NOTES ON COMPLETE LYAPUNOV FUNCTIONS

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ABSTRACT. In this note we presents a self-contained proof for the existence of complete Lyapunov function for semiflow admitting a Morse decomposition. The main references are C. Conley's CBMS lecture notes and the monograph by K.P. Rybakowski.

Definition 1. Let X be a complete metric space and $\varphi : [0,\infty) \times X$ be a semiflow, i.e. (i) $(t,u) \to \varphi(t,u_0)$ is continuous; (ii) $\varphi(0,u) = u$ for all $u \in X$; (iii) $\varphi(t,\varphi(s,u)) = \varphi(t+s,u)$ for $t,s \geq 0$.

(1) A function $\gamma : \mathbb{R} \to X$ is a *total trajectory* if $\gamma(t+t_0) = \varphi(t,\gamma(t_0))$ for all $t \geq 0$ and $t_0 \in \mathbb{R}$.

(2) A subset $A \subseteq X$ is said to be *invariant* if for each $u \in A$, there exists a total trajectory γ such that $\gamma(0) = u$.

(3) Define the *omega limit set* of a subset B of X by

$$\omega(B) := \cap_{t>0}\overline{\varphi([t,\infty), B)},$$

and define the *omega limit set of a point* $u \in X$ by $\omega(u) = \omega(\{u\})$.

(4) For u lying on some total trajectory γ, we define the *alpha limit set*

$$\alpha(u) = \alpha(\gamma) = \cap_{t<0}\overline{\gamma}((-\infty,t]).$$

(5) A invariant subset A is said to be an *attractor* if there exists a neighborhood U of A such that $\omega(U) = A$.

(6) For an attractor A, define the *repeller dual to A* by

$$A^* := \{u \in X : \omega(u) \cap A = \emptyset\}.$$

And the pair (A, A^*) is called a *attractor-repeller pair*.

(7) φ is *point-dissipative on X* if there exists a bounded set B_p of X such that $\omega(u) \subset B_p$ for all $u \in X$.

(8) φ is *eventually bounded on a set B* if $\varphi([t_0,\infty), B)$ is bounded for some $t_0 > 0$.

(9) φ is *asymptotically compact on B* for some subset $B \subset X$ if, for any $t_i \to \infty$ and $u_i \in B$, $\{\varphi(t_i,u_i)\}$ has a convergent subsqeuence.

(10) φ is *asymptotically smooth* if it is asymptotically compact on every forward invariant bounded closed set. [By Remark 2.26(b) of [3], a sufficient condition is: the mapping $u \mapsto \varphi(t,u)$ is compact for each $t > 0$.]

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(11) A nonempty, compact, invariant subset S is a *compact attractor of neighborhood of compact sets* if S is a compact subset of X and every compact set has a neighborhood U such that that $\omega(U) \subset S$.

Theorem 2 (Theorem 2.30 of [3])**.** *Assuming in addition that φ is point-dissipative, asymptotically smooth, and eventually bounded on every compact subset B of X, then φ has a compact attractor S of neighborhood of compact sets. In particular, there exists a neighborhood U of S such that $\omega(U) = S$.*

Definition 3 (Morse decomposition of the compact attractor)**.** Given a finite ordered collection $\{M_1, ..., M_m\}$ of pairwise disjoint compact invariant subsets of S. We say that $\{M_1, ..., M_m\}$ is a *Morse decomposition of the compact attractor S of X* (or simply, a *Morse decomposition of S*) if (i) for every $u \in X$ there is an i such that $\omega(u) \subset M_i$, and (ii) if u lies on some total trajectory γ, then $\alpha(u) \subset M_j$ for some $i < j \leq m$.

Our main theorem is as follows.

Theorem 4. *Given a Morse decomposition $\{M_1, ..., M_m\}$ of S. Then there exists a continuous function $V : X \to [0, \infty)$ such that*

- *$V^{-1}(i) = M_i$ for $1 \leq i \leq m$, and,*
- *For each $u \in X \setminus \bigcup_{i=1}^m M_i$, the mapping $t \mapsto V(\varphi(t, u))$ is strictly decreasing in $t \geq 0$,*

Proof. See Theorem 9. □

Proposition 5. *Given a Morse decomposition $\{M_1, ..., M_m\}$ of S. Set*

$$A_0 = \emptyset \quad \textit{and} \quad A_k = \{u \in S : \alpha(u) \subset \cup_{i=1}^k M_i\} \ \textit{for } 1 \leq k \leq m.$$

Then $A_0 \subseteq A_1 \subseteq ... \subset A_m$ is a sequence of attractors in S such that $A_i \cap A_{i-1}^* = M_i$.

