Dulac's lemma

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Dulac's lemma applies to homoclinic and heteroclinic cycles

Consider a DE

$$\dot{x} = F(x) \tag{1}$$

where $F : \bar{U} \to \mathbb{R}$ is C^1_{loc} in an open set U.

Proposition 1 (Dulac)**.** *Suppose that $F : U \to \mathbb{R}^2$ is C^1 on an open simply connected set $U \subset \mathbb{R}^2$. If there exists a C^1 function $g : U \to \mathbb{R}$ such that the divergence*

$$\nabla \cdot (gF)$$

does not change sign and is not identically zero on any open subset of U, then the equation

$$x' = F(x)$$

has no periodic solutions lying entirely within U. **In fact, it also rules out any homoclinic cycle or heterclinic cycle that lies entirely in U.**

Proof. Suppose there is a periodic solution $\gamma(t)$, then since U is simply connected, the orbit of γ forms the boundary $\partial\Omega$ of an open subset Ω of U. For $x \in \partial\Omega$, we have $x = \gamma(t)$ for some t, and we have

$$\dot{\gamma}(t) = F(x) \perp \mathbf{N}(x). \tag{2}$$

let $\mathbf{N}(x)$ be the unit outward normal vector with respect to Ω. By divergence theorem,

$$\iint_\Omega \nabla \cdot (g\mathbf{F})\, dA = \int_{\partial\Omega} (g\mathbf{F}) \cdot \mathbf{N}\, ds = 0.$$

Now, the left hand side is nonzero, but the right hand side must be zero thanks to (2). We arrive at a contradiction.

Finally, note that the above argument requires $\partial\Omega$ is piecewise C^1 (and $F \in C^1(\bar{\Omega})$), and that $\mathbf{F} \cdot \mathbf{N} = 0$ only, so the proof also applies for homoclinic orbits and heteroclinic cycles. ☐