HW 8 Solutions

SECTION 6.1

5) a) First we apply theorem 6.1.2 to $f(t) = t$: We note that if we fix $a > 0$, then $t \leq e^{at}$ for $t \geq t_0$ for some t_0 that depends on a. So $\mathcal{L}(f)$ exists for $S > 0$. (f is continuous in this case.)

Now we compute $\mathcal{L}(f)$:

$$\mathcal{L}(f)(S) = F(S) = \int_0^\infty t e^{-st}\,dt = \frac{te^{-st}}{-s}\Big|_0^\infty + \int_0^\infty \frac{e^{-st}}{s}\,dt$$

$$= \int_0^\infty \frac{e^{-st}}{s}\,dt = -\frac{e^{-st}}{s^2}\Big|_0^\infty = \frac{1}{s^2}$$

So $F(S) = \frac{1}{S^2}$ for $S > 0$.

b) As in a) we can use theorem 6.1.2 to conclude that $\mathcal{L}(t^2)$ exists and is defined for $S > 0$. Now for $S > 0$ we have;

$$F(S) = \mathcal{L}(t^2)(S) = \int_0^\infty t^2 e^{-st}\,dt = \frac{t^2 e^{-st}}{-s}\Big|_0^\infty + \frac{2}{S}\int_0^\infty t e^{-st}\,dt$$

$$= \frac{2}{S}\mathcal{L}(t)(S) = \frac{2}{S^3}.$$

c) Finally suppose that for some n, $\mathcal{L}(t^n)(s) = G(S) = \frac{n!}{S^{n+1}}$ for $S > 0$ and note that as in b); $\mathcal{L}(t^{n+1})(S) = F(S)$ is defined for $S > 0$.

Now for $s > 0$; $\mathcal{L}(t^{n+1})(S) = \int_0^\infty t^{n+1} e^{-st}\,dt = \frac{t^{n+1}e^{-st}}{-s}\Big|_0^\infty + \frac{n+1}{S}\int_0^\infty t^n e^{-st}\,dt$

$= \frac{n+1}{s}\mathcal{L}(t^n)(s) = \frac{n+1}{s}\cdot\frac{n!}{s^{n+1}} = \frac{(n+1)!}{s^{n+2}}$

So by a) we know that if $n=1$; $\mathcal{L}(t)(s) = \frac{1}{s^2}$ and by the previouse argument we have that if for some n, $\mathcal{L}(t^n)(s) = \frac{n!}{s^{n+1}}$; then $\mathcal{L}(t^{n+1}) = \frac{(n+1)!}{s^{n+2}}$; so for all n,

$$F(s) = \mathcal{L}(t^n)(s) = \frac{n!}{s^{n+1}} \quad \text{for } s>0.$$

9) $f(t) = e^{at}\cosh bt = \left(e^{(b+a)t} + e^{-(b-a)t}\right)/2$

We use the linearity of $\mathcal{L}$ and observe that the domain of the transform of a sum is the intersection of the domains of each of the summands; hence by example 5 and equation (6) we have

$$\mathcal{L}(f(t))(s) = \frac{1}{2}\mathcal{L}(e^{(b+a)t})(s) + \frac{1}{2}\mathcal{L}(e^{-(b-a)t})(s)$$

$$= \frac{1}{2}\left[\frac{1}{s-b-a} + \frac{1}{s+b-a}\right] \text{ for } s > \max\{a+b, a-b\}$$

so $\mathcal{L}(f(t))(s) = \frac{s-a}{s^2-2as+a^2-b^2} = \frac{s-a}{(s-a)^2-b^2}$

for $s > a+b$ and $s > a-b$; (which is the same as $s-a>b$ and $s-a>-b$ and hence it is the same as $s-a>|b|$.)

13) $f(t) = e^{at}\sin bt$

First we apply theorem 6.2.1 to observe that since $|e^{at}\sin bt| \le e^{at} \le e^{at+\tilde{s}t}$ for $\tilde{s}>0$; we have that $\mathcal{L}(f(t))(s)$ exists for $s>a$.

