HW 11 Solutions:

Section 4.1

9) The functions are linearly dependent since they are 4 polynomials of degree 2 and hence we can solve the system of 3 equations and 4 variables

$$\begin{aligned} & c_2 t^2 + c_3 2t^2 + c_4 t^2 = 0 \\ -c_1 2t & \quad - c_3 t + c_4 t = 0 \\ +c_1 \cdot 3 + c_2 & \quad\quad + c_4 = 0 \end{aligned}$$

By choosing $c_1 = 1$ we get $c_2 = -\frac{13}{12}$, $c_3 = \frac{3}{2}$, $c_4 = \frac{7}{2}$

(In fact we get these from solving

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -1 & 1 & 2 \\ 1 & 0 & 1 & -3 \end{array}\right].)$$

Hence $-\frac{13}{2}(t^2+1) + \frac{3}{2}(2t^2-t) + \frac{7}{2}(t^2+t+1) = 2t-3$.

17) $W(5, \sin^2 t, \cos 2t) = \begin{vmatrix} 5 & \sin^2 t & \cos 2t \\ 0 & 2\sin t\cos t & -2\sin 2t \\ 0 & 2(\cos^2 t - \sin^2 t) & -4\cos 2t \end{vmatrix}$

$= 5(-4\cos 2t \cdot 2\sin t\cos t + 2\sin(2t) \cdot 2(\cos^2 t - \sin^2 t))$

$= 5(-4\cos 2t \sin 2t + 4\sin 2t \cos 2t) = 0$

since $\sin 2t = 2\sin t\cos t$ and $\cos 2t = \cos^2 t - \sin^2 t$.

Observe that the the given functions are $f_1(t) = 5$; $f_2(t) = \sin^2 t$ and $f_3(t) = \cos 2t = \cos^2 t - \sin^2 t$ so clearly $f_1 = 10 f_2 + 5 f_3$; so we dont need to compute

$W(5, \sin^2 t, \cos 2t)$ to know that it is 0.

18) Let c_1, c_2 be constants and let y_1, y_2 be n times differentiable functions. We have

$$L(c_1 y_1 + c_2 y_2) = (c_1 y_1 + c_2 y_2)^{(n)} + P_1(t)(c_1 y_1 + c_2 y_2)^{(n-1)}$$
$$+ \cdots + P_n(c_1 y_1 + c_2 y_2)$$
$$= c_1 (y_1)^{(n)} + c_2 (y_2)^{(n)} + c_1 P_1(t)(y_1)^{(n-1)} + c_2 P_1(t)(y_2)^{(n-1)} + \cdots$$
$$\cdots + c_1 P_n(t) y_1 + c_2 P_n(t) y_2 = c_1 L(y_1) + c_2 L(y_2).$$

Hence if $c_1, c_2, \dots, c_n$ are constants and $y_1, \dots, y_n$ are such that $L(y_1) = 0, \dots, L(y_n) = 0$; then $L(c_1 y_1) = c_1 L(y_1) = 0$ and hence $c_1 y_1$ is a solution to $L(y) = 0$. Now suppose that for a fix natural number k we have shown that $L(c_1 y_1 + \cdots + c_k y_k) = 0$; then $L(c_1 y_1 + \cdots + c_k y_k + c_{k+1} y_{k+1}) = L(c_1 y_1 + \cdots + c_k y_k) + L(c_{k+1} y_{k+1}) = 0 + c_{k+1} L(y_{k+1}) = 0$. So by induction on n, for any n constants $c_1, \dots, c_n$ and any n solutions to $L(y) = 0$ $y_1, \dots, y_n$ we have $L(c_1 y_1 + \cdots + c_n y_n) = 0$.

19) a) $L(t^n) = a_0 n! + a_1 n! t + \cdots + a_n t^n$

b) $L(e^{rt}) = a_0 r^n e^{rt} + a_1 r^{n-1} e^{rt} + \cdots + a_n e^{rt}$
$$= (a_0 r^n + \cdots + a_n) e^{rt}.$$

c) We are looking for solutions of the form e^{rt}, hence we look at the characteristic polynomial and observe that $r^4-5r^2+4=0$ iff

$r=1,-1,2,-2$.

To see that $y_1=e^t$, $y_2=e^{-t}$, $y_3=e^{2t}$, $y_4=e^{-2t}$ is a fundamenthal set of solutions we only need to compute $W(y_1,y_2,y_3,y_4)(0)$ and check that it is not 0. For this we will use basic row operations.

$$W(y_1,y_2,y_3,y_4)(0)=\begin{vmatrix} 1 & 1 & 1 & 1\\ 1 & -1 & 2 & -2\\ 1 & 1 & 4 & 4\\ 1 & -1 & 8 & -8\end{vmatrix}$$

$$=\begin{vmatrix} 1 & 1 & 1 & 1\\ 0 & -2 & 1 & -3\\ 0 & 0 & 3 & 3\\ 0 & -2 & 7 & -9\end{vmatrix}=\begin{vmatrix} 1 & 1 & 1 & 1\\ 0 & -2 & 1 & -3\\ 0 & 0 & 3 & 3\\ 0 & 0 & 6 & -6\end{vmatrix}=\begin{vmatrix} 1 & 1 & 1 & 1\\ 0 & -2 & 1 & -3\\ 0 & 0 & 3 & 3\\ 0 & 0 & 0 & -12\end{vmatrix}\neq 0.$$

SECTION 4.2

3) $|-3|=3=r$ and and its angle with respect to the positive real axis is π, hence $-3=3e^{i\pi}$.

