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Discrete Counter Part of Proposition 6.2.1

Consider the discrete dynamical system generated by $G : U \to U$, where U is an open set in $\mathbb{R}^d$. We say that a fixed point b^* of G is unstable if for any neighborhood $\mathcal{N}$ of b^*, there exists $b \in \mathcal{N}$ and $n = n(b) \in \mathbb{N}$ such that

$$G^n(b) \notin \mathcal{N}.$$

Proposition 6.2.1' *Let $G : U \to U$ be C^1. Suppose b^* is a fixed point of G and DG has at least one eigenvalue λ such that $|\lambda| > 1$. Then b^* is unstable.*

Proof. We next transform to coordinates in which the fixed point is at the origin, and using similarity transformation, (replacing x and G by $T^{-1}x$ and $\tilde{G}(y) = T^{-1}G(Ty)$), we may assume

$$\tilde{G}(y) = By + r(y),$$

where $r(y) = o(|y|)$ and

$$B = \begin{pmatrix} B^{(1)} & 0 \\ 0 & B^{(2)} \end{pmatrix}.$$

so that $B^{(1)}$, of dimension n_1 (say), consists of Jordan blocks whose eigenvalues all exceed one in modulus, whereas none of the eigenvalues of $B^{(2)}$ exceeds one in modulus. We also assume that the Jordan blocks has the structure of (using Exercise 3(d) in Chapter 2 of the textbook)

$$J_\varepsilon = \begin{pmatrix} \lambda & \varepsilon & 0 & \cdots & 0 \\ 0 & \lambda & \varepsilon & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda & \varepsilon \\ 0 & 0 & \cdots & 0 & \lambda \end{pmatrix}. \tag{9.3}$$

Let the dimension of $B^{(2)}$ be n_2 so that $n_1 + n_2 = n$. The mapping, in the new variable $y = S^{-1}x$, takes the form

$$G(y) = \begin{bmatrix} B^{(1)}y^{(1)} + k^{(1)}(y^{(1)}, y^{(2)}) \\ B^{(2)}y^{(2)} + k^{(2)}(y^{(1)}, y^{(2)}) \end{bmatrix}.$$

Here the vector function $k(y)$ satisfies $|k(y)| = o(|y|)$. We may define a norm for $y^{(j)}$ by the formula

$$|y^{(j)}| = \sum_{l=1}^{n_j} |y_l^{(j)}|, \quad j = 1, 2.$$

In this formula the vertical bars $|\cdot|$ in the sum on the right-hand side represent the moduli of the (in general) complex numbers $y_l^{(j)}$ whereas those on the left-hand side of the equation indicate the norm of the vector $y^{(j)}$. This is the choice of norm made in equation (8.9) of Chapter 8 and used also in the proof of Theorem 9.1.1. The norm for the full vector $y = (y_1, y_2)^t$ is then

$$|y| = |y_1| + |y_2|.$$

This induces a norm for x: $\|x\| = |y|$ where $y = T^{-1}x$.

Let $\mu > 1$ be a lower bound for the eigenvalues of B_1. It is easy to see that $|B^{(1)}y^{(1)}| > (\mu - \epsilon)|y^{(1)}|$. **(This part uses** (9.3).**)** Similarly, we find $|B^{(2)}y^{(2)}| \leq (1+\epsilon)|y^{(2)}|$. By choosing $|y|$ sufficiently small we can ensure that

$$|k(y)| < \epsilon|y| = \epsilon(|y^{(1)}| + |y^{(2)}|).$$

Armed with these estimates, we can now show that the origin is unstable.

Assume, to the contrary, that the origin is Lyapunov stable. Denote by $y(0)$ the initial choice of y, and by $y(1), y(2), \ldots$ subsequent iterations under the mapping $\tilde{G}(y) = By + k(y)$. According to the assumption of Lyapunov stability, given any $\epsilon > 0$ there exists $\delta > 0$ such that, if $|y(0)| < \delta$, then $|y(n)| < \epsilon$ for all $n = 1, 2, \ldots$. With the aid of the estimates above we find that

$$|y^{(1)}(n+1)| \geq (\mu - \epsilon)|y^{(1)}(n)| - \epsilon(|y^{(1)}(n)| + |y^{(2)}(n)|),$$

and

$$|y^{(2)}(n+1)| \leq (1+\epsilon)|y^{(2)}(n)| + \epsilon(|y^{(1)}(n)| + |y^{(2)}(n)|).$$

so

$$|y^{(1)}(n+1)| - |y^{(2)}(n+1)| \geq (\mu - 3\epsilon)|y^{(1)}(n)| - (1+3\epsilon)|y^{(2)}(n)|.$$

We have not as yet specified ϵ: choose it less than $\frac{1}{6}(\mu - 1)$. Then

$$|y(n+1)| \geq |y^{(1)}(n+1)| - |y^{(2)}(n+1)| \geq (1+3\epsilon)\big(|y^{(1)}(n)| - |y^{(2)}(n)|\big).$$

Now choosing $|y^{(1)}(0)| - |y^{(2)}(0)| > 0$, we find that

$$|y(n)| \geq |y^{(1)}(n)| - |y^{(2)}(n)| \geq (1+3\epsilon)^n \big(|y^{(1)}(0)| - |y^{(2)}(0)|\big),$$

implying that the left-hand side grows without bound as $n \to \infty$. This contradicts the assumption of stability. $\square$

$\square$