NOTES ON COMPLETE LYAPUNOV FUNCTIONS

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ABSTRACT. In this note we presents a self-contained proof for the existence of complete Lyapunov function for semiflow admitting a Morse decomposition. The main references are C. Conley’s CBMS lecture notes and the monograph by K.P. Rybakowski.

Definition 1. Let $X$ be a complete metric space and $\varphi : [0, \infty) \times X$ be a semiflow, i.e. (i) $(t, u) \to \varphi(t, u_0)$ is continuous; (ii) $\varphi(0, u) = u$ for all $u \in X$; (iii) $\varphi(t, \varphi(s, u)) = \varphi(t + s, u)$ for $t, s \geq 0$.

1. A function $\gamma : \mathbb{R} \to X$ is a total trajectory if $\gamma(t + t_0) = \varphi(t, \gamma(t_0))$ for all $t \geq 0$ and $t_0 \in \mathbb{R}$.

2. A subset $A \subseteq X$ is said to be invariant if for each $u \in A$, there exists a total trajectory $\gamma$ such that $\gamma(0) = u$.

3. Define the omega limit set of a subset $B$ of $X$ by

$$\omega(B) := \cap_{t > 0} \varphi([t, \infty), B),$$

and define the omega limit set of a point $u \in X$ by $\omega(u) = \omega(\{u\})$.

4. For $u$ lying on some total trajectory $\gamma$, we define the alpha limit set

$$\alpha(u) = \alpha(\gamma) = \cap_{t < 0} \varphi((-\infty, t]).$$

5. A invariant subset $A$ is said to be an attractor if there exists a neighborhood $U$ of $A$ such that $\omega(U) = A$.

6. For an attractor $A$, define the repeller dual to $A$ by

$$A^* := \{u \in X : \omega(u) \cap A = \emptyset\}.$$

And the pair $(A, A^*)$ is called a attractor-repeller pair.

7. $\varphi$ is point-dissipative on $X$ if there exists a bounded set $B_p$ of $X$ such that $\omega(u) \subset B_p$ for all $u \in X$.

8. $\varphi$ is eventually bounded on a set $B$ if $\varphi([t_0, \infty), B)$ is bounded for some $t_0 > 0$.

9. $\varphi$ is asymptotically compact on $B$ for some subset $B \subset X$ if, for any $t_i \to \infty$ and $u_i \in B$, $\varphi(t_i, u_i)$ has a convergent subsequence.

10. $\varphi$ is asymptotically smooth if it is asymptotically compact on every forward invariant bounded closed set. [By Remark 2.26(b) of [3], a sufficient condition is: the mapping $u \mapsto \varphi(t, u)$ is compact for each $t > 0$.]

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(11) A nonempty, compact, invariant subset $S$ is a compact attractor of neighborhood of compact sets if $S$ is a compact subset of $X$ and every compact set has a neighborhood $U$ such that that $\omega(U) \subset S$.

**Theorem 2** (Theorem 2.30 of [3]). Assuming in addition that $\varphi$ is point-dissipative, asymptotically smooth, and eventually bounded on every compact subset $B$ of $X$, then $\varphi$ has a compact attractor $S$ of neighborhood of compact sets. In particular, there exists a neighborhood $U$ of $S$ such that $\omega(U) = S$.

**Definition 3** (Morse decomposition of the compact attractor). Given a finite ordered collection $\{M_1, ..., M_m\}$ of pairwise disjoint compact invariant subsets of $S$. We say that $\{M_1, ..., M_m\}$ is a Morse decomposition of the compact attractor $S$ of $X$ (or simply, a Morse decomposition of $S$) if (i) for every $u \in X$ there is an $i$ such that $\omega(u) \subset M_i$, and (ii) if $u$ lies on some total trajectory $\gamma$, then $\alpha(u) \subset M_j$ for some $i < j \leq m$.

Our main theorem is as follows.

