HW 10 SOLUTIONS

SECTION 6.5

4] $y'' - y = -20\,\delta(t-3)$ $\quad y(0)=1 \quad y'(0)=0$

a) We have

$$s^2F(s) - s\,y(0) - y'(0) - F(s) = -20\,e^{-3s}$$

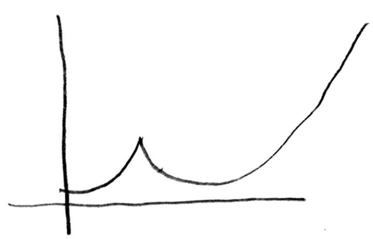
so

$$F(s) = \frac{s}{s^2-1} - \frac{20e^{-3s}}{s^2-1}$$

and hence

$$y = \cosh t - 20\,U_3(t)\sinh(t-3)$$

b)



10] $2y'' + y' + 4y = \delta(t-\pi/6)\sin t$ $\quad y(0)=0 \quad y'(0)=0$

We have

$$2s^2F(s) - 2s\,y(0) - 2y'(0) + sF(s) - y(0) + 4F(s)$$
$$= \int_0^\infty e^{-st}\,\delta\!\left(t-\tfrac{\pi}{6}\right)\sin t\,dt = e^{-\pi/6\,s}\sin\frac{\pi}{6} = \frac{e^{-\frac{\pi s}{6}}}{2}$$

so

$$F(s) = \frac{e^{-\pi s/6}}{2(2s^2+s+4)} - \frac{e^{-\pi s/6}}{4\left(s^2+\frac{s}{2}+2\right)}$$

$$= \frac{e^{-\pi s/6}}{4\left[(s+1/4)^2+\frac{31}{16}\right]} \quad \text{and hence}$$

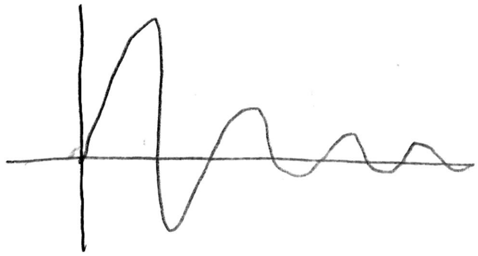
$$y = \mathcal{L}^{-1}(F(s)) = U_{\pi/6}(t)\,\mathcal{L}^{-1}\!\left(\frac{1}{4\left[(s+1/4)^2+\frac{31}{16}\right]}\right)\!\left(t-\frac{\pi}{6}\right)$$

$$= \frac{1}{4} U_{\pi/6}(t)\, e^{-\frac{1}{4}(t-\frac{\pi}{6})} \mathcal{L}^{-1}\left(\frac{1}{s^2+\frac{31}{16}}\right)\left(t-\frac{\pi}{6}\right)$$

$$= \frac{1}{4} U_{\pi/6}(t)\, e^{-(t-\frac{\pi}{6})/4} \frac{4}{\sqrt{31}} \sin \frac{\sqrt{31}}{4}\left(t-\frac{\pi}{6}\right)$$

$$= \frac{1}{\sqrt{31}} U_{\pi/6}(t) e^{-(t-\frac{\pi}{6})/4} \sin \frac{\sqrt{31}}{4}\left(t-\frac{\pi}{6}\right)$$

b)



14| First we will solve the initial value problem $y'' + \gamma y' + y = \delta(t-1)$ $y(0)=0$ and $y'(0)=0$. for $\gamma \in [0,1)$. Then we will answer a), b), c) and d).

For this first note

$$s^2F(s) + \gamma s F(s) + F(s) = e^{-s}$$

so $F(s) = \dfrac{e^{-s}}{s^2+\gamma s+1} = \dfrac{e^{-s}}{(s+\gamma/2)^2 + (1-\gamma^2/4)}$

and hence $y(t) = \mathcal{L}^{-1}(F(s))(t) =$

$$= U_1(t)\, \mathcal{L}^{-1}\left(\frac{1}{(s+\gamma/2)^2+(1-\gamma^2/4)}\right)(t-1)$$

$$= U_1(t)\, e^{-\gamma(t-1)/2} \mathcal{L}^{-1}\left(\frac{1}{s^2+(1-\gamma^2/4)}\right)(t-1)$$

$$= U_1(t)\, e^{-\gamma(t-1)/2} \sin \sqrt{1-\gamma^2/4}\,(t-1) \cdot \frac{1}{\sqrt{1-\gamma^2/4}}$$

Now we compute y' for t larger than 1,

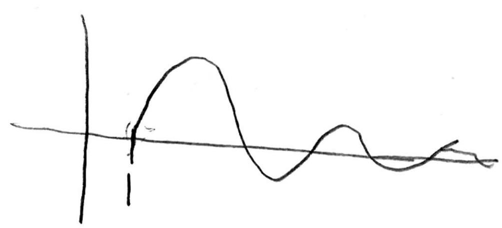
$$y'(t) = \frac{-\gamma}{\sqrt{4-\gamma^2}} e^{-\frac{\gamma}{2}(t-1)} \sin\sqrt{1-\frac{\gamma^2}{4}}\,(t-1)$$
$$+ e^{-\frac{\gamma}{2}(t-1)} \cos\sqrt{1-\frac{\gamma^2}{4}}\,(t-1)$$