Now the instructions say that we can use the formal properties of the integral (meaning that we only care about the form and not the meaning.) to obtain as in 9) by the linearity of $\mathcal{L}$ that for $s>a$;

$$\mathcal{L}(f(t))(s) = \frac{1}{2i}\mathcal{L}\left(e^{(ib+a)t}\right) - \frac{1}{2i}\mathcal{L}\left(e^{(-ib+a)t}\right)$$

$$= \frac{1}{2i}\left[\frac{1}{s-ib-a} - \frac{1}{s+ib-a}\right]$$

$$= \frac{1}{2i}\left[\frac{2ib}{(s-a)^2+(ib)^2}\right] = \frac{b}{(s-a)^2+b^2}$$

SECTION 6.2

2) $\frac{4}{(s-1)^3} = F(s)$

We know $\mathcal{L}(e^t g(t)) = G(s-1)$ whenever $\mathcal{L}(g(t)) = G(s)$, so since $\mathcal{L}(t^2) = \frac{2}{s^3}$ we have $\mathcal{L}(e^t 2t^2) = \frac{4}{(s-1)^3}$; so $\mathcal{L}^{-1}(F(s)) = 2e^t t^2$.

9] $F(s) = \dfrac{1-2s}{s^2+4s+5} = \dfrac{1-2s}{(s+2)^2+1}$

We use lines 9 and 10 of table 6.2.1. By letting $a=-2$ and $b=1$ we see that $\mathcal{L}(e^{at}\sin bt) = \dfrac{1}{(s+2)^2+1}$ and

$$\mathcal{L}(e^{at}\cos bt) = \frac{s+2}{(s+2)^2+1}$$

so $\mathcal{L}^{-1}(F(s)) = \mathcal{L}^{-1}\left[\dfrac{1+4}{(s+2)^2+1}\right] - 2\mathcal{L}^{-1}\left[\dfrac{s+2}{(s+2)^2+1}\right]$

$$= 5e^{-2t}\sin t - 2e^{-2t}\cos t$$

14] $y'' - 4y' + 4y = 0$; $y(0)=1$ $y'(0)=1$

First we use line 18 of table 6.2.1 and apply $\mathcal{L}$ to both sides of the equation. We get by leting $\mathcal{L}(y) = F(s)$,

$$s^2F(s) - sy(0) - y'(0) - 4sF(s) + 4f(0) + 4F(s) = 0$$

so $(s^2-4s+4)F(s) = s-3$ and hence

$$F(s) = \frac{s-3}{(s-2)^2} = \frac{1}{s-2} - \frac{1}{(s-2)^2}$$

Thus, by using lines 3 and 14 of table 6.2.1 we get $\mathcal{L}^{-1}(F(s)) = \mathcal{L}^{-1}\left(\frac{1}{s-2}\right) - \mathcal{L}^{-1}\left(\frac{1}{(s-2)^2}\right) = e^{2t}\mathcal{L}^{-1}\left(\frac{1}{s}\right) - e^{2t}\mathcal{L}^{-1}\left(\frac{1}{s^2}\right) = e^{2t} - te^{2t}$.

19] $y^{(4)} - 4y = 0$ $y(0)=1$ $y'(0)=0$
$y''(0) = -2$ $y^{(3)}(0) = 0$

We apply $\mathcal{L}$ on both sides of the equation

$$s^4F(s) - s^3y(0) - s^2y'(0) - sy''(0) - y^{(3)}(0) - 4F(s) = 0$$

so $(s^4-4)F(s) = s^3 - 2s$ and thus

$$F(s) = \frac{s(s^2-2)}{s^4-4} = \frac{s(s^2-2)}{(s^2+2)(s^2-2)} = \frac{s}{s^2+2}$$

So by line 6 we get that
$y = \cos\sqrt{2}t$.