**Theorem 4.** Given a Morse decomposition $\{M_1, ..., M_m\}$ of $S$. Then there exists a continuous function $V : X \to [0, \infty)$ such that

- $V^{-1}(i) = M_i$ for $1 \leq i \leq m$, and,
- For each $u \in X \setminus \bigcup_{i=1}^m M_i$, the mapping $t \mapsto V(\varphi(t, u))$ is strictly decreasing in $t \geq 0$.

**Proof.** See Theorem 9. \qed

**Proposition 5.** Given a Morse decomposition $\{M_1, ..., M_m\}$ of $S$. Set $A_0 = \emptyset$ and $A_k = \{u \in S : \alpha(u) \subset \bigcup_{i=1}^k M_i\}$ for $1 \leq k \leq m$.

Then $A_0 \subseteq A_1 \subseteq ... \subseteq A_m$ is a sequence of attractors in $S$ such that $A_i \cap A_{i-1} = M_i$.

**Proof.** The proof is taken from Theorem 3.1.8 in [2].

Step 1: The sets $A_k$ ($1 \leq k \leq m$) are closed.

Since by definition $A_m = S$, the set $A_m$ is closed (in fact compact). We now proceed inductively and assume $A_{k+1}$ to be closed for some $1 \leq k \leq m-1$. Let $u_j \in A_k$ with $u_j \to u$ for some $u \in S$. Then $u \in A_{k+1}$, since $A_k \subset A_{k+1}$ and $A_{k+1}$ is closed.

There are total trajectories $\gamma_j : \mathbb{R} \to S$ with $\gamma_j(0) = u_j$ and $\alpha(\gamma_j) \subset M_1 \cup ... \cup M_k$.

Using the compactness of $S$ we can pass to a subsequence and assume WLOG that $\lim_{j \to \infty} \gamma_j(t) \to \gamma(t)$ for each $t$, for some total trajectory $\sigma$ through $u$. We claim that $\alpha(\gamma) \subset (M_1 \cup ... \cup M_k)$. Indeed, since $\gamma_j(\mathbb{R}) \subset A_k \subset A_{k+1}$ and $A_{k+1}$ is closed, it follows that $\gamma(\mathbb{R}) \subset A_{k+1}$ and so $\alpha(\gamma) \subset A_{k+1}$. Observe that $M_i \cap A_{k+1} = \emptyset$ for $i > k + 1$ since $M_i$ is invariant. On the other hand, $\alpha(\gamma) \subset M_i$ for some $i$ by our assumptions and therefore $\alpha(\gamma) \subset M_1 \cup ... \cup M_k \cup M_{k+1}$. Consequently, either $\alpha(\gamma) \subset M_1 \cup ... \cup M_k$ in which case we are done, or else $\alpha(\gamma) \subset M_{k+1}$. In the latter case, let $V \supset M_{k+1}$ be an open neighborhood of $M_{k+1}$ such that $\gamma \cap M_i \neq \emptyset$ for $i \neq k + 1$. There is a sequence $t_v \to \infty$ and a $z \in M_{k+1}$ such that $\gamma(-t_v) \in V$ and $\text{dist}(\gamma(-t_v), z) \leq 1/v$ for all $v \in \mathbb{N}$. Therefore, for every $v$ there is a $j_v \geq v$ such that $\gamma_{j_v}(-t_v) \in V$ and $\text{dist}(\gamma_{j_v}(-t_v), z) \leq 2/v$. Since $(\alpha(\gamma_j) \cup \omega(\gamma_j)) \subset (M_1 \cup ... \cup M_k)$ for every $j$, there are $\tau_v \leq t_v \leq s_v$ such that...
\( \gamma_{j_\nu}(-s_\nu), \gamma_{j_\nu}(-\tau_\nu) \in \partial V \) and \( \gamma_{j_\nu}(-t) \in \overline{V} \) for \( t \in [\tau_\nu, s_\nu] \). The invariance of \( M_{k+1} \) now implies that \( t_\nu - \tau_\nu \to \infty \). Let \( \tilde{u}_\nu := \gamma_{j_\nu}(-s_\nu) \), then \( \tilde{u}_\nu \in S \) and since \( S \) is compact we may assume \( \tilde{u}_\nu \to \tilde{u} \in \partial V \). It then follows that \( \varphi(t, \tilde{u}) \in \overline{V} \) for all \( t \geq 0 \) and so \( \omega(\tilde{u}) \subseteq \overline{V} \) which implies by our hypotheses that \( \omega(\tilde{u}) \subseteq M_{k+1} \). Since \( \tilde{u}_\nu \in A_{k+1} \) and \( A_{k+1} \) is closed, we have \( \tilde{u} \in A_{k+1} \) and so there is a full solution \( \tilde{\gamma} : \mathbb{R} \to \overline{S} \) through \( \tilde{u} \) with \( \alpha(\tilde{\gamma}) \subseteq M_1 \cup \ldots \cup M_{k+1} \). The ordering of the sets \( M_i \) implies that \( (\alpha(\tilde{\gamma}) \cup \omega(\tilde{\gamma})) \subseteq M_{k+1} \). By definition of \( \{M_i\} \) being a Morse decomposition, we deduce \( \tilde{\gamma}(\mathbb{R}) \subseteq M_{k+1} \) and so \( \tilde{u} \in M_{k+1} \). This contradicts \( \tilde{u} \in \partial V \) as \( M_{k+1} \cap \partial V = \emptyset \). Step 1 is proved.