So if we let $y'(t) = 0$, $t>1$ we get

$$e^{-\frac{\gamma}{2}(t-1)} \cos\sqrt{1-\frac{\gamma^2}{4}}\,(t-1) = \frac{\gamma e^{-\frac{\gamma}{2}(t-1)}}{\sqrt{4-\gamma^2}} \sin\sqrt{1-\frac{\gamma^2}{4}}\,(t-1)$$

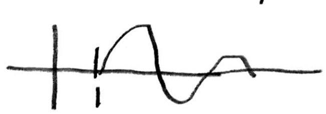
so $\cot\sqrt{1-\frac{\gamma^2}{4}}\,(t-1) = \frac{\gamma}{\sqrt{4-\gamma^2}} = \frac{1}{\sqrt{4/\gamma^2-1}}$

a) $y(t) = u_1(t)\, e^{-\frac{(t-1)}{4}} \sin\sqrt{\frac{15}{16}}\,(t-1) \cdot \sqrt{\frac{16}{15}}$



b) From the graph we know that we want $\sqrt{\frac{15}{16}}(t_1-1)$ to be as small as possible with $t_1>1$ and $\cot\sqrt{\frac{15}{16}}(t_1-1) = \frac{1}{\sqrt{15}}$. Hence $t_1 \cong 2.36$ and $y(t_1) \cong 0.71$

c) $y(t) = u_1(t)\, e^{-\frac{(t-1)}{8}} \sin\frac{\sqrt{63}}{8}(t-1) \cdot \frac{8}{\sqrt{63}}$



Now $t_1 \cong 2.46$ and $y(t_1) = 0.83$.

d) If $\gamma = 0$, we get that

$y(t) = u_1(t)\sin(t-1)$; so $t_1 = 1 + \frac{\pi}{2}$ and $y(t_1) = 1$

Now if we consider

$$\cot\left(\sqrt{1-\frac{\gamma^2}{4}}\,(t_1-1)\right) = \frac{1}{\sqrt{4/\gamma^2 - 1}}$$

and we let $\gamma \to 0$; we see that the right hand side approaches 0

so as $\gamma \to 0$, $\sqrt{1-\frac{\gamma^2}{4}}(t_1-1) \longrightarrow \frac{\pi}{2}$.

But $\lim_{\gamma \to 0} \sqrt{1-\frac{\gamma^2}{4}} = 1$; so $\lim_{\gamma\to 0} t_1 = 1 + \frac{\pi}{2}$

and by continuity, we see that

$\lim_{\gamma \to 0} Y_\gamma(t_1, \gamma) \to 1$ (This continuity is with respect to γ)

SECTION 6.6

6) $f(t) = \int_0^t (t-\tau)e^{\tau}\,d\tau = (x * e^x)(t)$

so $\mathcal{L}(f)(s) = \frac{1}{s^2}\,\frac{1}{s-1}$

8) $F(s) = \frac{1}{s^4} \cdot \frac{1}{s^2+1}$, so

$\mathcal{L}^{-1}(F)(t) = \frac{1}{6}(x^3 * \sin(x))(t)$

12| a) since $f(t)=t^m$ and $g(t)=t^n$ where m, n are positive integers, we have that

$$f*g(t)=\int_0^t (t-\tau)^m \tau^n d\tau$$

$$= t^{m+n+1}\int_0^t \left(1-\frac{\tau}{t}\right)^m \left(\frac{\tau}{t}\right)^n \frac{d\tau}{t}$$

so if we let $\tau = tU$ we get $\frac{d\tau}{t} = dU$ and hence $f*g(t) = t^{m+n+1}\int_0^1 (1-U)^m U^n dU$

b) We have

$$\frac{m!}{s^{m+1}} \cdot \frac{n!}{s^{n+1}} = \mathcal{L}(f*g)(s) = \int_0^1 (1-U)^m U^n dU \ \frac{m+n+1!}{s^{m+n+2}}$$

so $$\frac{m!\,n!}{(m+n+1)!} = \int_0^1 (1-U)^m U^n dU$$

c) Using the Γ function and following the same procedure we get

$$\frac{\Gamma(m+1)\Gamma(n+1)}{\Gamma(m+n+2)} = \int_0^1 (1-U)^m U^n dU$$

for $m, n > 0$.

16) $y'' + y' + \frac{5}{4}y = 1 - u_\pi(t) \qquad y(0) = 1;\ y'(0) = -1$

We have

$$s^2F(s) - sy(0) - y'(0) + sF(s) - y(0) + \frac{5}{4}F(s)$$

$$= \frac{1}{s} - \frac{e^{-\pi s}}{s}$$

so $$\left(s^2 + s + \frac{5}{4}\right)F(s) = \left(\frac{1}{s} - \frac{e^{-\pi s}}{s} + s\right)$$

and hence $$F(s) = \frac{1 - e^{-\pi s}}{s((s+1/2)^2+1)} + \frac{s}{(s+\frac{1}{2})^2+1}$$

$$= \left(\frac{1}{s} - \frac{e^{-\pi s}}{s}\right)\left(\frac{1}{(s+1/2)^2+1}\right)$$

$$+ \frac{s+1/2}{(s+1/2)^2+1} - \frac{1}{2}\,\frac{1}{(s+1/2)^2+1}$$

so $$y(t) = (1 - u_\pi(x)) * e^{-x/2}\sin(x)(t)$$

$$+ e^{-t/2}\cos t - \frac{1}{2}e^{-t/2}\sin t$$

$$= \int_0^t e^{-(t-x)/2}\sin(t-x)(1-u_\pi)(x)\,dx$$

$$+ e^{-t/2}\cos t - \frac{1}{2}e^{-t/2}\sin t.$$