Proof. The proof is taken from Theorem 3.1.8 in [2].
Step 1: The sets A_k $(1 \leq k \leq m)$ are closed.

Since by definition $A_m = S$, the set A_m is closed (in fact compact). We now proceed inductively and assume A_{k+1} to be closed for some $1 \leq k \leq m-1$. Let $u_j \in A_k$ with $u_j \to u$ for some $u \in S$. Then $u \in A_{k+1}$, since $A_k \subset A_{k+1}$ and A_{k+1} is closed. There are total trajectories $\gamma_j : \mathbb{R} \to S$ with $\gamma_j(0) = u_j$ and $\alpha(\gamma_j) \subset M_1 \cup ... \cup M_k$. Using the compactness of S we can pass to a subsequence and assume WLOG that $\lim_{j\to\infty} \gamma_j(t) \to \gamma(t)$ for each t, for some total trajectory σ through u. We claim that $\alpha(\gamma) \subset (M_1 \cup ... \cup M_k)$. Indeed, since $\gamma_j(\mathbb{R}) \subset A_k \subset A_{k+1}$ and A_{k+1} is closed, it follows that $\gamma(\mathbb{R}) \subset A_{k+1}$ and so $\alpha(\gamma) \subset A_{k+1}$. Observe that $M_i \cap A_{k+1} = \emptyset$ for $i > k + 1$ since M_i is invariant. On the other hand, $\alpha(\gamma) \subset M_i$ for some i by our assumptions and therefore $\alpha(\gamma) \subset M_1 \cup ... \cup M_k \cup M_{k+1}$. Consequently, either $\alpha(\gamma) \subset M_1 \cup ... \cup M_k$ in which case we are done, or else $\alpha(\gamma) \subset M_{k+1}$. In the latter case, let $V \supset M_{k+1}$ be an open neighborhood of M_{k+1} such that $\overline{V} \cap M_i \neq \emptyset$ for $i \neq k + 1$. There is a sequence $t_\nu \to \infty$ and a $z \in M_{k+1}$ such that $\gamma(-t_\nu) \in V$ and $\text{dist}(\gamma(-t_\nu), z) \leq 1/\nu$ for all $\nu \in \mathbb{N}$. Therefore, for every ν there is a $j_\nu \geq \nu$ such that $\gamma_{j_\nu}(-t_\nu) \in V$ and $\text{dist}(\gamma_{j_\nu}(-t_\nu), z) \leq 2/\nu$. Since $(\alpha(\gamma_j) \cup \omega(\gamma_j)) \subset (M_1 \cup .. \cup M_k)$ for every j, there are $\tau_\nu \leq t_\nu \leq s_\nu$ such that

$\gamma_{j_\nu}(-s_\nu), \gamma_{j_\nu}(-\tau_\nu) \in \partial V$ and $\gamma_{j_\nu}(-t) \in \overline{V}$ for $t \in [\tau_\nu, s_\nu]$. The invariance of M_{k+1} now implies that $t_\nu - \tau_\nu \to \infty$. Let $\tilde{u}_\nu := \gamma_{j_\nu}(-s_\nu)$, then $\tilde{u}_\nu \in S$ and since S is compact we may assume $\tilde{u}_\nu \to \tilde{u} \in \partial V$. It then follows that $\varphi(t, \tilde{u}) \in \overline{V}$ for all $t \geq 0$ and so $\omega(\tilde{u}) \in \overline{V}$ which implies by our hypotheses that $\omega(\tilde{u}) \subset M_{k+1}$. Since $\tilde{u}_\nu \in A_{k+1}$ and A_{k+1} is closed, we have $\tilde{u} \in A_{k+1}$ and so there is a full solution $\tilde{\gamma} : \mathbb{R} \to S$ through $\tilde{u}$ with $\alpha(\tilde{\gamma}) \subset M_1 \cup ... \cup M_{k+1}$. The ordering of the sets M_i implies that $(\alpha(\tilde{\gamma}) \cup \omega(\tilde{\gamma})) \subseteq M_{k+1}$. By definition of $\{M_i\}$ being a Morse decomposition, we deduce $\tilde{\gamma}(\mathbb{R}) \subset M_{k+1}$ and so $\tilde{u} \in M_{k+1}$. This contradicts $\tilde{u} \in \partial V$ as $M_{k+1} \cap \partial V = \emptyset$. Step 1 is proved.