Step 2: For \( 1 \leq k \leq m \), \( A_k \) is an attractor of certain neighborhood \( U_k \) in \( X \), i.e. \( \omega(U_k) = A_k \).

The claim is automatically true for \( k = m \) since \( A_m = S \) and, by Theorem 2, \( S \) attracts certain neighborhood \( U \) such that \( \omega(U) \subseteq S \). Hence, we proceed by induction and assume \( A_{k+1} \) to be an attractor in \( X \) for some \( k \leq m - 1 \). Choose a neighborhood \( U_{k+1} \cap A_{k+1} \) of \( A_{k+1} \) such that \( \omega(U_{k+1}) = A_{k+1} \). Since \( M_{k+1}, A_k \) are closed and disjoint subsets of the compact set \( A_{k+1} \), we can choose a neighborhood \( U_k \) of \( A_k \) and a neighborhood \( V \) of \( M_{k+1} \) such that \( U_k \cap V = \emptyset \) and \( \overline{U_k} \cup \overline{V} \subset U_{k+1} \). Since \( A_k \) is invariant and contained in \( U_k \) it is clear that \( A_k \subset \omega(U_k) \). It remains to show the reverse inclusion. Suppose \( \omega(U_k) \setminus A_k \neq \emptyset \), and choose \( u \in \omega(U_k) \setminus A_k \). Then there are sequences \( u_n \in U_k \) and \( t_n \to \infty \) such that \( \varphi(t_n, u_n) \to u \). We may assume that \( \varphi(t_n + t, u_n) \to \gamma(t) \) for every \( t \in \mathbb{R} \), where \( \gamma \) is total trajectory through \( u \). By induction assumption, \( \omega(U_k) \subseteq \omega(U_{k+1}) = A_{k+1} \), which implies \( \gamma(\mathbb{R}) \subseteq A_{k+1} \), whence, by step 1, \( \alpha(\gamma) \subseteq A_{k+1} \) and so \( \alpha(\gamma) \subseteq (M_1 \cup \ldots \cup M_{k+1}) \). But \( u \notin A_k \) and so \( \alpha(\gamma) \subseteq M_{k+1} \). There is a sequence \( \rho_v \to \infty \) and \( z \in M_{k+1} \) such that \( \gamma(-\rho_v) \in V \) and \( \text{dist}(\gamma(-\rho_v), z) \leq 1/v \) for every \( v \in \mathbb{N} \). Therefore, for every \( v \) there is \( n_v \geq v \) such that \( t_{n_v} \geq \rho_v + 1 \), \( \varphi(t_{n_v} - \rho_v, u_{n_v}) \in V \) and \( \text{dist}(\varphi(t_{n_v} - \rho_v, u_{n_v}), z) \leq 2/v \). We will show that by choosing \( U_k \) small enough, we can arrange that \( \omega(U_k) = A_k \). In fact, if this is not true, then there is a sequence \( \delta_v \to 0 \) such that \( \overline{U_{\delta_v}} \cap \overline{V} = \emptyset \), \( U_{\delta_v}(A_k) \subset U_{k+1} \) and \( \omega(U_{\delta_v}(A_k)) \setminus A_k \neq \emptyset \), where \( U_{\delta_v}(A_k) \) is the \( \delta_v \)-neighborhood of \( A_k \) in \( X \). Using what we have proved thus far, it is easily seen that there are sequences \( u_v \in U_{\delta_v}(A_k), s_v \geq 1 \) such that \( \varphi(s_v, u_v) \in V \) and \( \text{dist}(\varphi(s_v, u_v), M_{k+1}) \leq 2/v \). There are sequences \( \tau_v \leq s_v \leq \tau_v \leq \infty \) such that \( \varphi(\tau_v, u_v) \in \partial V \), \( \varphi(\tau_v, \tau_v, u_v) \in \overline{V} \) and either \( \tau_v = \infty \) or \( \varphi(\tau_v, u_v) \) is \( \partial V \). Set \( \hat{u}_v = \varphi(\tau_v, u_v) \). We may assume by the compactness of \( \varphi(1, U_{k+1}) \), that \( \hat{u}_v \to \hat{u} \) and the invariance of \( A_k \) and \( u_v \to A_k \) imply \( \tau_v \to \infty \), so \( \hat{u} \in \omega(U_{k+1}) = A_{k+1} \). On the other hand, \( \varphi(s_v, u_v) \to M_{k+1} \) and the invariance of \( M_{k+1} \) implies \( \tilde{\tau}_v \to \infty \) so \( \varphi([0, \infty), \hat{u}) \in \overline{V} \). Therefore \( \omega(\hat{u}) \subset M_{k+1} \) and \( \hat{u} \in A_{k+1} \). Now this obviously implies \( \hat{u} \in M_{k+1} \), a contradiction since \( \hat{u} \in \partial V \). Hence, indeed, \( U_k \) can be chosen such that \( \omega(U_k) = A_k \), i.e. \( A_k \) is an attractor (of a neighborhood in \( X \)).

