HW12 Solutions

SECTION 5.1

5| $\sum_{n=1}^{\infty} \frac{(2x+1)^n}{n^2} = \sum_{n=1}^{\infty} \frac{2^n}{n^2}\left(x+\frac{1}{2}\right)^n$

so $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2^{n+1}/(n+1)^2}{2^n/n^2}$

$= \lim_{n\to\infty} \frac{2n^2}{(n+1)^2} = 2$. so $R = \frac{1}{2}$

7| $\sum_{n=1}^{\infty} \frac{(-1)^n n^2 (x+2)^n}{3^n}$

so $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{(n+1)^2/3^{n+1}}{n^2/3^n} = \lim_{n\to\infty} \frac{(n+1)^2}{n^2}\frac{1}{3}$

$= \frac{1}{3}$. so $R = 3$.

9| We know that the nth derivative of $\sin(x)$ is given by:

$$\frac{d^n}{dx^n}\sin(x) = \begin{cases} \sin(x) & n=4K \\ \cos(x) & n=4K+1 \\ -\sin(x) & n=4K+2 \\ -\cos(x) & n=4K+3 \end{cases}$$

for $K=0,\dots$. Equivalently,

$$\frac{d^n}{dx^n}\sin(x) = \begin{cases} (-1)^{n/2}\sin(x) & \text{if } n \text{ is even} \\ (-1)^{(n-1)/2}\cos(x) & \text{if } n \text{ is odd} \end{cases}$$

Hence $\sum_{n=0}^{\infty} \left.\frac{d^n \sin(x)}{dx^n}\right|_{x=0} \cdot \frac{1}{n!} x^n$

$= \sum_{n=0}^{\infty} \left.\frac{d^{2n}\sin(x)}{dx^n}\right|_{x=0} \frac{x^{2n}}{2n!} + \sum_{n=0}^{\infty} \left.\frac{d^{2n+1}\sin(x)}{dx^{2n+1}}\right|_{x=0} \frac{x^{2n+1}}{(2n+1)!}$

$$= 0 + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

Now $R = \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} = \lim_{n\to\infty} 2n+1 = \infty.$

13| $\ln(x)$ $x_0 = 1$

We know that $\frac{1}{1+(x-1)} = \sum_{n=0}^{\infty} (-1)^n (x-1)^n$

(It is a geometric series with $R=1$.)

& know that $\int^t \frac{1}{1+(x-1)}\, dx = \ln(t)$

so $\ln(t) = \int_1^t \sum_{n=0}^{\infty} (-1)^n (x-1)^n \, dt$

$$= \sum_{n=0}^{\infty} \int_1^t (-1)^n (x-1)^n dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} (x-1)^{n+1}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (x-1)^n.$$

Alternatively we can compute $f^{(n)}(1)$ for each n and get the same result.

Now $R = \lim_{n\to\infty} |a_n|/|a_{n+1}| = \lim_{n\to\infty} n/n+1 = 1.$

25| $\sum_{m=2}^{\infty} m(m-1) a_m x^{m-2} + x \sum_{k=1}^{\infty} k a_k x^{k-1}$

$$= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{k=1}^{\infty} k a_k x^k$$

$$= 2a_2 + \sum_{n=1}^{\infty} \left[(n+2)(n+1) a_{n+2} + n a_n\right] x^n$$

$$= \sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} + n a_n \right] x^n$$

28| $\sum_{n=1}^{\infty} n a_n x^{n-1} + 2\sum_{n=0}^{\infty} a_n x^n = 0$

so $\sum_{n=0}^{\infty} \left((n+1) a_{n+1} + 2a_n \right) x^n = 0$

Thus for each n, we must have

$(n+1) a_{n+1} + 2a_n = 0$; so $a_{n+1} = \frac{-2a_n}{n+1}$.

So if $a_0 = 1$; then for $n \geq 1$; $a_n = \frac{(-2)^n}{n!}$.

So $\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{(-2)^n}{n!} x^n = \sum_{n=0}^{\infty} \frac{(-2x)^n}{n!} = e^{-2x}$.

SECTION 5.2

3| a) We set $y = \sum_{n=0}^{\infty} a_n (x-1)^n$ so

$$\sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2} - x\sum_{n=1}^{\infty} n a_n (x-1)^{n-1} - \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

so

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x-1)^n - x\sum_{n=0}^{\infty} (n+1) a_{n+1} (x-1)^n - \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

and thus since $x = (x-1)+1$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x-1)^n - \sum_{n=0}^{\infty} (n+1) a_{n+1} (x-1)^{n+1} - \sum_{n=0}^{\infty} (n+1) a_{n+1} (x-1)^n - \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

Therefore

$$(2a_2 - a_1 - a_0) + \sum_{n=1}^{\infty}\Big[(n+2)(n+1)a_{n+2} - n a_n - (n+1)a_{n+1} - a_n\Big](x-1)^n = 0$$

so $2a_2 = a_1 + a_0$ and

$$a_{n+2} = \frac{(n+1)(a_n + a_{n+1})}{(n+1)\ (n+2)} = \frac{a_n + a_{n+1}}{n+2}$$

b) Then if $a_0 = 1,\ a_1 = 0$ we get

$$a_2 = \frac{1}{2}\ ;\ a_3 = \frac{1}{3!}\ ;\ a_4 = \frac{4}{4!} = \frac{1}{3!}$$

and if $a_0 = 0$ and $a_1 = 1$ we get

$$a_2 = \frac{1}{2},\ a_3 = \frac{1}{2},\ a_4 = \frac{1}{4}$$

c) Now $W(y_1, y_2)(1) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$

so

$$y_1 \simeq 1 + \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3!} + \frac{(x-1)^4}{3!} + \cdots$$

$$y_2 \simeq x + \frac{(x-1)^2}{2} + \frac{(x-1)^3}{2} + \frac{(x-1)^4}{4} + \cdots$$

are linearly independent.

7| a) $y'' + xy' + 2y = 0 \quad x_0 = 0$

We have that

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{n=1}^{\infty} n a_n x^n + 2\sum_{n=0}^{\infty} a_n x^n = 0$$

so

$$2(a_2 + a_0) + \sum_{n=1}^{\infty} \left[(n+2)(n+1)a_{n+2} + (n+2)a_n\right] x^n = 0$$

Therefor

$a_2 = \frac{-a_0}{2}$ and $a_{n+2} = \frac{-a_n}{n+1}$

b) If $a_0 = 1$ and $a_1 = 0$ we get

$a_0 = 1;\ a_2 = -1;\ a_4 = \frac{1}{3};\ a_6 = \frac{-1}{3 \cdot 5}$

and if $a_0 = 0$ and $a_1 = 1$ we get

$a_0 = 0,\ a_1 = 1,\ a_3 = \frac{-1}{2};\ a_5 = \frac{1}{2 \cdot 4}$

$a_7 = \frac{-1}{2 \cdot 4 \cdot 6}$

c and d)

we have $W(y_1, y_2)(0) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$

so $y_1 = \sum_{n=0}^{\infty} \frac{(-1)^n 2^n n!}{(2n)!} x^{2n}$ and

$$y_2 = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} x^{2n+1}$$

are linearly independent.