Step 2: For $1 \leq k \leq m$, A_k is an attractor of certain neighborhood U_k in X, i.e. $\omega(U_k) = A_k$

The claim is automatically true for $k = m$ since $A_m = S$ and, by Theorem 2, S attracts certain neighborhood U such that $\omega(U) \subset S$. Hence, We proceed by induction and assume A_{k+1} to be an attractor in X for some $k \leq m-1$. Choose a neighborhood $U_{k+1} \supset A_{k+1}$ of A_{k+1} such that $\omega(U_{k+1}) = A_{k+1}$. Since M_{k+1}, A_k are closed and disjoint subsets of the compact set A_{k+1}, we can choose a neighborhood U_k of A_k and a neighborhood V of M_{k+1} such that $\overline{U_k} \cap \overline{V} = \emptyset$ and $\overline{U_k} \cup \overline{V} \subset U_{k+1}$. Since A_k is invariant and contained in U_k it is clear that $A_k \subset \omega(U_k)$. It remains to show the reverse inclusion. Suppose $\omega(U_k) \setminus A_k \neq \emptyset$, and choose $u \in \omega(U_k) \setminus A_k$. Then there are sequences $u_n \in U_k$ and $t_n \to \infty$ such that $\varphi(t_n, u_n) \to u$. We may assume that $\varphi(t_n + t, u_n) \to \gamma(t)$ for every $t \in \mathbb{R}$, where γ is total trajectory through u. By induction assumption, $\omega(U_k) \subset \omega(U_{k+1}) = A_{k+1}$, which implies $\gamma(\mathbb{R}) \subset A_{k+1}$, whence, by step 1, $\alpha(\gamma) \subseteq A_{k+1}$ and so $\alpha(\gamma) \subset (M_1 \cup ... \cup M_{k+1})$. But $u \notin A_k$ and so $\alpha(\gamma) \subseteq M_{k+1}$. There is a sequence $\rho_\nu \to \infty$ and $z \in M_{k+1}$ such that $\gamma(-\rho_\nu) \in V$ and $\mathrm{dist}(\gamma(-\rho_\nu), z) \leq 1/\nu$ for every $\nu \in \mathbb{N}$. Therefore, for every ν there is $n_\nu \geq \nu$ such that $t_{n_\nu} \geq \rho_\nu + 1$, $\varphi(t_{n_\nu} - \rho_\nu, u_{n_\nu}) \in V$ and $\mathrm{dist}(\varphi(t_{n_\nu} - \rho_\nu, u_{n_\nu}), z) \leq 2/\nu$. We will show that by choosing U_k small enough, we can arrange that $\omega(U_k) = A_k$. In fact, if this is not true, then there is a sequence $\delta_\nu \to 0$ such that $\overline{U_{\delta_\nu}} \cap \overline{V} = \emptyset$, $U_{\delta_\nu}(A_k) \subset U_{k+1}$ and $\omega(U_{\delta_\nu}(A_k)) \setminus A_k \neq \emptyset$, where $U_{\delta_\nu}(A_k)$ is the δ_ν-neighborhood of A_k in X. Using what we have proved thus far, it is easily seenthat there are sequences $u_\nu \in U_{\delta_\nu}(A_k)$, $s_\nu \geq 1$ such that $\varphi(s_\nu, u_\nu) \in V$ and $\mathrm{dist}(\varphi(s_\nu, u_\nu), M_{k+1}) \leq 2/\nu$. There are sequences $\tau_\nu \leq s_\nu < \tilde{\tau}_\nu \leq \infty$ such that $\varphi(\tau_\nu, u_\nu) \in \partial V$, $\varphi([\tau_\nu, \tilde{\tau}_\nu), u_\nu) \subset \overline{V}$ and either $\tilde{\tau}_\nu = \infty$ or $\varphi(\tilde{\tau}_\nu, u_\nu) \in \partial V$. Set $\hat{u}_\nu = \varphi(\tau_\nu, u_\nu)$. We may assume by the compactness of $\varphi(1, \overline{U}_{k+1})$, that $\hat{u}_\nu \to \hat{u}$. The invariance of of A_k and $u_\nu \to A_k$ imply $\tau_\nu \to \infty$, so $\hat{u} \in \omega(U_{k+1}) = A_{k+1}$. On the other hand, $\varphi(s_\nu, u_\nu) \to M_{k+1}$ and the invariance of M_{k+1} implies $\tilde{\tau}_\nu \to \infty$ so $\varphi([0, \infty), \hat{u}) \subset \overline{V}$. Therefore $\omega(\hat{u}) \subset M_{k+1}$ and $\hat{u} \in A_{k+1}$. Now this obvoously implies $\hat{u} \in M_{k+1}$, a contradiction since $\hat{u} \in \partial V$. Hence, indeed, U_k can be chosen such that $\omega(U_k) = A_k$, i.e. A_k is an attractor (of a neighborhood in X).

