Rudin

Ch 1 #2. Proposition: There is no rational number whose square is 12

Pf: suppose the contrary. Let $r = \frac{m}{n}$ be a fraction in its lowest terms such that $r^2 = 12$ or

$$m^2 = 12n^2$$

Then m^2 is even, so m is even; $m = 2k$ and $m^2 = 4k^2$ and hence $4k^2 = 12n^2$ or

$$k^2 = 3n^2.$$

Since m and n have no common factors, we know that k and n have no common factors.

But now 3 is a divisor of k^2 and hence (since 3 is prime) 3 must divide k. [Here we argue that if a prime p divides ab then p must divide a or b.]

But then $k = 3l$, $k^2 = 9l^2 = 3n^2$ and repeating the argument we see that 3 divides n. This contradicts the assumption that k and n have no common factors.

Ch 1 #4 E nonempty, $E \subset S$ (ordered set). Let α be a lower bound for E and β an upper bound; show $\alpha \leq \beta$.

Pf If E is non empty, then $\exists\, x \in E$. Since α is a lower bound, $\alpha \leq x$. Since β is an upper bound, $x \leq \beta$. Hence, since the order relation is transitive,

$$\alpha \leq x \leq \beta$$

as required.

Strictly speaking, what you know from 1.5 (ii) is $x < y$ and $y < z \Rightarrow x < z$. For completeness, you should extend this by proving that $x \leq y$ and $y \leq z$ implies $x \leq z$.

#7 Fix $b>1$, $y>0$. We prove $\exists!\ x\in\mathbb{R}$ s.t. $b^x=y$. We use prob 6, which defines b^x.

Pf

(a) Show $b^n-1\geq n(b-1)$

Pf The binomial theorem gives $(b^n-1)=(b-1)(b^{n-1}+b^{n-2}+\dots+b+1)$

In the second parenthesis there are n terms, and each is ≥ 1 since $b>1$. Hence $b^{n-1}+\dots+1\geq n$. (In fact, a strict inequality if $n>1$.)

(b) Replace b by $b^{1/n}$ in the expression in (a); in recitation we showed that $b>1 \Rightarrow b^{1/n}>1$ for a positive integer n. Then

$$b-1\geq n(b^{1/n}-1).$$

(c) If $t>1$ and $n>\frac{b-1}{t-1}$, we can multiply by $t-1>0$: $n(t-1)>b-1$; divide by n: $t-1>\frac{b-1}{n}\geq b^{1/n}-1$ by (b), and add 1 to both sides:

$$t>b^{1/n}.$$

(Each step preserves the order relation.)

(d) Given w with $b^w<y$, define $t=\frac{y}{b^w}=y\cdot b^{-w}$; $t>1$ since $b^w<y$.

Now define n as in (c), so $t>b^{1/n}$. Then $b^w t=y$ and since $t>b^{1/n}$, then

$$b^w\cdot b^{1/n}<y \quad \text{or} \quad b^{w+\frac{1}{n}}<y.$$

(e) Given w with $b^w>y$, let $t=\frac{b^w}{y}>1$ and define n as in (c). Then $\frac{b^w}{t}=y$ and $t>b^{1/n}$ means $\frac{1}{t}<\frac{1}{b^{1/n}}=b^{-1/n}$ so $b^{w-1/n}>y$.

(f) Let $A=\{w \mid b^w<y\}$. $\boxed{\text{First, show } A\neq\emptyset.}$ If $y>1$, then $0\in A$, since $b^0=1<y$. In this case, $\boxed{A \text{ is also bounded above}}$, because by the Archimedean principle, there is an integer $N>\frac{y+1}{b-1}$, and then by (a), $b^N\geq N(b-1)+1=y$, and any $w\geq N$ is an upper bound. But if $0<y<1$, then 0 is an upper bound and the Archimedean construction for $1/y$ gives an element in A. Now show $x=\sup A$ satisfies $b^x=y$.

Now, since x is a real number, and b^x is a real number, either $b^x=y$ or $b^x>y$ or $b^x<y$.

Suppose $b^x<y$. Then from (d), there is an n such that $b^{x+\frac{1}{n}}<y$. Hence $x+\frac{1}{n}\in A$; but $x+\frac{1}{n}>x$, so (i) is violated. So we cannot have $b^x<y$.

Suppose $b^x>y$. Then from (e), there is an n such that $b^{x-1/n}>y$. But then $x-\frac{1}{n}=\gamma$ is also an upper bound for A, because if $w\in A$ then $b^w<y<b^{x-1/n}$ so $\frac{b^{x-1/n}}{b^w}>1$ or $b^{x-\frac{1}{n}-w}>1$. We observe that if $b>1$ then $b^\alpha>1 \Leftrightarrow \alpha>0$ (this was essentially proved in prob 6), so $w<x-\frac{1}{n}$ for any $w\in A$.

Since we cannot have $b^x<y$ or $b^x>y$, we must have $b^x=y$.

(g) Suppose there are two real numbers, with $x<z$, such that $b^x=y=b^z$. Then we have $b^{z-x}=1$ and as noted in (f) this means $z-x=0$.