Step 3: \( M_j = (A_j \cap A_{j-1}^*) \).

We first show \( M_j \subset A_j \cap A_{j-1}^* \). Indeed, if \( u \in M_j \), then there is a solution \( \gamma : \mathbb{R} \to M_j \) through \( u \) and therefore \( u \in A_j \). Suppose \( u \notin A_{j-1}^* \). Then \( \omega(u) \subset A_{j-1}^* \).
and therefore $\omega(u) \subset M_k$ for some $k \leq j - 1$. Since $u \in M_j$, we get $\omega(u) \subset M_j$ and hence $\omega(u) \subseteq (M_k \cap M_j) = \emptyset$, which is impossible. Hence $M_j \subset (A_j \cap A_{j-1}^*)$.

Next, we show $(A_j \cap A_{j-1}^*) \subset M_j$. If $u \in A_j \cap A_{j-1}^*$, then there is a solution $\gamma : \mathbb{R} \to S$ through $u$ such that $\alpha(\gamma) \subset (M_1 \cup \ldots \cup M_j)$. From $u \in A_{j-1}^*$ we conclude $\omega(u) \cap (M_1 \cup \ldots \cup M_{j-1}) = \emptyset$, and hence $\omega(u) \subset M_k$ for some $k \geq j$. Now the assumptions of the proposition imply $k = j$ and $\gamma(\mathbb{R}) \subset M_j$, and so $u \in M_j$, completing the proof. 

\textbf{Lemma 6.} Let $\varphi$ be a semiflow in a complete metric space $X$ satisfying the assumptions of Theorem 2. Let $S$ be the compact global attractor of neighborhoods of compact sets in $X$, then there is a continuous real-valued function $g_0$ in a neighborhood $U$ of $S$ such that $g_0^{-1}(0) = A$, $g_0(1) = A^*$ and $g_0$ is strictly decreasing on orbits that are not contained in $S$.