Step 3: $M_j = (A_j \cap A^*_{j-1})$.

We first show $M_j \subset A_j \cap A^*_{j-1}$. Indeed, if $u \in M_j$, then there is a solution $\gamma : \mathbb{R} \to M_j$ through u and therefore $u \in A_j$. Suppose $u \notin A^*_{j-1}$. Then $\omega(u) \subset A_{j-1}$

and therefore $\omega(u) \subset M_k$ for some $k \leq j-1$. Since $u \in M_j$, we get $\omega(u) \subset M_j$ and hence $\omega(u) \subseteq (M_k \cap M_j) = \emptyset$, which is impossible. Hence $M_j \subset (A_j \cap A^*_{j-1})$.

Next, we show $(A_j \cap A^*_{j-1}) \subset M_j$. If $u \in A_j \cap A^*_{j-1}$, then there is a solution $\gamma : \mathbb{R} \to S$ through u such that $\alpha(\gamma) \subset (M_1 \cup ... \cup M_j)$. From $u \in A^*_{j-1}$ we conclude $\omega(u) \cap (M_1 \cup ... \cup M_{j-1}) = \emptyset$, and hence $\omega(u) \subset M_k$ for some $k \geq j$. Now the assumptions of the proposition imply $k = j$ and $\gamma(\mathbb{R}) \subset M_j$, and so $u \in M_j$, completing the proof. □

Lemma 6. *Let φ be a semiflow in a complete metric space X satisfying the assumptions of Theorem 2. Let S be the compact global attractor of neighborhoods of compact sets in X, then there is a continuous real-valued function g_0 in a neighborhood U of S such that $g_0^{-1}(0) = A$, $g_0^{-1}(1) = A^*$ and g_0 is strictly decreasing on orbits that are not contained in S.*

Proof. This proof is due to Ch. II, Result 5.1B in [1]. Define $l : X \to [0, +\infty)$ by

$$l(u) = \mathrm{dist}(u, S).$$

Then l is continuous and $l^{-1}(0) = S$. Define $k : X \to [0, \infty)$ by $k(u) = \sup\{l(\varphi(t, u)) : t \geq 0\}$. Then $k^{-1}(0) = S$, and k is non-increasing on orbits.

Also, k is continuous as will now be shown. Note that $+\infty > k(u) \geq l(u)$. Since S is a compact attractor of neighborhoods of compact sets, and S is compact, there exists $\epsilon_0 > 0$ such that $\omega(U_\epsilon) \subset S$ for all $\epsilon \in (0, \epsilon_0)$. For each ϵ, we claim that there is a neighborhood U of S such that $\sup_U k < \epsilon$. If not, then there exists $\epsilon > 0$ and a sequence $u_j \in X$ and $t_j > 0$ such that $u_j \to z$ for some $z \in S$ and $l(\varphi(t_j, u_j)) \geq \epsilon$. By continuous dependence, we can assume that $t_j \to \infty$. But this contradicts that $\cap_{t\geq 0}\varphi([t, \infty), U_\epsilon) \subset A$ for all sufficiently small ϵ-neighborhood U_ϵ of S. Therefore k is continuous at points of S. Given $u \notin S$, let U be a neighborhood of S such that $\sup_U l < l(u)$. Since S attracts certain neighborhood of every compact subsets, we can choose a neighborhood U' of u such that $\omega(U') \subset S$. By shrinking the neighborhood further, we may assume $\sup_U l < \inf_{U'} l$ as well. We claim that, there is some $\bar{t} > 0$ such that $\overline{\varphi([\bar{t}, \infty), U')} \subset U$. If not, then there exists $\epsilon > 0$ and a sequence $u_j \in X$ and $t_j > 0$ such that $u_j \to z$ for some $z \in S$ and $l(\varphi(t_j, u_j)) \geq \epsilon$. By continuous dependence, we can assume that $t_j \to \infty$. But this contradicts that $\cap_{t\geq 0}\varphi([t, \infty), U_\epsilon) \subset S$ for all sufficiently small ϵ-neighborhood U_ϵ of S. With this choice of $\bar{t}$, if $u' \in U'$ then $k(u') := \sup_{\varphi([0,\infty),u')} l = \sup_{\varphi([0,\bar{t}],u')} l$. Now k is continuous at u because $\sup_{\varphi([0,\bar{t}],u')} l$ depends continuously on u'.

