In this note we give a proof of the equivalence of the following two definitions of a connected set. Let E be a subset of a metric space X.

Definition 1. (i) *Two sets A, B are said to be separated if*

$$\bar{A} \cap B = \emptyset \quad \text{and} \quad A \cap \bar{B} = \emptyset.$$

(ii) *A set E is said to be connected if it is not the union of two sets A, B that are non-empty and separated.*

Proposition 1. *A set E is disconnected if and only if there exist disjoint open sets A_*, B_* such that*

$$A_* \cap E \neq \emptyset, \quad B_* \cap E \neq \emptyset, \quad \text{and} \quad E \subset (A_* \cup B_*). \tag{1}$$

Proof. Suppose there exist disjoint open sets A_*, B_* such that (1) holds. Define

$$A = A_* \cap E, \quad \text{and} \quad B = B_* \cap E,$$

then (1) says that A, B both non-empty. It remains to show that A, B are separated. We first show $\bar{A} \cap B = \emptyset$. Now, by Theorem 2.23 B^c is a closed set. Since B^c also contains A (by disjointness of A, B), Theorem 2.27(c) says that $\bar{A} \subset B^c$, i.e. $\bar{A} \cap B = \emptyset$. By symmetry, one can also argue that $A \cap \bar{B} = \emptyset$. Hence $E = A \cup B$, for some non-empty separated sets A, B. This proves that E is disconnected in the sense of (ii).

Conversely, suppose E is disconnected in the sense of (ii). Then there are non-empty separated sets A, B, such that $E = A \cup B$. Since $\bar{A} \cap B = \emptyset$ and $E = A \cup B$, we have $A = E \setminus (\bar{B})$ and $B = E \setminus (\bar{A})$. Then by Theorem 2.30, A, B are open sets relative to E, and that there are open sets A_1, B_1 relative to X such that $A = A_1 \cap E$ and $B = B_1 \cap E$. Next, define

$$A_* = A_1, \quad \text{and} \quad B_* = B_1 \setminus \overline{A_1}.$$

It remains to show that A_*, B_* are (i) open, (ii) disjoint, (iii) $E \subset (A_* \cup B_*)$, (iv) $E \cap A_* \neq \emptyset$, (iv) $E \cap B_* \neq \emptyset$.

For (i), $A_* = A_1$ is open, and that $B_* = B_1 \cap (\overline{A_1})^c$ is open (Theorem 2.24(c)). Claim (ii) follows from the fact that $B_* \subset (\overline{A_1})^c \subset (A_1)^c = (A_*)^c$. For (iii), observe that by definition of $A_* = A_1$, $A \subset A_*$, and that

$$A = E \cap A_1 \Rightarrow \bar{A} = \bar{E} \cap \bar{A}_1 \Rightarrow \bar{A}^c = \bar{E}^c \cup \bar{A}_1^c \Rightarrow E \cap \bar{A}^c = E \cap \bar{A}_1^c, \tag{2}$$

where we used Exercise 7(a) in the first implication, and Theorem 2.22 in the second implication. Using (2) in the third equality below, we have

$$B_* \supset B_* \cap E = (B_1 \cap \bar{A_1}^c) \cap E = (B_1 \cap E) \cap (E \cap \bar{A_1}^c) = B \cap (E \cap \bar{A}^c) = B.$$

i.e. $E = (A \cup B) \subset (A_* \cup B_*)$. This give (iii). For (iv), $A_* \cap E = A_1 \cap E = A \neq \emptyset$. And that by (2),

$$B_* \cap E = (B_1 \cap E) \cap (\bar{A_1}^c \cap E) = B \cap (E \cap \bar{A}^c) = B \neq \emptyset.$$

This gives (v) and finishes the proof. $\square$