\textbf{Proof.} This proof is due to Ch. II, Result 5.1B in [1]. Define $l : X \to [0, +\infty)$ by

$$l(u) = \text{dist}(u, S).$$

Then $l$ is continuous and $l^{-1}(0) = S$. Define $k : X \to [0, \infty)$ by $k(u) = \sup\{l(\varphi(t, u)) : t \geq 0\}$. Then $k^{-1}(0) = S$, and $k$ is non-increasing on orbits.

Also, $k$ is continuous as will now be shown. Note that $+\infty > k(u) \geq l(u)$. Since $S$ is a compact attractor of neighborhoods of compact sets, and $S$ is compact, there exists $\varepsilon_0 > 0$ such that $\omega(U_\varepsilon) \subset S$ for all $\varepsilon \in (0, \varepsilon_0)$. For each $\varepsilon$, we claim that there is a neighborhood $U$ of $S$ such that $\sup_{U_\varepsilon} k < \varepsilon$. If not, then there exists $\varepsilon > 0$ and a sequence $u_j \in X$ and $t_j > 0$ such that $u_j \to z$ for some $z \in S$ and $l(\varphi(t_j, u_j)) \geq \varepsilon$. By continuous dependence, we can assume that $t_j \to \infty$. But this contradicts that $\sup_{t \geq 0} \varphi([t, \infty), U_\varepsilon) \subset A$ for all sufficiently small $\varepsilon$-neighborhood $U_\varepsilon$ of $S$. Therefore $k$ is continuous at points of $S$. Given $u \notin S$, let $U$ be a neighborhood of $S$ such that $l(U) < \infty$. Since $S$ attracts certain neighborhood of every compact subsets, we can choose a neighborhood $U'$ of $u$ such that $\omega(U') \subset S$. By shrinking the neighborhood further, we may assume $\sup_{U'} l < \inf_{U_\varepsilon} l$ as well. We claim that, there is some $\tilde{t} > 0$ such that $\varphi([t, \infty), U_\varepsilon') \subset U$. If not, then there exists $\varepsilon > 0$ and a sequence $u_j \in X$ and $t_j > 0$ such that $u_j \to z$ for some $z \in S$ and $\varphi(t_j, u_j) \geq \varepsilon$. By continuous dependence, we can assume that $t_j \to \infty$. But this contradicts that $\sup_{t \geq 0} \varphi([t, \infty), U_\varepsilon) \subset S$ for all sufficiently small $\varepsilon$-neighborhood $U_\varepsilon$ of $S$. With this choice of $\tilde{t}$, if $u' \in U'$ then $k(u') := \sup_{\varphi([0, \tilde{t}], u')} l = \sup_{\varphi([0, \tilde{t}], u')} l$. Now $k$ is continuous at $u$ because $\sup_{\varphi([0, \tilde{t}], u')} l$ depends continuously on $u'$.

The function $g_0$ is defined by $g_0(0) = \int_0^{\infty} e^{-\tau} k(\varphi(\tau, u)) d\tau$. The function $g_0$ is well defined since the semiflow $\varphi$ has precompact and thus bounded trajectories. Because $k$ does, $g_0$ satisfies the conditions $g_0^{-1}(0) = S, g_0$ is continuous and $g_0$ is nonincreasing on orbits. If $u \notin S$ and $t > 0$, then

$$g_0(\varphi(t, u)) - g(u) = \int_0^{\infty} e^{-\tau} (k(\varphi(\tau + t, u)) - k(\varphi(\tau, u))) d\tau.$$
Lemma 7. Let \( \varphi \) be a semiflow in a complete metric space \( X \) such that \( \varphi(t, \cdot) \) is compact for each \( t > 0 \). Let \( S \) be the compact global attractor of bounded sets in \( X \), and let \( A, A^* \subset S \). If \((A, A^*)\) is an attractor-repeller pair, then there is a continuous real-valued function \( g \) in a neighborhood \( U \) of \( S \) such that \( g^{-1}(0) = A \), \( g^{-1}(1) = A^* \) and \( g \) is strictly decreasing on orbits that are not contained in \( A, A^* \).