The function g_0 is defined by $g_0(u) = \int_0^\infty e^{-\tau} k(\varphi(\tau, u))\, d\tau$. The function g_0 is well defined since the semiflow φ has precompact and thus bounded trajectories. Because k does, g_0 satisfies the conditions $g_0^{-1}(0) = S$, g_0 is continuous and g_0 is nonincreasing on orbits. If $u \notin S$ and $t > 0$, then

$$g_0(\varphi(t, u)) - g(u) = \int_0^\infty e^{-\tau}(k(\varphi(\tau + t, u)) - k(\varphi(\tau, u))\, d\tau.$$

Since, $\varphi(0, u) \notin S$ and $\varphi(t, u) \to S$ as $t \to \infty$, we deduce $k(\varphi(0, u)) > 0$ and $k(\varphi(t, u)) \to 0$ as $t \to \infty$. Hence The last integral is strictly negative because the nonpositive integrand is not identically zero. This concludes the proof. □

Lemma 7. *Let φ be a semiflow in a complete metric space X such that $\varphi(t,\cdot)$ is compact for each $t > 0$. Let S be the compact global attractor of bounded sets in X, and let $A, A^* \subset S$. If (A, A^*) is an attractor-repeller pair, then there is a continuous real-valued function g in a neighborhood U of S such that $g^{-1}(0) = A$, $g^{-1}(1) = A^*$ and g is strictly decreasing on orbits that are not contained in A, A^*.*

Proof. Define $l : X \to [0, +\infty)$ by

$$l(u) = \frac{\operatorname{dist}(u, A)}{\operatorname{dist}(u, A) + \operatorname{dist}(u, A^*)}$$

Then l is continuous, $l^{-1}(0) = A$ and $l^{-1}(1) = A^*$. Define $k : S \to [0, \infty)$ by $k(u) = \sup\{l(\varphi(t, u)) : t \geq 0\}$. Then $k^{-1}(0) = A$, $k^{-1}(1) \cap S = A^*$,

$$k^{-1}(1) = \{u \in X : \omega(u) \subset A^*\},$$

and k is non-increasing on orbits.

Also, k is continuous as will now be shown. Since $1 \geq k(u) \geq l(u)$ for all u, and l is continuous, we deduce that k is continuous in $k^{-1}(1)$. For each ϵ sufficiently small, the neighborhood $U = U_\epsilon(A)$ satisfies $\omega(U) = A$. For each ϵ, we claim that there is a neighborhood U of A such that $\sup_U k < \epsilon$. If not, then there exists $\epsilon > 0$ and a sequence $u_j \in X$ and $t_j > 0$ such that $u_j \to z$ for some $z \in A$ and $l(\varphi(t_j, u_j) \geq \epsilon$. By continuous dependence, we can assume that $t_j \to \infty$. But this contradicts that $\cap_{t \geq 0} \varphi([t, \infty), U_\epsilon) \subset A$ for all sufficiently small ϵ-neighborhood U_ϵ of A. Therefore k is continuous at points of A. Given $u \in k^{-1}((0, 1))$, then $\omega(u) \subset A$. Let U be a neighborhood of A such that $\sup_U l < l(u)$. Choose a bounded neighborhood $U' = U'_\delta(u)$ of u, then there exists $\bar{t} > 0$ such that $\varphi(\bar{t}, \overline{U'}) \subset U$, and hence $\omega(U') \subset A$ (here we use the fact that U' is bounded so that $\varphi(t, \overline{U'})$ is compact for any $t > 0$. Therefore, U' is disjoint from $k^{-1}(1)$. With this choice of $\bar{t}$, if $u' \in U'$ then $k(u') := \sup_{\varphi([0,\infty),u')} l = \sup_{\varphi([0,\bar{t}],u')} l$. Now k is continuous at u because $\sup_{\varphi([0,\bar{t}],u')} l$ depends continuously on u'.

