# HW 7 SOLUTIONS

Worksheet problems:

1) a) We know that $(D-3)e^{3t}=0$ and $(D-5)e^{5t}=0$; so $(D-5)(D-3)(e^{3t}+e^{5t})=0$ hence the operator is $(D-3)(D-5)$

b) as in a) $(D-i)e^{it}=0$ and $(D+i)e^{-it}=0$; so $(D-i)(D+i)\left(\frac{e^{it}+e^{-it}}{2i}\right)=0$ and hence by Euler's formula $(D^2+1)\sin t=0$.

The operator is $D^2+1$.

c) $(D-1)e^t=0$ and $(D-1)(te^t)=e^t$, so $(D-1)^2(te^t)=0$ and hence by linearity $(D-1)^2(e^t+te^t)=0$.

$(D-1)^2$ is the operator.

d) As in b), since $\cos t=\frac{e^{it}+e^{-it}}{2}$; $e^t\cos t=\frac{e^{it+t}+e^{-it+t}}{2}$; so

$$(D-(1+i))(D-(1-i))(e^t\cos t)=0$$

so as in c)

$$(D-(1+i))(D-(1-i))(te^t\cos t)$$
$$=(D-(1+i))\left[e^t\cos t+tD(e^t\cos t)-(1-i)(te^t\cos t)\right]$$

$D(e^t\cos t) - (1+i)e^t\cos t + tD^2(e^t\cos t) + D(e^t\cos t)$
$- (1+i)(tD(e^t\cos t)) - (1-i)e^t\cos t - (1-i)tD(e^t\cos t)$
$+ 2te^t\cos t$
so by rearranging terms and noting that $(D^2-2D+2)(e^t\cos t) = 0$ we get

$$= t(D^2(e^t\cos t) - 2D(e^t\cos t) + e^t\cos t)$$
$$+ 2D(e^t\cos t) - 2(e^t\cos t) = D^2(e^t\cos t)$$

so $(D-(1+i))^2(D-(1-i))^2$ is the desired operator. (We showed
$(D-(1+i))^2(D-(1-i))^2 = (D-(1+i))(D-(1-i))D^2e^t\cos t$
$= D^2(D-(1+i))(D-(1-i))(e^t\cos t) = 0$.)

2) a) $y'' = -9\sin 3t$

b) $y'' + 9y = 0$

c) $D^2 + 9 = (D-3i)(D+3i)$

so $(D-3i)(D+3i)y = 0$.

3) a) $(D-1)^2(D-i)(D+i)y = 0$; so
$y = c_1e^t + c_2te^t + c_3\cos t + c_4\sin t$

b) $(D-(1+3i))(D-(1-3i))(D-i)(D+i)(D+2)^2 y = 0$
so $y = c_1e^t\sin 3t + c_2e^t\cos 3t + c_3\sin t + c_4\cos t$
$+ c_5e^{-2t} + c_6te^{-2t}$.

## Section 3-8

5) We need to substitute on equation (7).

We have $m = \frac{4}{32} = \frac{1}{8}$; $K = 4/(1.5) \cdot 12$

$= 32$

so

$$\frac{1}{8}u'' + 32u = 2\cos 3t, \ u(0) = \frac{1}{6}; \ u'(0) = 0.$$

11) We have $K = \frac{8 \cdot 12}{6} = 16$, $\gamma = 0.25$ and $m = \frac{1}{4}$ the external force is $4\cos 2t$

a) we consider $\frac{1}{4}u'' + \frac{1}{4}u' + 16u = 4\cos 2t$, and look for the particular solution of this equation by leting

$u = A\cos 2t + B\sin 2t$ so that

$u' = -2A\sin 2t + 2B\cos 2t$ and

$u'' = -4A\cos 2t - 4B\sin 2t$

so $16A + B/2 - A = 4$ and

$16B - A/2 - B = 0$

and hence $30B = A$; so $450.5B = 4$

so $B = \frac{4}{450.5}$ and $A = \frac{120}{450.5}$

so the steady state response is

# Section 3.7

7) We have by substituting

$K = 3\,lb/\tfrac{1}{4}ft = 12$; $m = 3/32$ in equation 7, chapter 3.7 that

$$\frac{3}{32}u'' + 12u = 0$$

with $u(0) = -1$ and $u'(0) = 2$.

Hence $u = c_1 \cos\sqrt{128}\,t + c_2 \sin\sqrt{128}\,t$

so $c_1 = -1/12$ and $c_2 = \dfrac{2}{\sqrt{128}} = \dfrac{1}{4\sqrt{2}}$

so $u = \dfrac{-1}{12}\cos\sqrt{128}\,t + \dfrac{1}{4\sqrt{2}}\sin\sqrt{128}\,t$.

so frequency $= \sqrt{128}$

Period $= 2\pi/\sqrt{128} = \pi/\sqrt{32}$

Amplitude $= \sqrt{1/144 + 1/32} = \sqrt{11/288}$

Phase $= \tan^{-1}\left(\dfrac{1/4\sqrt{2}}{-1/12}\right) = \tan^{-1} -3/\sqrt{2}$

on the second quadrant.

$\boxed{17}$ We need to find $\gamma = 2\sqrt{KM}$

so $\gamma = 2\sqrt{8/(1.5/12)\; 8/32} = 2\sqrt{16} = 8$

so $\gamma = 8\ \dfrac{lb \cdot s}{ft}$

$$U = \frac{240}{901}\cos 2t + \frac{8}{901}\sin 2t.$$

b) We fix $k=16$ and $\gamma = 0.25$ and observe that

$$R = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}}$$

$$= \frac{4}{\sqrt{m^2\left(\frac{256}{m^2} - 16\right)^2 + 1}}$$

$$= \frac{4}{\sqrt{\frac{(256-16m^2)^2}{m^2} + 1}} \qquad \text{(Eq 11, pg 224)}$$

So $R$ is maximized whenever $256 - 16m^2 = 0$; that is when $m = 4$ $(m \geq 0.)$.