Thm 2.27(a) If X is a metric space and $E \subset X$. Then $\overline{E} = \{ y \in X : ~~N \cap E~~ \; N_r(y) \cap E \neq \phi \quad \forall r > 0 \}$ is closed.

Pf. $\boxed{\underline{\text{Fact}} \quad x \in \overline{E}^c \text{ iff } N_r(x) \cap E = \phi \text{ for some } r > 0}$

~~It Assume~~ It remains to show that $\overline{E}^c$ is open.

Let $y \in \overline{E}^c$, then by Fact, there is r_1 s.t.

$$N_{r_1}(y) \cap E = \phi.$$

Claim. $N_{r_1}(y) \subset \overline{E}^c$.

To see the Claim, let $x \in N_{r_1}(y)$

then ~~h=N~~ $h = d(x,y) < r_1$

We have $N_{r_1-h}(x) \subset N_{r_1}(y)$

and $(N_{r_1-h}(x) \cap E) \subset (N_{r_1}(y) \cap E) \subset \phi$.

Hence $E \cap N_{r_1-h}(x) = \phi$ and we have found a neighd. of x that is disjoint from E.

$\overset{\text{By Fact}}{\Longrightarrow} x \in \overline{E}^c \quad \forall x \in N_{r_1}(y)$.

$\Rightarrow N_{r_1}(y) \subset \overline{E}^c$ This proves the Claim.

Hence, for each $y \in \overline{E}^c$, there is a neighd. of y s.t. $N_{r_1}(y) \subset \overline{E}^c$

This shows that y is an interior pt. of $\overline{E}^c$.

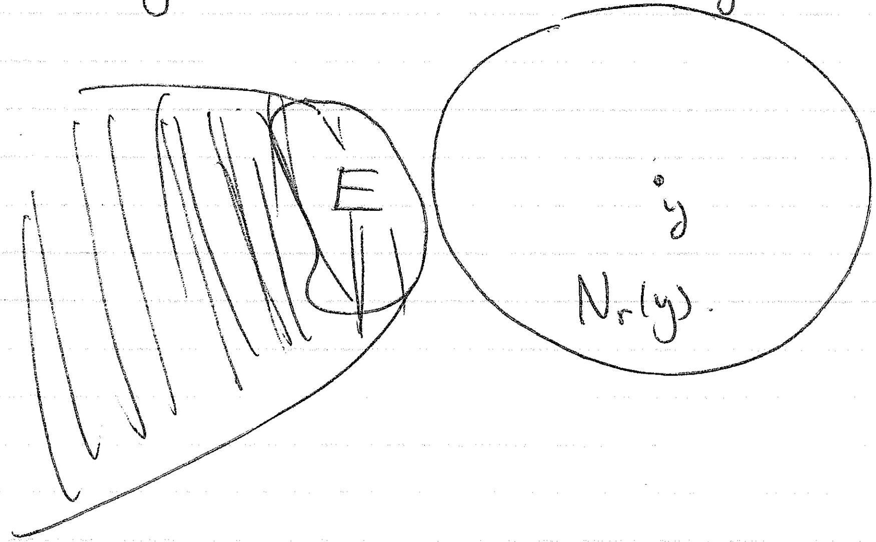
i.e. $\overline{E}^c$ is open

\#

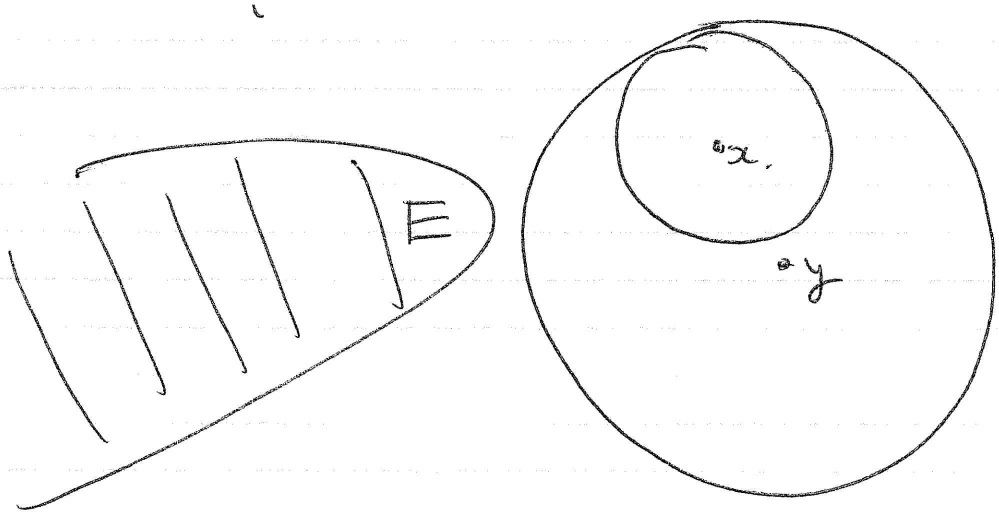
IDEA: Not a proof! If $y \in \bar{E}^c$

then $N_r(y) \cap E = \phi$ for some r.

i.e. y is "bounded away" from E.



But any $x \in N_r(y)$ is also "bounded away" from E!



Hence $x \in \bar{E}^c$ $\forall x \in N_r(y)$

and $\bar{E}^c$ is open.