Define the function g_1 by $g_1(u) = \int_0^\infty e^{-\tau} k(\varphi(\tau, u))\, d\tau$. Because k does, g_1 satisfies the conditions $g_1^{-1}(0) = A$, g_1 is continuous and g_1 is nonincreasing on orbits. Now,

$$g_1(\varphi(t, u)) - g_1(u) = \int_0^\infty e^{-\tau} (k(\varphi(\tau + t, u)) - k(\varphi(\tau, u))\, d\tau \qquad \text{for } t > 0. \quad (1)$$

If $u \in S \setminus (A \cup A^*)$, then $0 < k(\varphi(t, u)) < 1$ for $t \geq 0$ and $\lim_{t\to\infty} k(\varphi(t, u)) = 0$, so that the last integral of (1) is strictly negative, so that $t \mapsto g_1(\varphi(t, u))$ is strictly decreasing for $t \geq 0$.

Finally, let $g : X \to [0, \infty)$ be defined by $g = g_0 + g_1$, then $g^{-1}(0) = A$, $g^{-1}(1) = A^*$, g is continuous in X, and g is nonincreasing on orbits. In fact, since g_0 is strictly decreasing for orbits initiating from $u \in X \setminus S$, and g_1 is strictly decreasing for orbits initiating from $u \in S \setminus (A \cup A^*)$, we conclude that g is strictly decreasing in orbits initiating from $X \setminus (A \cup A^*)$. This concludes the proof. □

Proof of Theorem 4. By Proposition 5, there exists m attractor-repeller pairs (A_j, A_j^*) $(1 \leq j \leq m)$ such that $M_j = A_j \cap A_{j-1}^*$ for $1 \leq j \leq m$ (here $A_0 = \emptyset$). For each

$1 \le j \le m$, let g_j be the Lyapunov function corresponding to the attractor-repeller pair (A_j, A_j^*), as guaranteed by Lemma 7. Then $V(u) : \sum_{j=1}^m g_j(u)$ satisfies all the desired properties. □

Lemma 8 (Ch. II, Result 6.4A of [1])**.** *If S is compact there are at most countably many attractor-repeller pairs in S.*

Proof. Since S is compact, the family of compact subsets of S with the Hausdorff metric is also a compact metric space. An attractor-repeller pair can be considered a point in the product of this subset space with itself.

Let (A, A^*) be such a pair and let U and U^* be disjoint open (in S) sets about A and A^* respectively. Then $(A.A^*)$ is the unique attractor-repeller pair with $A \subset U$ and $A^* \subset U^*$.

Now (U, U^*) determines an open set in the product of the subset space with itself which contains only one attractor-repeller pair. Thus the set of attractor-repeller pairs is at most countable. □

Recall that a subset A of S is said to be *internally chain transitive* with respect to the semiflow φ if, for two points $u_0, v_0 \in A$, and any $\delta > 0, T > 0$, there is a finite sequence

$$C_{\delta,T} = \{u^{(1)} = u_0, u^{(2)}, ..., u^{(m)} = v_0; t_1, ..., t_{m-1}\}$$

with $u^{(j)} \in A$ and $t_j \ge T$, such that $\|\varphi(t_j, u^{(j)}) - u^{(j+1)}\| < \delta$ for all $1 \le i \le m-1$. The sequence $C_{\delta,T}$ is called a (δ, T)-chain connecting u_0 and v_0. Define the *chain recurrent set* $R(S)$ to be the set of all $u_0 \in S$ such that for any $T \gg 1$, and $\delta \ll 1$ there exists a (δ, T)-chain connecting u_0 to itself.

Theorem 9 (Ch. II, Result 6.4B of [1])**.** *There exists a continuous function $G : X \to [0, \infty)$ which is constant on each connected component of the chain recurrent set, and strictly decreasing on orbits outside the chain recurrent set.*

Proof. Let $\{(A_i, A_i^*)\}_i$ be an enumeration of the attractor-repeller pairs, and let g_i be given by Lemma 7. Define $G(u) = \sum_{i=1}^\infty 3^{-i} g_i(u)$. □

Remark 10. Define a critical value of G to be one achieved on the chain recurrent set. Since each $g_i\big|_S$ is either zero or one at a point of the chain recurrent set, each critical value of G lies in the "middle third" Cantor set, and in particular the critical values are nowhere dense. Furthermore, each critical value of G determines a unique component of the chain recurrent set: because u and u' lie in the same component of $R(S)$ if and only if u is chained to u' and vice versa, and this is true if and only if u and u' are in $R(S)$ and each attractor containing u also contains u'.

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