Proof. Define \( l : X \to [0, +\infty) \) by

\[
 l(u) = \frac{\text{dist}(u, A)}{\text{dist}(u, A) + \text{dist}(u, A^*)}
\]

Then \( l \) is continuous, \( l^{-1}(0) = A \) and \( l^{-1}(1) = A^* \). Define \( k : S \to [0, +\infty) \) by

\[
 k(u) = \sup\{l(\varphi(t, u)) : t \geq 0\}. \quad \text{Then} \quad k^{-1}(0) = A, k^{-1}(1) \cap S = A^*,
\]

and \( k \) is non-increasing on orbits.

Also, \( k \) is continuous as will now be shown. Since \( 1 \geq k(u) \geq l(u) \) for all \( u \), and \( l \) is continuous, we deduce that \( k \) is continuous in \( k^{-1}(1) \). For each \( \varepsilon \) sufficiently small, the neighborhood \( U = U_\varepsilon(A) \) satisfies \( \omega(U) = A \). For each \( \varepsilon \), we claim that there is a neighborhood \( U \) of \( A \) such that \( \sup_U k < \varepsilon \). If not, then there exists \( \varepsilon > 0 \) and a sequence \( u_j \in X \) and \( t_j > 0 \) such that \( u_j \to z \) for some \( z \in A \) and \( l(\varphi(t_j, u_j)) \geq \varepsilon \). By continuous dependence, we can assume that \( t_j \to \infty \). But this contradicts that \( \cap_{t \geq 0} \varphi([t, \infty), U_\varepsilon) \subset A \) for all sufficiently small \( \varepsilon \)-neighborhood \( U_\varepsilon \) of \( A \). Therefore \( k \) is continuous at points of \( A \). Given \( u \in k^{-1}((0, 1)) \), then \( \omega(u) \subset A \). Let \( U \) be a neighborhood of \( A \) such that \( \sup_U l < l(u) \). Choose a bounded neighborhood \( U' = U'_\delta(u) \) of \( u \), then there exists \( \delta > 0 \) such that \( \varphi(\delta, U') \subset U \), and hence \( \omega(U') \subset A \) (here we use the fact that \( U' \) is bounded so that \( \varphi(t, U') \) is compact for any \( t > 0 \). Therefore, \( U' \) is disjoint from \( k^{-1}(1) \). With this choice of \( \delta \), if \( u' \in U' \) then \( k(u') := \sup_{\varphi([0, \infty), u')} l = \sup_{\varphi([0, \delta], u')} l \). Now \( k \) is continuous at \( u \) because \( \sup_{\varphi([0, \delta], u')} l \) depends continuously on \( u' \).

Define the function \( g_1 \) by \( g_1(u) = \int_0^\infty e^{-t} k(\varphi(t, u)) \, d\tau \). Because \( k \) does, \( g_1 \) satisfies the conditions \( g_1^{-1}(0) = A \), \( g_1 \) is continuous and \( g_1 \) is non-increasing on orbits. Now,

\[
 g_1(\varphi(t, u)) - g_1(u) = \int_0^\infty e^{-\tau} (k(\varphi(\tau + t, u)) - k(\varphi(\tau, u))) \, d\tau \quad \text{for} \quad t > 0. \quad (1)
\]

If \( u \in S \setminus (A \cup A^*) \), then \( 0 < k(\varphi(t, u)) < 1 \) for \( t \geq 0 \) and \( \lim_{t \to \infty} k(\varphi(t, u)) = 0 \), so that the last integral of \( (1) \) is strictly negative, so that \( t \mapsto g_1(\varphi(t, u)) \) is strictly decreasing for \( t \geq 0 \).

