Example 2 $S = \mathbb{Q}$, $E = \{x \in \mathbb{Q} : x^2 < 2\}$.

Then (a) E is bounded from above, but

(b) E does not have a ~~greatest~~ l.u.b. in $\mathbb{Q}$.

Proof (a) Claim ~~$x < 2 \quad \forall x \in E$~~.

If $x \in E$, then $x < 2$.

Suppose $x \geq 2$, then $x^2 \geq 2x \geq 2 \cdot 2 = 4$.

then $x \notin E$.

So the contrapositive statement holds,

i.e. $x \in E \Rightarrow x < 2$.

This shows the bounded from above of E.

(b) ~~We have already seen that $x^2 \neq 2 \quad \forall x \in \mathbb{Q}$~~.

Suppose to the contrary that $\alpha = \sup E \in \mathbb{Q}$ exists, Then either (i) $\alpha^2 > 2$ or (ii) $\alpha^2 < 2$.

(Since we have already seen that $x^2 \neq 2 \quad \forall x \in \mathbb{Q}$).

We will derive a contradiction for each case.

Case (i) $\alpha^2 > 2$,

Define $y = \frac{1}{2}(\alpha + \frac{2}{\alpha})$ (Note, α is an u.b. and $0 \in E$, so $\alpha > 0$)

then $0 < y < \alpha$ (as $y - \alpha = \frac{1}{2}(\frac{2}{\alpha} - \alpha) = \frac{1}{2\alpha}(2 - \alpha^2) < 0$)

and $y^2 \geq 2$ (as $y^2 = \left[\frac{1}{2}(\alpha + \frac{2}{\alpha})\right]^2 \geq \alpha \cdot \frac{2}{\alpha} = 2$ using $\left(\frac{a+b}{2}\right)^2 \geq ab \Leftrightarrow \left(\frac{a-b}{2}\right)^2 \geq 0$)

So $y^2 - x^2 > 0 \quad \forall x \in E$

$(y-x)(y+x) > 0 \quad \forall x \in E$

$y - x > 0 \quad \forall x \in E$ cancell a positive factor $y + x > 0$.

Hence y ~~is a~~ furnishes a smaller u.b. than α.

This contradicts the fact that α is the least u.b.

Case (ii) $\alpha^2 < 2$,

Then $\tilde{\alpha} = \frac{2}{\alpha}$ satisfies $\tilde{\alpha}^2 > 2$

Repeating previous argument, there exists $\tilde{y} \in \mathbb{Q}$ s.t.

$0 < \tilde{y} < \tilde{\alpha}$ and $\tilde{y}^2 > 2$

So $y = 2/\tilde{y}$ satisfies $y > \alpha$ and $y^2 < 2$

which means that α is not an upper bound!

Contradictions!

#.