HW 6 SOLUTION

SECTION 3.4

16| $y'' - y' + 0.25y = 0 \quad y(0) = 2 \quad y'(0) = b$

The characteristic polynomial is $p(r) = (r - 0.5)^2$ so the general solution is given by $y = c_1 e^{0.5t} + t c_2 e^{0.5t}$ so

$$2 = c_1$$

and

$$b = 0.5c_1 + c_2$$

thus $c_1 = 2$ and $c_2 = b - 1$

Therefore the solution is given by

$$y = 2e^{0.5t} + t(b-1)e^{0.5t}$$

so if $b \geq 1$ the solution is increasing and if $b < 1$ the solution is eventually decreasing.

20| a) Given the differential equation $y'' + 2ay' + a^2 y = 0$ we have that its characteristic polynomial is $p(r) = (r + a)^2$; so $y = e^{-at}$ is a solution to this equation.

b) By Abel's formula we know that $W(y_1, y_2)(t) = ce^{-\int 2a\,dt} = ce^{-2at}$

c) From b) we have that

$$e^{-at}y' + ae^{-at}y = ce^{-2at}$$

so $y' + ay = ce^{-at}$

and hence $\frac{d}{dt}(e^{at}y) = c$

Therefore $e^{at}y = ct$ and

hence $y = cte^{-at}$

<u>SECTION 3.5</u>

20) $y'' + 2y' + 5y = 4e^{-t}\cos 2t \quad y(0) = 1$

$y'(0) = 0$

First we find the general solution for the associated homogeneous equation, whose characteristic polynomial is $P(r) = r^2 + 2r + 5r = (r+1+2i)(r+1-2i)$

so the solution to this homogeneous problem is $y = c_1e^{-t}\cos 2t + c_2e^{-t}\sin 2t$

Now we apply the method of undetermined coefficients to find the particular solution y_p. We know y_p is of the form $y_p = Ate^{-t}\cos 2t + Bte^{-t}\sin 2t$

$$\begin{aligned} y_p' &= Ae^{-t}\cos 2t + Be^{-t}\sin 2t \\ &\quad - Ate^{-t}\cos 2t - Bte^{-t}\sin 2t \\ &\quad + 2Bte^{-t}\cos 2t - 2Ate^{-t}\sin 2t \\ &= Ae^{-t}\cos 2t + Be^{-t}\sin 2t \\ &\quad + (2B-A)te^{-t}\cos 2t + (-2A-B)te^{-t}\sin 2t \ldots \end{aligned}$$

$$y_p'' = -Ae^{-t}\cos 2t - Be^{-t}\sin 2t$$
$$+2Be^{-t}\cos 2t - 2Ae^{-t}\sin 2t$$
$$+(2B-A)e^{-t}\cos 2t + (-2A-B)e^{-t}\sin 2t$$
$$-(2B-A)te^{-t}\cos 2t - (-2A-B)te^{-t}\sin 2t$$
$$-(2B-A)e^{-t}\sin 2t \cdot 2t + (-2A-B)2te^{-t}\cos 2t$$
$$= (4B-2A)e^{-t}\cos 2t + (-4A-2B)e^{-t}\sin 2t$$
$$(-3A-4B)te^{-t}\cos 2t + (-3B+4A)te^{-t}\sin 2t$$

so $(-3A-4B) + (4B-2A) + 5A = 0$

$-3B + 4A + (-4A-2B) + 5B = 0$

$4B - 2A + 2A = 4$

$-4A - 2B + 2B = 0$

so $A = 0$ and $B = 1$

Hence $y_p = te^{-t}\sin 2t$ and

$$y = c_1 e^{-t}\cos 2t + c_2 e^{-t}\sin 2t + te^{-t}\sin 2t$$

Now $1 = y(0) = c_1$

$$y' = -e^{-t}\cos 2t - 2e^{-t}\sin 2t$$
$$- c_2 e^{-t}\sin 2t + c_2 e^{-t}2\cos 2t$$
$$+ e^{-t}\sin 2t - te^{-t}\sin 2t$$
$$+ 2e^{-t}t\cos 2t$$

