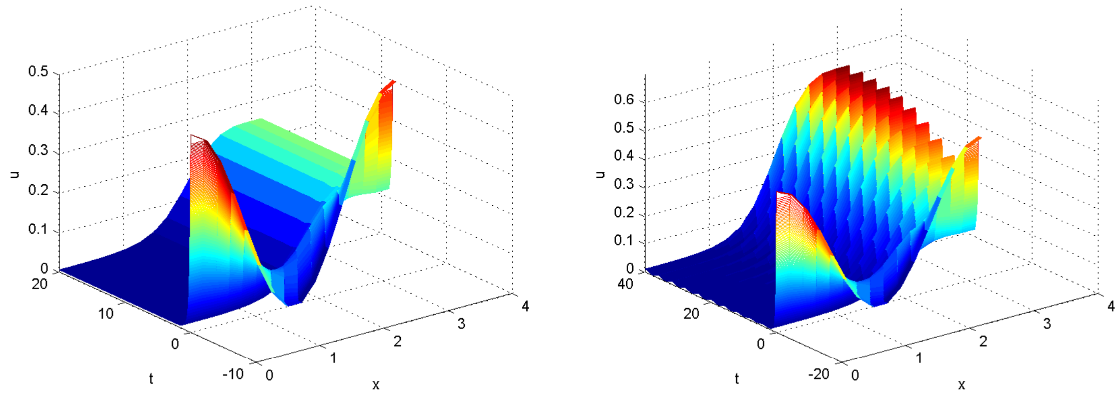

Figure 2: Eq. (4.2) occurs Hopf bifurcation with the homogeneous Neumann boundary condition. Here we choose $m(x) = -\sin 2x$, $\Omega = (0, \pi)$, $\lambda = 4$ and $\alpha = -1$. (Left): $\tau = 0.5$; (Right): $\tau = 1$.

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