7) First we note that $1=e^{i2\pi k}$ for $k=0,\pm1,\cdots$; hence $1^{1/3}=e^{i\frac{2\pi k}{3}}$; that is $1^{1/3}=e^{i\frac{2\pi\cdot 0}{3}}=1$ or $1^{1/3}=e^{i2\pi/3}$ or $1^{1/3}=e^{i4\pi/3}$. (note that all the other values of k fall into one of these three cases.)

8) First we note $|1-i|^2=(1-i)(1+i)=2$; so $|1-i|=\sqrt{2}$. Also $1-i$ is in the 4th quadrant and and $\tan^{-1}(1)=\frac{\pi}{4}$; so $1-i=\sqrt{2}\,e^{-i(\pi/4+2\pi k)}$ for $k=0,\pm1,\pm2,\cdots$. Hence $(1-i)^{1/2}=\sqrt[4]{2}\,e^{-i(\frac{\pi}{8})}, \sqrt[4]{2}\,e^{-i(\pi-\pi/8)}$.

15| $y^{(6)}+y=0$

The characteristic polynomial is

$$r^6+1=p(r)=(r-e^{i\pi/6})(r-e^{-i\pi/6})(r-e^{i\frac{\pi}{6}+\frac{i\pi}{3}})(r-e^{-i(\frac{\pi}{6}+\frac{\pi}{3}}$$
$$\times(r-e^{i(\frac{\pi}{6}+\frac{2\pi}{3})})(r-e^{-i(\frac{\pi}{6}+\frac{2\pi}{3})})$$

so the general solution is

$$y=e^{\sqrt{3}/2\,t}\left(c_1\cos\frac{t}{2}+c_2\sin\frac{t}{2}\right)$$
$$+\left(c_3\cos t+c_4\sin t\right)$$
$$+e^{-\sqrt{3}/2\,t}\left(c_5\cos\frac{t}{2}+c_6\sin\frac{t}{2}\right).$$

23| $y'''-5y''+3y'+y=0$

The characteristic polynomial is

$$P(r)=r^3-5r^2+3r+1=(r-1)(r^2-4r-1)$$
$$=(r-1)(r-(2+\sqrt{5}))(r-(2-\sqrt{5}))$$

so $y=c_1e^t+c_2e^{(2+\sqrt{5})t}+c_3e^{(2-\sqrt{5})t}$

34| $4y'''+y'+5y=0$ $\quad y(0)=2;\ y'(0)=1;\ y''(0)=-1$

The characteristic polynomial is

$$P(r)=4r^3+r+5=(r+1)(4r^2-4r+5)$$
$$=(r+1)(r-(\tfrac{1}{2}+i))(r-(\tfrac{1}{2}-i))\,4$$

so $y=c_1e^{-t}+c_2e^{t/2}\cos t+c_3e^{t/2}\sin t$

Now $2=y(0)=c_1+c_2$

$$1=y'(0)=-c_1+\frac{c_2}{2}+c_3$$
$$-1=y''(0)=c_1-\frac{3}{4}c_2+c_3$$

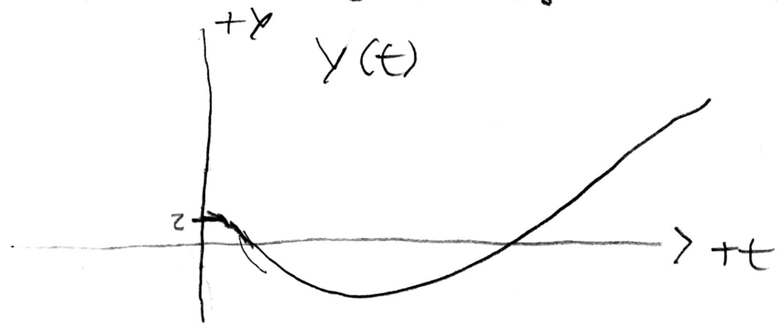
so $2-c_2=c_1$; $-3+\frac{7}{4}c_2=c_3$ and

$1=-2+c_2+\frac{c_2}{2}-3+\frac{7}{4}c_2$. so

$c_2=\frac{24}{13}$; $c_3=\frac{3}{13}$ and $c_1=\frac{2}{13}$. so

$$y=\frac{2}{13}e^{-t}+\frac{24}{13}e^{t/2}\cos t+\frac{3}{13}e^{t/2}\sin t$$

Note that y is oscilating and its amplitud increases as $t\to\infty$.



Section 4.3

3| $y'''+y''+y'+y=e^{-t}+4t$

First we find the general solution by looking at the characteristic polynomial

$P(r)=r^3+r^2+r+1=\frac{r^4-1}{r-1}=(r+1)(r+i)(r-i)$

so the general solution is $y=c_1e^{-t}+c_2\cos t+c_3\sin t$.