$0 = y'(0) = -1 + 2c_2$ so $c_2 = 1/2$

Hence $y = e^{-t}\cos 2t + \frac{1}{2}e^{-t}\sin 2t + te^{-t}\sin 2t$

29) a) $y(t) = ve^{-t}$

$y'(t) = v'e^{-t} - e^{-t}v$

$y''(t) = v''e^{-t} - 2e^{-t}v' + e^{-t}v$

so $v''e^{-t} - 2e^{-t}v' + e^{-t}v - 3v'e^{-t} + 3e^{-t}v$
$- 4ve^{-t} = 2e^{-t}$

and $v''e^{-t} - 5e^{-t}v' = 2e^{-t}$, thus
$v'' - 5v' = 2$

b) Consider $w(t) = v'(t)$; $w'(t) = v''(t)$ so
$w'(t) - 5w = 2$; so $e^{-5t}w = -\frac{2}{5}e^{-5t}$

so $w = -2/5$

c) Now $\int w\,dt = -\frac{2}{5}t = v(t)$ so
if we let $y_p = -\frac{2}{5}te^{-t}$ we have
from a) that y_p is the particular
solution of $y'' - 3y' - 4y = 2e^{-t}$

so $y = -\frac{2}{5}e^{-t}t + c_1e^{-t} + c_2e^{4t}$

is the general solution for the
given problem.

SECTION 3.6:

3] $y'' + 2y' + y = 3e^{-t}$ (*)

First note that the characteristic polynomial of $y'' + 2y' + y = 0$ is $(r+1)^2$; so the general solution for $y'' + 2y' + y = 0$ is $y = c_1 e^{-t} + c_2 te^{-t}$

Now we see that $W(e^{-t}, te^{-t}) =$

$$e^{-2t} - te^{-2t} + te^{-2t} = e^{-2t}$$

We compute the particular solution

$$Y_p = -e^{-t}\int \frac{3e^{-s}se^{-s}}{e^{-2s}}\,ds + te^{-t}\int \frac{3e^{-s}e^{-s}}{e^{-2s}}\,ds$$

$$= -e^{-t}\frac{3}{2}t^2 + te^{-t}3t = \frac{3}{2}t^2e^{-t}$$

Hence the general solution to (*) is

$$y = \frac{3}{2}t^2e^{-t} + c_1e^{-t} + c_2te^{-t}$$

Finally we will find Y_p using undetermined coefficients. For this propose $Y_p = At^2e^{-t}$. Then $Y_p' = 2Ate^{-t} - At^2e^{-t}$ and $Y_p'' = 2Ae^{-t} - 4Ate^{-t} + At^2e^{-t}$; so

$(2A-2A)t^2e^{-t} = 0$, $(4A-4A)te^{-t} = 0$ and $2Ae^{-t} = 3e^{-t}$; so $A = \frac{3}{2}$ as needed.

13) $t^2y''-2y=3t^2-1 \quad t>0$

$y_1(t)=t^2; \quad y_2(t)=t^{-1}$

Observe that we can apply the method of variation of parameters on $(0,\infty)$.

Now we check y_1, y_2 are solutions of $t^2y''-2y=0$

$y_1=t^2 \qquad y_2=t^{-1}$

$y_1'=2t \qquad y_2'=-t^{-2}$

$y_1''=2 \qquad y''=+2t^{-3}$

so $t^2y_1''-2y_1=2t^2-2t^2=0$ and

$t^2y_2''-2y_2=\frac{2}{t}-\frac{2}{t}=0.$

Now we compute $W(y_1(t),y_2(t))=-1-2=-3$

so $y_p=-t^2\int\frac{(3-1/s^2)/s}{-3}ds+\frac{1}{t}\int\frac{3s^2-1}{-3}ds$

$=t^2\ln t+\frac{1}{6}-\frac{t^2}{3}+\frac{1}{3}$

$=t^2\ln(t)-\frac{t^2}{3}+\frac{1}{2}$

19) $(1-x)y'' + xy' - y = g(x) \quad 0 < x < 1$

$y_1(x) = e^x; \; y_2(x) = x.$

First we show that y_1, y_2 are solutions of $(1-x)y'' + xy' - y = 0$

$$y_1 = e^x \qquad y_2 = x$$
$$y_1' = e^x \qquad y_2' = 1$$
$$y'' = e^x \qquad y_2'' = 0$$

so since

$$(1-x)e^x + xe^x - e^x = 0$$

and

$$(1-x)\cdot 0 + x\cdot 1 - x = 0$$

we are done.

Now we compute $W(y_1, y_2)(t) = e^x - xe^x$

so

$$y_p = -e^x \int_{1/2}^{x} \frac{g(s)/(1-s)}{e^s(1-s)} s\,ds + x \int_{1/2}^{x} \frac{g(s)/(1-s)}{e^s(1-s)} e^s\,ds$$

$$= -e^x \int_{1/2}^{x} \frac{s g(s)}{e^s(1-s)^2}\,ds + x \int_{1/2}^{x} \frac{g(s)}{(1-s)^2}\,ds$$