HW 9 Solutions

Section 6.3

13| By theorem 6.1.2; $\mathcal{L}(f)(s)$ is defined for all $s>0$. so we fix $s>0$ and compute

$$\mathcal{L}(f)(s) = \int_0^2 f(t)e^{-st}\,dt + \int_2^\infty (t-2)^2 e^{-st}\,dt$$

$$= \int_0^2 0\,e^{-st}\,dt + \int_2^\infty (t-2)^2 e^{-st}\,dt = \int_2^\infty (t-2)^2 e^{-st}\,dt$$

so by substituting $t-2$ by u (so $dt = du$) we get

$$\mathcal{L}(f)(s) = \int_0^\infty u^2 e^{-su} e^{-2s}\,du = e^{-2s}\mathcal{L}(u^2)(s)$$

$$= \frac{2e^{-2s}}{s^3}$$

14| We will use table 6.2.1 to find $\mathcal{L}(f)$ where $f(t) = \begin{cases} 0 & t<1 \\ t^2-2t+2 & t \geq 1 \end{cases}$.

First observe that $f(t) = g(t) + h(t)$ where $g(t) = \begin{cases} 0 & t<1 \\ (t-1)^2 & t \geq 1 \end{cases}$ and $h(t) = \begin{cases} 0 & t<1 \\ 1 & t \geq 1 \end{cases}$

so since $g(t) = u_1(t)(t-1)^2$ and $h(t) = u_1(t)$ we have

$$\mathcal{L}(f)(s) = \mathcal{L}(g)(s) + \mathcal{L}(h(s)) = \frac{2e^{-s}}{s^3} + \frac{e^{-s}}{s}$$

for $s>0$ (Use lines 13 and 3 of table 6.2.1.)

19 $F(S) = \dfrac{3!}{(S-2)^4}$

To find $\mathcal{L}^{-1}(F)$ we use table 6.2.1 and apply line 14 and the fact that $\mathcal{L}(t^3)(S) = \dfrac{3!}{S^4}$ to get that $\mathcal{L}(e^{2t}t^3) = \dfrac{3!}{(S-2)^4}$; so $\mathcal{L}^{-1}(F(S)) = e^{2t}t^3$

SECTION 6.4:

21 $h(t) = U_\pi(t) - U_{2\pi}(t)$ and its graph so by taking $\mathcal{L}$ in both sides of the equation $y'' + 2y' + 2y = h(t)$ $y(0)=0$; $y'(0)=1$

we get $S^2F(S) - Sy(0) - y'(0) + 2SF(S) - 2y(0)$
$+ 2F(S) = \dfrac{e^{-\pi S}}{S} - \dfrac{e^{-2\pi S}}{S}$

thus
$$(S^2+2S+2)F(S) = \frac{e^{-\pi S}}{S} - \frac{e^{-2\pi S}}{S} + 1$$
$$F(S) = (e^{-\pi S} - e^{-2\pi S})\frac{1}{S(S^2+2S+2)} + \frac{1}{S^2+2S+2}$$

Now we use partial fractions to write
$$\frac{1}{S(S^2+2S+2)} = \frac{A}{S} + \frac{BS+C}{S^2+2S+2}$$
First we multiply both sides by S and let $S=0$ to get $A = 1/2$; so
$$\frac{-S^2/2 - S}{S(S^2+2S+2)} = \frac{BS+C}{S^2+2S+2}$$; so $B = -1/2$ and $C=-1$

so $$\frac{1}{S(S^2+2S+2)} = \frac{1/2}{S} - \frac{1}{2}\,\frac{S+2}{S^2+2S+2}$$

Thus

$$F(s) = \left(e^{-\pi s} - e^{-2\pi s}\right)\left[\frac{1/2}{s} - \frac{1}{2}\frac{(s+1)}{(s+1)^2+1} - \frac{1}{2}\frac{1}{(s+1)^2+1}\right]$$

$$+ \frac{1}{(s+1)^2+1}$$

so

$$Y = \frac{1}{2}\Big[U_\pi(t) - U_{2\pi}(t) - U_\pi(t)e^{-(t-\pi)}\cos(t-\pi)$$

$$+ U_{2\pi}(t)e^{-(t-2\pi)}\cos(t-2\pi)$$

$$- U_\pi(t)e^{-(t-\pi)}\sin(t-\pi)$$

$$+ U_{2\pi}(t)e^{-(t-2\pi)}\sin(t-2\pi)\Big] + e^{-t}\sin(t)$$

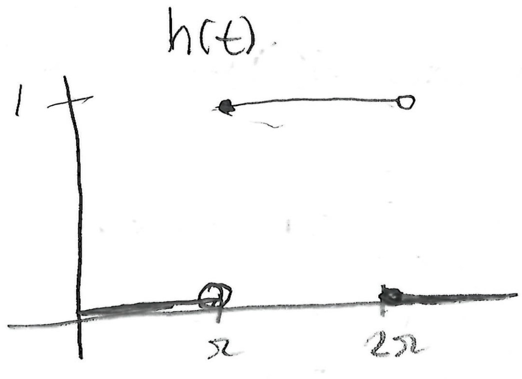
and hence (use $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \sin\beta\cos\alpha$
$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

