Rudin Ch. 2 # 22 – 26

#22 A metric space is <u>separable</u> if it contains a countable dense subset. Show that $\mathbb{R}^k$ is separable.

<u>Sol.</u> Let $\mathbb{Q}^k := \{(x_1, \dots, x_k) \in \mathbb{R}^k : x_i \in \mathbb{Q} \ \forall i\}$.

Then $\mathbb{Q}^k$ is countable (Thm 2.13).

We claim that $\mathbb{Q}^k$ is dense.

Let $(x_1, \dots, x_k) \in \mathbb{R}^k \setminus \mathbb{Q}^k$ and $r > 0$ be given.

For each $i = 1, \dots, k$, choose $p_i \in \mathbb{Q}$ s.t. $0 < |x_i - p_i| < r/\sqrt{k}$. then $(p_1, \dots, p_k) \in \mathbb{Q}^k$ and

$$|(x_1, \dots, x_k) - (p_1, \dots, p_k)| = \sqrt{\sum_{i=1}^{k} (x_i - p_i)^2} < r$$

i.e. $(p_1, \dots, p_k) \in N_r(x) \setminus \{x\}$.

This shows that x is a limit pt of $\mathbb{Q}^k$.

Since $x \in \mathbb{R}^k$ is arbitrary, $\mathbb{Q}^k$ is dense.

As $\mathbb{Q}^k$ is a countable, dense subset of $\mathbb{R}^k$, $\mathbb{R}^k$ is separable.

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#23 A coll'n $\{V_\alpha\}$ of open subsets of X is said to be a base for X if: For every $x \in X$ any every open set $G \subset X$ s.t. $x \in G$, $\exists\, \alpha$ s.t. $x \in V_\alpha \subset G$. Prove that every separable metric space has a countable base.

Sol Let X be a separable metric space.
Let $A \subset X$ be a countable dense subset.
For each $a \in A$, $n \in \mathbb{N}$, define

$$V_{a,n} = N_{\frac{1}{n}}(a).$$

We claim that $\{V_{a,n}\}_{\substack{a\in A\\ n\in\mathbb{N}}}$ is a countable base.
First it is countable by Thm 2.13.
Second for each $x \in X$, and open set $G \ni x$,

① $\exists\, n_1 \in \mathbb{N}$ s.t. $x \in N_{\frac{2}{n_1}}(x) \subset G$.

② By density of A, either (i) $x \in A$ or
(ii) given n_1 as above, $\exists\, a \in A$ such that
$a \in N_{1/n_1}(x) \setminus \{x\}$.

③ If (i) holds, then $x \in V_{x,n_1}(x) \subset G$.

If (ii) holds, then $x \in V_{a,n_1} \subset N_{\frac{2}{n_1}}(x) \subset G$

This shows that $\{V_{a,n}\}$ is a countable base.

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#24 Let X be a metric space in which every infinite subset has a limit point. Prove that X is separable.

Sol. Claim For each $\delta > 0$, $\exists$ a finite subset $\{x_1, \dots, x_N\}$ of X such that $X \subset \bigcup_{i=1}^{N} N_\delta(x_i)$.

To show the claim, take an arbitrary $x_1 \in X$.
Having chosen $x_1, \dots, x_k$ (for some $k \geq 1$),
if $X \setminus \bigcup_{i=1}^{k} N_\delta(x_i) = \phi$ then we are done.
Otherwise choose $x_{k+1} \in X \setminus \bigcup_{i=1}^{k} N_\delta(x_i)$.

We claim that this process must stop at some $k = N$ i.e. $\exists N$ s.t. $X \setminus \bigcup_{i=1}^{N} N_\delta(x_i) = \phi$.
Assume to the contrary, then there is a sequence $\{x_i\}$ such that, by construction,
$$d(x_i, x_j) \geq \delta \quad \text{if } i \neq j.$$
But then $\{x_i\}$ has no limit point,
as for any $x \in X$, $N_\delta(x)$ contains at most one point of $\{x_i\}$, so that by Thm 2.20, $\{x_i\}$ has no limit pt.

This contradicts the assumption that every infinite subset of X has a limit pt.
Hence, Claim is proved.

(Sol #24 cont.)

Now, take $\delta = \frac{1}{n}$, $n = 1, 2, 3$. We deduce that for each n, $\exists$ a finite subset A_n s.t.

$$X \subset \bigcup_{x \in A_n} N_{\frac{1}{n}}(x).$$

We claim that $A = \bigcup_{n \in \mathbb{N}} A_n$ is a countable dense subset of X.

① A is at most countable since it is a countable union of finite sets. (Thm 2.12)

② Let $x \in X \setminus A$.

then given $r > 0$, choose n s.t. $\frac{1}{n} < r$.

By def. of A_n, $\exists\ p \in A_n \subset A$

s.t. $x \in N_{\frac{1}{n}}(p) \Leftrightarrow p \in N_{\frac{1}{n}}(x) \subset N_r(x)$

Since r is arbitrary, x is thus a limit point of A.

This shows that A is dense.

Hence, X is separable.

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#26 Let X be a metric space in which every infinite subset has a limit pt. Prove that X is compact.

Sol By #24, X is separable.

By #23, X has a countable base $\{V_n\}$

Claim 1 Every open cover of X has a countable subcover.

Let $\{G_\alpha\}$ be an open cover of X

Then for each $x \in X$, $\exists \alpha$ s.t. $x \in G_\alpha$.

By property of base, $\exists n$ s.t. $x \in V_n \subset G_\alpha$.

i.e. (i) X is covered by a subset of $\{V_n\}$,

(ii) each such V_n is contained in some G_α

Here $\exists$ a countable set $\{\alpha_i\}$ such that

$$X \subset \bigcup_{i \in \mathbb{N}} G_{\alpha_i}.$$

If $\{G_{\alpha_i}\}$ has a finite subcover, then we are done.

Suppose to the contrary that $\forall n$,

$$F_n = (G_{\alpha_1} \cup \cdots \cup G_{\alpha_n})^c \neq \phi.$$

Choose $x_n \in F_n$.

By assumption, $\{x_n\}$ has a limit pt. $x \in X$.

and $x \in G_{\alpha_{n_0}}$ $\exists n_0$

By property of limit pt. $G_{\alpha_{n_0}}$ contains infinitely many points in x_n. In particular, $\exists n' > n_0$ s.t. $x_{n'} \in G_{\alpha_{n_0}}$

But this contradicts the fact that $x_{n'} \notin (G_{\alpha_1} \cup \cdots \cup G_{\alpha_{n'}})$

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Prop. Let K be compact, then every infinite subset of K has a limit pt.

pf If K is compact and $\{y_n\} \subset K$.

Suppose for contradiction that $\{y_n\}$ has no limit pt. Then for each $x \in X$, $\exists r_x > 0$ s.t. $N_{r_x}(x) \cap \{y_n\}$ is finite.

Now, $\{N_{r_x}(x)\}_{x \in K}$ is an open cover of K so that by compactness, $\exists x_1 \ldots x_m$ s.t.
$$\{y_n\} \subset K \subset \bigcup_{j=1}^{m} N_{r_{x_j}}(x_j).$$

But then the RHS contains only finitely many pts of $\{y_n\}$! This is a contradiction.

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