Now we observe that the particular solution has the form $Y_p=Ate^{-t}+Bt+C$ so since

$y'=-Ate^{-t}+Ae^{-t}+B$; $y''=Ate^{-t}-2Ae^{-t}$ and

$y'''=-Ate^{-t}+3Ae^{-t}$ we have

$B=4$, $C+B=0$; $A-2A+3A=1$; so $A=\frac{1}{2}$. Hence

$$y=c_1e^{-t}+c_2\cos t+c_3\sin t+4(t-1)+\frac{te^{-t}}{2}$$

5] $y'''' - 4y'' = t^2 + e^t$.

First note that the characteristic polynomia is $r^4 - 4r^2 = r^2(r-2)(r+2)$; so

$$y = c_1 + c_2 t + c_3 e^{2t} + c_4 e^{-2t} + y_p.$$

To find y_p we note that it has the form $y_p = (At^2 + Bt + C)t^2 + De^t$. Since

$y_p'' = 12At^2 + 6Bt + 2C + De^t$ and

$y_p'''' = 24A + De^t$ we get that

$D - 4D = 1$; $-4 \cdot 12A = 1$; $-4 \cdot 6B = 0$

$-4 \cdot 2C + 24A = 0$. So $D = -\frac{1}{3}$; $A = \frac{-1}{48}$

and $C = \frac{-1}{16}$. Thus

$$y = c_1 + c_2 t + c_3 e^{2t} + c_4 e^{-2t} + \left(\frac{-1}{48} t^4 - \frac{1}{16} t^2\right) - \frac{1}{3} e^t.$$

9] $y''' + 4y' = t$ $\quad y(0) = y'(0) = 0$; $y''(0) = 1$.

The characteristic polynomial is $P(r) = r^3 + 4r = r(r^2+4) = r(r-2i)(r+2i)$ so the solution $y(t)$ has the form

$$y = c_1 + c_2 \cos 2t + c_3 \sin 2t + At^2 + Bt$$

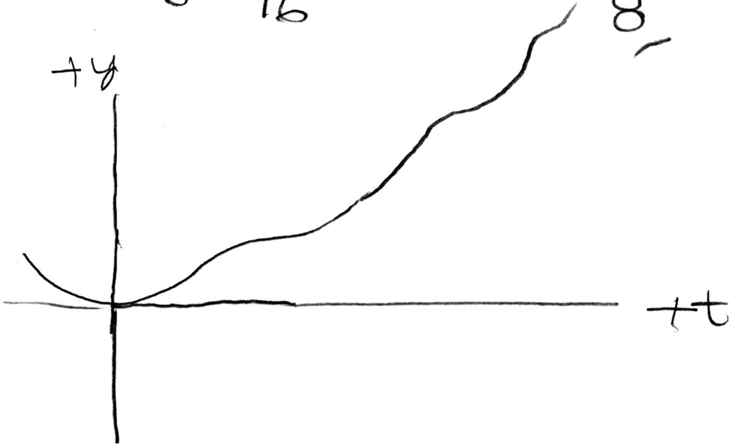
If we let $y_p = At^2 + Bt$ we get $y_p' = 2At + B$ and $y_p''' = 0$. So $8A = 1$ and $4B = 0$. Thus $A = \frac{1}{8}$.

Now $0 = y(0) = c_1 + c_2$ and $0 = y'(0) = 2c_3$ so $c_3 = 0$ and $c_1 = -c_2$.

Now, from $y''(0)=1$, we get $-4c_2+\frac{1}{4}=1$

so $c_2=-\frac{3}{16}$ and therefore

$$Y=\frac{3}{16}-\frac{3}{16}\cos 2t+\frac{t^2}{8}$$



13] The characteristic polynomial is r^3-2r^2+r $=r(r^2-2r+1)$; so the solution to

$y'''-2y''+y'=t^3+2e^t$ has the form

$Y=c_1+c_2e^t+c_3te^t+At^2e^t+(Bt^3+Ct^2+Dt+E)t$

so $Y(t)=At^2e^t+(Bt^3+Ct^2+Dt+E)t$

15] $y^{(4)}-2y''+y=e^t+\sin t.$

Characteristic polynomial is $P(r)=r^4-2r^2+1$
$=(r^2-1)^2=(r-1)^2(r+1)^2$; so $y(t)=c_1e^t+c_2te^t+c_3e^{-t}+c_4te^{-t}$
$+At^2e^t+B\sin t+C\cos t$

so $Y(t)=At^2e^t+B\sin t+C\cos t$

17] The characteristic polynomial is $P(r)=r^4-r^3-r^2+r$
$=r(r^3-r^2-r+1)=r(r+1)(r-1)^2$ so
$Y(t)=c_1+c_2e^{-t}+c_3e^t+c_4te^t+(At^2+Bt+C)t$
$+(Dt+E)\sin t+(Ft+G)\cos t$

so $Y(t)=(At^2+Bt+C)t+(Dt+E)\sin t$
$+(Ft+G)\cos t$