Finally, let \( g : X \to [0, +\infty) \) be defined by \( g = g_0 + g_1 \), then \( g^{-1}(0) = A \), \( g^{-1}(1) = A^* \). \( g \) is continuous in \( X \), and \( g \) is non-increasing on orbits. In fact, since \( g_0 \) is strictly decreasing for orbits initiating from \( u \in X \setminus S \), and \( g_1 \) is strictly decreasing for orbits initiating from \( u \in S \setminus (A \cup A^*) \), we conclude that \( g \) is strictly decreasing in orbits initiating from \( X \setminus (A \cup A^*) \). This concludes the proof. \( \square \)

Proof of Theorem 4. By Proposition 5, there exists \( m \) attractor-repeller pairs \((A_j, A^*_j)\) \((1 \leq j \leq m)\) such that \( M_j = A_j \cap A^*_{j-1} \) for \( 1 \leq j \leq m \) (here \( A_0 = \emptyset \)). For each
1 ≤ j ≤ m, let \( g_j \) be the Lyapunov function corresponding to the attractor-repeller pair \((A_j, A_j^*)\), as guaranteed by Lemma 7. Then \( V(u) : \sum_{j=1}^{m} g_j(u) \) satisfies all the desired properties. \( \square \)

**Lemma 8** (Ch. II, Result 6.4A of [1]). If \( S \) is compact there are at most countably many attractor-repeller pairs in \( S \).

**Proof.** Since \( S \) is compact, the family of compact subsets of \( S \) with the Hausdorff metric is also a compact metric space. An attractor-repeller pair can be considered a point in the product of this subset space with itself.

Let \((A, A^*)\) be such a pair and let \( U \) and \( U^* \) be disjoint open (in \( S \)) sets about \( A \) and \( A^* \) respectively. Then \((A, A^*)\) is the unique attractor-repeller pair with \( A \subset U \) and \( A^* \subset U^* \).

Now \((U, U^*)\) determines an open set in the product of the subset space with itself which contains only one attractor-repeller pair. Thus the set of attractor-repeller pairs is at most countable. \( \square \)

Recall that a subset \( A \) of \( S \) is said to be **internally chain transitive** with respect to the semiflow \( \varphi \) if, for two points \( u_0, v_0 \in A \), and any \( \delta > 0, T > 0 \), there is a finite sequence

\[
C_{\delta, T} = \{u^{(1)} = u_0, u^{(2)}, \ldots, u^{(m)} = v_0; t_1, \ldots, t_{m-1}\}
\]

with \( u^{(j)} \in A \) and \( t_j \geq T \), such that \( \|\varphi(t_j, u^{(j)}) - u^{(j+1)}\| < \delta \) for all \( 1 \leq i \leq m - 1 \).

The sequence \( C_{\delta, T} \) is called a \((\delta, T)\)-chain connecting \( u_0 \) and \( v_0 \). Define the **chain recurrent set** \( R(S) \) to be the set of all \( u_0 \in S \) such that for any \( T \gg 1 \), and \( \delta \ll 1 \), there exists a \((\delta, T)\)-chain connecting \( u_0 \) to itself.

**Theorem 9** (Ch. II, Result 6.4B of [1]). There exists a continuous function \( G : X \to [0, \infty) \) which is constant on each connected component of the chain recurrent set, and strictly decreasing on orbits outside the chain recurrent set.

**Proof.** Let \( \{(A_i, A_i^*)\}_i \) be an enumeration of the attractor-repeller pairs, and let \( g_i \) be given by Lemma 7. Define \( G(u) = \sum_{i=1}^{m} 3^{-i} g_i(u) \). \( \square \)

**Remark 10.** Define a critical value of \( G \) to be one achieved on the chain recurrent set. Since each \( g_i |_{S} \) is either zero or one at a point of the chain recurrent set, each critical value of \( G \) lies in the "middle third" Cantor set, and in particular the critical values are nowhere dense. Furthermore, each critical value of \( G \) determines a unique component of the chain recurrent set: because \( u \) and \( u' \) lie in the same component of \( R(S) \) if and only if \( u \) is chained to \( u' \) and vice versa, and this is true if and only if \( u \) and \( u' \) are in \( R(S) \) and each attractor containing \( u \) also contains \( u' \).

**References**