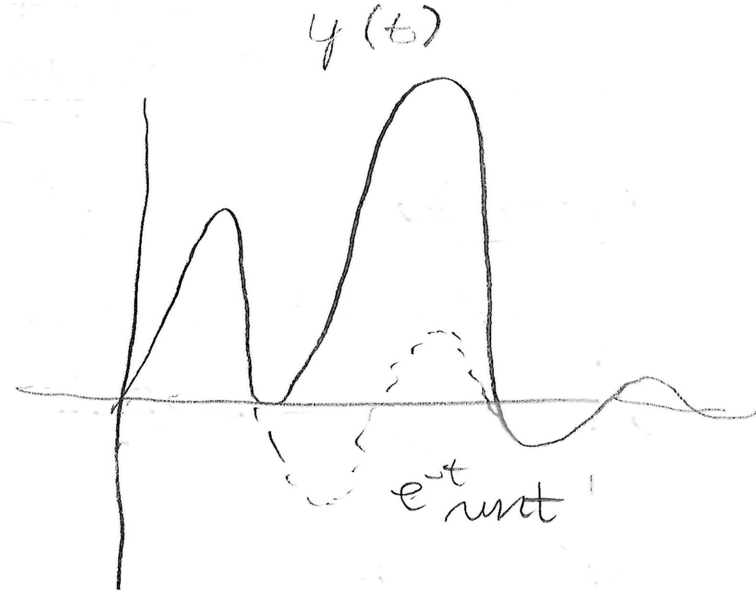
$$Y = \frac{1}{2}U_\pi(t)\left[1 + e^{-(t-\pi)}\cos t + e^{-(t-\pi)}\sin(t)\right]$$

$$+ \frac{1}{2}U_{2\pi}(t)\left[-1 + e^{-(t-2\pi)}\cos t + e^{-(t-2\pi)}\sin(t)\right]$$

$$+ e^{-t}\sin t$$

Graphs:





q) $y'' + y = g(t)$ $\quad y(0) = 0 \quad y'(0) = 1$

$$g(t) = \begin{cases} t/2 & 0 \le t < 6 \\ 3 & t \ge 6 \end{cases}$$

$$s^2F(s) - sy(0) - y'(0) + F(s) = \frac{1}{2s^2} - \frac{e^{-6s}}{2s^2}$$

because $g(t) = t/2 - U_6(t)\left(\frac{t-6}{2}\right)$

so $F(s) = \frac{1}{2s^2}\,\frac{1}{s^2+1}\left(1 - e^{-6s}\right) + \frac{1}{s^2+1}$

Now note that $\frac{1}{2s^2}\,\frac{1}{s^2+1} = \frac{1}{2}\left[\frac{1}{s^2} - \frac{1}{s^2+1}\right]$

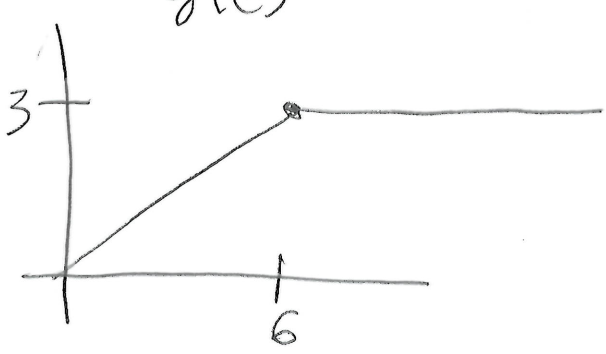
(Use partial fractions or see the similarity with $\frac{1}{x(x+1)}$.)

so $F(s) = \frac{1}{2}\,\frac{1-e^{-6s}}{s^2} - \frac{1(1-e^{-6s})}{2(s^2+1)} + \frac{1}{s^2+1}$

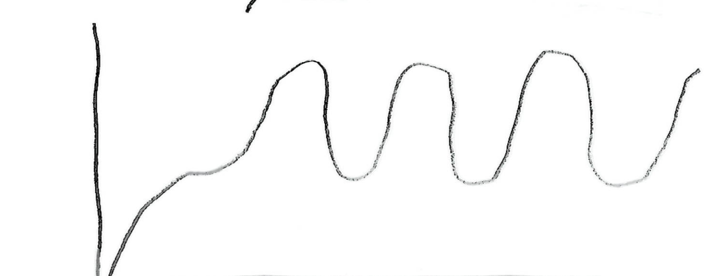
so $y = \frac{1}{2}t - U_6(t)\frac{(t-6)}{2} - \frac{\sin(t)}{2}$
$+ \frac{U_6(t)\sin(t-6)}{2} + \sin(t).$

Graphs:

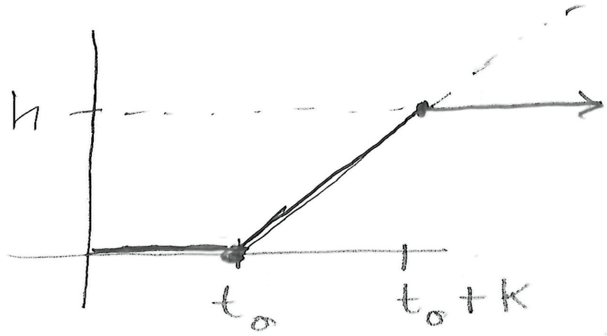
g(t)



y(t)



14



$$y = U_{t_0}(t)\left(\frac{h}{k}(t-t_0)\right) - U_{t_0+k}(t)\left(\frac{h}{k}(t-t_0-k)\right)$$